ON SOME NEW EXAMPLES OF CAMERON-LIEBLER LINE CLASSES

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ABSTRACT

Cameron-Liebler line classes are sets of lines in $\mathrm{PG}(3,q)$ having many nice combinatorial properties; among them, a Cameron-Liebler line class $\mathcal{L}$ shares precisely x lines with any spread of the space for some non-negative integer x, called the parameter of the set. These objects were originally studied as generalizations of symmetric tactical decompositions of $\mathrm{PG}(3,q)$, as well as of subgroups of $\mathrm{P\Gamma L}(4,q)$ having equally many orbits on points and lines of $\mathrm{PG}(3,q)$. They have connections to many other combinatorial objects, including blocking sets in $\mathrm{PG}(2,q)$, certain error-correcting codes, and strongly regular graphs.

We construct many new examples of Cameron-Liebler line classes, each stabilized by a cyclic group of order q^2+q+1 having a semi-regular action on the lines. In particular, new examples are constructed in $\mathrm{PG}(3,q)$ having parameter $\frac{1}{2}(q^2-1)$ for all values of $q \equiv 5$ or $9 \bmod 12$ with $q < 200$; with parameter $\frac{1}{3}(q+1)^2$ (found in collaboration with Jan de Beule, Klaus Metsch, and Jeroen Demeyer) for all values of $q \equiv 2 \bmod 3$ with $2 < q \leq 128$; with parameter 336 in $\mathrm{PG}(3,27)$; and with parameter 495 in $\mathrm{PG}(3,32)$. The new examples with parameter $\frac{1}{2}(q^2-1)$ when $q \equiv 9 \bmod 12$ are of particular interest. These induce a symmetric tactical decomposition of $\mathrm{PG}(3,q)$ having four classes on points and lines, one of the line classes being the set of lines in a common plane. This decomposition can be used to construct examples of two-intersection sets in the affine plane $\mathrm{AG}(2,q)$. Since the only previously known examples of two-intersection sets in affine planes of odd order are in planes of order 9, our example in $\mathrm{AG}(2,81)$ is new.

The form and content of this abstract are approved. I recommend its publication.

Approved: Stanley E. Payne

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1. Introduction

1.1 Overview

The focus of this dissertation is to construct new examples of Cameron-Liebler line classes admitting a certain cyclic automorphism group. These line classes have many different characterizations. Most notably, a Cameron-Liebler line class $\mathcal{L}$ has the property that, for some integer x called the *parameter*, $\mathcal{L}$ shares precisely x lines with every spread of the space. Cameron-Liebler line classes are also of interest to other areas of mathematics including group theory and coding theory. These sets of lines were originally studied in relation to a group theory problem regarding collineation groups of $\mathrm{PG}(3,q)$ having the same number of orbits on points and on lines. They also serve as generalizations of the notion of a symmetric tactical decomposition of $\mathrm{PG}(3,q)$; i.e., a tactical decomposition having the same number of point classes and line classes. Through the Klein correspondence, a Cameron-Liebler line class is equivalent to a set of points of the hyperbolic quadric $\mathcal{Q}^+(5,q)$ called a *tight set*. Tight sets of this quadric often determine two-intersection sets of the underlying projective space $\mathrm{PG}(5,q)$, that is, sets of points having two intersection numbers with respect to hyperplanes. Two-intersection sets can then be used to construct error correcting codes with codewords having precisely two nonzero weights which, in turn, give rise to examples of strongly regular graphs.

After reviewing the geometry of $\mathrm{PG}(3,q)$ and $\mathcal{Q}^+(5,q)$, as well as their relationship through the Klein correspondence, we survey the known results on Cameron-Liebler line classes, including the known examples as well as some non-existence results. We show the equivalence of these objects with tight sets of $\mathcal{Q}^+(5,q)$ and give results on when these sets determine two-intersection sets of the underlying $\mathrm{PG}(5,q)$. We also look at the construction of two-weight codes and strongly regular graphs from these two-intersection sets. Once we have developed this background material, we develop tools which are used to construct new examples. The main tools are an eigenvector method for finding tight sets and results on tactical decompositions which facilitate this method. Since we are primarily working from the point of view of $\mathcal{Q}^+(5,q)$, an algebraic model for this space is introduced. This allows us to give a concise notation for a cyclic group of order q^2+q+1 acting semi-regularly on the space, which is contained in the stabilizer of each new example constructed.

We use the tools we develop to construct several new examples of Cameron-Liebler line classes in $\mathrm{PG}(3,q)$, including many having parameters $\frac{1}{2}(q^2-1)$ and $\frac{1}{3}(q+1)^2$. We also describe other examples in $\mathrm{PG}(3,27)$ and $\mathrm{PG}(3,32)$. For all of these new examples, we detail structural information such as automorphism groups and intersection numbers with planes of $\mathrm{PG}(3,q)$, as well as the related two-intersection sets of $\mathrm{PG}(5,q)$, two-weight codes, and strongly regular graphs. Furthermore, when $q=9$ or 81, the new examples with parameter $\frac{1}{2}(q^2-1)$ are line partitions of a symmetric tactical decomposition of $\mathrm{PG}(3,q)$ having four parts on points and on lines; we give a construction from this decomposition of two-intersection sets in $\mathrm{AG}(2,q)$. Few examples of two-intersection sets of odd order affine planes are known; in fact, the only previously known examples are in planes of order 9. Thus, our example in $\mathrm{AG}(2,81)$ is new.

1.2 Finite fields

A finite field always has order $q=p^e$, where p is a prime. This field, which is unique up to isomorphism, will be denoted $\mathbb{F}_q$ and has *characteristic* p; i.e., $\sum_{i=1}^{p} x = 0$ for every $x \in \mathbb{F}_q$ (and p is the smallest integer for which this is true). The multiplicative group $\mathbb{F}_q^*$ of nonzero elements of $\mathbb{F}_q$ is a cyclic group of order $q-1$; an element $\alpha \in \mathbb{F}_q^*$ having order $q-1$ is called a *primitive element*, and $\langle\alpha\rangle = \mathbb{F}_q^*$ for such an element.

Let $K=\mathbb{F}_q$, where $q=p^h$. A subset F of K which is also a field under the same operations is called a *subfield* of K; we write $F \leq K$. K contains a unique subfield isomorphic to $\mathbb{F}_{p^e}$ for each e dividing h, consisting of $\{a \in K : a^{p^e} = a\}$. The intersection of all subfields of K is called the *prime subfield* of K and is isomorphic to $\mathbb{F}_p$. We can construct a larger field $E = \mathbb{F}_{q^d}$ from K by considering an irreducible polynomial $f(x)$ in $K[x]$ of degree d; in this case

$$E = K[x]/(f(x)) = \{a_0 + a_1x + \ldots + a_{d-1}x^{d-1} |\ a_i \in K,\ f(x) = 0\}$$

is a finite field of order q^d containing K as a subfield. We say that E is an *extension field* of K.

A map $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ is called an *automorphism* of $\mathbb{F}_q$ if σ is a permutation of the elements such that $(x+y)^\sigma = x^\sigma + y^\sigma$, and $(xy)^\sigma = (x^\sigma)(y^\sigma)$ for all x, y in $\mathbb{F}_q$. We write $\mathrm{Aut}(\mathbb{F}_q)$ for the group of automorphisms of $\mathbb{F}_q$. If $q = p^e$, p prime, then $\mathrm{Aut}(F_q)$ is cyclic, isomorphic to $\mathbb{Z}_e$, and is generated by the *Frobenius automorphism* $\phi : x \mapsto x^p$.

Let $q = p^e$, $F = \mathbb{F}_q$, and $K = \mathbb{F}_{q^h}$ with $F \leq K$; then $\mathrm{Aut}(K/F)$, the group of automorphisms of K fixing every element of F, has order h and is generated by $\phi^e : x \mapsto x^{p^e} = x^q$. We define the *relative trace map from K to F*, $\mathrm{T}_{K/F} : K \mapsto F$, by

$$\mathrm{T}_{K/F}(a) = \sum_{\sigma \in \mathrm{Aut}(K/F)} a^\sigma = a + a^q + a^{q^2} + \ldots + a^{q^{(h-1)}}.$$

This map has the following properties:

1. $\mathrm{T}_{K/F}(a+b) = \mathrm{T}_{K/F}(a) + \mathrm{T}_{K/F}(b)$ for all $a,\ b \in K$.

2. $\mathrm{T}_{K/F}(ca) = c\mathrm{T}_{K/F}(a)$ for all $c \in F$, $a \in K$.

3. $\mathrm{T}_{K/F}(a) = ha$ for all $a \in F$.

4. $\mathrm{T}_{K/F}(a^\sigma) = \mathrm{T}_{K/F}(a)$ for all $a \in K$ and for all $\sigma \in \mathrm{Aut}(K/F)$.

Notice that the first two items imply that, if K is viewed as a vector space over F, then $\mathrm{T}_{K/F}$ is a linear transformation from K to F. We actually have more; the map $\mathrm{T}_{K/F}$ maps K *onto* F, and in fact, every linear map from K into F takes the form $L_b(a) = \mathrm{T}_{K/F}(ba)$ for some $b \in K$.

1.3 The projective geometry $\mathrm{PG}(n, q)$

Much of this material is treated thoroughly in [25] or in [18]. There are also examples of projective planes which are not of the form $\mathrm{PG}(2, q)$; for details on these examples, see [19].

Definition 1.1 *Let $F = \mathbb{F}_q$ be a finite field of order q and V be a vector space of dimension $n + 1$ over F. We define the geometry $\mathrm{PG}(n, q)$ as follows:*

- *The one-dimensional vector subspaces of V are the points of $\mathrm{PG}(n, q)$.*
- *The $(d + 1)$-dimensional vector subspaces of V are the d-dimensional subspaces of $\mathrm{PG}(n, q)$.*
- *Incidence is defined in terms of containment of the corresponding vector subspaces.*

The word "dimension" is used in two ways with a different meaning; when it is not clear which we mean from context, we will specify *projective* dimension when we are talking about the dimension in $\mathrm{PG}(n, q)$, or *vector space* dimension when we are talking about a subspace of V. This definition of a projective space allows us to associate vectors in V with points of $\mathrm{PG}(n, q)$; namely, a nonzero vector $\mathbf{v} \in V$ represents a point of $\mathrm{PG}(n, q)$, with nonzero vectors $\mathbf{v}$, $\mathbf{w}$ representing the same point if and only if $\mathbf{v} = c\mathbf{w}$ for some $c \in F^*$.

We will call a subspace of $\mathrm{PG}(n, q)$ having dimension $n - 1$ a *hyperplane*; the set of points on a hyperplane can be described as those satisfying a homogeneous linear equation. We write the coefficients the equation describing a hyperplane H as $\mathbf{h} = [x_0, \ldots,\ x_n]$, with the convention that a point $\mathbf{u}$ is in H if and only if $\mathbf{u}\mathbf{h}^T = 0$. We will speak of a set of points or of hyperplanes as being *linearly independent* if the corresponding vectors are linearly independent in the vector space.

It is frequently useful to describe a subspace U of $\mathrm{PG}(n, q)$ as either the intersection or the span of other subspaces. Given two subspaces U_1 and U_2 of $\mathrm{PG}(n, q)$, the intersection $U_1 \cap U_2$ is again a subspace of $\mathrm{PG}(n,q)$. We define the *span* of U_1 and U_2 to be the smallest subspace of $\mathrm{PG}(n,q)$ containing both U_1 and U_2; we denote this by $\langle U_1,\ U_2 \rangle$. In general, the projective dimension of $\langle U_1,\ U_2 \rangle$ is given by

$$\dim \langle U_1,\ U_2 \rangle = \dim U_1 + \dim U_2 - \dim (U_1 \cap U_2).$$

The span of two distinct points, for example, is the line containing both of them. Given a subspace U of dimension d and a hyperplane H, we have that $U \cap H$ is either equal to U, or has dimension $d-1$. Thus we can describe a d-dimensional subspace U of $\mathrm{PG}(n,q)$ as either the span of $d+1$ linearly independent points, or as the intersection of $n-d$ linearly independent hyperplanes. If $\mathbf{h}_1, \ldots,\ \mathbf{h}_{n-d}$ are the vectors containing the coefficients of the equations for these hyperplanes, then we can associate U with the left null space of the matrix

$$\begin{bmatrix} \mathbf{h}_1^T \ldots \mathbf{h}_{n-d}^T \end{bmatrix}.$$

1.4 Collineations and dualities

A bijection on the points of $\mathrm{PG}(n,q)$ which preserves the lines is called a *collineation*; that is, a map $\theta : \mathrm{PG}(n,q) \to \mathrm{PG}(n,q)$ such that for all lines ℓ of $\mathrm{PG}(n,q)$, the image $\theta(\ell)$ is also a line of $\mathrm{PG}(n,q)$. This necessarily implies that any d-dimensional subspace of $\mathrm{PG}(n,q)$ gets mapped by θ to another d-dimensional subspace. Since we view the subspaces of $\mathrm{PG}(n,q)$ as corresponding to subspaces of an $(n+1)$-dimensional vector space V over $\mathbb{F}_q$, we can describe a collineation of $\mathrm{PG}(n,q)$ in terms of its action on V. In particular, any matrix $A \in GL(n+1,q)$ can be used to define a collineation $L_A : \mathbf{x} \mapsto \mathbf{x}A$ of $\mathrm{PG}(n,q)$. Collineations of this type are called *homographies*. Note that, for any $\lambda \in \mathbb{F}_q^*$, the matrices A and λA define the same map on $\mathrm{PG}(n,q)$; we say these two maps are *projectively equivalent*. The group

$$PGL(n+1,\ q) = GL(n+1,\ q)/Z(GL(n+1,q))$$

is called the *projective linear group*, and acts faithfully on $\mathrm{PG}(n,q)$. It is worth noting that, for a hyperplane H represented by $\mathbf{h}$ and a matrix $A \in PGL(n+1,q)$, we have that $\mathbf{x} \in H$ if and only if $(\mathbf{x}A)(A^{-1}\mathbf{h}^T) = 0$, so the hyperplane H gets mapped by L_A to a new hyperplane defined by $A^{-1}\mathbf{h}^T$. Automorphisms of $\mathbb{F}_q$ also give examples of collineations of $\mathrm{PG}(n,q)$. Given $\sigma \in \mathrm{Aut}(F_q)$, the map on $\mathrm{PG}(n,q)$ induced by

$$\sigma : V \to V : \ (x_0, \ldots,\ x_n) \mapsto (x_0^\sigma, \ldots,\ x_n^\sigma)$$

is called an *automorphic collineation*.

The Fundamental Theorem of Projective Geometry tells us that any collineation of $\mathrm{PG}(n,q)$ can be obtained by composing an automorphic collineation with a homography. Such a map is of the form

$$L_A \circ \sigma : \mathbf{x} \mapsto L_A(\mathbf{x}^\sigma) = \mathbf{x}^\sigma A,$$

where $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ and $A \in PGL(n+1,q)$, and is called a *projective semilinear* map. The group of these maps is denoted

$$P\Gamma L(n+1,\ q) = PGL(n+1,q) \rtimes \mathrm{Aut}(\mathbb{F}_q).$$

Associated with a projective geometry $\mathcal{S}$ is the so-called *dual geometry* $\mathcal{S}^*$; this geometry's points and hyperplanes are, respectively, the hyperplanes and points of $\mathcal{S}$. A projective geometry of the form $\mathrm{PG}(n,q)$ is isomorphic to its dual geometry, and we call an isomorphism from the points of $\mathrm{PG}(n,q)$ onto the hyperplanes a *reciprocity*. One important example is the map sending a point $\mathbf{x}$ to the hyperplane determined by $\mathbf{x}^T$. By our earlier comments about the Fundamental Theorem of Projective Geometry, any reciprocity can be written in the form $\mathbf{x} \to (\mathbf{x}^\sigma A)^T$, where $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ and $A \in PGL(n+1,q)$. If we have a reciprocity θ that is an involution, that is, if $\theta^2 = 1$, then we call θ a *polarity*.

1.5 Combinatorics of $\mathrm{PG}(n,q)$

Theorem 1.2 *[18] In* PG(n, q)*, there are*

$$\begin{array}{rl} \frac{(q^{n+1}-1)}{(q-1)} & \textit{points,} \\ \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2(q+1)} & \textit{lines, and} \\ \frac{\prod_{i=n-d+1}^{n+1}(q^i-1)}{\prod_{i=1}^{r+1}(q^i-1)} & d\textit{-dimensional subspaces.} \end{array}$$

Given $d < s$, the number of s dimensional subspaces containing a fixed d-dimensional subspace is given by

$$\frac{\displaystyle\prod_{i=s-d+1}^{n-d}(q^i - 1)}{\displaystyle\prod_{i=1}^{n-s}(q^i - 1)}$$

It is useful to note that, since PG(n, q) is self dual, the number of d-dimensional subspaces is the same as the number of $(n-d)$-dimensional subspaces. In particular, there are the same number of hyperplanes as there are points. Also, the number of hyperplanes through a fixed point is the same as the number of points in a fixed hyperplane.

We now give some consideration to the orders of various groups associated with PG(n, q) [16].

Theorem 1.3 *If $q = p^e$, then*

$$\begin{array}{rll} GL(n+1, q) & \textit{has order} & q^{\frac{1}{2}n(n+1)}\displaystyle\prod_{k=1}^{n+1}(q^k - 1), \\ PGL(n+1, q) & \textit{has order} & q^{\frac{1}{2}n(n+1)}\displaystyle\prod_{k=2}^{n+1}(q^k - 1), \textit{ and} \\ P\Gamma L(n+1, q) & \textit{has order} & eq^{\frac{1}{2}n(n+1)}\displaystyle\prod_{k=2}^{n+1}(q^k - 1). \end{array}$$

The projective space we will be primarily interested in will be PG$(3, q)$. Specializing the above results to this specific case, we see that PG$(3, q)$ contains q^3+q^2+q+1 points as well as planes, and $(q^2 + q + 1)(q^2 + 1)$ lines. Through each point there are $q^2 + q + 1$ lines and $q^2 + q + 1$ planes; also, given a line, there are $q + 1$ planes containing that line.

The groups acting on $\mathrm{PG}(3,q)$ are $PGL(4,q)$ and $P\Gamma L(4,q)$; if $q=p^e$, the orders of these are

$$
\begin{aligned}
&PGL(4,q) \ \text{ has order } \ \ q^6(q^2-1)(q^3-1)(q^4-1) \ \text{ and} \\
&P\Gamma L(4,q) \ \text{ has order } \ eq^6(q^2-1)(q^3-1)(q^4-1)\,.
\end{aligned}
$$

A set $\mathcal{R}$ of $q+1$ mutually skew lines in $\mathrm{PG}(3,q)$ is called a *regulus* provided

1. through every point of every line of $\mathcal{R}$ there is a transversal of the lines of $\mathcal{R}$ (that is, a line meeting each of the lines of $\mathcal{R}$); and,

2. through every point of every transversal there is a line of $\mathcal{R}$.

