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# Proof Theory

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*This text is based on lecture notes in Dutch written by Jonathan Peck en Robbert Gurdeep Singh when they followed a course on proof theory by Andreas Weiermann*

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Contents

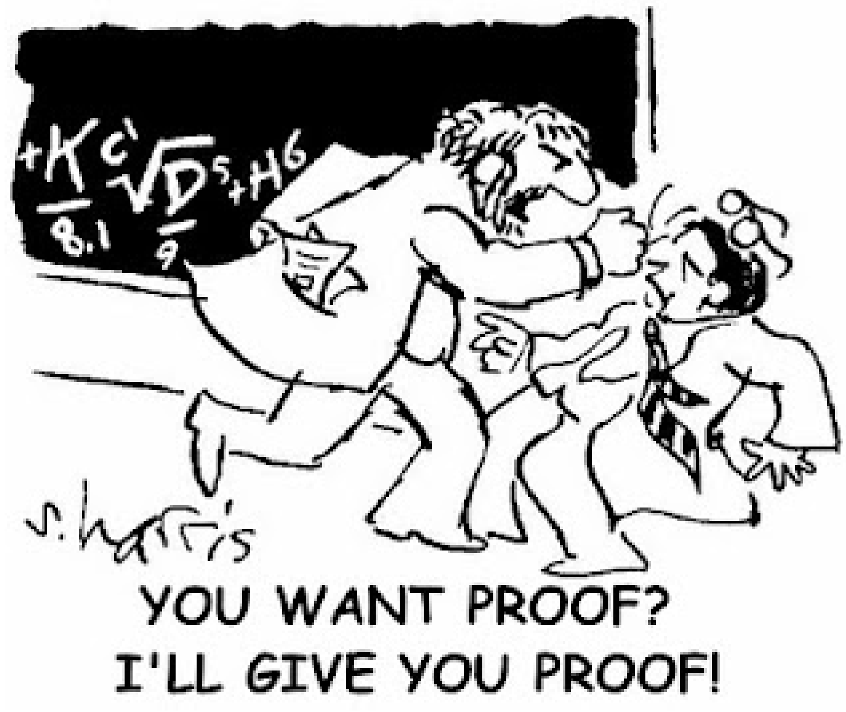
# Inleiding

These notes are based on a latesfile produced by Robbert Gurdeep Singh and Jonathan Peck during a course proof theory in the first semester of the academic year 2016-2017, taught by Andreas Weiermann.

In this course we are going to study the limits of proof formalisms, in particular formalisms bases on the sequent calculus. During this excursion we study ordinals, subrecursive hierarchies and formal systems. It is important to note that the results are based on a well thought combination of systems which allow for a straight forward verification of many simple steps. It will be a good exercise for the student to do some proofs by himself or herself to get a feeling for the automatism. Some proofs are thus left out. To start with let us consider the Russell set $R$

$$R = \{x | x \notin x\}$$

Is $R$ a member of this set or not? This leads to a classical paradoxon which is at the heart of proof theory. How can we safeguard that proof formalism are free of contradictions? Another question (going back to Kreisel) is: what extra information besides truth do we got from the verification of a fact in a formal system.



CHAPTER 1

# Gentzen's Hauptsatz

We recall some basic concepts from predicate logic and will cover Gentzen's classical cut elimination theorem. At the end we give some applications to first order predicate logic.

To set the stage properly we have to recall some basic notions from logic. Our exposition follows worked out lecture notes by Justus Diller by whom the lecturer started learning about logic.

## 1. First order languages

**Definition 1.1.** *A first order language $L$ is determined by:*

- *a countably infinite set of free variables* $\mathrm{FV}(L) = \{a_1, a_2, a_3, \dots\}$,
- *a countably infinite set of bound variables* $\mathrm{BV}(L) = \{x_1, x_2, x_3, \dots\}$,
- *a set of constants $L_C$,*
- *a set of function symbols $L_F$ together with an assignment of arities $\#f > 0$ for all $f \in L_F$,*
- *a set of relation symbols $L_R$ together with an assignment of arities $\#R > 0$ for all $R \in L_R$, among which in all cases the symbol $=$ for identity with arity $2$,*
- *logical symbols: $\bot$, $\to$ en $\forall$.*

Note that we distinguish between free and bound variables. This is basically a matter of taste but this choice helps avoiding certain pitfalls which could show up later otherwise.

**Definition 1.2.** *The set of $L$-terms $T(L)$ is inductively generated by the following clauses*

1. *if $a \in \mathrm{FV}(L)$ then $a \in T(L)$,*
2. *if $c \in L_C$ then $c \in T(L)$,*
3. *if $f \in L_F$ with $\#f = n$ and if $t_1, \dots, t_n \in T(L)$ then $f(t_1, \dots, t_n) \in T(L)$. Note that this case includes the case when $R$ is the symbol for equality.*

**Definition 1.3.** *The set of prime formulas $P(L)$ of $L$ is given by:*

1. *$\bot \in P(L)$,*
2. *if $R \in L_R$ with $\#R = n$ and $t_1, \dots, t_n \in T(L)$ then $R(t_1, \dots, t_n) \in P(L)$.*

**Definition 1.4.** *The set of formulas $F(L)$ is inductively generated by the following clauses:*

1. *If $\phi$ is a prime formula of L then $\phi \in F(L)$,*
2. *if $\phi, \psi \in F(L)$ then $\phi \to \psi \in F(L)$,*
3. *if $\phi(a) \in F(L)$ and if the bound variable $x$ does not occur in $\phi$ then $\forall x \phi(x) \in F(L)$.*

$\mathrm{FV}(\phi)$ *denotes the set of free variables which occur in $\phi$ and* $\mathrm{BV}(\phi)$ *denotes the set of bound variables which occur in $\phi$. An $L$-formula $\phi$ is called an $L$-sentence if* $\mathrm{FV}(\phi) = \emptyset$.