It is clear that the set of transversals of $\mathcal{R}$ is also a regulus which we call $\mathcal{R}_{\mathrm{opp}}$, the *opposite* regulus of $\mathcal{R}$. Any three skew lines in $\mathrm{PG}(3,q)$ determine a unique regulus. A *spread* of $\mathrm{PG}(3,q)$ is a set of q^2+1 lines of the space that partitions the points. A spread $\mathcal{S}$ is called *regular* if, given any three skew lines in $\mathcal{S}$, the regulus determined by those three lines is also contained in $\mathcal{S}$. Spreads can be defined in higher dimensional spaces as well, and are of considerable interest, as they can be used to construct examples of projective planes. Their classification is an important problem in finite geometry that is beyond the scope of this thesis.

A *k-arc* $\mathcal{K}$ of $\mathrm{PG}(2,q)$ (or any projective plane of order q) is a set of k points such that no three are collinear. Thus any line ℓ of $\mathrm{PG}(2,q)$ meets $\mathcal{K}$ in 0, 1, or 2 points; we call these lines *external*, *tangent*, or *secant* to $\mathcal{K}$ respectively. A k-arc must have $k \le q+2$. A k-arc $\mathcal{K}$ which is not contained in any $(k+1)$-arc is called *maximal*. A $(q+1)$-arc is called an *oval*, and if q is odd, any $(q+1)$-arc is maximal. However, if q is even, every tangent line to an oval $\mathcal{K}$ passes through a common point N which we call the *nucleus* of $\mathcal{K}$. In this situation, $\mathcal{K}\cup\{N\}$ is a $(q+2)$-arc, which we call a *hyperoval*. Given a plane π embedded in $\mathrm{PG}(3,q)$ containing an oval or hyperoval $\mathcal{O}$, and a point p not in π, we define a *cone* over $\mathcal{O}$ to be the set of points on the lines joining p to points of $\mathcal{O}$. The lines are called the *generators* of the cone, and p is called the *vertex* of the cone.

1.6 Bilinear and quadratic forms

Let V be a vector space of dimension $n+1$ over $F = \mathbb{F}_q$ A *bilinear form* on V is a function $\mathrm{B} : V \times V \to F$ that is linear in each argument; that is,

$$\mathrm{B}(a\mathbf{u}+\mathbf{v},\mathbf{w}) = a\mathrm{B}(\mathbf{u},\mathbf{w}) + \mathrm{B}(\mathbf{v},\mathbf{w}) \text{ and}$$
$$\mathrm{B}(\mathbf{u},a\mathbf{v}+\mathbf{w}) = a\mathrm{B}(\mathbf{u},\mathbf{v}) + \mathrm{B}(\mathbf{u},\mathbf{w})$$

for all $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$ and all $a \in F$.

A bilinear form B is said to be *symmetric* if $\mathrm{B}(\mathbf{u},\mathbf{v}) = \mathrm{B}(\mathbf{v},\mathbf{u})$ for all $\mathbf{u},\ \mathbf{v} \in V$, and *alternating* if $\mathrm{B}(\mathbf{u},\mathbf{u}) = 0$ for all $\mathbf{u} \in V$. We are strictly interested in *reflexive* bilinear forms, that is, those for which $\mathrm{B}(\mathbf{u},\mathbf{v}) = 0$ implies $\mathrm{B}(\mathbf{v},\ \mathbf{u}) = 0$. Every reflexive bilinear form is either symmetric or alternating.

A *quadratic form* on a vector space V is a map $Q : V \to F$ defined by a homogeneous degree 2 polynomial in the coordinates of V relative to some basis. Equivalently, we call $Q : V \to F$ a quadratic form if

$$Q(a\mathbf{u}) = a^2 Q(\mathbf{u}) \text{ for all } a \in F \text{ and } \mathbf{u} \in V \text{ and}$$
$$\mathrm{B} : (\mathbf{u},\mathbf{v}) \mapsto Q(\mathbf{u}+\mathbf{v}) - Q(\mathbf{u}) - Q(\mathbf{v}) \text{ gives a bilinear form on } V.$$

It is clear that B is symmetric if q is odd and alternating if q is even. This associated bilinear form, B, is called the *polar form* of Q. A vector space equipped with a quadratic form is called an *orthogonal space*.

Given a bilinear form B and an ordered basis $\mathcal{B} = \{\mathbf{v}_0, \ldots,\ \mathbf{v}_n\}$ for V, we put $b_{ij} = \mathrm{B}(\mathbf{v}_i, \mathbf{v}_j)$. The *Gram matrix relative to* $\mathcal{B}$ is then defined by $\hat{B} = [b_{ij}]$. This matrix has the property that, if $[\mathbf{u}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{B}}$ are coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$ relative to $\mathcal{B}$, then $\mathrm{B}(\mathbf{u},\mathbf{v}) = [\mathbf{u}]_{\mathcal{B}}\hat{\mathrm{B}}[\mathbf{v}]_{\mathcal{B}}^T$. Any two Gram matrices of a bilinear form B have the same rank, which we define to be the *rank* of B. If Q is a quadratic form on V, we define the upper triangular matrix

$$A = (a_{ij}), \text{ where } a_{ij} = \begin{cases} \mathrm{B}(\mathbf{v}_i,\ \mathbf{v}_j), & i < j; \\ Q(\mathbf{v}_i), & i = j; \\ 0 & i > j. \end{cases}$$

We then have that $Q(\mathbf{u}) = [\mathbf{u}]_{\mathcal{B}} A [\mathbf{u}]_{\mathcal{B}}^T$, and the Gram matrix for the polar form of Q with respect to $\mathcal{B}$ is then $\hat{B} = A + A^T$.

1.7 Orthogonality and totally isotropic subspaces

Let B be a reflexive bilinear form on $\mathrm{PG}(n, q)$. We define an *orthogonality* relationship on the points of $\mathrm{PG}(n, q)$ by $\mathbf{u} \perp \mathbf{v}$ if $\mathrm{B}(\mathbf{u}, \mathbf{v}) = 0$. If $S \subset V$, we define S "perp" to be

$$S^{\perp} = \{\mathbf{v} \in V |\ \mathbf{v} \perp \mathbf{s}\ \forall \mathbf{s} \in S\}.$$

A point $\mathbf{v}$ of $\mathrm{PG}(n, q)$ is called *singular* with respect to a bilinear form B if $\mathbf{v}^{\perp} = V$; it is called *singular* with respect to a quadratic form Q if it is singular with respect to the associated bilinear form and $Q(\mathbf{v}) = 0$. We say B or Q is *degenerate* if there is a singular point, and *nondegenerate* otherwise.

We place a special significance on points $\mathbf{v}$ for which $\mathrm{B}(\mathbf{v}, \mathbf{v}) = \mathbf{0}$ or $Q(\mathbf{v}) = 0$. Such a point $\mathbf{v}$ is called *isotropic* with respect to to bilinear form B or the quadratic form Q, respectively. We call a subspace W of V isotropic if it contains an isotropic point, *anisotropic* otherwise, and *totally isotropic* if $\mathrm{B}(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{u}, \mathbf{v} \in W$ (for a bilinear form), or if $Q(\mathbf{v}) = 0$ for all $\mathbf{v} \in W$ (for a quadratic form). If a subspace is totally isotropic with respect to a quadratic form Q, then it is also totally isotropic with respect to the associated bilinear form, though the converse only holds if q is odd. The set of isotropic points in $\mathrm{PG}(n, q)$ with respect to a nondegenerate quadratic form is called a *quadric*, and has the property that any line of $\mathrm{PG}(n, q)$ containing more than two points of a quadric must be completely contained in the quadric.

If B is nondegenerate form, the orthogonality relation can be used to define a polarity $\sigma : U \mapsto U^{\perp}$ of $\mathrm{PG}(n, q)$. In this case, if U and W are subspaces of V with

$U \leq W$, then $W^\perp \leq U^\perp$; furthermore, for any subspace U of V, $\dim U + \dim U^\perp = \dim V$. A point $\mathbf{x}$ is said to be *isotropic* with respect to the polarity if $\mathbf{x} \subset \mathbf{x}^\perp$, and a subspace U is said to be *totally isotropic* with respect to the polarity if $U \leq U^\perp$. This is in agreement with the notions of being isotropic or totally isotropic with respect to the bilinear form. When q is odd, the polar form of a nondegenerate quadratic form is necessarily nondegenerate. Thus in this case the notions of being (totally) isotropic with respect to the quadratic form, the bilinear form, and the associated polarity all agree.

The situation is more complicated when q is even, since it is possible for the polar form of Q to be degenerate even when Q is nondegenerate. In this case, we do not have a polarity associated with the quadratic form. Even if the polar form B of Q is nondegenerate, we have $\mathrm{B}(\mathbf{u}, \mathbf{u}) = 0$ for every $\mathbf{u} \in \mathrm{PG}(n, q)$, so *every* point of $\mathrm{PG}(n, q)$ is incident with its image under the induced polarity (such a polarity is called a *null polarity*). Thus the set of points which are isotropic with respect to this polarity does not agree with the set of points which are isotropic with respect to the quadratic form.

1.8 Orthogonal polar spaces in $\mathrm{PG}(n, q)$

Definition 1.4 *A* polar space of rank r *is an incidence geometry consisting of a set of points, lines, projective planes, ..., $(r-1)$-dimensional projective spaces called* subspaces *such that*

1. *Any two subspaces intersect in a subspace.*

2. *If U is a subspace of dimension $r-1$ and $\mathbf{p}$ is a point not in U, there is a unique subspace W containing $\mathbf{p}$ with $U \cap W$ having dimension $r-2$; it consists of all points of U which are joined to $\mathbf{p}$ by some line.*

3. *There are two disjoint subspaces of dimension $r-1$.*

The $(r-1)$-dimensional subspaces are called maximals *of the polar space.*

The *finite classical polar spaces* are the examples naturally embedded in a projective space $\mathrm{PG}(n,q)$; they are defined by a nondegenerate quadratic or sesquilinear form on the space. A result of Tits [33] proved that any polar space with rank at least 3 is classical. Rank 2 polar spaces are a special case. They are called generalized quadrangles, and there are nonclassical examples of these; see [26] for a detailed treatment.

Let $F = \mathbb{F}_q$, and V be a $(n+1)$-dimensional vector space over F. Take Q to be a nondegenerate quadratic form on V with polar form B. The geometry consisting of the totally isotropic subspaces of $\mathrm{PG}(n,q)$ with respect to Q is an example of a classical polar space; we call an example arising in this way an *orthogonal polar space.*

Note: We have now introduced three very closely related terms, an **orthogonal space**, a **quadric**, and an **orthogonal polar space**.

- The vector space V along with a nondegenerate quadratic form is an **orthogonal space**.
- The set of isotropic points of $\mathrm{PG}(n,q)$ with respect to the quadratic form is called a **quadric**.
- The geometry of totally isotropic subspaces with respect to the quadratic form is called an **orthogonal polar space**, in this context the polar space can either be considered as embedded in $\mathrm{PG}(n,q)$ or as a geometry in its own right.

The most general collineation of $\mathrm{PG}(n,q)$ preserving a quadric is called a *semi-similarity*; this is a map σ such that, for some $a \in \mathbb{F}_q^*$ and some $\tau \in \mathrm{Aut}(\mathbb{F}_q)$,

$$Q(\sigma(\mathbf{x})) = a(Q(\mathbf{x}))^\tau \cdot$$

We call σ a *similarity* if $\tau = 1$, and we call σ an *isometry* if $a = 1$ and $\tau = 1$. The following important theorem is known as Witt's Extension Theorem:

Theorem 1.5 *If U, $W \leq V$, and $\sigma : U \to W$ an isometry, then there is an isometry $\tau : V \to V$ such that $\tau|_U = \sigma$.*

Corollary 1.6 *Any two maximals of V have the same dimension.*

The vector space dimension of a maximal is called the *Witt index* of the polar space. The Witt index of a nondegenerate form is less than or equal to $\frac{1}{2}\dim V$, since a totally isotropic subspace W is contained in $W^{\perp}$.

We define two distinct points $\mathbf{u}$, $\mathbf{v}$ of the quadric to be a *hyperbolic pair* if $\mathrm{B}(\mathbf{u}, \mathbf{v}) = 1$. We then call $\langle \mathbf{u}, \mathbf{v} \rangle$ a *hyperbolic line*. Note that this is a line of $\mathrm{PG}(n, q)$ containing precisely two points of the quadric.

Theorem 1.7 *Any nondegenerate orthogonal space of Witt index r over $\mathbb{F}_q$ is isometric to one of the following:*

1. *A hyperbolic quadric $\mathcal{Q}^+(2r-1, q)$ is the orthogonal direct sum of r hyperbolic lines.*

2. *A parabolic quadric $\mathcal{Q}(2r, q)$ is the orthogonal direct sum of r hyperbolic lines and a one-dimensional anisotropic space. These fall into two isometry classes and one similarity class when q is odd, and one isometry class when q is even.*

3. *An elliptic quadric $\mathcal{Q}^-(2r+1, q)$ is the orthogonal direct sum of r hyperbolic lines and a two-dimensional anisotropic space.*

The group of isometries of $\mathcal{Q}^+(2r-1, q)$, $\mathcal{Q}(2r, q)$, or $\mathcal{Q}^-(2r+1, q)$ is denoted $O^+(2r, q)$, $O(2r+1, q)$, or $O^-(2r+2, q)$, respectively. For the projective versions of these groups, we prefix this with P, PG, or $P\Gamma$ depending on whether we want the group of isometries, similarities, or semi-similarities, respectively.

If we have a set of points $\mathcal{O}$ in a polar space such that every maximal of the polar space meets $\mathcal{O}$ in a unique point, then we call $\mathcal{O}$ an *ovoid* of the polar space. The classification of ovoids in classical polar spaces is an important open problem in finite geometry; we are primarily interested in these objects because of how they interact with other objects in the space.

1.9 $\mathcal{Q}^+(5,q)$ and the Klein correspondence

The 5-dimensional hyperbolic orthogonal space $\mathcal{Q}^+(5,q)$ plays an important role, as this geometry is closely related to $\mathrm{PG}(3,q)$. This quadric is made up of the orthogonal direct sum of three hyperbolic lines, and the standard associated quadratic form is given by

$$Q : (x_0, x_1, x_2, x_3, x_4, x_5) \mapsto x_0x_1 + x_2x_3 + x_4x_5,$$

which is described by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Gram matrix for the polar form B with respect to the standard basis is then $\hat{B} = A + A^T$.

Another way to think of the structure of this polar space is given by taking one point from each of the three hyperbolic pairs. Since the hyperbolic lines they determine are pairwise orthogonal, these three points are also pairwise orthogonal and so span a totally isotropic plane π_1, necessarily a maximal of the polar space. The three remaining points from the hyperbolic pairs then span a totally isotropic plane π_2 which is disjoint from π_1.

The geometries $\mathrm{PG}(3,q)$ and $\mathcal{Q}^+(5,q)$ are closely related through a mapping known as the *Klein correspondence*. This refers to a bijection from the lines of $\mathrm{PG}(3,q)$

to the points of $\mathcal{Q}^+(5,q)$ such that two lines of $\mathrm{PG}(3,q)$ intersect if and only if their images are collinear in $\mathcal{Q}^+(5,q)$. To define the bijection, we will first establish a way to describe lines of $\mathrm{PG}(3,q)$ using *Plücker coordinates.* Let $\mathbf{x} = (x_0,\ x_1,\ x_2,\ x_3)$ and $\mathbf{y} = (y_0,\ y_1,\ y_2,\ y_3)$ be distinct points on a line ℓ of $\mathrm{PG}(3,q)$. Define $G(\ell) = (p_{01}, p_{23}, p_{02}, p_{31}, p_{03}, p_{12})$, where

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \quad \text{for } 0 \leq i < j \leq 3.$$

If we consider $G(\ell) \in \mathrm{PG}(5,q)$, any choice of two points on ℓ determine the same point. Furthermore, it can be seen that $p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$. Thus, G takes lines of $\mathrm{PG}(3,q)$ to points of $\mathcal{Q}^+(5,q)$.

Now, given a point $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5) \in \mathcal{Q}^+(5,q)$, so $a_0a_1 + a_2a_3 + a_4a_5 = 0$, the matrix

$$\begin{bmatrix} 0 & a_1 & a_3 & a_5 \\ -a_1 & 0 & a_4 & -a_2 \\ -a_3 & -a_4 & 0 & a_0 \\ -a_5 & a_2 & -a_0 & 0 \end{bmatrix}$$

has rank two. Therefore the map

$$H(\mathbf{a}) = \text{row} \begin{bmatrix} 0 & a_1 & a_3 & a_5 \\ -a_1 & 0 & a_4 & -a_2 \\ -a_3 & -a_4 & 0 & a_0 \\ -a_5 & a_2 & -a_0 & 0 \end{bmatrix}$$

takes the point $\mathbf{a}$ to a line of $\mathrm{PG}(3,q)$. It can be verified that $H(G(\ell)) = \ell$. Thus G and H give a bijection between the lines of $\mathrm{PG}(3,q)$ and the points of $\mathcal{Q}^+(5,q)$. We refer to this bijection as the *Klein correspondence.*

Here we detail some important properties of the Klein correspondence.

Theorem 1.8 *Two lines ℓ and ℓ' of* $\mathrm{PG}(3,q)$ *are concurrent if and only if their corresponding points L and L' are collinear in* $\mathcal{Q}^+(5,q)$.

Corollary 1.9 *The set of lines in a spread of* $\mathrm{PG}(3,q)$ *correspond to an ovoid of* $\mathcal{Q}^+(5,q)$*.*

Corollary 1.10 *Let* ℓ *and* ℓ' *be two concurrent lines in* $\mathrm{PG}(3,q)$ *with corresponding points* L *and* L' *in* $\mathcal{Q}^+(5,q)$*. Then the lines of the flat pencil of lines in* $\mathrm{PG}(3,q)$ *determined by* ℓ *and* ℓ' *correspond to the line of* $\mathcal{Q}^+(5,q)$ *through* L *and* L'*. Conversely, each line of* $\mathcal{Q}^+(5,q)$ *corresponds to a set of lines in* $\mathrm{PG}(3,q)$ *lying in a flat pencil.*

Corollary 1.11 *The set of points in a totally isotropic plane of* $\mathcal{Q}^+(5,q)$ *corresponds to a set of* q^2+q+1 *lines in* $\mathrm{PG}(3,q)$*, any two of which are concurrent. Thus, they correspond to either the set of lines through a common point* p*, denoted star*(p)*, or the set of lines in a common plane* π*, denoted line*(π)*.*

We call a totally isotropic plane of $\mathcal{Q}^+(5,q)$ a *Latin plane* if it corresponds to star(p) for some point $p \in \mathrm{PG}(3,q)$, and a *Greek plane* if it corresponds to line(π) for some plane π of $\mathrm{PG}(3,q)$.

Corollary 1.12 *Any two distinct planes of the same type in* $\mathcal{Q}^+(5,q)$ *intersect in a* 0*-dimensional subspace (a single point). Any two planes of different types are either disjoint, or meet in a line of* $\mathcal{Q}^+(5,q)$*. Thus two planes are of the same type if and only if their intersection has even dimension.*

These correspondences allow us to count the following:

Corollary 1.13 $\mathcal{Q}^+(5,q)$ *contains*

1. $(q^2+1)(q^2+q+1)$ *points;*

2. $(q^3+q^2+q+1)(q^2+q+1)$ *lines;*

3. $2(q^3+q^2+q+1)$ *planes;*

4. $q(q+1)^2$ *points collinear to a given point;*

5. $(q+1)^2$ *lines containing a given point;*

6. $2(q+1)$ *planes containing a given point;*

7. 2 *planes containing a given line.*

The Klein correspondence also gives us a connection between the groups $P\Gamma L(4,q)$ acting on $\mathrm{PG}(3,q)$ and $P\Gamma O^+(6,q)$ acting on $\mathcal{Q}^+(5,q)$. Specifically, any element of $P\Gamma L(4,q)$ induces an action on the points of $\mathcal{Q}^+(5,q)$ preserving collinearity, and so $P\Gamma O^+(6,q)$ has a subgroup isomorphic to $P\Gamma L(4,q)$. Any map on $\mathcal{Q}^+(5,q)$ arising in this fashion maps Greek planes to Greek planes and Latin planes to Latin planes. Any correlation of $\mathrm{PG}(3,q)$ sends lines to lines, and so also induces an action on the points of $\mathcal{Q}^+(5,q)$ preserving collinearity. A map arising in this fashion interchanges the Greek and Latin planes. These are known to be the only automorphisms of $\mathcal{Q}^+(5,q)$.

Theorem 1.14 *The structure of the projective similarity and semi-similarity groups of $\mathcal{Q}^+(5,q)$ are as follows:*

- $PGO^+(6,q) \simeq PGL(4,q) \rtimes \mathbb{Z}_2$.
- $P\Gamma O^+(6,q) \simeq P\Gamma L(4,q) \rtimes \mathbb{Z}_2$.

Using this connection between the lines of $\mathrm{PG}(3,q)$ and the points of $\mathcal{Q}^+(5,q)$ can be helpful, especially when dealing with combinatorics of sets of lines in $\mathrm{PG}(3,q)$. In addition to having many theoretical results to apply, it is more computationally convenient to deal with sets of points. For this reason, much of our work in this thesis is done in the context of $\mathcal{Q}^+(5,q)$.

2. Cameron-Liebler line classes

In this chapter, we will survey many of the known results on Cameron-Liebler line classes. This includes non-existence results, known constructions, and a discussion of the images of these line sets in $\mathcal{Q}^+(5,q)$ under the Klein correspondence.

2.1 Definitions and history

Here we detail sets of lines in $\mathrm{PG}(3,q)$ having some special combinatorial properties. These sets of lines were originally studied by Cameron and Liebler [8], who called them "special" line classes, in connection with the study of collineation groups of $\mathrm{PG}(3,q)$ having the same number of orbits on points and lines. Such a group induces a symmetric tactical decomposition of the incidence structure of points and lines in $\mathrm{PG}(3,q)$, and they showed that a line class from such a decomposition has nice intersection properties with respect to reguli and spreads of the space. They abstracted the concept of sets of lines with these properties, hoping it would lead to the classification of symmetric tactical decompositions and collineation groups of $\mathrm{PG}(3,q)$ with this orbit structure. However, this problem proved interesting in a more general setting than originally envisioned.

Definition 2.1 *Let A be the point-line incidence matrix of $\mathrm{PG}(3,q)$ with respect to some ordering of the points and lines, and let $\mathcal{L}$ be a set of lines in $\mathrm{PG}(3,q)$ with characteristic vector $\chi = \chi_{\mathcal{L}}$. We will write $(\chi)_\ell$ for the entry of χ corresponding to the line ℓ. The following statements are all equivalent; if they hold, $\mathcal{L}$ is called a Cameron-Liebler line class [8], [27].*

1. $\chi_{\mathcal{L}} \in row(A)$.

2. $\chi_{\mathcal{L}} \in (null(A^T))^\perp$

3. $|\mathcal{R} \cap \mathcal{L}| = |\mathcal{R}_{opp} \cap \mathcal{L}|$ *for every regulus $\mathcal{R}$ and its opposite $\mathcal{R}_{opp}$.*

4. *There exists $x \in \mathbb{Z}^+$ such that $|\mathcal{L} \cap \mathcal{S}| = x$ for every spread $\mathcal{S}$.*

5. *There exists* $x \in \mathbb{Z}^+$ *such that* $|\mathcal{L} \cap \mathcal{S}| = x$ *for every* regular *spread* $\mathcal{S}$.

6. *There exists* $x \in \mathbb{Z}^+$ *such that, for every incident point-plane pair* (p, π),

$$|star(p) \cap \mathcal{L}| + |line(\pi) \cap \mathcal{L}| = x + (q+1)|pencil(p, \pi) \cap \mathcal{L}|.$$

7. *There exists* $x \in \mathbb{Z}^+$ *such that, for every line* ℓ *in* $\mathrm{PG}(3, q)$,

$$|\{lines\ m \in \mathcal{L}\ meeting\ \ell, m \neq \ell\}| = x(q+1) + (q^2+1)(\chi)_\ell.$$

8. *There exists* $x \in \mathbb{Z}^+$ *such that, for every pair* ℓ, m *of skew lines in* $\mathrm{PG}(3, q)$,

$$|\{n \in \mathcal{L} :\ n\ is\ a\ transversal\ to\ \ell, m\}| = x + q[(\chi)_\ell + (\chi)_m].$$

The value x must satisfy $0 \leq x \leq q^2 + 1$, and will necessarily be the same in each instance; we call x the *parameter* of the line class. If $\mathcal{L}$ is a Cameron-Liebler line class with parameter x, then $|\mathcal{L}| = x(q^2+q+1)$. The complement $\mathcal{L}'$ of a Cameron-Liebler line class $\mathcal{L}$ with parameter x is also a Cameron-Liebler line class having parameter q^2+1-x, and the union of two disjoint Cameron-Liebler line classes with parameters x_1 and x_2 is a Cameron-Liebler line class with parameter x_1+x_2. A Cameron-Liebler line class is said to be *irreducible* if it does not properly contain any other line class as a subset.

2.2 Tight sets of $\mathcal{Q}^+(5, q)$

To investigate the existence of Cameron-Liebler line classes, it is frequently useful to translate their definition to the setting of $\mathcal{Q}^+(5, q)$ using the Klein correspondence. In this context, part 7 of Definition 2.1 has an especially interesting interpretation; $\mathcal{L}$ is a Cameron-Liebler line class if and only if its image M in $\mathcal{Q}^+(5, q)$ has the following property:

There exists $x \in \mathbb{Z}^+$ such that, for every point p in $\mathcal{Q}^+(5, q)$,

$$|p^\perp \cap M| = x(q+1) + q^2(\chi)_p,$$

where $\chi = \chi_M$ is the characteristic vector of M.

Definition 2.2 *Let $\mathcal{S}$ be a polar space of rank $r \geq 3$ over $\mathbb{F}_q$. Then a set $\mathcal{T}$ of points in $\mathcal{S}$ is an x-*tight set *if for all points $p \in \mathcal{S}$*

$$|p^\perp \cap \mathcal{T}| = \begin{cases} x\frac{q^{r-1}-1}{q-1} + q^{r-1} & \text{if } p \in \mathcal{T} \\ x\frac{q^{r-1}-1}{q-1} & \text{if } p \notin \mathcal{T} \end{cases}$$

Adapting this definition for the rank 3 polar space $\mathcal{Q}^+(5,q)$, we see that a Cameron-Liebler line class with parameter x is equivalent to an x-tight set of $\mathcal{Q}^+(5,q)$.

Point sets in polar spaces having precisely two intersection numbers with respect to perps of points are called *intriguing* by Bamberg, Kelly, Law and Penttila [2]. There are two types of intriguing sets in finite polar spaces, and they can be characterized in terms of their intersection numbers. If $\mathcal{I}$ is an intriguing set of a polar space having intersection numbers h_1 for perps of points inside $\mathcal{I}$ and h_2 for perps of points outside $\mathcal{I}$, then $\mathcal{I}$ is a tight set if $h_1 > h_2$. A tight set of points in a finite polar space can also be defined as a set of points $\mathcal{T}$ such that each point of the space is, on average, collinear with as many points in $\mathcal{T}$ as possible. These sets were originally studied in generalized quadrangles by Payne [24] and their definition was later extended to more general polar spaces by Drudge [12]. An intriguing set with $h_1 < h_2$ is called an m-ovoid for some m. These are generalizations of the concept of an ovoid in a polar space. The study of intriguing sets in finite polar spaces is an active area of research with many open problems; for details, see [2].

In addition to the equivalent characterizations of Cameron-Liebler line classes carrying over under the Klein correspondence, we have a few additional properties in this context that do not have a good interpretation in $\mathrm{PG}(3,q)$.

Theorem 2.3 *Let M be a set of points in $\mathcal{Q}^+(5,q) \subset \mathrm{PG}(5,q)$. The following are equivalent [22].*

1. *M corresponds to a Cameron-Liebler line class of $\mathrm{PG}(3,q)$ with parameter x.*

2. *M is an x-tight set of $\mathcal{Q}^+(5,q)$.*

3. *There exists* $x \in \mathbb{Z}^+$ *such that* $|M| = x(q^2+q+1)$*, every tangent hyperplane to* $\mathcal{Q}^+(5,q)$ *at a point of* M *meets* M *in* $q^2 + x(q+1)$ *points, and every other hyperplane of* $\mathrm{PG}(5,q)$ *meets* M *in* $x(q+1)$ *points.*

4. *There exists* $x \in \mathbb{Z}^+$ *such that* $|\ell^\perp \cap M| = q|\ell \cap M| + x$ *for every line* ℓ *of* $\mathrm{PG}(5,q)$.

5. *There exists* $x \in \mathbb{Z}^+$ *such that* $|\ell^\perp \cap M| = q|\ell \cap M| + x$ *for every line* ℓ *of one of the four line types in* $\mathrm{PG}(3,q)$ *(external, tangent, secant, totally isotropic).*

It is important to note that the last three characterizations are stronger than their related versions in $\mathrm{PG}(3,q)$. Part 3 in particular states that, in addition to knowing the intersection numbers for tangent hyperplanes of $\mathcal{Q}^+(5,q)$, we also know that every nontangent hyperplane section of $\mathcal{Q}^+(5,q)$ meets an x-tight set in $x(q+1)$ points. This property is important enough that we state it on its own, as it will be used in the next section to construct related combinatorial objects.

Theorem 2.4 *Let* $\mathcal{T}$ *be a proper* x*-tight set of* $\mathcal{Q}^+(5,q)$ *that spans the ambient projective space. Then the set of points covered by* $\mathcal{T}$ *has two intersection numbers with respect to hyperplanes of* $\mathrm{PG}(5,q)$*. These numbers are*

$$h_1 = (q^2+1) + x(q+1) \text{ and}$$
$$h_2 = x(q+1).$$

2.3 Two-intersection sets, two-weight codes, and strongly regular graphs

Tight sets of $\mathcal{Q}^+(5,q)$ are related to many other combinatorial objects; here we investigate some properties of these objects.

Definition 2.5 *A set of points* $\mathcal{S}$ *of* $\mathrm{PG}(n,q)$ *is called a* two-intersection set *with* intersection numbers h_1 *and* h_2 *if every hyperplane of* $\mathrm{PG}(n,q)$ *intersects* $\mathcal{S}$ *in either* h_1 *or* h_2 *points. Such a set is also sometimes called a* set of type (h_1, h_2).

From the previous theorem, an x-tight set of $\mathcal{Q}^+(5,q)$ whose points span $\mathrm{PG}(5,q)$ is a two-intersection set of $\mathrm{PG}(5,q)$. These sets are related to a wide range of other combinatorial objects. We begin by detailing results on an important class of linear codes.

An $[n,\ k]_q$ *code* C is a k-dimensional subspace of the vector space $V = \mathbb{F}_q^n$. Vectors in C are called *codewords*, and the *weight* $\mathrm{wt}(\mathbf{v})$ of a codeword $\mathbf{v}$ is the number of nonzero entries of $\mathbf{v}$. A *two-weight code* C is a code whose codewords have precisely two nonzero weights. Given a code C, we define the *dual code*

$$C^{\perp} = \{\mathbf{v} \in V |\ \mathbf{v}\mathbf{c}^T = 0\ \forall \mathbf{c} \in C\}.$$

We have that $C^{\perp}$ is an $[n, n-k]_q$ code.

Let C be an $[n,k]_q$ code; there exist linear functionals $f_i : \mathbb{F}_q^k \to \mathbb{F}_q$ such that $C = \{(f_1(\mathbf{v}),\ \ldots,\ f_n(\mathbf{v})) :\ \mathbf{v} \in \mathbb{F}_q^k\}$. Since $(\mathbf{u},\mathbf{v}) \mapsto \mathbf{u}\mathbf{v}^T$ is a nondegenerate bilinear form, there exist $\mathbf{u}_1,\ \ldots,\ \mathbf{u}_n \in \mathbb{F}_q^k$ such that $f_i(\mathbf{v}) = \mathbf{v}\mathbf{u}_i^T$ for all $\mathbf{v} \in \mathbb{F}_q^k$. Thus, we have that $C = \{(\mathbf{v}\mathbf{u}_1^T,\ \ldots,\ \mathbf{v}\mathbf{u}_n^T) |\ \mathbf{v} \in V\}$, and since $\dim(C) = k$, the $\mathbf{u}_i$ span $\mathbb{F}_q^k$. We say C is *projective* if no two of the $\mathbf{u}_i$ represent the same point in $\mathrm{PG}(k-1,q)$.

Let $\Omega \subset V \setminus \{0\}$. We say Ω is a $\{\lambda_1, \lambda_2\}$ *difference set* if, for every $\mathbf{v} \in V \setminus \{0\}$, the number of pairs $(\mathbf{x},\mathbf{y}) \in \Omega^2$ such that $\mathbf{x} - \mathbf{y} = \mathbf{v}$ is λ_1 if $\mathbf{v} \in \Omega$, and λ_2 if $\mathbf{v} \notin \Omega$. If $-\Omega = \Omega$, we define a graph $G(\Omega)$ whose vertices are the vectors in V, with $\mathbf{u}$ and $\mathbf{v}$ adjacent if and only if $\mathbf{u} - \mathbf{v} \in \Omega$.

Definition 2.6 *A* strongly regular graph *with parameters* $(v,\ k,\ \lambda,\ \mu)$ *is a connected k-regular (simple, undirected) graph on v vertices, not null or complete, such that any two adjacent vertices share λ common neighbors, and any two nonadjacent vertices share μ common neighbors.*

We now give a result connecting the concepts of two intersection sets, two-weight codes, $\{\lambda_1, \lambda_2\}$ difference sets, and strongly regular graphs due to Calderbank and Kantor [7].

Theorem 2.7 *Let $V = \mathbb{F}_q^{n+1}$, $\mathcal{O} = \{\mathbf{y}_i \mid 1 \leq i \leq r\}$ be a set of vectors which span V (so $r \geq n+1$) and are pairwise independent, and let $\Omega = \{c\mathbf{y}_i \mid c \in \mathbb{F}_q^*\}$ be the set of nonzero scalar multiples of the $\mathbf{y}_i$; then the following statements are equivalent:*

1. *$\mathcal{O}$ is a set of type $(r - w_1, r - w_2)$ in $\mathrm{PG}(n, q)$ for some w_1, w_2;*
2. *$C = \{(\mathbf{x} \cdot \mathbf{y}_1, \ \ldots, \ \mathbf{x} \cdot \mathbf{y}_k) | \ \mathbf{x} \in V\}$ is a projective two-weight $[r, n+1]_q$ code with nonzero weights w_1 and w_2;*
3. *Ω is a $\{\lambda_1, \lambda_2\}$ difference set for some λ_1, λ_2;*
4. *$G(\Omega)$ is a strongly regular graph with parameters $(q^{n+1}, r(q-1), \lambda, \mu)$, where for some w_1, w_2 we have*

   $\lambda = r^2(q-1)^2 + 3r(q-1) - qw_1w_2 - r(q-1)(w_1 + w_2) + q^2(w_1 + w_2)$ *and*
   $\mu = \frac{q^2 w_1 w_2}{q^{n+1}}$.

This means that if $\mathcal{L} = \{\mathbf{y}_1, \ldots, \mathbf{y}_{x(q^2+q+1)}\}$ is an x-tight set of $\mathcal{Q}^+(5, q)$ which spans $\mathrm{PG}(5, q)$, then

1. $\mathcal{L}$ is a set of type $((q^2+1) + x(q+1), x(q+1))$ in $\mathrm{PG}(5, q)$;
2. the points of $\mathcal{L}$ define a projective two-weight $[x(q^2+q+1), 6]_q$ code with weights $(q-1)q^2 - 1$ and xq^2;
3. $\Omega = \{c\mathbf{y}_i \mid c \in \mathbb{F}_q^*\}$ is a $\{\lambda_1, \lambda_2\}$ difference set for some λ_1, λ_2; and
4. $G(\Omega)$ is strongly regular with parameters

$$(q^6, x(q^3-1), x(x-3) + q^3, x(x-1)).$$

2.4 Trivial examples

There are a few examples which trivially satisfy the necessary requirements to be a Cameron-Liebler line class.

1. The empty set $\emptyset$ is a Cameron-Liebler line class with parameter 0.

2. The set star($\mathbf{p}$) of lines through a common point $\mathbf{p}$ of $\mathrm{PG}(3, q)$ is a Cameron-Liebler line class with parameter 1 corresponding to a 1-tight set of $\mathcal{Q}^+(5, q)$ consisting of the set of points in a (Latin) plane.

3. The set line(π) of lines in a plane π of $\mathrm{PG}(3, q)$ is a Cameron-Liebler line class with parameter 1 corresponding to a 1-tight set of $\mathcal{Q}^+(5, q)$ consisting of the set of points in a (Greek) plane (this is equivalent to the previous example in $\mathcal{Q}^+(5, q)$).

4. The set star($\mathbf{p}$)$\cup$line(π), where π is a plane of $\mathrm{PG}(3, q)$ and $\mathbf{p}$ is a point not in π, is a Cameron-Liebler line class with parameter 2 corresponding to a 2-tight set of $\mathcal{Q}^+(5, q)$ which is a union of two disjoint planes (one Latin and one Greek).

5. The complements of the above sets are Cameron-Liebler line classes with parameters $q^2 + 1$, q^2, q^2, $q^2 - 1$ respectively.