As usual we use the following abbreviations:

$$\neg\phi \equiv \phi \to \bot \qquad \top \equiv \neg\bot \qquad \phi \vee \psi \equiv \neg\phi \to \psi$$
$$\phi \wedge \psi \equiv \neg(\phi \to \neg\psi) \qquad \phi \leftrightarrow \psi \equiv (\phi \to \psi) \wedge (\psi \to \phi) \qquad \exists x F(x) \equiv \neg(\forall x \neg F(x))$$
$$s \neq t \equiv \neg(s = t)$$

**Definition 1.5.** *A sequent is an ordered pair $\Gamma : \Delta$ of finite sets of L-formulas. We write $\Gamma, \Delta$ for $\Gamma \cup \Delta$ and $\Gamma, \phi$ for $\Gamma \cup \{\phi\}$. A sequent $\Gamma : \Delta$ represents informally the statement "if all formulas in $\Gamma$ are true then there exists a formula from $\Delta$ which is true".*

## 2. Theories

**Definition 1.6.** *An $L$-theory $T$ is an ordered pair $(L(T), Ax(T))$ where $L(T)$ is a first order language and $Ax(T)$ is a set of $L$-sentences (called the set of axioms of $T$).*

**Definition 1.7.** *The semantics of a first order language $L$ is given by an $L$-structure $S = (|S|, S_C, S_F, S_R)$ which is given by*
- *a non empty set $|S|$ called the domain of $S$,*
- *a set $S_C$ of interpretations of symbols for the constants in $L_F$, i.e. for all $c \in L_C$ exists a $c_S \in |S|$,*
- *a set $S_F$ of interpretations of function symbols in $L_F$, i.e. for all $f \in L_F$ with $\#f = n$ there exists an $f_S \in S_F$ such that $f_S : |S|^n \to |S|$,*
- *a set $S_R$ of interpretations of relation symbols in $L_F$, i.e. for all $R \in L_F$ with $\#f = n$ there exists an $R_S \in S_F$ such that $R_S \subseteq |S|^n$.*

*Given an $L$-structure $S$ we define $L_S$, the language of $S$, by adding to $L$ a new constant $c_s$ for every element $s \in |S|$. It is understood that in this context the interpretation of $c_s$ is equal to $s$.*
*For closed $L_S$-terms t we define its interpretation $S(t)$ recursively as follows.*
1. $S(c) = c_S$,
2. $S(c_s) = s$,
3. $S(f(t_1, \ldots, t_n)) = f_S(S(t_1), \ldots, S(t_n))$.

*For closed $L_S$-formulas $\phi$ we define its interpretation $S(\phi) \in \{\top, \bot\}$ recursively as follows.*
1. $S(s = t) = \top \iff S(s) = S(t)$,
2. $S(R(t_1, \ldots, t_n)) = \top \iff (S(t_1), \ldots, S(t_n)) \in R_S$,
3. $S(\bot) = \bot$,
4. $S(\phi \to \psi) = \top \iff S(\phi) = \bot$ *of* $S(\psi) = \top$,
5. $S(\forall x \phi(x)) \iff S(\phi(c_s)) = \top$ *voor alle* $s \in |S|$.
   *For closed sequents $\Gamma : \Delta$ we define its interpretation $S(\Gamma : \Delta)$ as follows: $S(\Gamma : \Delta) = \top \iff$ there exists a $\psi \in \Delta$ such that $S(\psi) = \top$ as long as $S(\phi) = \top$ for all $\phi \in \Gamma$.*

**Definition 1.8.** *An $S$-assignment is a mapping $\sigma : FV(L) \to \{c_s : s \in |S|\}$. $\Gamma^\sigma$ denotes the result of replacing all occurrences of free variables in $\Gamma$ through hun images onder $\sigma$.*

**Definition 1.9.** *Let $S$ be an $L$-structure.*
1. $S \models \phi$ *if* $S(\phi^\sigma) = \top$ *for all assignments* $\sigma$.
2. $S \models \Gamma : \Delta$ *if* $S(\Gamma^\sigma : \Delta^\sigma) = \top$ *for all assignments* $\sigma$.

## 3. The sequent calculus

Let us now define the axioms and rules of the Gentzen caluculus. It is a specific feature of such a calculus that it derives sequents of formulas instead of single formulas. This calculus comes with a so called cut rule which moldels the modus pones proof rule.

The insight that one can eliminate all applications of the cut rule in derivations is one of the major achievements of Gerhard Gentzen. His cut elimination procedure and variations and refinements thereof are cental tools used in proof theory even nowadays.

**Definition 1.10.** *The sequent caluculus has as logical axioms: For every prime formule $P$:*

$$\Gamma, P : P, \Delta$$

$$\Gamma, \bot : \Delta$$

*The sequent calculus comes with the following derivation rules:*

$$\frac{\Gamma, \phi : \psi, \Delta}{\Gamma : \phi \to \psi, \Delta} \to S$$

$$\frac{\Gamma : \phi, \Delta \qquad \Gamma, \psi : \Delta}{\Gamma, \phi \to \psi : \Delta} \to A$$

$$\frac{\Gamma : \phi(a), \Delta}{\Gamma : \forall x \phi(x), \Delta} \forall S$$

*where $a \notin \mathrm{FV}(\Gamma : \forall x \phi(x), \Delta)$*

$$\frac{\Gamma, \phi(t) : \Delta}{\Gamma, \forall x \phi(x) : \Delta} \forall A$$

*where $t$ is an arbitrary term,*

$$\frac{\Gamma, t = t : \Delta}{\Gamma : \Delta} = I$$

$$\frac{\Gamma, f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n) : \Delta}{\Gamma, t_1 = s_1, \ldots, t_n = s_n : \Delta} = F$$

$$\frac{\Gamma, R(t_1, \ldots, t_n) : \Delta}{\Gamma, R(s_1, \ldots, s_n), t_1 = s_1, \ldots, t_n = s_n : \Delta} = P$$

$$\frac{\Gamma : \psi, \Delta \qquad \Gamma, \psi : \Delta}{\Gamma : \Delta} \textit{CUT}$$