We call the Cameron-Liebler line classes in this list *trivial*.

2.5 Non-existence results

Cameron and Liebler conjectured that there were no nontrivial examples of these line classes, and proved this conjecture for classes with parameter ≤ 2. Many other results followed, leading to some interesting connections with various geometric objects.

Many of the early non-existence results relied strictly on counting arguments; specifically, we can think of sets of the type star($\mathbf{p}$) or line(π) for a point $\mathbf{p}$ or a plane π as being essentially the same, and refer to these as *cliques*. The equivalent definitions for a Cameron-Liebler line class allow us to perform some analysis on the potential intersection numbers with respect to cliques of a hypothetical line class with a given parameter x. Using these arguments, Penttila [27] was able to rule out several parameters in specific cases, and Bruen and Drudge [5] were able to rule out the existence of line classes with parameter $2 < x \leq \sqrt{q}$.

These methods were greatly improved in 1999 by Drudge [13] when he showed that if the intersection of an indecomposable Cameron-Liebler line class $\mathcal{L}$ with parameter $x > 2$ and some clique $\mathcal{C}$ has $x < |\mathcal{L} \cap \mathcal{C}| \leq x + q$ then $\mathcal{L} \cap \mathcal{C}$ forms a *blocking set* in $\mathcal{C}$ (in this context, a set of lines not containing any pencil, such that every point is on at least one of the lines; the normal definition is dual to this). Blocking sets are well studied, and there are many results on their minimum possible size. This gives a powerful tool for investigating the feasibility of certain parameters for Cameron-Liebler line classes. Drudge used this method to rule out the case where $2 < x \leq \frac{1}{2}(q+1)$ when q is prime, and also gave the first nontrivial example of a Cameron-Liebler line class having parameter 5 in $\mathrm{PG}(3,3)$. Soon after, he and Bruen [6] constructed an infinite family of examples of line classes having parameter $\frac{1}{2}(q^2+1)$ for any odd q.

In 2004, Govaerts and Storme [15] used these blocking set techniques to improve the nonexistence result, eliminating the possibility of $2 < x \leq q$ when q was an odd prime. Soon after, Govaerts and Penttila [14] were able to rule out a few parameters in $\mathrm{PG}(3,4)$ by considering intersections with multiple blocking sets. In this same paper, they constructed the first example of a Cameron-Liebler line class for even q, an example with parameter 7 in $\mathrm{PG}(3,q)$. Multiple blocking sets can also be viewed as a special case of a more general class of combinatorial objects called minihypers. De Beule, Hallez, and Storme [10] used known results on these objects in $\mathcal{Q}^+(5,q)$ to show that, when q is not prime, we cannot have $2 < x < \frac{q}{2}$.

While the previous results are of considerable interest, in that they relate Cameron-Liebler line classes to other well-studied objects, the most recent and strongest nonexistence result is notable in that it uses primarily geometric arguments. In 2010, Metsch [22] looked at how an x-tight set of $\mathcal{Q}^+(5,q)$ could potentially intersect a parabolic $\mathcal{Q}(4,q)$ embedded in the quadric. He was able to use this technique to show the following:

Theorem 2.8 *A Cameron-Liebler line class $\mathcal{L}$ with parameter $x \leq q$ exists only for $x \leq 2$, and corresponds in $\mathcal{Q}^+(5,q)$ to the union of x skew planes.*

This shows that any nontrivial example must have parameter $q < x \leq q^2 - q$.

2.6 Known examples

We now give constructions for the known examples of Cameron-Liebler line classes.

2.6.1 Bruen and Drudge examples

Drudge constructed the first known nontrivial example of a Cameron-Liebler line class in his 1998 doctoral dissertation [12]. His original example was in $\mathrm{PG}(3,3)$ and had parameter $x = 5$. Not long after, he and Bruen [6] generalized this construction to an infinite family of examples in $\mathrm{PG}(3,q)$ having parameter $x = \frac{1}{2}(q^2+1)$ for every odd q. Here we describe their construction.

Let q be odd, and $\mathcal{O} = \mathcal{Q}^-(3,q) \subset \mathrm{PG}(3,q)$ be an elliptic quadric with quadratic form Q. The rank 1 geometry $\mathcal{O}$ has q^2+1 isotropic points and no totally isotropic lines, so no three points of the quadric are collinear; therefore every line of $\mathrm{PG}(3,q)$ contains 0, 1, or 2 points of $\mathcal{O}$. These lines are called *external*, *tangent*, or *secant*, respectively. Each point $\mathbf{p} \in \mathcal{O}$ lies on q^2 secants to $\mathcal{O}$, and so lies on $q+1$ tangent lines. Let $\mathcal{L}_{\mathbf{p}}$ be a set of $\frac{1}{2}(q+1)$ of the tangent lines to $\mathcal{O}$ through $\mathbf{p}$, and let $\mathcal{S}$ be the set of secant lines to $\mathcal{O}$; then

$$\mathcal{L} = (\cup_{\mathbf{p}\in\mathcal{O}}\mathcal{L}_{\mathbf{p}}) \cup \mathcal{S}$$

is a set of

$$\frac{1}{2}(q^2+1)(q+1) + \frac{1}{2}(q^2+1)(q^2) = \frac{1}{2}(q^2+1)(q^2+q+1)$$

lines of $\mathrm{PG}(3,q)$, which is the number of lines in a Cameron-Liebler line class with parameter $\frac{1}{2}(q^2+1)$. The goal is to select the sets $\mathcal{L}_{\mathbf{p}}$ in such a way that $\mathcal{L}$ is in fact a Cameron-Liebler line class.

Every plane of $\mathrm{PG}(3,q)$ is either tangent to $\mathcal{O}$, and so contains a unique point of $\mathcal{O}$, or else intersects $\mathcal{O}$ in a conic. The nontangent plane sections of $\mathcal{O}$ can be used to associate the points and nontangent plane sections of $\mathcal{O}$ with points and circles of the inversive plane $\mathrm{IP}(q)$ [23], so that each circle of $\mathrm{IP}(q)$ corresponds to a section of $\mathcal{O}$ by a nontangent plane π containing $q+1$ tangent lines to $\mathcal{O}$. An *intersecting pencil* of circles is the set of $(q+1)$ circles through two common points of $\mathrm{IP}(q)$, and a *tangent pencil* of circles is a (maximal) set of q mutually tangent circles on a given point of $\mathrm{IP}(q)$.

An equivalence relation $\sim$ can be defined on the circles of $\mathrm{IP}(q)$ by

$$\mathcal{C}_1 \sim \mathcal{C}_2 \iff \exists \mathcal{C} \text{ such that } \mathcal{C} \text{ is tangent to both } \mathcal{C}_1 \text{ and } \mathcal{C}_2.$$

The circles of $\mathrm{IP}(q)$ fall into precisely two equivalence classes under this relation according to whether $Q(\pi^{\perp})$ is a square or nonsquare, where π is the plane containing the circle in question. Let $\mathcal{A}$ be one of these equivalence classes; $\mathcal{A}$ contains exactly $\frac{1}{2}$ of the circles in each intersecting pencil and either all or none of the circles in each tangent pencil. Thus if we define $\mathcal{L}_{\mathbf{p}}$ to be the set of tangent lines at $\mathbf{p}$ contained in a plane section which corresponds to a circle in $\mathcal{A}$, $\mathcal{L}_{\mathbf{p}}$ contains $\frac{1}{2}(q+1)$ of the tangent lines to $\mathcal{O}$ at $\mathbf{p}$.

Bruen and Drudge show that $\mathcal{L}$ is a Cameron-Liebler line class with parameter $\frac{1}{2}(q^2+1)$ by showing the set of lines in $\mathcal{L}$ has a certain "matching" property with respect to the external lines to $\mathcal{O}$ which are the intersection of two tangent planes.

2.6.2 Penttila and Govaerts example in $\mathrm{PG}(3,4)$

Another known example of a nontrivial Cameron-Liebler was constructed by Penttila and Govaerts [14]. This is an example in $\mathrm{PG}(3,4)$ with parameter $x=7$, and was the first known nontrivial example when q is even. So far there has not been a generalization of this construction.

Let π be a plane in $\mathrm{PG}(3,4)$ containing a hyperoval $\mathcal{O}$ and let $\mathbf{p}$ be a point not in π. Define $\mathcal{C}$ to be the cone with base $\mathcal{O}$ and vertex $\mathbf{p}$, with G the set of generators of $\mathcal{C}$, S the set of secants to $\mathcal{C}$ which do not contain a point of $\mathcal{O}$, and E the set of lines in π which are external to $\mathcal{O}$.

Theorem 2.9 *The set $\mathcal{L} = G \cup S \cup E$ is a Cameron-Liebler line class with parameter 7.*

Proof: There are seven types of lines in $\mathrm{PG}(3,4)$ with respect to the cone $\mathcal{C}$ and the distinguished plane π containing $\mathcal{O}$.

1. Generators of $\mathcal{C}$; this is the set $G \subset \mathcal{L}$.
2. Secants to $\mathcal{C}$ which are skew to $\mathcal{O}$; this is the set $S \subset \mathcal{L}$.
3. Lines in π which are skew to $\mathcal{O}$; this is the set $E \subset \mathcal{L}$.
4. Lines through $\mathbf{p}$ not contained in $\mathcal{C}$.
5. Secants to $\mathcal{C}$ which meet a single point of $\mathcal{O}$.
6. Secants to $\mathcal{O}$.
7. Lines skew to $\mathcal{C}$ which are not contained in π.

The points are of 5 types. Here we count the number of lines of each type through a point of each type.

1. $\{\mathbf{p}\}$; of the 21 lines through $\mathbf{p}$, 6 are of type 1, and 15 are of type 4.
2. Points on $\mathcal{C} \setminus (\{\mathbf{p}\} \cup \pi)$; of the 21 lines through such a point, 1 is of type 1, 15 are of type 2, and 5 are of type 5.

3. Points on $\mathcal{O}$; of the 21 lines through such a point, 1 is of type 1, 15 are of type 5, and 5 are of type 6.

4. Points on $\pi \setminus \mathcal{O}$; of the 21 lines through such a point, 9 are of type 2, 2 are of type 3, 1 is of type 4, 3 are of type 6, and 6 are of type 7.

5. Points on $\mathrm{PG}(3,4) \setminus (\mathcal{C} \cup \pi)$; of the 21 lines on such a point, 9 are of type 2, 1 is of type 4, 6 are of type 5, and 5 are of type 7.

From this, we can count that a line in $\mathcal{L}$ meets 50 other lines of $\mathcal{L}$, and a line not in $\mathcal{L}$ meets 35 lines of $\mathcal{L}$. Thus $\mathcal{L}$ is a Cameron-Liebler line class with parameter 7. ■

Unfortunately this construction does not generalize to other values of q in any obvious way, as we do not get the correct number of lines for a Cameron-Liebler line class unless $q = 4$.

3. Methodology

Here we describe some algebraic techniques which we will use to search for new examples of Cameron-Liebler line classes of $\mathrm{PG}(3,q)$. We will search for these as tight sets of $\mathcal{Q}^+(5,q)$; as such, we will develop a model of this quadric which will be convenient for our computational work.

3.1 An eigenvector method for tight sets

Our search for new Cameron-Liebler line classes will be conducted in the context of searching for new x-tight sets of $\mathcal{Q}^+(5,q)$. An eigenvector method will be used to search for these objects, which is due to the following result of Bamberg, Kelly, Law and Penttila [2].

Theorem 3.1 *Let $\mathcal{L}$ be a set of points in $\mathcal{Q}^+(5,q)$ with characteristic vector χ and let A be the collinearity matrix of $\mathcal{Q}^+(5,q)$. Then $\mathcal{L}$ is an x-tight set if and only if*

$$(\chi - \frac{x}{q^2+1}\mathbf{j})A = (q^2-1)(\chi - \frac{x}{q^2+1}\mathbf{j}),$$

where $\mathbf{j}$ is the all-ones vector.

Proof: By definition, $\mathcal{L}$ is an x-tight set if and only if, for $\mathbf{p} \in \mathcal{L}$, $\mathbf{p}$ is collinear with $(q^2-1)+(q+1)x$ other points of $\mathcal{Q}^+(5,q)$ and, for $\mathbf{p} \notin \mathcal{L}$, $\mathbf{p}$ is collinear with $(q+1)x$ points of $\mathcal{Q}^+(5,q)$. Thus $\mathcal{L}$ is an x-tight set if and only if

$$\chi A = (q^2-1)\chi + x(q+1)\mathbf{j}.$$

Since $\mathbf{j}A = q(q+1)^2\mathbf{j}$, the above formula follows immediately. ■

In $\mathcal{Q}^+(5,q)$, there exist two disjoint totally isotropic planes π_1 and π_2; our goal is to find tight sets which are disjoint from $(\pi_1 \cup \pi_2)$. The above method will be slightly modified to account for this. We will let A' be the submatrix of A obtained by throwing away the rows and columns corresponding to points in $(\pi_1 \cup \pi_2)$.

Theorem 3.2 *Let $\mathcal{L}$ be a set of points of $\mathcal{Q}^+(5,q)$ disjoint from π_1 and π_2 and let χ' be the vector obtained from the characteristic vector of $\mathcal{L}$ by removing entries* *corresponding to points of π_1 and π_2. Then $\mathcal{L}$ is an x-tight set of $\mathcal{Q}^+(5,q)$ if and only if*

$$(\chi' - \frac{x}{q^2-1}\mathbf{j})A' = (q^2-1)(\chi - \frac{x}{q^2-1}\mathbf{j}).$$

Proof: Denote the eigenspace of A corresponding to the eigenvalue (q^2-1) by E, and the eigenspace of A' corresponding to the eigenvalue (q^2-1) by E'. Since $\pi_1 \cup \pi_2$ is a 2-tight set,

$$\mu = \chi_{(\pi_1 \cup \pi_2)} - \frac{2}{q^2+1}\mathbf{j} \in E.$$

Let $\mathcal{L}$ be a set of points of $\mathcal{Q}^+(5,q)$ disjoint from π_1 and π_2 with characteristic vector χ; then $\mathcal{L}$ is a x-tight set if and only if

$$\chi - \frac{x}{q^2+1}\mathbf{j} \in E \iff \mathbf{v} = (\chi - \frac{x}{q^2+1}\mathbf{j}) + \frac{x}{q^2-q}\mu \in E.$$

The entries of $\mathbf{v}$ corresponding to points of $(\pi_1 \cup \pi_2)$ are 0, and the entry corresponding to a point $\mathbf{p} \notin (\pi_1 \cup \pi_2)$ is given by

$$(\chi)_{\mathbf{p}} - \frac{x}{q^2+1} - \frac{x}{q^2-1}\frac{2}{q^2+1} = (\chi)_{\mathbf{p}} - \frac{x}{q^2-1}.$$

Thus if we obtain a new vector $\mathbf{v}'$ from $\mathbf{v}$ by throwing away entries corresponding to points in $(\pi_1 \cup \pi_2)$, and a new vector χ' from χ in the same manner,

$$\mathbf{v}' = \chi' - \frac{x}{q^2-1}\mathbf{j} \in E' \iff \chi - \frac{x}{q^2+1}\mathbf{j} \in E \iff \mathcal{L} \text{ is an } x\text{-tight set.}$$

∎

3.2 Tactical Decompositions

For any incidence structure, a *tactical decomposition* is a partition of the points into point classes and the blocks into block classes such that the number of points in a point class which lie on a block depends only on the class in which the block lies, and similarly with points and blocks interchanged. Examples can be obtained by taking as point and block classes the orbits of some collineation group acting on the structure.

The idea of a tactical decomposition can also be extended to matrices. Let $A = [a_{ij}]$ be a matrix, along with a partition of the row indices into subsets R_1, …, R_t and a partition of the column indices into subsets C_1, …, $C_{t'}$. We will call this a *tactical decomposition* of A if for every (i, j), $1 \le i \le t$, $1 \le j \le t'$, the submatrix $[a_{h,\ell}]$ ($h \in R_i$, $\ell \in C_j$) has constant column sums c_{ij} and row sums r_{ij}. A tactical decomposition of an incidence structure corresponds to a tactical decomposition of its incidence matrix. The row and column sum matrices of A are defined to to be $R_A = [r_{ij}]$ and $C_A = [c_{ij}]$ respectively.

Utilizing a tactical decomposition makes finding eigenvectors corresponding to x-tight sets easier, as an eigenvector of the column sum matrix C_A obtained from the decomposition can be used to recover an eigenvector of A. The following result comes from the theory of the interlacing of eigenvalues, which was introduced by Higman and Sims, used by Payne in the study of generalized quadrangles, and further developed by Haemers; see [17] for a detailed survey.

Theorem 3.3 *Suppose the matrix A can be partitioned as*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix} \tag{3.1}$$

with each A_{ii} square, $1 \le i \le k$, and each A_{ij} having constant column sum c_{ij}; then any eigenvalue of the column sum matrix $C_A = [c_{ij}]$ is also an eigenvalue of A.

Proof: An eigenvector of C_A can be expanded according to the partition of A (by duplicating the entries corresponding to each part) to construct an eigenvector of A. ■

To apply this theorem to the task of finding eigenvectors of the collinearity matrix of $\mathcal{Q}^+(5, q)$, we define an incidence structure $\mathcal{H}$ with both "points" and "blocks" being given by the points of $\mathcal{Q}^+(5, q)$, and incidence being given by collinearity. Thus the incidence matrix A of $\mathcal{H}$ is given by the collinearity matrix of $\mathcal{Q}^+(5,q)$. Furthermore, any automorphism of $\mathcal{Q}^+(5,q)$ determines an automorphism of $\mathcal{H}$ in an obvious way. The matrix A is symmetric, and any tactical decomposition arising from an automorphism group of $\mathcal{Q}^+(5,q)$ will induce the same partition on the rows of A and the columns of A. The following theorem gives us a nice relationship between the row and column sums in arising from such a tactical decomposition.

Theorem 3.4 *Let A be a symmetric matrix and let O_1, …, O_k be the parts of a tactical decomposition of A (so the row and column partition is the same) with $|O_i| = o_i$; then $r_{ij} = c_{ji}$, and $o_i r_{ij} = o_j c_{ij}$.*

Proof: If A_{ij} is the submatrix associated with the row part corresponding to O_i and the column part corresponding to O_j, then $A_{ij} = A_{ji}^T$, thus $r_{ij} = c_{ji}$ for all i, j. Also, each of the o_i rows of A_{ij} has row sum r_{ij}, and each of the o_j columns has column sum c_{ij}. Summing over all entries of A_{ij} in two ways gives $o_i r_{ij} = o_j c_{ij}$. ■

Corollary 3.5 *Let A be a symmetric matrix with a tactical decomposition having the same parts for rows and columns, with part i containing o_i rows/columns; then we have the following relationship between the row and column sum matrices:*

$$R_A^T = [r_{ji}] = [c_{ij}] = [\frac{o_i}{o_j} r_{ij}] = C_A.$$

3.3 A model of $\mathcal{Q}^+(5,q)$

We now describe a model for $\mathcal{Q}^+(5,q)$ which gives us a range of algebraic tools to use in searching for tight sets. Let $F = \mathbb{F}_q$, $E = \mathbb{F}_{q^3}$, and

$$\mathrm{T} = \mathrm{T}_{E/F} : x \mapsto x^{q^2} + x^q + x.$$

We consider $\mathcal{Q}^+(5,q)$ to have $V = E^2$ as its underlying vector space, considered over F and equipped with the quadratic form

$$Q : (x,y) \mapsto \mathrm{T}(xy).$$

The polar form B of Q is then given by

$$\mathrm{B}((u_1,u_2),\ (v_1,v_2)) = \mathrm{T}(u_1v_2) + \mathrm{T}(u_2v_1).$$

This form is nondegenerate, since if $(v_1, v_2) \in V$ has $\mathrm{B}((v_1,v_2),\ (x,y)) = 0$ for all $(x, y) \in V$, then $\mathrm{T}(v_1y) + \mathrm{T}(v_2x) = 0$ for all $(x, y) \in V$. Setting $x = 0$ forces

$$\mathrm{T}(v_1y) = v_1y + v_1^q y^q + v_1^{q^2} y^{q^2} = 0 \text{ for all } y \in E^*,$$

thus $v_1 = 0$. Likewise, setting $y = 0$ can be seen to force $v_2 = 0$, and so $(v_1, v_2) = (0, 0)$.

It can be also be seen that

$$\pi_1 = \{(x, 0) :\ x \in E^*\} \text{ and}$$
$$\pi_2 = \{(0, y) :\ y \in E^*\}$$

are totally isotropic planes with respect to this form. This shows that the quadric defined by Q has Witt index 3, and so is hyperbolic.

3.4 The general method

Theorem 3.6 *Let $q \not\equiv 1 \bmod 3$. Take $\mu \in E^*$ with $|\mu| = q^2 + q + 1$, and define the map g on $\mathcal{Q}^+(5, q)$ by*

$$g : (x, y) \mapsto (\mu x, \mu^{-1} y);$$

then the group $C = \langle g \rangle \leq PGO^+(6, q)$ and has $|C| = q^2 + q + 1$. This group acts semi-regularly on the points of $\mathcal{Q}^+(5, q)$ and stabilizes the totally isotropic planes π_1 and π_2.

Proof: It is clear that g is an isometry of $\mathcal{Q}^+(5, q)$ having order $q^2 + q + 1$. To see that C acts semi-regularly on the points of $\mathcal{Q}^+(5, q)$, notice that $g^i\left((x, y)\right) = (x, y)$ implies that $\mu^i \in F$. But

$$(q^2 + q + 1, q - 1) = (q - 1, 3) = 1,$$

since $q \not\equiv 1 \bmod 3$. Thus this can only happen when $\mu^i = 1$, and so the identity is the only element of this group fixing a point. ∎

If α is a primitive element of E, we can without loss of generality assume that $\mu = \alpha^{q-1}$. The semi-regular action of C on $\mathcal{Q}^+(5, q)$ gives us the following result.

Theorem 3.7 *Let A be the collinearity matrix of $\mathcal{Q}^+(5, q)$, $q \not\equiv 1 \bmod 3$, with a tactical decomposition induced by the action of the cyclic group C defined above; then the row of each submatrix of the decomposition is the same as the column sum. Thus the decomposition matrix (which is the same for row sums and column sums) is symmetric.*

Proof: This follows directly from 3.4 since all orbits have the same size. ∎

Since each orbit has size $q^2 + q + 1$, a union of x orbits contains the right number of points to be an x-tight set of $\mathcal{Q}^+(5, q)$. Our goal will be to find ways of combining these orbits which will result in an x-tight set. We accomplish this by considering large subgroups $G \leq N_{P\Gamma O^+(6,q)}(C)$ having relatively few orbits on the points of $\mathcal{Q}^+(5, q)$. The orbits of such a group are unions of orbits of C. We use such a group G to induce a tactical decomposition on the points of $\mathcal{Q}^+(5, q)$, and then use this decomposition to form the column sum matrix B of the collinearity matrix A, after throwing away the entries corresponding to points in π_1 and π_2. The eigenspace of B for the eigenvalue $q^2 - 1$ is then searched for eigenvectors having a form corresponding to an x-tight set of $\mathcal{Q}^+(5, q)$. Whenever new examples show a pattern, e.g. a common formula for x in terms of q or a similar stabilizing group, algebraic, geometric, and combinatorial details are analyzed in an attempt to find a construction for a new infinite family of tight sets.

4. New examples

Throughout this chapter, we let $q \not\equiv 1 \bmod 3$, $E = \mathbb{F}_{q^3}$ with $E^* = \langle \alpha \rangle$, and $F = \mathbb{F}_q \leq E$ with $F^* = \langle \omega \rangle$ where $\omega = \alpha^{q^2+q+1}$. The hyperbolic quadric $\mathcal{Q}^+(5,q)$ is defined over the vector space $V = E^2$ (considered over F) and has quadratic form $Q((x,y)) = T(xy)$, where $T = T_{F/E}$, and polar form $\mathrm{B}(\mathbf{u},\mathbf{v}) = T(u_1v_2) + T(u_2v_1)$ as described in Chapter 3. Put $\mu = \alpha^{q-1}$, and define the cyclic group $C = \langle g \rangle$, where

$$g : (x,y) \mapsto (\mu x, \mu^{-1} y);$$

then C acts semi-regularly on the points of $\mathcal{Q}^+(5,q)$ and stabilizes the disjoint totally isotropic planes $\pi_1 = \{(x,0) : x \in E^*\}$ and $\pi_2 = \{(0,y) : y \in E^*\}$.

Below is a summary of the new examples of Cameron-Liebler line classes which are described in this chapter. Notice that here we consider the parameter of the line class to be smaller than that of its complement, and we take the line class to be disjoint from $(\pi_1 \cup \pi_2)$; thus a new example with parameter x described below also gives new line classes with parameter $x+1$, $x+2$, $(q^2+1)-x$, q^2-x, and $(q^2-1)-x$.

x	q	$\mathrm{Aut}(\mathcal{L})$
$\frac{1}{2}(q^2-1)$	$q \equiv 5$ or $9 \bmod 12$ and $q < 200$	$(\mathbb{Z}_{q^2+q+1} \times \mathbb{Z}_{\frac{1}{4}(q-1)}) \rtimes \mathbb{Z}_3$
$\frac{1}{3}(q+1)^2$	$q \equiv 2 \bmod 3$ and $q < 150$	$\mathbb{Z}_{q^2+q+1} \rtimes \mathbb{Z}_3$
336	$q = 27$	$(\mathbb{Z}_{q^2+q+1} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$
495	$q = 32$	$\mathbb{Z}_{q^2+q+1} \rtimes \mathbb{Z}_{15}$

Table 4.1: *Parameters and automorphism groups of the new examples of Cameron-Liebler line classes constructed.*

4.1 New examples with parameter $\frac{1}{2}(q^2-1)$

Here we describe a construction giving many new examples of tight sets in $\mathcal{Q}^+(5,q)$ having parameter $\frac{1}{2}(q^2-1)$. This construction requires us to have $q \equiv 5$ or $9 \bmod 12$, and has resulted in new tight sets for all such $q < 200$.

4.1.1 The construction

Let $\mathcal{S} = \{x : x \in E^* | \ T(x) = 0\}$; then the orbits of C on the points of $\mathcal{Q}^+(5,q)$ are $\pi_1 = (1,0)^C$, $\pi_2 = (0,1)^C$, and $(1,x)^C$ for each $x \in \mathcal{S}$. We also let the group $H = \langle h \rangle$, where

$$h : (x,y) \mapsto (x, \omega^4 y),$$

act on the space, and put $G = \langle C, H \rangle$.

Lemma 4.1 *The group H defined above centralizes C, and intersects C trivially.*

Proof: To show that H centralizes C, we only need to show that g and h commute; we have that

$$\begin{aligned} h(g((x,y))) &= h((\mu x, \mu^{-1}y)) = (\mu x, \mu^{-1}\omega^4 y) \quad \text{and} \\ g(h((x,y))) &= g((x, \omega^4 y)) \qquad = (\mu x, \mu^{-1}\omega^4 y). \end{aligned}$$

Since the powers of μ are pairwise independent over F, it is clear that $H \cap C$ contains only the identity. ■

Corollary 4.2 *The group G defined above is equal to $C \times H$, and so $|G| = \frac{1}{4}(q-1)(q^2+q+1)$.*

Furthermore, it can be seen that G acts semiregularly on the points of $\mathcal{Q}^+(5,q) \setminus (\pi_1 \cup \pi_2)$, as H acts semi-regularly on those points, and induces a semi-regular action on those orbits of C as well.

Let S_k be the subset of $\mathcal{S}$ containing the elements with $\log_\alpha(x) \equiv k \bmod 4$ for $0 \le k \le 3$. For $x \in \mathcal{S}$, put $\bar{x} = \{\omega^{4t}x : \ 0 \le t < (q-1)/4\}$; for shorthand we will write $(1, \bar{x}) = \{(1, x') : \ x' \in \bar{x}\}$. Now we define

$$\kappa(x, y) := \left|(1, x)^\perp \cap (1, \bar{y})^C\right|.$$

In terms of the tactical decomposition induced by G on $\mathcal{Q}^+(5, q)$, $\kappa(x, y)$ is the number of points in $(1, \bar{y})^C$ collinear with any given point of $(1, \bar{x})^C$. Thus $\kappa(x', y') = \kappa(x, y)$ for all $x' \in \bar{x}$ and $y' \in \bar{y}$.

Let A be the matrix obtained from the tactical decomposition induced by G on the collinearity matrix of $\mathcal{Q}^+(5, q)$ after throwing away the entries corresponding to points in $\pi_1 \cup \pi_2$. We use a specific ordering of the orbits of G to define A. Notice that S_0 contains $\frac{1}{4}(q^2 - 1)$ elements of E^*, and so contains $q + 1$ equivalence classes of the form $\bar{x}$. Let x_0, …, x_q be representatives from these $q + 1$ orbits. We order the orbits as

$$(1, \bar{x}_0)^C, \ \ldots, \ (1, \bar{x}_q)^C, \ (1, \omega\bar{x}_0)^C, \ldots, (1, \omega\bar{x}_q)^C,$$
$$(1, \omega^2\bar{x}_0)^C, \ldots, (1, \omega^2\bar{x}_q)^C, (1, \omega^3\bar{x}_0)^C, \ldots, (1, \omega^3\bar{x}_q)^C.$$

Now A can be described as follows:

$$A = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ A_3 & A_0 & A_1 & A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_1 & A_2 & A_3 & A_0 \end{bmatrix},$$

where $A_k = \left(\kappa(x_i, \omega^k x_j)\right)_{0 \le i,j \le q}$ for $0 \le k \le 3$. This matrix is block-circulant, which allows us to apply the following result on eigenvectors of block-circulant matrices due to Garry Tee [30].

Theorem 4.3 *Let ζ be any fourth root of unity, and A be a block-circulant matrix as defined above, with blocks A_0, A_1, A_2, A_3 each having size $n/4$. Take a vector $\mathbf{v} \in \mathbb{R}^{\frac{n}{4}}$. Then the vector*

$$\mathbf{w} = [\mathbf{v}\ \zeta\mathbf{v}\ \zeta^2\mathbf{v}\ \zeta^3\mathbf{v}]$$

is an eigenvector of A for λ if and only if $\mathbf{v}$ *is an eigenvector of* $A_0+\zeta A_1+\zeta^2 A_2+\zeta^3 a_3$ *for* λ.

We now investigate some properties of κ in order to better understand the structure of A.

Lemma 4.4 *For $x, y \in \mathcal{S}$ (not necessarily distinct), $\kappa(x,y) = \kappa(y,x)$.*

Proof: This follows directly from Theorem 3.4, along with the fact that $(1,\bar{x})^C$ and $(1,\bar{y})^C$ are the same size. ■

Corollary 4.5 *A is symmetric; thus A_0 and A_2 are symmetric, and $A_1 = A_3^T$.*

Lemma 4.6 *For $x, y \in S_0$ (not necessarily distinct), $\kappa(x, \omega^k y) = \kappa(y, \omega^k x)$ for $0 \leq k \leq 3$.*

Proof: First we notice that $\langle \mu \rangle$ contains q^2+q+1 distinct elements of E^*, no two differing by a multiple in F. Thus, for any $z \in E^*$, there exists an integer $0 \leq j < q^2+q+1$ such that $\mu^j = \omega^s z$ for some $0 \leq s < (q-1)$. Since $\mu = \alpha^{q-1}$ and $q \equiv 1 \bmod 4$,

$$\log_\alpha(\omega^s z) = \log_\alpha(\mu^j) = \log_\alpha(\alpha^{j(q-1)}) \equiv 0 \bmod 4.$$

Now take x, y in S_0; we have that

$$\kappa(x, \omega^k y) = \sum_{0 \leq t < (q-1)/4} |\{i :\ 0 \leq i < q^2+q+1 |\ T(\mu^i x + \mu^{-i}\omega^{4t}\omega^k y) = 0\}|.$$

There exists (a unique) j with $0 \leq j < q^2+q+1$ such that $\mu^j = \omega^{4s}x^{-1}y$ for some s (since $\log_\alpha(xy^{-1}) \equiv 0 \bmod 4$), and so

$$\mu^{i+j}x + \mu^{-(i+j)}\omega^{4t}\omega^k y = \mu^i\omega^{4s}y + \mu^{-i}\omega^{4(t-s)}\omega^k x.$$

From this we can see (by relabeling indices) that

$$\begin{aligned}\kappa(x,\omega^k y) &= \sum_{0\le t<(q-1)/4} |\{i:\ 0\le i<q^2+q+1|\ T(\mu^i x+\mu^{-i}\omega^{4t}\omega^k y)=0\}| \\ &= \sum_{0\le t<(q-1)/4} |\{i:\ 0\le i<q^2+q+1|\ T(\mu^i \omega^{4s} y+\mu^{-i}\omega^{4t}\omega^k x)=0\}| \\ &= \kappa(\omega^{4s}y,\omega^k x)=\kappa(y,\omega^k x).\end{aligned}$$

■

Lemma 4.7 *For $x, y \in S_0$ (not necessarily distinct), $\kappa(x, \omega y) = \kappa(x, \omega^3 y)$.*

Proof: We have that $\kappa(x,\omega^3 y)=\kappa(\omega x,\omega^4 y)=\kappa(\omega x,y)=\kappa(y,\omega x)=\kappa(x,\omega y)$. ■

Corollary 4.8 *$A_1 = A_3$, and so all blocks of A are symmetric.*

The following two lemmas have been verified computationally. It seems reasonable to attempt to prove that they will hold true for all values of $q \equiv 5$ or $9 \bmod 12$, although it is not clear why this would occur.

Lemma 4.9 *If $q < 200$, then for all $x \in S_0$, $\kappa(x,x) - \kappa(x,\omega^2 x) = -1$.*

Proof: Computed with Magma, see Appendix B. ■

Lemma 4.10 *If $x < 200$, then for all $x, y \in S_0$ with $x \neq y$, $\kappa(x,y) - \kappa(x,\omega^2 y) = \pm q$.*

Proof: Computed with Magma, see Appendix B. ■

Lemma 4.11 *If $x < 200$, then for $x, y, z \in S_0$, distinct, $\kappa(x,y) - \kappa(x,\omega^2 y) = \kappa(x,z) - \kappa(x,\omega^2 z)$ if and only if $\kappa(y,z) - \kappa(y,\omega^2 z) = q$.*

Proof: Computed with Magma, see Appendix B. ■

Theorem 4.12 *If $q < 200$ is a prime power congruent to either 5 or $9 \bmod 12$, then there exists an x-tight set $\mathcal{L}$ in $\mathcal{Q}^+(5,q)$ with $x = \frac{1}{2}(q^2-1)$ in $\mathrm{PG}(3,q)$ stabilized by a cyclic group of order q^2+q+1 acting semi-regularly on the points. We also have* *that $\mathcal{L}$ is disjoint from a trivial 2-tight set consisting of a union of two skew totally isotropic planes.*

Proof: We have that i is a fourth root of unity. The matrix

$$H = A_0 + iA_1 + i^2 A_2 + i^3 A_3 = A_0 + iA_1 - A_2 - iA_3 = A_0 - A_2$$

has a nice form; all of the diagonal entries are -1, and all other entries are $\pm q$. Furthermore, there is some partition of $\{x_0, \ldots, x_q\}$, into parts L_1 and L_2 such that $H_{ij} = q$ if and only if $i \neq j$ and x_i, x_j are in the same part; say $|L_1| = a$ and $|L_2| = b$, with $a + b = q + 1$. If we take K to be the adjacency matrix of the graph $K_{L_1} \oplus K_{L_2}$ (where K_{L_1} and K_{L_2} are the complete graphs on the sets L_1 and L_2, respectively) and K' be the adjacency matrix of the complement of this graph, then $H = qK - qK' - I$. We will form the vector $\mathbf{v} = \chi_{L_1} - \frac{1}{2}\mathbf{j} = \frac{1}{2}\chi_{L_1} - \frac{1}{2}\chi_{L_2}$. Notice that $\frac{x}{(q^2-1)} = \frac{1}{2}$ if we let $x = \frac{q^2-1}{2}$. Now we have that

$$\begin{aligned}
(\chi_{L_1} - \chi_{L_2})H &= (\chi_{L_1} - \chi_{L_2})(qK - qK' - I) \\
&= ((a-1)q\chi_{L_1} - aq\chi_{L_2} - \chi_{L_1}) - ((b-1)q\chi_{L_2} - bq\chi_{L_1} - \chi_{L_2}) \\
&= ((a+b-1)q - 1)\,\chi_{L_1} - ((a+b-1)q - 1)\,\chi_{L_2} \\
&= (q^2 - 1)(\chi_{L_1} - \chi_{L_2}),
\end{aligned}$$

thus $\mathbf{v}$ is an eigenvector of H having the desired form. We can use $\mathbf{v}$ to construct an eigenvector of A in four different ways, namely

$$\pm[\,\mathbf{v} \; -\mathbf{v} \; -\mathbf{v} \quad \mathbf{v}\,]$$
$$\pm[\,\mathbf{v} \quad \mathbf{v} \; -\mathbf{v} \; -\mathbf{v}\,]$$

each of which in turn can be used to construct an eigenvector of the collinearity matrix M of $\mathcal{Q}^+(5, q)$ corresponding to a $\frac{1}{2}(q^2 - 1)$-tight set. ■

4.1.2 Some details of these examples

Our examples with parameter $\frac{1}{2}(q^2 - 1)$ have a group isomorphic to $\mathbb{Z}_{q^2+q+1} \times \mathbb{Z}_{\frac{1}{4}(q-1)}$ acting on them, by construction. By observing details about the orbits used in the construction, we notice that these examples are also stabilized by $\mathrm{Aut}(\mathbb{F}_{q^3})$, thus they are stabilized by a group isomorphic to $(\mathbb{Z}_{q^2+q+1} \times \mathbb{Z}_{\frac{1}{4}(q-1)}) \rtimes \mathbb{Z}_3$. For those examples small enough to compute their full stabilizer in $P\Gamma L(4, q)$, which are those with $q \leq 41$, this is in fact the full group.