*The formula $\psi$ in the cut rule is called CUT-formula. Note that the CUT-rule is the only rule where premisses in a rule may show up which do not necessarily occur in the conclusion of the rule in some traceable way.*

**Definition 1.11.** *Let $T$ be an $L$-theory, then for all $\phi \in Ax(T)$, we add the following proof rule*

$$\frac{\Gamma, \phi : \Delta}{\Gamma : \Delta} T$$

*This rule is called $T$-rule.*

**Definition 1.12.** *A derivation in $T$ is inductively defined as follows:*
(1) *every logical axiom $\Gamma : \Delta$ is a $T$-derivation of $\Gamma : \Delta$.*
(2) *If $H_i$ are $T$-derivations of $\Gamma_i : \Delta_i$ ($i = 1, \ldots, n$) and if*
$$\frac{\Gamma_1 : \Delta_1 \qquad \cdots \qquad \Gamma_n : \Delta_n}{\Gamma : \Delta}$$
*is a derivation rule or a $T$-rule, then*
$$\frac{H_1 \qquad \cdots \qquad H_n}{\Gamma : \Delta}$$
*is a $T$-derivation.*

*We write $T \vdash \Gamma : \Delta$ if there exists a $T$-derivation of $\Gamma : \Delta$. We write $\vdash \Gamma : \Delta$ for an $\emptyset$-derivation of $\Gamma : \Delta$. In such a situation the derivation does not contain an application of the $T$-rule.*

THEOREM 1.13 (Correctness). *Let $T$ be an $L$-theory. If $T \vdash \Gamma : \Delta$ then $T \models \Gamma : \Delta$.*

PROOF. By induction on the length of the derivation of $\Gamma : \Delta$. □

**Definition 1.14.** *We define the notion of direct sub formula as follows:*
(1) *If $P \in L_R$ with $\#P = n$ then $P(t_1, \ldots, t_n)$ is a direct sub formula of $P(s_1, \ldots, s_n)$ for all terms $t_1, \ldots, t_n, s_1, \ldots, s_n$,*
(2) *$\perp$ has no direct sub formula,*
(3) *$\phi$ and $\psi$ are direct sub formulas of $\phi \to \psi$,*
(4) *$\phi(t)$ is a direct subformule of $\forall x \phi(x)$ for all terms $t$.*

*Using this notion we define the notion of sub formula as follows*
(1) *$\phi$ is a sub formule of $\phi$*
(2) *if $\phi$ is a sub formule of $\psi$ and if $\psi$ is a direct sub formule of $\chi$ then $\phi$ is a sub formule of $\chi$.*

This definition is not completely intuitive since for example $P(b)$ is a directe sub formule of $P(a)$ even if $a$ and $b$ are different.

**Lemma 1.15** (Sub formule-property)**.** *In a $\emptyset$-derivation where the CUT rule has not been applied all sequents consist of sub formules of formulas from the conclusion or of equations.*

PROOF. By induction on the length of the derivation of $\Gamma : \Delta$. □

**Definition 1.16.** *We define the complexity $|\phi|$ of a formula $\phi$ as follows:*
(1) $|\perp| = 0 = |P(t_1, \ldots, t_n)|$,
(2) $|\phi \to \psi| = \max\{|\phi|, |\psi|\} + 1$,
(3) $|\forall x \phi(x)| = |\phi(x)| + 1$.

**Definition 1.17.** *We write $\left|\frac{n}{r}\right. \Gamma : \Delta$ if there is a derivation of $\Gamma : \Delta$ such that:*
(1) *The derivation $\Gamma : \Delta$ has height not exceeding $n$*
(2) *Vor every cut formula $\phi$ which occurs in the derivation of $\Gamma : \Delta$ we have $|\phi| < r$.*

*Note that $\left|\frac{n}{0}\right. \Gamma : \Delta$ implies that there is a cut free derivation of $\Gamma : \Delta$.*

**3.1. Cut-elimination.** The cut rule is special in the sense that it is the only rule where a formula possibly can disappear from the conclusion in a derivation. So application of this rule destroy the sub formula property and so in presence of this rule it is usually very difficult to extract information from proofs. It turns out that one can replace applications of the cut rule by alternative derivations. The price to pay is that the derivations become much longer. In this section we learn how this can be achieved.

We assume that we will deal with derivations in pure predicate logic (where no additional axioms are around). So we do not consider applications of the $T$-rule in this section. When we analyze Peano arithmetic later we will learn how in special situations (partial) cut elimination can be achieved even in the presence of applications of the $T$-rule.

THEOREM 1.18. *Let $\phi$ be an $L$ formula. Then* $\vdash^{2|\phi|}_{0} \Gamma, \phi : \phi, \Delta$.

PROOF. By induction on the complexity of $\phi$.

If $|\phi| = 0$ then $\phi$ is a prime formula priem, and $\vdash \Gamma, \phi : \phi, \Delta$ is an axiom.

If $|\phi| > 0$ then there are two possibilities:

- $\phi \equiv \psi \to \chi$. The induction hypothesis yields
$$\vdash^{2|\psi|}_{0} \Gamma, \psi : \psi, \Delta \text{ and}$$
$$\vdash^{2|\chi|}_{0} \Gamma, \chi : \chi, \Delta.$$
By applying the derivation rules $\to A$ and $\to S$ we find
$$\vdash^{2|\psi\to\chi|}_{0} \Gamma, \phi : \phi, \Delta.$$
- $\phi \equiv \forall x\psi(x)$. The induction hypothesis yields
$$\vdash^{2|\psi(a)|}_{0} \Gamma, \psi(a) : \psi(a), \Delta.$$
By applying $\forall A$ and $\forall S$ we find
$$\vdash^{2|\phi|}_{0} \Gamma, \phi : \phi, \Delta.$$

□

THEOREM 1.19 (Substitution theorem). *Assume that* $\vdash^{n}_{r} \Gamma(b) : \Delta(b)$ *where* $b \notin \{a\} \cup \mathrm{FV}(\Gamma(a) : \Delta(a))$. *Then we have* $\vdash^{n}_{r} \Gamma(s) : \Delta(s)$ *for all terms $s$.*

PROOF. By induction on $n$.

If $n = 0$ then $P(b) \in \Gamma(b) \cap \Delta(b)$ or $\bot \in \Gamma(b)$. Then $P(s) \in \Gamma(s) \cap \Delta(s)$ or $\bot \in \Gamma(s)$, hence $\Gamma(s) : \Delta(s)$ is an axiom.

If $n > 0$ then we consider the situation when $\Gamma(b) : \Delta(b)$ is the result of an application of a derivation rule and we distinguish cases accordingly:

- Suppose the last applied derivation rule was $(\rightarrow A)$. Then $\phi(b) \rightarrow \psi(b) \in \Gamma(b)$ and

$$\vdash^{n_1}_{r} \Gamma(b) : \psi(b), \Delta(b),$$
$$\vdash^{n_2}_{r} \Gamma(b), \chi(b) : \Delta(b).$$

The induction hypothesis yields

$$\vdash^{n_1}_{r} \Gamma(s) : \psi(s), \Delta(s) \text{ and}$$
$$\vdash^{n_2}_{r} \Gamma(s), \chi(s) : \Delta(s).$$

The assertion follows by applying $(\rightarrow A)$.

- Suppose the last applied derivation rule was $(\forall S)$. Then $\forall x \psi(x, b) \in \Delta(b)$ and

$$\vdash^{n_1}_{r} \Gamma(b) : \psi(a, b), \Delta(b)$$

where $a \notin \mathrm{FV}(\Gamma(b) : \Delta(b))$. The induction hypothesis yields

$$\vdash^{n_1}_{r} \Gamma(s) : \psi(c, s), \Delta(s)$$

for $c \notin \mathrm{FV}(s)$. The assertion follows by applying $(\forall S)$.

The other cases are similar. □

**Lemma 1.20** (Weakening lemma)**.** *If* $\vdash^{n}_{k} \Gamma : \Delta$ *then* $\vdash^{n'}_{k} \Gamma' : \Delta'$ *where* $\Gamma \subseteq \Gamma', \Delta \subseteq \Delta' \cup \{\bot\}$ *and* $n \leq n'$.

PROOF. By induction on $n$. Note that the treatment of $\forall S$ needs some extra care due to the variable condition. □

In the sequel applications of the weakening lemma will not be mentioned explicit. So cases where the principal formula of a derivation rule is present in the assumption or not can be treated simultaneously by assuming w.l.o.g. that the principal formula of a derivation rule is present in the assumption.

**Lemma 1.21** (Inversielemma)**.**
(1) *If* $\vdash^{n}_{k} \Gamma : \phi \rightarrow \psi, \Delta$ *then* $\vdash^{n}_{k} \Gamma, \phi : \psi, \Delta$.
(2) *If* $\vdash^{n}_{k} \Gamma : \Delta, \forall x \phi(x)$ *then* $\vdash^{n}_{k} \Gamma : \Delta, \phi(t)$ *where t is an arbitrary term.*

PROOF. Both assertions are proved by induction on $n$.

(1) Assume first that $n = 0$, then $\vdash^{n}_{k} \Gamma : \phi \rightarrow \psi, \Delta$ is an axiom. This yields that $\Gamma, \phi : \psi, \Delta$ is an axiom, too. If $n > 0$ we consider the situation when the conclusion is the result of an application of a derivation rule and we distinguish cases accordingly:

- $(\rightarrow S)$. Then $\chi \rightarrow \xi \in \Delta, \phi \rightarrow \psi$ and

$$\vdash^{n_1}_{k} \Gamma, \chi : \xi, \phi \rightarrow \psi, \Delta$$

for some $n_1 < n$. The induction hypothesis yields

$$\vdash^{n_1}_{k} \Gamma, \chi, \phi : \xi, \psi, \Delta$$

If $\phi \to \psi$ is the principal formule of the last rule (hence $\phi \to \psi \equiv \chi \to \xi$) then the assertion is clear. Otherwise we find

$$\vdash^{n}_{k} \Gamma, \phi : \psi, \Delta$$

by applying the inference rule $(\to S)$.

- $(\forall S)$. Then $\forall x\chi(x) \in \Delta, \phi \to \psi$ and

$$\vdash^{n_1}_{k} \Gamma : \Delta, \forall x\chi(x), \chi(a), \phi \to \psi$$

where $n_1 < n$. The induction hypothesis yields

$$\vdash^{n_1}_{k} \Gamma, \phi : \psi, \chi(a), \forall x\chi(x), \Delta.$$

Another application of $(\forall S)$ yields the assertion.

The other cases can be dealt with similarly.

(2) If $n = 0$, then $\Gamma : \Delta, \forall x\phi(x)$ is an axiom and hence $\Gamma : \Delta, \phi(t)$ is an axiom, too. If $n > 0$ we consider the situation when the conclusion is the result of an application of a derivation rule and we distinguish cases accordingly:

- $(\forall S)$. Then

$$\vdash^{n_1}_{k} \Gamma : \psi(a), \forall x\psi(x), \forall x\phi(x), \Delta$$

with $n_1 < n$. If $\forall x\phi(x)$ was the principal formula (hence $\forall x\phi(x) \equiv \forall x\psi(x)$), then we obtain

$$\vdash^{n_1}_{k} \Gamma : \phi(a), \forall x\phi(x), \Delta$$

The substitution lemma yields

$$\vdash^{n_1}_{k} \Gamma : \phi(t), \forall x\phi(x), \Delta$$

The induction hypothesis yields

$$\vdash^{n_1}_{k} \Gamma : \phi(t), \Delta$$

If $\forall x\phi(x)$ was not the principal formule, then the induction hypothesis yields

$$\vdash^{n_1}_{k} \Gamma : \psi(a), \forall x\psi(x), \phi(t), \Delta$$

An application of the rule $(\forall S)$ yields

$$\vdash^{n_1}_{k} \Gamma : \forall x\psi(x), \phi(t), \Delta$$

- The remaining cases are similar.