We can also compute intersection numbers of these line classes in $\mathrm{PG}(3, q)$ with respect to planes and point stars of $\mathrm{PG}(3, q)$; this becomes prohibitively expensive, computationally, when $q > 32$. Here we include details for some small values of q, as well as $q = 81$; this special case was of particular interest, see Chapter 5, so a considerable amount of time was dedicated to computing these values.

In this table, we include the intersection numbers with respect to planes of $\mathrm{PG}(3, q)$; the examples considered here are all isomorphic to their dual, and so have the same intersection numbers with the same multiplicities with respect to the point stars of $\mathrm{PG}(3, q)$.

q	x	Intersection numbers, with multiplicity; we have $r = q^2 + q + 1$
5	12	$\mathbf{0}^{(1)}$, $6^{(r)}$, $\mathbf{12}^{(2r)}$, $18^{(r)}$, $24^{(r)}$
9	40	$\mathbf{0}^{(1)}$, $30^{(4r)}$, $\mathbf{40}^{(r)}$, $60^{(4r)}$
17	144	$\mathbf{0}^{(1)}$, $108^{(4r)}$, $126^{(4r)}$, $\mathbf{144}^{(r)}$, $180^{(4r)}$, $198^{(4r)}$
29	420	$\mathbf{0}^{(1)}$, $330^{(7r)}$, $390^{(7r)}$, $\mathbf{420}^{(r)}$, $480^{(7r)}$, $540^{(7r)}$
81	3280	$\mathbf{0}^{(1)}$, $2952^{(40r)}$, $\mathbf{3280}^{(r)}$, $3690^{(40r)}$

Table 4.2: *Intersection numbers of line classes with parameter $\frac{1}{2}(q^2 - 1)$ with the planes of* $\mathrm{PG}(3, q)$.

The numbers in bold represent the intersection numbers for the plane corresponding to π_1 in $\mathcal{Q}^+(5, q)$, which is disjoint from our line class, and the planes through the point which corresponds to π_2 in $\mathcal{Q}^+(5, q)$, which shares $\frac{1}{2}(q^2 - 1)$ lines with our line class. It is worth noting that the number of lines shared by each plane with the line class is divisible by $(q+1)$. The multiplicities being divisible by q^2+q+1 is a side effect of C acting semi-regularly on the planes of $\mathrm{PG}(3,q)$ not corresponding to π_1.

The new examples of tight sets in $\mathcal{Q}^+(5,q)$ give examples of two-intersection sets, two-weight codes, and strongly regular graphs, as detailed in Chapter 2. While there are tables of known strongly regular graphs, these examples on q^6 vertices are too large to appear. Furthermore, if there were known examples, checking isomorphism would most likely be unreasonable.

4.2 New examples with parameter $\frac{1}{3}(q+1)^2$

Here we give details on many new examples which have been constructed in joint work with Jan De Beule, Klaus Metsch, and Jeroen Demeyer, having parameter $\frac{1}{3}(q+1)^2$. These examples have been constructed for all values of $q \equiv 2 \bmod 3$ which are computationally feasible. Unfortunately, in general these examples do not exhibit much symmetry; all of the examples found have $C \rtimes \mathrm{Aut}(\mathbb{F}_{q^3})$ as their full stabilizing group. When q is prime, this does not give much to work with. These are found through more of a search than a construction; first we put together the tactical decomposition matrix for $\mathcal{Q}^+(5,q)\backslash(\pi_1 \cup \pi_2)$ with respect to the group, then we search over the eigenspace for appropriate eigenvectors (see Appendix A for details about the algorithms used). With a small stabilizer, there are lots of orbits; for example, if q is prime, there are $\frac{1}{3}(q^2-1)$ orbits to consider. As such, forming the matrix for the tactical decomposition is a large task. Furthermore, we do not currently have a good method for reducing the size of the eigenspace to search over, so finding these examples is computationally infeasible if the eigenspace of the tactical decomposition matrix is too large (12 dimensions starts to push the limits of our computing power).

An important subcase of these examples occurs when $q=2^e$, where $e>1$ and odd. In this case, we have a slightly larger stabilizing group to work with. These examples are also of particular interest since there is only one previously known Cameron-Liebler line class in $\mathrm{PG}(3,q)$ for q even (see Chapter 2 for this construction). New examples with this parameter have been found for $q \in \{8, 32, 128\}$, as well as for odd primes $q \leq 100$ which are congruent to 2 mod 3, and for $q = 125$. In all of the cases where it is feasible to compute the stabilizer group ($q \leq 32$), we have that $C \rtimes \mathrm{Aut}(\mathbb{F}_{q^3})$ is the full group. Below, we describe how some of these line classes intersect planes of $\mathrm{PG}(3, q)$. All of the examples considered below are isomorphic to their dual, and so have the same intersection numbers with the same multiplicities with respect to point stars of $\mathrm{PG}(3, q)$.

q	x	Intersection numbers, with multiplicity; we have $r = q^2 + q + 1$
5	12	$\mathbf{0}^{(1)},\ 6^{(r)},\ \mathbf{12}^{(2r)},\ 18^{(r)},\ 24^{(r)}$
8	27	$\mathbf{0}^{(1)},\ 18^{(3r)},\ \mathbf{27}^{(r)},\ 36^{(3r)},\ 54^{(3r)}$
11	48	$\mathbf{0}^{(1)},\ 24^{(r)},\ 36^{(2r)},\ \mathbf{48}^{(4r)},\ 60^{(2r)},\ 72^{(r)},\ 96^{(r)}$
17	108	$\mathbf{0}^{(1)},\ 72^{(3r)},\ 90^{(3r)},\ \mathbf{108}^{(4r)},\ 126^{(3r)},\ 144^{(3r)},\ 216^{(r)}$
23	192	$\mathbf{0}^{(1)},\ 120^{(r)},\ 144^{(4r)},\ 168^{(3r)},\ \mathbf{192}^{(6r)},\ 216^{(3r)},\ 240^{(4r)},\ 264^{(r)},\ 384^{(r)}$
32	363	$\mathbf{0}^{(1)},\ 264^{(5r)},\ 330^{(10r)},\ \mathbf{363}^{(r)},\ 396^{(10r)},\ 462^{(5r)},\ 726^{(r)}$

Table 4.3: *Intersection numbers of line classes with parameter $\frac{1}{3}(q+1)^2$ with the planes of* $\mathrm{PG}(3, q)$.

4.3 Some other new examples

We also have a couple of other new examples which do not currently seem to fit into a nice grouping. These examples have been found by assuming a group acting on the points of $\mathcal{Q}^+(5, q)$ (usually a subgroup of $N_{O^+(6,q)}(C)$), forming the orbits of $\mathcal{Q}^+(5, q) \setminus (\pi_1 \cup \pi_2)$ and the associated matrix for the tactical decomposition, and searching over all possible parameters. The number of possible parameters can be very large, especially if the orbits are not all the same size. It was computationally feasible when $q \leq 23$ to assume that the examples we were looking for admitted the group $C \rtimes \mathrm{Aut}(\mathbb{F}_{q^3}/\mathbb{F}_q)$ as a stabilizer, though these searches did not yield any new examples.

For $q = 27$, the variation in the orbit sizes gives a large number of possible parameters, and there is a relatively large eigenspace to consider. Thus, considering a small stabilizing group was not feasible. By assuming the group $C \rtimes \mathrm{Aut}(\mathbb{F}_{q^3})$ stabilized the examples, we were able to find a new tight set with parameter 336. This example is also stabilized by the map $(x, y) \mapsto (x, -y)$, and so has full stabilizer isomorphic to $(\mathbb{Z}_{q^2+q+1} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$. Restricting our first search with $C \rtimes \mathrm{Aut}(\mathbb{F}_{q^3}/\mathbb{F}_q)$ by assuming our parameter was divisible by $q + 1$, we found no other new examples.

With $q = 32$, all of the point orbits of the group $C \rtimes \mathrm{Aut}(\mathbb{F}_{q^3})$ on $\mathcal{Q}^+(5, q)$ have the same size, so the search is feasible assuming this stabilizer. We are able to find two new examples having parameter 495, each having $C \rtimes \mathrm{Aut}(\mathbb{F}_{32^3}/\mathbb{F}_2)$ as their full stabilizer. In this case, these two examples are isomorphic as tight sets, but not as Cameron-Liebler line classes. In $\mathrm{PG}(3, q)$, they are dual to one another.

Below we detail how these examples intersect the planes of $\mathrm{PG}(3, q)$. Note that only the first example is self-dual; in this case, the intersection numbers and multiplicities for point stars of $\mathrm{PG}(3, q)$ are the same as for planes. For the two examples with $q = 32$, the plane intersection numbers of one example are the point star intersection numbers for the other, and vice-versa.

q	x	Intersection numbers, with multiplicity; we have $r = q^2 + q + 1$
27	336	$\mathbf{0}^{(1)},\ 252^{(6r)},\ \mathbf{336}^{(13r)},\ 420^{(6r)},\ 504^{(2r)}$
32	495	$\mathbf{0}^{(1)},\ 330^{(r)},\ 396^{(5r)},\ 462^{(10r)},\ \mathbf{495}^{(r)},\ 528^{(5r)},\ 594^{(5r)}$
32	495	$\mathbf{0}^{(1)},\ 396^{(10r)},\ \mathbf{495}^{(r)},\ 528^{(15r)},\ 660^{(6r)}$

Table 4.4: *Intersection numbers of some other new line classes.*

5. Planar two-intersection sets

A set of type (m, n) in a projective or affine plane is a set $\mathcal{K}$ of points such that every line of the plane contains either m or n points of $\mathcal{K}$; we require that $m < n$, and we want both values to occur. For projective planes, there are many examples of these types of sets with q both even and odd, however the situation is quite different for affine planes. When q is even, we obtain a set of type $(0, 2)$ in $\mathrm{AG}(2, q)$ from a hyperoval of the corresponding projective plane, and similarly a set of type $(0, n)$ from a maximal arc of degree n. Examples of sets of type (m, n) in affine planes of odd order, on the other hand, are extremely scarce; the only previously known examples exist in affine planes of order 9 [28].

Here we examine some combinatorial properties of these point sets in both affine and projective planes, and review the current state of the art for the affine situation. We then look at how affine sets of type (m, n) can be constructed from some of the Cameron-Liebler line classes found in Chapter 3.

5.1 Projective examples

We can deduce some information about sets of type (m, n) in projective planes through elementary counting.

Lemma 5.1 *Let $\mathcal{K}$ be a set of type (m, n) in a projective plane π of order q. Let t_m and t_n be the number of m-secants and n-secants to $\mathcal{K}$, and let $k = |\mathcal{K}|$; then*

$$t_m + t_n = q^2 + q + 1, \tag{5.1}$$

$$mt_m + nt_n = k(q + 1), \text{ and} \tag{5.2}$$

$$m(m-1)t_m + n(n-1)t_n = k(k-1). \tag{5.3}$$

Proof: Each of the $q^2 + q + 1$ lines in π is either a m-secant or an n-secant, giving us (5.1). Counting over all secants, t_m meet m points of $\mathcal{K}$, and t_n meet n points of $\mathcal{K}$; this counts each of the k points of $\mathcal{K}$ $q + 1$ times, giving (5.2). To obtain (5.3), we notice that each of the t_m m-secants contains $m(m-1)$ ordered pairs of points in

$\mathcal{K}$, and each of the t_n n-secants contains $n(n-1)$ such pairs. Each of the $k(k-1)$ ordered pairs of points in $\mathcal{K}$ is counted once in this manner. ■

Corollary 5.2 *If we have a set $\mathcal{K}$ of type (m,n) in a projective plane of order q, then $k = |\mathcal{K}|$ must satisfy*

$$k^2 - k(q(n+m-1)+n+m) + mn(q^2+q+1) = 0. \tag{5.4}$$

If we take a fixed point $\mathbf{p} \in \mathcal{K}$, and let ρ_m and ρ_n be the number of m-secants and n-secants through $\mathbf{p}$, respectively, we see that

$$\rho_m + \rho_n = q+1 \text{ and}$$
$$(m-1)\rho_m + (n-1)\rho_n = k-1.$$

From this, we see that

$$\rho_m = (n(q+1)-k-q)/(n-m) \text{ and}$$
$$\rho_n = (k+q-m(q+1))/(n-m),$$

and so ρ_m and ρ_n do not depend on our choice of $\mathbf{p}$. Likewise, if we take a fixed point $\mathbf{q} \notin \mathcal{K}$, and let σ_m and σ_n be the number of m-secants and n-secants through $\mathbf{q}$, we see that

$$\sigma_m + \sigma_n = q+1 \text{ and}$$
$$m\sigma_m + n\sigma_n = k.$$

Again, these values can be seen to be independent of our choice of $\mathbf{q}$; we have that

$$\sigma_m = (n(q+1)-k)/(n-m) \text{ and}$$
$$\sigma_n = (k-m(q+1))/(n-m).$$

From these numbers we see that, given a set of type (m,n) in a projective plane of order q, we can construct three other related sets with two intersection numbers (for a proof, see [18]).

Theorem 5.3 *Let $\mathcal{K}$ be a set of type (m, n) in a projective plane π of order q, with $|\mathcal{K}| = k$.*

1. *The complement of $\mathcal{K}$ is a set of type $(q + 1 - n, q + 1 - m)$ in π containing $q^2 + q + 1 - k$ points.*

2. *The set of m-secants to $\mathcal{K}$ is a set of type (ρ_m, σ_m) in the dual plane to π containing t_m points.*

3. *The set of n-secants to $\mathcal{K}$ is a set of type (ρ_m, σ_m) in the dual plane to π containing t_n points.*

Notice that $\rho_n - \sigma_n = q/(n - m)$ and so it is necessary for $n - m$ to divide q. If $n - m = q$, then $\mathcal{K}$ can be seen to be either the set of points on a common line, or the complement of this; the examples having $n - m = 1$ are dual to these, and we consider the examples in either of these situations to be trivial.

One major class of examples are the sets of type $(0, n)$. These examples are also known as maximal arcs of degree n, or as $(qn - q + n, n)$-arcs (as they necessarily contain $qn - q + n$ points). The prototypical examples are given by hyperovals, which are sets of type $(0, 2)$; other families of maximal arcs of degree larger than 2 have been described by Denniston [11], Thas [31] [32], and Mathon [21]. Maximal arcs of degree 2^a are known to exist in $\mathrm{PG}(2, 2^e)$ for all pairs (a, e) with $0 < a < e$, and it was proven by Ball, Blokhuis and Mazzoca [1] that there are no examples in $\mathrm{PG}(2, q)$ for odd q.

5.2 Affine examples

We will be concerned with sets of type (m, n) in affine planes. Through elementary counting, we get formulas very similar to those in Lemma 5.1, but giving different results.

Lemma 5.4 *Let $\mathcal{K}$ be a set of type (m,n) in an affine plane π of order q. Let t_m and t_n be the number or m-secants and n-secants to $\mathcal{K}$, and let $k = |\mathcal{K}|$; then*

$$t_m + t_n = q^2 + q, \tag{5.5}$$

$$mt_m + nt_n = k(q+1), \text{ and} \tag{5.6}$$

$$m(m-1)t_m + n(n-1)t_n = k(k-1). \tag{5.7}$$

These modified formulas lead to the following alternate version of Corollary 5.2.

Corollary 5.5 *If we have a set $\mathcal{K}$ of type (m,n) in an affine plane of order q, then $k = |\mathcal{K}|$ must satisfy*

$$k^2 - k(q(n+m-1) + n + m) + mnq(q+1) = 0.$$

We again get constant values ρ_m and ρ_n for the number of m-secants and n-secants through a point in $\mathcal{K}$, and σ_m and σ_n for the number of m-secants and n-secants through a point not in $\mathcal{K}$, given by the formulas

$$\rho_m = (n(q+1) - k - q)/(n-m),$$

$$\rho_n = (k + q - m(q+1))/(n-m),$$

$$\sigma_m = (n(q+1) - k)/(n-m), \text{ and}$$

$$\sigma_n = (k - m(q+1))/(n-m).$$

This tells us that, as for the projective case, we must have $n-m$ dividing q. However, since the dual of an affine plane is not again an affine plane, we do not have results about the m-secants or n-secants forming another planar set with two intersection numbers.

There are very few known examples of sets of type (m,n) in affine planes. For planes of even order, we can obtain an example from a set of type $(0,n)$ in a projective plane, by choosing an external line to the set as the line at infinity to form the affine plane. However, sets of this type do not exist in projective planes of odd order.

In affine planes of odd order, the only previously known examples of sets of type (m, n) are sets of type $(3, 6)$ in planes of order 9. These sets were found through an exhaustive computer search, see [28], and examples were found in each of the four projective planes of order 9.

The size k of a set of type $(3, 6)$ in a plane of order 9 must satisfy

$$k^2 - 81k + 1620 = 0$$

which has solutions $k_1 = 36$ and $k_2 = 45$. The complement of a set of type $(3, 6)$ in a plane of order 9 will again be a set of type $(3, 6)$, and the complement of a set of size k_1 will contain k_2 points. The 45-sets of type $(3, 6)$ have $\rho_3 = 2$, $\rho_6 = 8$, $\sigma_3 = 5$, and $\sigma_6 = 5$.

5.3 Constructions from Cameron-Liebler line classes

We now describe a method of constructing some of the known sets of type $(3, 6)$ in $\mathrm{AG}(2, 9)$ starting with a Cameron-Liebler line class with parameter 40 in $\mathrm{PG}(3, 9)$. We then generalize this method to give a new example in $\mathrm{AG}(2, 81)$.

5.3.1 A two-intersection set in $\mathrm{AG}(2, 9)$

Take a Cameron-Liebler line class $\mathcal{L}_1$ of parameter 40 in $\mathrm{PG}(3, 9)$, as constructed in the Chapter 4. This set of lines is disjoint from a trivial Cameron-Liebler line class with parameter 2, which we will consider to be $\mathrm{star}(\mathbf{p}) \cup \mathrm{line}(\pi)$, where $\mathbf{p}$ is a point in $\mathrm{PG}(3, q)$ and π is a plane not containing $\mathbf{p}$. This line class induces a symmetric tactical decomposition on $\mathrm{PG}(3, 9)$ having four classes of points and lines, as follows: the four line classes are

1. $\mathrm{star}(\mathbf{p})$,
2. $\mathrm{line}(\pi)$,
3. $\mathcal{L}_1$, and
4. $\mathcal{L}_2 = \mathrm{line}(\mathrm{PG}(3, q)) \setminus (\mathrm{line}(\pi) \cup \mathrm{star}(\mathbf{p}) \cup \mathcal{L}_1)$.

Each point of $\mathrm{PG}(3,q) \setminus (\{\mathbf{p}\} \cup \pi)$ lies on either 30 or 60 lines of $\mathcal{L}_1$ (see Table 4.2), and so the four point classes of the tactical decomposition are

1. $\{\mathbf{p}\}$,

2. π,

3. $\mathcal{P}_1 = \{\mathbf{u} \in \mathrm{PG}(3,q) \ : \ \mathrm{star}(\mathbf{u}) \cap \mathcal{L}_1 = 30\}$, and

4. $\mathcal{P}_2 = \{\mathbf{v} \in \mathrm{PG}(3,q) \ : \ \mathrm{star}(\mathbf{v}) \cap \mathcal{L}_1 = 60\}$.

The numbers of lines in each given line class through a fixed point in each given point class can be found in Table 5.1, and the numbers of points in each given point class on a fixed line in each given line class can be found in Table 5.2.

	star($\mathbf{p}$)	line(π)	$\mathcal{L}_1$	$\mathcal{L}_2$
$\{\mathbf{p}\}$	91	0	0	0
π	1	10	40	40
$\mathcal{P}_1$	1	0	30	60
$\mathcal{P}_2$	1	0	60	30

Table 5.1: *Lines per point for the symmetric tactical decomposition induced on* $\mathrm{PG}(3,9)$ *by a Cameron-Liebler line class of parameter* 40.

	star($\mathbf{p}$)	line(π)	$\mathcal{L}_1$	$\mathcal{L}_2$
$\{\mathbf{p}\}$	1	0	0	0
π	1	10	1	1
$\mathcal{P}_1$	4	0	3	6
$\mathcal{P}_2$	4	0	6	3

Table 5.2: *Points per line for the symmetric tactical decomposition induced on* $\mathrm{PG}(3,9)$ *by a Cameron-Liebler line class of parameter* 40.

Now, if we take a plane π' of PG$(3,9)$ not equal to π and not containing $\mathbf{p}$, π' contains precisely one line of line(π) and no lines of star$(\mathbf{p})$. Furthermore, π' will contain either 30 or 60 lines of $\mathcal{L}_1$, and so 60 or 30 lines of $\mathcal{L}_2$ (see Table 4.2). Without loss of generality, we may assume that π' contains 30 lines of $\mathcal{L}_1$ and 60 lines of $\mathcal{L}_2$. As for the various point classes, π' does not contain $\mathbf{p}$, and contains 10 points of π, all on a common line. Under our assumptions, π' also contains 30 points of $\mathcal{P}_1$ and 60 points of $\mathcal{P}_2$. In fact, we have a symmetric tactical decomposition of π' having 3 classes on points and lines induced by our tactical decomposition of the larger space. By taking $\pi' \cap \pi$ to be the line at infinity and removing it (along with all of its points) from π', we obtain an affine plane AG$(2,9)$. All of the points of this affine plane are in $\mathcal{P}_1$ or $\mathcal{P}_2$, and all of the lines are in $\mathcal{L}_1$ or $\mathcal{L}_2$. It can be easily verified that $\pi \cap \mathcal{P}_1$ is a set of type $(3,6)$ in this plane containing 30 points. This set admits a stabilizer isomorphic to $\mathbb{Z}_3$. As the sets of type (m,n) in AG$(2,9)$ were completely classified in [28], this set is not new.

5.3.2 A new two-intersection set in AG$(2,81)$

We are also able to follow the above procedure with a Cameron-Liebler line class $\mathcal{L}_1$ of parameter 3280 in PG$(3,81)$ constructed as in Chapter 4. In this case, the line classes are formed as before. As for the point classes, each point of PG$(3,81)$ is on either 2952 lines of $\mathcal{L}_1$, or on 3690 points of $\mathcal{L}_1$. We define $\mathcal{P}_1$ to be the set of points on 2952 lines of $\mathcal{L}_1$. These point and line classes give a symmetric tactical decomposition of PG$(3,81)$; the numbers of lines in each given line class through a fixed point in each given point class can be found in Table 5.3, and the numbers of points in each given point class on a fixed line in each given line class can be found in Table 5.4.

We let π' be a plane of PG$(3,81)$ not equal to π, and not containing $\mathbf{p}$. Then π' contains one line of line(π) and no lines of star$(\mathbf{p})$. Also, π' will contain either 2952 lines of $\mathcal{L}_1$ or 3690 lines of $\mathcal{L}_1$ (see Table 4.2); without loss of generality, assume π'

	star($\mathbf{p}$)	line(π)	$\mathcal{L}_1$	$\mathcal{L}_2$
$\{\mathbf{p}\}$	6643	0	0	0
π	1	82	3280	3280
$\mathcal{P}_1$	1	0	2952	3690
$\mathcal{P}_2$	1	0	3690	2952

Table 5.3: *Lines per point for the symmetric tactical decomposition induced on* $\mathrm{PG}(3,81)$ *by a Cameron-Liebler line class of parameter* 3280.

	star($\mathbf{p}$)	line(π)	$\mathcal{L}_1$	$\mathcal{L}_2$
$\{\mathbf{p}\}$	1	0	0	0
π	1	82	1	1
$\mathcal{P}_1$	40	0	36	45
$\mathcal{P}_2$	40	0	45	36

Table 5.4: *Points per line for the symmetric tactical decomposition induced on* $\mathrm{PG}(3,81)$ *by a Cameron-Liebler line class of parameter* 3280.

contains 2952 lines of $\mathcal{L}_1$. The point set of π' is again disjoint from $\{\mathbf{p}\}$, and contains 82 points of π, all on a common line. By taking this line, which is $\pi' \cap \pi$, to be the line at infinity and removing it along with all of its points from π', we obtain an affine plane $\mathrm{AG}(2,81)$. All of the points of this affine plane are in $\mathcal{P}_1$ or $\mathcal{P}_2$, and all of the lines are in $\mathcal{L}_1$ or $\mathcal{L}_2$. It is clear that $\mathcal{P}_1$ is a set of type $(36,45)$ containing 2952 points. There are no previously known examples of sets of type (m,n) in $\mathrm{AG}(2,81)$, so this example is new. Using Magma, the stabilizer of this set is computed and is isomorphic to $\mathbb{Z}_6$.

5.3.3 A family of examples in $\mathrm{AG}(2,3^{2e})$?

The combinatorics of our Cameron-Liebler line classes of parameter $\frac{1}{2}(q^2-1)$ seem to be especially nice over fields of order 3^{2e}, inducing a symmetric tactical decomposition on the space having four classes of lines and of points. A future direction of research related to this observation is to focus on proving the existence of an infinite family of Cameron-Liebler line classes having this parameter in the specific case where $q = 3^{2e}$, and examining the intersection numbers with respect to the planes and point stars of $\mathrm{PG}(3, q)$. If these line classes always induce such a tactical decomposition, with one of the classes being a plane and another a point star, then we will be able to construct an infinite family of sets of type (m, n) in $\mathrm{AG}(2, 3^{2e})$.

Assume we have a Cameron-Liebler line class $\mathcal{L}_1$ with parameter $\frac{1}{2}(3^{6e} - 1)$ in $\mathrm{PG}(3, 3^{2e})$ which is disjoint from a trivial Cameron-Liebler line class $\mathrm{star}(\mathbf{p}) \cup \mathrm{line}(\pi)$ with parameter 2 (so $\mathbf{p} \notin \pi$), and that this line class induces a symmetric tactical decomposition of $\mathrm{PG}(3, q)$ as above having point classes $\{\mathbf{p}\}$, π, $\mathcal{P}_1$, $\mathcal{P}_2$, and line classes $\mathrm{star}(\mathbf{p})$, $\mathrm{line}(\pi)$, $\mathcal{L}_1$, $\mathcal{L}_2$. Take a plane π' distinct from π and not containing $\mathbf{p}$. The points in the affine plane $\pi' \setminus (\pi' \cap \pi)$ are all in $\mathcal{P}_1$ or $\mathcal{P}_2$, and the lines are all in $\mathcal{L}_1$ or $\mathcal{L}_2$. If we let $\mathcal{K} = \mathcal{P}_1 \cap \pi'$, then the lines of the affine plane have precisely two intersection numbers with $\mathcal{K}$ depending on whether they are in $\mathcal{L}_1$ or $\mathcal{L}_2$. Without loss of generality we will assume that each line of $\mathcal{L}_1 \cap \mathrm{star}(\pi')$ meets m points of $\mathcal{K}$. Let A and B be such that each point in $\mathcal{P}_1$ is on A lines of $\mathcal{L}_1$ and B lines of $\mathcal{L}_2$; thus each point in $\mathcal{P}_2$ is on B lines of $\mathcal{L}_1$ and A lines of $\mathcal{L}_2$.

The most likely possibility for $n - m$, and the situation for our earlier examples, is that $n - m = 3^e$. Assume that this is the case, so that $n = m + 3^e$. By Definition 2.1, we have

$$\frac{1}{2}(3^{6e} - 1) + (3^{2e} + 1)\rho_m = |\mathcal{L}_1 \cap \pi'| + A \text{ and}$$
$$\frac{1}{2}(3^{6e} - 1) + (3^{2e} + 1)\sigma_m = |\mathcal{L}_2 \cap \pi'| + A,$$

by applying the result to $\mathcal{L}_1$ using the incident point-plane pair $(\mathbf{u}, \pi')$ with $\mathbf{u} \in \mathcal{P}_1$, and to $\mathcal{L}_2$ using the incident point-plane pair $(\mathbf{v}, \pi')$ with $\mathbf{v} \in \mathcal{P}_2$. Since $\sigma_m - \rho_m = 3^e$, we see that

$$|\mathcal{L}_2 \cap \pi'| - |\mathcal{L}_1 \cap \pi'| = (3^{2e} + 1)(\sigma - \rho) = (3^{2e} + 1)3^e$$

which, along with the fact that

$$|\mathcal{L}_2 \cap \pi'| + |\mathcal{L}_1 \cap \pi'| = 3^{4e} + 3^{2e} = (3^{2e} + 1)3^{2e},$$

tells us that

$$\begin{aligned}|\mathcal{L}_1 \cap \pi'| &= \frac{1}{2}(3^{2e} - 3^e)(3^{2e} + 1) \text{ and} \\ |\mathcal{L}_2 \cap \pi'| &= \frac{1}{2}(3^{2e} + 3^e)(3^{2e} + 1).\end{aligned}$$

This allows us to solve for

$$\begin{aligned}m &= \frac{1}{2}(3^{2e} - 3^e) \text{ and} \\ n &= \frac{1}{2}(3^{2e} + 3^e).\end{aligned}$$

Conjecture 5.6 *For any $e \geq 1$, there exist sets of type $(\frac{1}{2}(3^{2e} - 3^e), \frac{1}{2}(3^{2e} + 3^e))$ in* $\mathrm{AG}(2, 3^{2e})$.

Our hope is that, in the future, we will be able to prove that we have an infinite family of Cameron-Liebler line classes in $\mathrm{PG}(3, 3^{2e})$ which induce tactical decompositions of the space, allowing us to show the existence of these two-intersection sets.

APPENDIX A. Algorithms

Here we detail some of the algorithms that facilitate our findings.

A.1 CLaut Matrix

- We have
  - *C* is our cyclic group of order q^2+q+1 whose orbits on $\mathcal{Q}^+(5,q)\setminus(\pi_1\cup\pi_2)$ are represented by elements of OREP$=\{x : x \in \mathbb{F}_{q^3}^* \mid T(x)=0\}$ (considered as an ordered set for consistency).
  - $Z_1 = \mathrm{Aut}(\mathbb{F}_{q^3}/\mathbb{F}_p)$ and $Z_2 = \mathbb{F}_q^*$; these groups, considered on $\mathcal{Q}^+(5,q)$, normalize C, so we only consider their action on OREP). Z_1 is assumed to stabilize our tight set but Z_2 is not.
  - XBLOCK and XO are the orbits of Z_1 and Z_2, respectively; the elements of each orbit are also ordered.
- Want to store, for each $x \in$OREP, indexed pairs $<i_1,j_1>$ and $<i_2,j_2>$ so that $x=$XBLOCK$[i_1][j_1]$=XO$[i_2][j_2]$.
- We now form a set *R* such that, for each $x \in$OREP, we have a unique $r \in$*R* such that $\omega^i r^{p^k} = x$ for some i, k.
- Form the structure *S*$=[s_r]$, where $s_r = [|(1,r)^\perp \cap (1,x)^C| \ : \ x \in$OREP$]$.
- Form the array *O1P*, where *O1P*$[i][j] =k$, where k is such that $\omega^j r_i \in$XBLOCK$[k]$.
- Each row and column of our matrix A corresponds to an orbit on $\mathcal{Q}^+(5,q)\setminus(\pi_1\cup\pi_2)$ under $G=<C,\ Z_1>$. We consider them to be ordered according to the ordering of the orbits XBLOCK of Z_1. We find A_{ij} as follows:

– Let a, b be such that $\omega^a r_b \in$ *Z*$_1$[*i*].

– Then A_{ij} is formed by summing over $s_{r_b}[k]$ as k ranges over the values satisfying *O1P*[*a*][*k*]$= j$.

In other words, for each *Z*$_1$[*i*], we can find an x in *Z*$_1$[*i*], an r in *R*, and an integer a such that $x = \omega^a r$. We know how many elements of $(1, y)^C$ are collinear with $(1, r)$, so we consider how the map $y \mapsto \omega^a y$ on OREP permutes the associated orbits of *C*, and consider which of these orbits have their representatives in *Z*$_1$[*j*], and add them up.

APPENDIX B. Programs

We have structured the code to examine tight sets of $\mathcal{Q}^+(5,q)$ in a modular fashion. That is, we have a shell program that contains the parameters we may wish to modify, and from there we load the files containing specific methods, and then call the specific functions we are interested in. Here is the code for our basic shell program; we comment out any parts we are not interested in before submitting the job to the cluster. Any restrictions for the parameters, either from the requirements of the algorithms or for the sake of computational speed, will be mentioned when we discuss the individual pieces of the code.

B.1 CLshell.mgm

We have variables p, h, and t which can be modified; p does not need to be prime. The idea is that we will have $q = p^h$, and $/Au(F_{p^3h}/F_p)$ will be assumed to act on a t-tight set found by the search. We usually define t in terms of q, but we must take care that t is an integer.

```
p := 81;
h := 1;
```

`CLpreamble.mgm` sets up our basic infrastructure.

```
load "CLpreamble.mgm";
t := Rationals() ! (1/2)*(q^2 -1);
```

`CLaut.mgm` holds the basic search algorithm; when $h = 1$, it will find any of our examples having parameter t, although it is very slow. Using larger values of h (while leaving q fixed) makes things much faster, but will only find examples stabilized by this larger group.

```
load "CLaut.mgm";
FindL(t, ~L);
```

`CLbcirc.mgm` is specialized for $h = 1$, $p \equiv 1 \bmod 4$, and $t = (1/2)(q^2 - 1)$. It is very fast and memory efficient.

```
load "CLbcirc.mgm";
```

We have that L contains the set of orbit representatives for each tight set found.

```
print "There were", #L,
"line classes found with parameter", t;
```

`CLvspace.mgm` contains definitions for the vector spaces $V = E^2$, $W = F^6$, and the map $\phi : V \to W$. This map uses a basis for W which gives the standard orthogonal form. The vector space $U = F^4$ is also defined, along with maps $\delta : U \to W$ and $\gamma : W \to U$ which map a point to a line of $\mathrm{PG}(3, q)$ via the Klein correspondence.

```
load "CLvspace.mgm";
```

`CLpg3q.mgm` defines the function LU(), which maps a set of trace zero elements of E^* to the set of lines of $\mathrm{PG}(3, q)$ corresponding to their orbits under C. Also defines the groups $GL = P\Gamma L(4, q)$ and $CU \simeq C$ acting on lines of $\mathrm{PG}(3, q)$.

```
load "CLpg3q.mgm";
S := Stabilizer(GL, LU(L[1]));
print "The stabilizer of L is as follows:\n", S;
```

`CLint.mgm` is used to compute intersection numbers. The orbit representatives are expanded to the full pointset throuth LW(), and intersection numbers with stars and planes are computed using intPlane() and intStar(), respectively.

```
load "CLint.mgm";
LL := LW(L[1]);
print "The intersection numbers (with multiplicity) of L
with point stars of the space are as follows:\n",
intStar(LL);
print "The intersection numbers with respect to
```

```
planes of the space are as follows:\n",
intPlane(LL);
```

`CLint81.mgm` gives an alternate way to compute intersection numbers. It is much slower, but more memory efficient. We use it for the case where $p = 81$, since the computations are impossible otherwise.

```
load "CL81int.mgm";
print "The intersection numbers (with multiplicity) of L
with point stars of the space are as follows:\n",
intStar(LW(L[1]));
print "The intersection numbers with respect to
planes of the space are as follows:\n",
intStar(LWd(L[1]));
```

`MNset.mgm` will give the intersection of the set in $\mathrm{PG}(3, q)$ with a plane piPrime as described in Chapter 5. The set K will also be given, which should be a two-intersection set of $\mathrm{AG}(2, q)$ when $q = 3^{2e}$.

```
load "MNsetTH2.mgm";
K := MN(L[1]);
S := KStab(K);
print "The stabilizer of K is as follows:\n", S;
```

B.2 CLpreamble.mgm

This code is required for everything that follows.