□

**Lemma 1.22** (Reductielemma). *Let $\phi$ be a formula with $|\phi| \leq r$. Assume $\vdash^{n}_{r} \Gamma, \phi : \Delta$ and $\vdash^{m}_{r} \Gamma : \phi, \Delta$. Then*

$$\vdash^{2n+m}_{r} \Gamma : \Delta$$

PROOF. We distinguish two cases:

- $\phi$ is a prime formule. Then we perform induction on $m$. If $m = 0$ then $\Gamma : \phi, \Delta$ is an axiom en we have to deal with three cases:

- There is a prime formula $\psi \not\equiv \phi$ with $\psi \in \Gamma$ and $\psi \in \Delta$. Then $\Gamma : \Delta$ is an axiom.
- $\bot \in \Gamma$. Then $\Gamma : \Delta$ is an axiom.
- $\phi \in \Gamma$. Then $\vdash^{n}_{k} \Gamma : \Delta$ follows from the assumptions since $.\Gamma, \phi$ is equal to $\Gamma$.

If $m > 0$ then $\Gamma : \phi, \Delta$ is the conclusion of a derivation rule. Let us assume the premisses

$$\vdash^{m_i}_{r} \Gamma_i : \phi, \Delta_i$$

where $m_i < m$ for all $i$. The induction hypothesis yields

$$\vdash^{2n+m_i}_{r} \Gamma_i : \Delta_i.$$

An application of the same rule yields

$$\vdash^{2n+m}_{r} \Gamma : \Delta.$$

- $\phi$ is not a prime formula. If $\phi \in \Gamma$ then the assertion follows from the assumptions. Assume therefore that $\phi \notin \Gamma$. Now we perform induction on $n$. Assume first that $n = 0$. Then $\Gamma, \phi : \Delta$ is an axiom where $\phi$ not a principal formula. Then $\Gamma : \Delta$ is an axiom, too and the assertion follows. Assume $n > 0$. We consider the situation when $\Gamma : \Delta$ is the result of an application of a derivation rule and we distinguish three cases:
  - $\phi$ is not the principal formula of the last rule. Then we have the premisses

    $$\vdash^{n_i}_{r} \Gamma_i, \phi : \Delta_i$$

    with $n_i < n$. The induction hypothesis yields

    $$\vdash^{2n_i+m}_{r} \Gamma, \Gamma_i : \Delta_i, \Delta$$

    Applying the same rule yields

    $$\vdash^{2n+m}_{r} \Gamma : \Delta$$

  - $\phi \equiv \psi \to \chi$ is the principal formula of the last rule. Then this rule is $(\to A)$. We have the following premisses

    $$\vdash^{n_1}_{r} \Gamma, \phi : \psi, \Delta \qquad\qquad \vdash^{n_2}_{r} \Gamma, \phi, \chi : \Delta$$

    with $n_1, n_2 < n$. The induction hypothesis yields

    $$\vdash^{2n_1+m}_{r} \Gamma, \phi : \psi, \Delta \qquad\qquad \vdash^{2n_2+m}_{r} \Gamma, \phi, \chi : \Delta$$

    Inversion applied to $\vdash^{m}_{r} \Gamma : \phi, \Delta$ yields $\vdash^{m}_{r} \Gamma, \psi : \chi, \Delta$. We have $|\psi|, |\chi| < |\phi| \leq r$. An application of the cut rule yields

    $$\frac{\vdash^{m}_{r} \Gamma, \psi : \chi, \Delta \qquad \vdash^{2n_1+m}_{r} \Gamma : \psi, \chi, \Delta}{\vdash^{k}_{r} \Gamma : \chi, \Delta}\ \text{CUT}$$

    with $k = 2n_1 + m + 1$. We hence obtain

    $$\frac{\vdash^{k}_{r} \Gamma : \chi, \Delta \qquad \vdash^{n_2}_{r} \Gamma, \chi : \Delta}{\vdash^{l}_{r} \Gamma : \Delta}\ \text{CUT}$$

    for $l = k + 1$. The assertion follows because $2n + m \geq l$.

- Assume that $\phi \equiv \forall x\psi(x)$ and that $\phi$ is the principal formula of the last rule. Then this rule is $(\forall A)$. We have then a premise

$$\vdash_r^{n_1} \Gamma, \psi(t), \forall x\psi(x) : \Delta$$

with $n_1 < n$. The induction hypothesis yields

$$\vdash_r^{2n_1+m} \Gamma, \psi(t) : \Delta.$$

Inversion applied to $\vdash_r^m \Gamma : \phi, \Delta$ yields $\vdash_r^m \Gamma : \psi(t), \Delta$. Because of $|\psi(t)| < |\phi| \leq r$ we obtain the conclusion as follows by a cut

$$\dfrac{\vdash_r^{2n_1+m} \Gamma, \psi(t) : \Delta \qquad \vdash_r^m \Gamma : \psi(t), \Delta}{\vdash_r^{2n+m} \Gamma : \Delta} \text{CUT}$$

□

This proof can be fine tuned in several ways.