We let $F = \mathbb{F}_q$ and $E = \mathbb{F}_{q^3}$, with primitive elements ω and α, respectively, and view V as E^2, with bilinear form $(x, y) \mapsto T(xy)$, where $T = \mathrm{Tr}(E/F)$.

```
q := p^h;
λ := (q^2-1);
```

```
F<ω> := FiniteField(q);
if not IsPrimitive(ω) then
    ω := PrimitiveElement(F);
end if;
E<α> := ext<F|3>;
T := func<x| Trace(E ! x, F)>;
```

It is useful to have these ordered. This ordering makes *Fstar*[*i*]= $\omega^{(i+1)}$, likewise *Estar*[*i*]= $\alpha^{(i+1)}$

```
Fstar := {@ ω^i : i in [0..q-2]@};
Estar := {@ α^i : i in [0..q^3-2]@};
```

μ is a element of E with order $r = q^2 + q + 1$.

```
μ := α^(q-1);
r := q^2 +q +1;
```

ORep is the set of nonzero elements with trace 0; fills the role of $\mathcal{S}$ in the thesis.

```
ORep := {@ x: x in Estar| T(x) eq 0 @};
```

L is a placeholder sequence; it will contain subsets of *ORep* corresponding to t-tight sets of the quadric.

```
L := [PowerSet(ORep)|];
```

Instead of defining our cyclic group acting on $V = E^2$, we define *C* to be a permutation group on *Estar*, generated by *C*.1: $x \mapsto \mu * x$.

```
C := PermutationGroup<Estar | [μ*x : x in Estar]>;
```

We define the group Z_1= $\mathrm{Aut}(E/\mathbb{F}_p)$; to save memory, we define this group acting on just the trace zero elements *ORep* $\subset$ *Estar*. This requires some consideration to the order in which we apply groups to look at orbits.

```
Gr := Sym(ORep);
Z_1 := sub<Gr | [x^p : x in ORep]>;
```

```
o := Z1.1;
```

The group Z_2 generated by $z : x \mapsto \omega * x$ permutes the orbits of *C*; the orbits of this group on the trace zero elements of *Estar* are used to efficiently form the tactical decomposition.

```
Z2 := sub<Gr | [ω*x : x in ORep]>;
z := Z2.1;
```

B.3 CLaut.mgm

The method used here is the most general search technique. Forming the matrix for the tactical decomposition can be very computationally expensive.

```
Xblock := Orbits(Z1);
XO := Orbits(Z2);
n := #Xblock;
```

The following record format is used to keep track of important information about the trace zero elements of *Estar*, such as their location in the cycles induced by specific group elements on representative trace zero elements. See Appendix A for details.

```
TZero := recformat
            <rep : ORep,
             O1 : car<{i: i in [1..#Xblock]}, {j: j in [1..(3*h)]}>,
             O2 : car<{i: i in [1..#XO]}, {j : j in [1..(q−1)]}>>;
TZ := [rec<TZero | rep := ORep[i]> : i in [1..#ORep]];
for i in [1..#Xblock] do
    for j in [1..#Xblock[i]] do
        TZ[Index(ORep, Xblock[i][j])]`O1 := <i,j>;
    end for;
end for;
for i in [1..#XO] do
```

```
        for j in [1..#XO[i]] do
            TZ[Index(ORep, XO[i][j])]`O2 := <i,j>;
        end for;
    end for;
```

It is efficient to sort the records according to the size of the orbit of the trace zero element under Z_1; we maintain separate sequences of the trace zero elements, and the records associated with them, each in the same order. The memory used to store this redundant information is made up for in the time saved accessing the information.

```
    forward m;
    Sort(~TZ, func<x,y | #Xblock[x`O1[1]] - #Xblock[y`O1[1]]>, ~m);
    ORep := {@ ORep[i^m] : i in [1..#ORep]@};
```

For details on the formation of A, see Appendix A.

```
    O2block := {@{@ TZ[j]`O1[1]
                    : j in [Index(ORep, x) : x in XO[TZ[i]`O2[1]]]@}
                    : i in [1..#ORep] @};
    R := {@ Index(ORep, Xblock[O2block[i][1]][1]) : i in [1..#O2block]@};
    Cy := [[Trace(E ! x, F) : x in Cycle(C.1, XO[i][1])] : i in [1..#XO]];
    yC := [Rotate(Reverse(Cy[i]), 1): i in [1..#Cy]];
    OO2 := [[Fstar[x[2]]*yC[x[1]][j] : j in [1..#Cy[x[1]]]]
                where x := TZ[R[i]]`O2 : i in [1..#R]];
    M2 := func<i,j | #{k : k in [1..r]
                            | OO2[i][k] + Fstar[x[2]]*Cy[x[1]][k] eq 0
                            where x := TZ[j]`O2}>;
    S := [PowerSequence(Rationals())| ];
    for i in [1..#R] do
        s := [M2(i,j) : j in [1..#ORep]];
        Append(~S, s);
```

```
end for;
O1P := [ [TZ[Index(ORep, x*ORep[j])]`O_1[1]: j in [1..#ORep]]: x in Fstar];
M := function(i,j)
        d := exists(k,s)
        {<m,n> : m in [1..#R], n in [1..#Fstar]
          | TZ[Index(ORep, ORep[R[m]]*Fstar[n])]`O_1[1] eq i };
        return &+[ S[k][x] : x in [1..#O1P[s]] | O1P[s][x] eq j];
end function;
A := SymmetricMatrix(Rationals(), &cat[[M(i,j) :i in [1..j]] : j in [1..n]])
          - ScalarMatrix(Rationals(), n, 1);
```

This loop adjusts the values of the matrix if the orbits are not all the same size, which occurs when $q \equiv 0 \bmod 3$ or when 3 divides h.

```
s := [#Xblock[i] : i in [1..#Xblock]];
for i in [1..(n-1)] do
        for j in [(i+1)..n] do
                A[i,j] := A[i,j]/(s[j]/s[i]);
        end for;
end for;
```

We now find the eigenspace for λ, and define a function FindL() which will search for a tight set with a given parameter t and return the sets of orbit representatives (trace zero elements of *Estar*) corresponding to any tight sets found.

```
Ba := Basis(Eigenspace(A, λ));
FindL := procedure(t, ~L)
        ρ := Rationals() ! t/(q^2 -1);
        e := (1-ρ);
        f := -ρ;
```

Since the basis vectors for the eigenspace for λ are normalized, a vector with all entries equal to e or f must be a linear combination of these basis vectors with all weights equal to e or f; we search over all such linear combinations for tight sets.

```
        for c in CartesianPower({0,1}, #Ba) do
            s := [ (c[i] eq 1) select e else f : i in [1..#Ba]];
            v := &+[s[i] * Ba[i] : i in [1..#Ba]];
            if forall{i:i in [1..n] | v[i] in {e,f}} then
                Append(~L, &join{Xblock[i]: i in [1..n] | v[i] eq e});
                print v;
            end if;
        end for;
    end procedure;
```

B.4 CLbcirc.mgm

This code searches for tight sets of $\mathcal{Q}^+(5,q)$ as described in Chapter 4 when $q \equiv 1 \bmod 4$, by forming a block circulant matrix for the tactical decomposition. The requirements for the parameters in the shell are as follows:

$$p \equiv 1 \bmod 4,$$
$$e = 1, \text{ and}$$
$$t = \text{Rationals}((1/2) * (q^2 - 1)).$$

We must also have loaded the CLpreamble.mgm file.

We define the subgroup $Z_3 = \langle z^4 \rangle \leq Z_2$; groups are defined to act on the trace zero elements of E to save memory.

```
    Z3 := sub<Z2 | z^4>;
```

Xblock contains the orbits on the trace zero elements under $\langle Z_1, Z_3 \rangle$. We order these orbits according to whether $\log_\alpha x \equiv 0, 1, 2, 3 \bmod 4$, as described in Chapter 4.

This way, $XO[i][k]$ is the orbit given by $\omega^{(k-1)} * XO[i][1]$ for $1 \leq k \leq 4$, where $XO[i][1]$ is an orbit containing elements with $\log_\alpha x \equiv 0 \bmod 4$.

```
Xblock := Orbits(sub<Gr | Z_1, Z_3>);
n := #Xblock;
Sort(~Xblock, func<x,y | #x - #y>);
Sort(~Xblock, func<x,y | (Log(x[1]) mod 4) - (Log(y[1]) mod 4)>);
XO := {@ Xblock[i]^Z_2 : i in [1..n]@};
n := n div 4;
```

The following algorithm is used to generate the matrix corresponding to the tactical decomposition of $\mathcal{Q}^+(5,q)$ induced by the group $\langle\ C,\ Z_1,\ Z_3\ \rangle$; see the details in Appendix A.

```
Cy := [[[Trace(E ! x, F) : x in Cycle(C.1, y)] :
            y in XO[i][1]] : i in [1..#XO]];
OO2 := [Rotate(Reverse(Cy[i][1]), 1) : i in [1..n]];
M := func<i,j,k | &+[#{z : z in [1..r]
                    | OO2[i][z] + Fstar[k]*Cy[j][x][z] eq 0}
                        : x in [1..#Cy[j]] ] >;
Ablock := [ SymmetricMatrix
              (Rationals(), &cat[[M(i,j,k): i in [1..j]] : j in [1..n]])
              : k in [1..3] ];
if ((q mod 3) eq 0) then
    s := 3;
    for i in [1..3] do
        for j in [2..n] do
            Ablock[i][1,j] := Ablock[i][1,j]/s;
        end for;
    end for;
```

```
end if;
Ablock[1] := Ablock[1] − ScalarMatrix(Rationals(), n, 1);
Ablock[4] := Ablock[2];
```

The eigenvectors of the block symmetric matrix are now constructed over the cyclotomic field $R[\zeta]$:=$\mathbb{Q}[i]$. We are only interested in real valued eigenvectors. For these examples, for all values of q which are feasible computationally, we get a one-dimensional eigenspace of $H[1]$ = $Ablock[1]-Ablock[3]$, giving eigenvectors which correspond to $(1/2)(q^2-1)$-tight sets (in essentially four equivalent ways).

```
R<ζ> := CyclotomicField(4);
H := [&+[(ζ^j)^(i−1)* Ablock[i] : i in [1..4]]: j in [1..4]];
H[1];
v := Basis(Eigenspace(H[1], (q^2 −1)))[1];
A := HorizontalJoin([VerticalJoin([Ablock[1 + ((i−j) mod 4)]
                                  : i in [0..3]]) : j in [0..3]]);
```

This loop puts together a set of the orbit representatives for each of the four t-tight sets. (Need some explanation of the method, which is based on conjecture.) The sequence L consists of a sequence of sets of orbit representatives, each corresponding to a t-tight set.

```
ρ := Rationals() ! t/(q^2 −1);
e := (1−ρ);
f := −ρ;
Sp := &cat([[<j,k> : k in [1..n]]: j in [1..4]]);
v := Vector([Rationals() | (Integers() ! Ablock[2][i,1] mod 2) − ρ
                : i in [1..n]]);
for c in CartesianPower({1,2}, 2) do
    u := Vector(&cat([ Eltseq(((−1)^c[1])*v),
```

```
                    EltSeq(((-1)^c[2])*v),
                    EltSeq(((-1)^(c[1]+1))*v),
                    EltSeq(((-1)^(c[2]+1))*v)]));
        if (u*A eq λ*u) then
            Append(~L, &join{XO[Sp[i][2]][Sp[i][1]]
                                : i in [1..#Xblock] | u[i] eq e});
            print u;
        end if;
    end for;
```

B.5 CLvspace.mgm

This code includes definitions of various vector spaces, as well as maps to implement the Klein correspondence. It is required for all of the code in the proceeding sections.

We begin by defining our vector spaces V and W, and our bilinear form.

```
V := VectorSpace(E, 2);
W, φ := VectorSpace(V, F);
B := func<u,v | T(u[1]*v[2]) + T(u[2]*v[1])>;
Q := func<v | T(v[1]*v[2])>;
BW := func<u,v | B((φ^-1)(u), (φ^-1)(v))>;
QW := func<v | Q((φ^-1)(v))>;
OForm := Matrix(F, [[BW(Basis(W)[i], Basis(W)[j])
                        : i in [1..6]]: j in [1..6]]);
```

We now find a basis for W under which we can use the standard Plucker coordinates for the Klein correspondence.

```
Ba := [Basis(W)[1]];
```

```
Ba := Append(Ba, a) where a := rep{x: x in W
                              | QW(x) eq 0 and BW(Ba[1], x) ne 0};
for i in [1..3] do
    Ba[2*i-1] := (BW(Ba[2*i-1], Ba[2*i]))^(-1)*Ba[2*i-1];
    Ba[2*i] := QW(Ba[2*i])*Ba[2*i-1] + Ba[2*i];
    if (i ne 3) then
        Ba := Append(Ba, a)
        where a := rep{x : x in Nullspace
            (OForm*Transpose(Matrix([Ba[k] : k in [1..2*i]])))
                | x ne 0 and QW(x) eq 0};
        Ba := Append(Ba, a)
        where a := rep{x : x in Nullspace
            (OForm*Transpose(Matrix([Ba[k] : k in [1..2*i]])))
                | x ne 0 and QW(x) eq 0
                    and BW(Ba[2*i+1], x) ne 0};
    end if;
end for;
Ba := [Ba[1], Ba[3], Ba[5], Ba[6], Ba[4], Ba[2]];
WBa := (Matrix(F, 6, 6, Ba))^(-1);
```

We redefine ϕ to map $V \to W$ in such a way that we can use the standard orthogonal form for the quadric.

```
φ := map<V -> W | v |-> φ(v)*WBa, w |-> (φ^-1)(w*WBa^-1)>;
WForm := Matrix(F, [[BW(Ba[i], Ba[j]): i in [1..6]] : j in [1..6]]);
BW := func<u, v | B((φ^-1)(u), (φ^-1)(v))>;
QW := func<v | Q((φ^-1)(v))>;
```

We now define the vector space U which will underlie $\mathrm{PG}(3, q)$; $\delta : U \to W$ and $\gamma : W \to U$ give the Klein correspondence.

```
U := VectorSpace(F, 4);

δ := func<x | RowSpace(Matrix(F, 4, 4,[[ 0, x[1], x[2], x[3]],
                                        [-x[1], 0, x[4], -x[5]],
                                        [-x[2], -x[4], 0, x[6]],
                                        [-x[3], x[5], -x[6], 0]]))>;

pK := func<line, j, k | Determinant(Matrix(F, 2, 2,
                        [[Basis(line)[1][j+1], Basis(line)[1][k+1]],
                        [Basis(line)[2][j+1], Basis(line)[2][k+1]]]))>;

γ := func<line | Basis(sub<W|
            [pK(line, 0, 1), pK(line, 0, 2), pK(line, 0, 3),
             pK(line, 1, 2), pK(line, 3, 1), pK(line, 2, 3)]>)[1]>;
```

B.6 CLpg3q.mgm

This code requires `CLvspace.mgm` in order to run.

We set up the group acting on $\mathrm{PG}(3, q)$, and it's action on the lines.

```
G, P := PGammaL(U);

PointsU := func<line |
            {Index(P, Normalize(Basis(line)[2]))} join
            {Index(P, Normalize(Basis(line)[1] + x*Basis(line)[2]))
             : x in F}>;

Lines := PointsU(sub<U| P[1], P[2]>)^G;

Lines := GSet(G, Lines);

ρ, GL := Action(G, Lines);
```

Here we define the action of our cyclic group on $\mathrm{PG}(3, q)$.

```
CU := (ρ^-1)(sub<GL | [PointsU(δ(φ(V !
            [ v[1] ne 0 select v[1]^C.1 else 0,
              v[2] ne 0 select v[2]^C.-1 else 0])))
```

```
        where v is (φ^-1)(γ(sub<U | P[l[1]], P[l[2]]>))
        where l is rep{<a,b> : a, b in Lines[i] | a ne b}
            : i in [1..#Lines]]>);
```

The function LU() maps our orbit representatives to a set of lines in $\mathrm{PG}(3, q)$.

```
LU := func<L | &join{@ (PointsU(δ(φ(V ! [1,x]))))^CU : x in L @} >;
```

B.7 CLint.mgm

This code requires `CLvspace.mgm` in order to run.

```
H, N := PGOPlus(W);
Hprime := Subgroups(H : IndexEqual := 2)[1]`subgroup;

CW := sub<H | [Index(N, Normalize(φ(V ! [μ*z[1], (μ^-1)*z[2]])))
                        where z is (φ^-1)(x) : x in N]>;

LW := func< L | &join{ Index(N, Normalize(φ(V ! [1,x])))^CW : x in L}>;

intStar := function(WCL)
        π1 := { Index(N, Normalize(φ(V ! [x,0]))) : x in Estar };
        Stars := π1^Hprime;
        return {* #(WCL meet π) : π in Stars*};
end function;

intPlane := function(WCL)
        π2 := { Index(N, Normalize(φ(V ! [0,y]))) : y in Estar };
        Planes := π2^Hprime;
        return {* #(WCL meet π) : π in Planes*};
end function;
```

B.8 CL81int.mgm

This code requires CLvspace.mgm in order to run.

```
G, P := PGammaL(U);

aC := [μ^k : k in [1..r]];

Ca := Reverse(aC);

Rotate(~aC, 1);

CW := func<x | {@ V ! [aC[i], Ca[i] *x] : i in [1..r] @} >;

ULines := Parent(δ(φ(V ! [1, ORep[1]])*WBa));

LW := function(LRep)
    LL := [ ULines |];
    for x in LRep do
        for v in CW(x) do
            Append(~LL, δ(φ(v)));
        end for;
    end for;
    return LL;
end function;

LWd := function(LRep)
    LL := [ ULines |];
    for x in LRep do
        for v in CW(x) do
            Append(~LL, δ(φ(V ! [v[2], v[1]])));
        end for;
    end for;
    return LL;
end function;
```

```
intStar := function(UCL)
    INT := {* Integers()| *};
    for x in P do
        xINT := #{v : v in UCL | x in v};
        Include(~INT, xINT);
    end for;
    return INT;
end function;

intPlane := function(UCL)
    INT := {* Integers()| *};
    Nv := [Nullspace(Transpose(Matrix([Basis(v)[1], Basis(v)[2]])))
            : v in UCL];
    for x in P do
        xINT := #{i : i in [1..#Nv] | x in Nv[i]};
        Include(~INT, xINT);
    end for;
    return INT;
end function;
```

B.9 MNset.mgm

This code requires `CLvspace.mgm` in order to run.

```
aC := [μ^k : k in [1..r]];
Ca := Reverse(aC);
Rotate(~aC,1);
CW := func<x | {@ φ(V ! [aC[i], Ca[i] *x]) : i in [1..r] @} >;

ULines := Parent(δ(φ(V ! [1, ORep[1]])*WBa));
```

```
π := Nullspace(Transpose(Matrix(U ! [1,0,0,0])));
piW := sub<W | γ(sub<U| Basis(π)[1], Basis(π)[2]>),
                    γ(sub<U| Basis(π)[1], Basis(π)[3]>),
                    γ(sub<U| Basis(π)[2], Basis(π)[3]>)>;

LWpi := function(LRep)
    LL := [ ULines |];
    for x in LRep do
        for v in CW(x) do
            if v in piW then
                Append(~LL, δ(v));
            end if;
        end for;
    end for;
    return LL;
end function;

MN := function(UCL)
    UCLpi := LWpi(UCL);

    l := {x : x in π |
        #{v : v in UCLpi | x in v} eq (#UCLpi div (q+1))};
    l := sub< π | l >;
    Bp := ExtendBasis(l, π);
    Bp := [Bp[3], Bp[1], Bp[2]];

    H,N := PGL(3,q);

    mn := {* #{v : v in UCLpi | x in v} : x in π*};
    m := Min(mn);
```

```
    K := {Normalize(
              Vector(F, Coordinates(VectorSpaceWithBasis(Bp), x)))
                   : x in π |#{v : v in UCLpi | x in v} eq m};
    K := {[x[2], x[3]] : x in K};
    return K;
end function;

KStab := function(mnSet)
    H, J := AGammaL(2, q);
    K := {Index(J, x) : x in mnSet};
    S := Stabilizer(H, K);
    return S;
end function;
```

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