**Exercise 1.23.** *Consider the following variant of the rank function.*
(1) $|\phi| := 0$ *if* $\phi$ *is atomic.*
(2) $|\phi| := \max\{|\psi| + 1, |\chi|\}$ *if* $\phi = \psi \to \chi$.
(3) $|\phi| := |\psi| + 1$ *if* $\phi = \forall x\psi$.
*Prove the following refinement of the reduction lemma: Let* $\phi$ *be a formula with* $|\phi| \leq r$. *Assume* $\vdash_r^n \Gamma, \phi : \Delta$ *and* $\vdash_r^m \Gamma : \phi, \Delta$. *Then*

$$\vdash_r^{n+m} \Gamma : \Delta$$

**Exercise 1.24.** *Consider the following variant of the rank function.*
(1) $|\phi| := 0$ *if* $\phi$ *is atomic.*
(2) $|\phi| := \max\{|\psi|\}$ *if* $\phi = \psi \to \bot$.
(3) $|\phi| := \max\{|\psi| + 1, |\chi|\}$ *if* $\phi = \psi \to \chi$ *and* $\chi \neq \bot$.
(4) $|\phi| := |\psi| + 1$ *if* $\phi = \forall x\psi$.
*Prove the following refinement of the reduction lemma: Let* $\phi$ *be a formula with* $|\phi| \leq r$. *Assume* $\vdash_r^n \Gamma, \phi : \Delta$ *and* $\vdash_r^m \Gamma : \phi, \Delta$. *Then*

$$\vdash_r^{n+m} \Gamma : \Delta$$

THEOREM 1.25 (Cut-elimination). *If* $\vdash_{r+1}^n \Gamma : \Delta$ *then* $\vdash_r^{3^n} \Gamma : \Delta$.

PROOF. By induction on $n$. If $n = 0$ then $\Gamma : \Delta$ is an axiom en the assertion follows immediately. If $n > 0$ then there are two cases:

- $\Gamma : \Delta$ was not derived by an application of the cut rule. The we have the premises

$$\vdash_{r+1}^{n_i} \Gamma_i : \Delta_i$$

where $n_i < n$. The induction hypothesis yields

$$\vdash_r^{3^{n_i}} \Gamma_i : \Delta_i$$

Applying the same rule yields

$$\vdash_r^{3^n} \Gamma : \Delta$$

- $\Gamma : \Delta$ was the conclusion of a cut rule. Then we have the premises.

$$\vdash^{n_1}_{r+1} \Gamma, \phi : \Delta \qquad\qquad \vdash^{n_2}_{r+1} \Gamma : \phi, \Delta$$

with $n_1, n_2 < n$ en $|\phi| \leq r$. The induction hypothesis yields

$$\vdash^{3^{n_1}}_{r} \Gamma, \phi : \Delta \qquad\qquad \vdash^{3^{n_2}}_{r} \Gamma : \phi, \Delta$$

The reduction lemma then implies

$$\vdash^{2 \cdot 3^{n_1} + 3^{n_2}}_{r} \Gamma : \Delta$$

so that we arrive at

$$\vdash^{3^n}_{r} \Gamma : \Delta$$

□

Iterated applications of the cut elimination theorem yields the following classical result.

THEOREM 1.26 (Gentzen's Hauptsatz). *If* $\vdash^{n}_{r} \Gamma : \Delta$ *then there exists an* $m$ *such that* $\vdash^{m}_{0} \Gamma : \Delta$ *(more specifically we can choose* $m = 3_r(n)$ *where* $3_0(n) = n$ *and* $3_{r+1}(n) = 3^{3_r(n)}$.

**Exercise 1.27.** *Consider the propositional fragment of the Genten calculus where all formulas are quantifier free. Prove the following refined version of the Cut elimination theorem for propositional logic.*

THEOREM 1.28 (Cut-elimination). *If* $\vdash^{n}_{r+1} \Gamma : \Delta$ *then* $\vdash^{3 \cdot n}_{r} \Gamma : \Delta$.

## 4. Applications of Gentzen's theorem

**Definition 1.29.** *A formula* $\phi$ *is called universal if* $\phi$ *has the form* $\forall x_1, \ldots, x_n \psi(x_1, \ldots, x_n)$ *where* $n \geq 0$ *and* $\psi(a_1, \ldots, a_n)$ *is quantifier free.*

**Definition 1.30.** *A theory* $T$ *is called open if all* $T$*-axiom's are universal.*

**Definition 1.31.** *A sequent* $\Gamma : \Delta$ *is called existential if* $\Gamma$ *consists of universal formulas only and* $\Delta$ *consists only of quantifier free formulas. If* $\Gamma$ *consists of universal formulas the we call a set formulas* $\Gamma^H$ *a Herbrand instantiation of* $\Gamma$ *if for all universal formulas* $\forall x \phi(x)$ *in* $\Gamma$ *the set* $\Gamma^H$ *contains finitely many instances* $\phi(t_1), \ldots, \phi(t_k)$ *and moreover we demand that* $\Gamma^H$ *must not contain other formulas. Here* $x$ *denotes a tupel* $x = x_1, \ldots, x_n$ *of variables and the* $t_i$ *denote* $n$*-tuples of terms.*

Notice that $\Gamma^H$ is quantifier free. Moreover note that $\Gamma = \Gamma^H$ if $\Gamma$ is quantifier free. If $\Gamma^{H_1}$ and $\Gamma^{H_2}$ are Herbrand instantiations of $\Gamma$ then $\Gamma^{H_1} \cup \Gamma^{H_2}$ is a Herbrand instantiation of $\Gamma$, too.

**Lemma 1.32.** *Let* $\Gamma : \Delta$ *be an existential sequent. If* $\vdash^{n}_{0} \Gamma : \Delta$ *then there exists a Herbrand instantiation* $\Gamma^H$ *of* $\Gamma$ *such that* $\vdash^{n}_{0} \Gamma^H : \Delta$.

PROOF. By induction on $n$. If $n = 0$ then $\Gamma : \Delta$ is an axiom. Define

$$\Gamma^H = \{\phi \in \Gamma \mid \phi \text{ kwantorvrij}\}$$

Then $\Gamma^H$ is an Herbrand instantiation of $\Gamma$ and moreover $\Gamma^H : \Delta$ is an axiom.

If $n > 0$, then we have to consider the following cases depending on the last applied derivation rule:

- $(\forall A)$. Then we have the premises

  $$\vert\frac{m}{0}\, \Gamma, \forall x_1\phi, \phi(t_1) : \Delta$$

  where $\forall x_1\phi \in \Delta$ and $m < n$. the sequent $\Gamma, \phi(t_1) : \Delta$ is an existential sequent en the induction hypothesis can be applied. So we find a Herbrand instantiation $(\Gamma, \phi(t_1) : \Delta)^{H'} \subseteq \Gamma^{H'}, \phi(t_1)^{H'}$ such that:

  $$\vert\frac{m}{0}\, (\Gamma, \forall x_1\phi)^{H'}, \phi(t_1)^{H'} : \Delta$$

  The set $\Gamma^{H'}, \phi(t_1)^{H'}$ is a Herbrand instantiation of $\Gamma$ and the assertion follows.

- $(\rightarrow S)$. Then we have the premises

  $$\vert\frac{m}{0}\, \Gamma, \phi : \psi, \Delta$$

  where $\phi \rightarrow \psi \in \Delta$ and $m < n$. Since $\Delta$ is quantifier free the sequent $\Gamma, \phi : \psi, \Delta$ is an existential sequent en the induction hypothesis can be applied. So we find a Herbrand instantiation $(\Gamma, \phi)^H \subseteq \Gamma^H, \phi$ such that:

  $$\vert\frac{m}{0}\, \Gamma^H, \phi : \psi, \Delta$$

  An application of $(\rightarrow S)$ yields

  $$\vert\frac{m+1}{0}\, \Gamma^H : \phi \rightarrow \psi, \Delta$$

- The other cases can be treated similarly.

□

**Lemma 1.33.** *Let $\phi$ be a universal formula and $\phi^H$ be a Herbrand instantiation of $\phi$. If*

$$T \vdash \Gamma, \phi^H : \Delta$$

*then*

$$T \vdash \Gamma, \phi : \Delta$$

PROOF. Because $\phi$ is universal, we conclude that $\phi \equiv \forall \boldsymbol{x}\psi(\boldsymbol{x})$ where $\psi$ is quantifier free. A Herbrand instantiation of $\phi$ consists of a set

$$\phi^H = \{\psi(\boldsymbol{t}_1), \ldots, \psi(\boldsymbol{t}_k)\}$$

where every $\boldsymbol{t}_i$ is a term tupel $(t_{1i}, \ldots, t_{ni})$ is. We thence have a premis

$$T \vdash \Gamma, \psi(\boldsymbol{t}_1), \ldots, \psi(\boldsymbol{t}_k) : \Delta$$

After appying the rule $(\forall A)$ several times we finally arrive at

$$T \vdash \Gamma, \forall \boldsymbol{x}\psi(\boldsymbol{x}) : \Delta.$$

□

THEOREM 1.34 (Herbrand' theorem). *Let $T$ be open and $T \vdash \exists \boldsymbol{x}\phi(\boldsymbol{x})$ where $\phi$ is quantifier free. Then there exist finitely many term tupels $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k$ of the same length as $\boldsymbol{x}$ such that $T \vdash \phi(\boldsymbol{t}_1), \ldots, \phi(\boldsymbol{t}_k)$.*

PROOF. Let $T$ be open and assume $T \vdash \exists \boldsymbol{x}\phi(\boldsymbol{x})$. The formula $\exists \boldsymbol{x}\phi(\boldsymbol{x})$ is the same as $\neg\forall \boldsymbol{x}\neg\phi(\boldsymbol{x})$ From $T \vdash \neg\forall \boldsymbol{x}\neg\phi(\boldsymbol{x})$ we conclude $T \vdash \forall \boldsymbol{x}\neg\phi(\boldsymbol{x}) : \perp$ There exist finitely many axioms $\psi_1, \ldots, \psi_n$ van $T$ such that

$$\vdash \psi_1, \ldots, \psi_n, \forall \boldsymbol{x}\neg\phi(\boldsymbol{x}) : \perp$$

Gentzen's Hauptsatz implies that there exists a cut free derivation of this sequent:

$$\vdash_0 \psi_1, \ldots, \psi_n, \forall \boldsymbol{x}\neg\phi(\boldsymbol{x}) : \perp$$

lemma 1.32 yields

$$\vdash_0 \psi_1^H, \ldots, \psi_n^H, (\forall \boldsymbol{x}\neg\phi(\boldsymbol{x}))^H : \perp$$

From lemma 1.33 we obtain

$$\vdash_0 \psi_1, \ldots, \psi_n, (\forall \boldsymbol{x}\neg\phi(\boldsymbol{x}))^H : \perp$$

This yields

$$T \vdash (\forall \boldsymbol{x}\neg\phi(\boldsymbol{x}))^H : \perp$$

Assume that $(\forall \boldsymbol{x}\neg\phi(\boldsymbol{x}))^H = \{\neg\phi(\boldsymbol{t}_1), \ldots, \neg\phi(\boldsymbol{t}_k)\}$. Then we see

$$T \vdash \neg\phi(\boldsymbol{t}_1), \ldots, \neg\phi(\boldsymbol{t}_k) : \perp$$

We finally arrive at

$$T \vdash \phi(\boldsymbol{t}_1), \ldots, \phi(\boldsymbol{t}_k)$$

□

Please not that this proof heavily depends on Gentzen's Hauptsatz. If the derivation would contain applications of the cut rule we would not be able to apply the induction hypothesis to cut formulas of possibly large complexity.

**Exercise 1.35.** *Suppose that $L$ is a first order language and let $\Gamma : \Delta$ be an $L$-sequent. Let $\langle \Gamma : \Delta \rangle$ be the set of variables occuring in $\Gamma : \Delta$ and the predicate symbols occuring in $\Gamma : \Delta$ (with the exception of $=$). We put $\langle B \rangle := \langle \emptyset : B \rangle$. An $L$-formula$B$ is called interpolant for the sequents $\Gamma_1 : \Delta_1$ and $\Gamma_2 : \Delta_2$, if*

(1) $\vdash \Gamma_1, B : \Delta_1$ *and* $\vdash \Gamma_2 : B, \Delta_2$ *and*
(2) $\langle B \rangle \subseteq \langle \Gamma_1 : \Delta_1 \rangle \cap \langle \Gamma_2 : \Delta_2 \rangle$

*Let us define positive and negative occurrence of predicate symbols in formulas and sequent as follows: We say that $P$ occurs positively in $P(t_1, \dots, t_n)$. If $P$ occurs positivel (negatively) in a formula $B$, then $P$ occurs positievel (resp. negatively) in $A \to B$ and $\Gamma : B, \Delta$ and negatively (resp. positively) in $B \to A$ and $\Gamma, B : \Delta$. If $P$ occurs positively (negatively) in $B$, then $P$ occurs positively (resp. negatively) in $\forall x B$.*
*Suppose that $\Gamma : \Delta$ is an $L$-sequent. Let $\langle \Gamma : \Delta \rangle^+$ ($\langle \Gamma : \Delta \rangle^-$) be the set of free variables occurring in $\Gamma : \Delta$ plus the in in $\Gamma : \Delta$ positively (resp. negatively) occurring predicate symbols (with exception of $=$). Moreover, let $\langle B \rangle^s := \langle \emptyset : B \rangle^s$ for $s \in \{+, -\}$.*
*An $L$-formula $B$ is called a signed interpolant for the sequents $\Gamma_1 : \Delta_1$ and $\Gamma_2 : \Delta_2$, if*

(1) $\vdash \Gamma_1, B : \Delta_1$ *and* $\vdash \Gamma_2 : B, \Delta_2$ *and*
(2) $\langle B \rangle^+ \subseteq \langle \Gamma_1 : \Delta_1 \rangle^+ \cap \langle \Gamma_2 : \Delta_2 \rangle^-$ *and*
(3) $\langle B \rangle^- \subseteq \langle \Gamma_1 : \Delta_1 \rangle^- \cap \langle \Gamma_2 : \Delta_2 \rangle^+$.

*Prove the following assertions:*

(1) *If $B$ is a signed interpolant for the sequents $\Gamma_1 : \Delta_1$ and $\Gamma_2 : \Delta_2$ then $\neg B$ is a signed interpolant for the sequents $\Gamma_2 : \Delta_2$ and $\Gamma_1 : \Delta_1$.*
(2) *If $\vdash \Gamma_1, \Gamma_2 : \Delta_1, \Delta_2$ then there exists a signed interpolant for the sequents $\Gamma_1 : \Delta_1$ and $\Gamma_2 : \Delta_2$. (Hint: Assume that the derivation is cut free.)*

CHAPTER 2

# Ordinals

## 1. Postulates

We will work informally in naive set theory while taking care that our argumentation is justifiable as usual in axiomatic set theory $ZFC$.

**Definition 2.1.** *The class of ordinals is denoted by* $\mathsf{On}$ *and it satisfies the following postulates (which can be proved in* $ZFC$*):*

(1) $(\mathsf{On}, <)$ *is linearly ordered (= totally ordered).*
(2) *if* $\emptyset \nsubseteq C \subseteq \mathsf{On}$ *then* $C$ *has a minimal element, denoted by* $\min C$.
(3) *The class* $\{\xi \in \mathsf{On} \mid \xi < \alpha\}$ *is a set for all* $\alpha \in \mathsf{On}$.
(4) *for every set* $A \subseteq \mathsf{On}$ *there exists a* $\gamma \in \mathsf{On}$ *such that* $\alpha < \gamma$ *for all* $\alpha \in A$.

*We use the following abbreviations:*

$$0 = \min \mathsf{On}, \qquad \alpha' = \min\{\xi \in \mathsf{On} \mid \alpha < \xi\}.$$

Note that $\alpha'$ plays the role of a successor function.

## 2. Properties

THEOREM 2.2 (Transfinite induction). *If* $\forall \alpha \in \mathsf{On} : (\forall \xi < \alpha : \phi(\xi)) \to \phi(\alpha)$ *then* $\forall \alpha \in \mathsf{On} : \phi(\alpha)$.

PROOF. Assume that $\forall \alpha \in \mathsf{On} : (\forall \xi < \alpha : \phi(\xi)) \to \phi(\alpha)$ and assume that there exists a $\beta \in \mathsf{On}$ such that $\neg\phi(\beta)$. Define

$$C = \{\xi \in \mathsf{On} \mid \neg\phi(\xi)\}$$

This set is non empty since $\beta \in C$. Put $\alpha_0 = \min C$. Then $\neg\phi(\alpha_0)$. By assumption there exists an $\alpha_1 < \alpha_0$ such that $\neg\phi(\alpha_1)$ since otherwise $\phi(\alpha_0)$. Contradicton with the minimality of $\alpha_0$. □

**Definition 2.3.** *The class of limit ordinals (denoted by* $Lim$*) is defined by*

$$\alpha \in \mathsf{Lim} \iff \alpha \neq 0 \wedge \forall \xi < \alpha : \xi' < \alpha$$

*The least limit ordinal is* $\omega = \min \mathsf{Lim}$.

**Definition 2.4.** *For* $A \subseteq \mathsf{On}$ *define*

$$\sup A = \min\{\xi \in \mathsf{On} \mid \forall \alpha \in A : \alpha \leq \xi\}$$

*In particular we have* $\sup \emptyset = 0$.

**Lemma 2.5.** *Assume that* $A \neq \emptyset$ *and that* $A \subseteq \mathsf{On}$ *is a set. If* $\sup A \notin A$ *then* $\sup A \in \mathsf{Lim}$.

PROOF. Let $\alpha = \sup A$. There exists a $\beta < \alpha$ in $A$ since $A \neq \emptyset$. We have to show that $\beta' < \alpha$. Because of $\beta < \alpha$ and $\alpha = \sup A$ there exists a $\xi \in A$ such that $\beta < \xi \leq \alpha$. The assumption $\sup A \notin A$ now yields $\beta' \leq \xi < \alpha$. □

**Definition 2.6.** *A binary relation $R$ is called well founded if every non empty set contains an $R$-minimal element.*

**Lemma 2.7.**

(1) *If $<$ is well founded then there does not exist an infinite descending chain of elements in* $\mathsf{On}$.

(2) *If $< \subseteq A \times A$ and $\exists F : A \to \mathsf{On}$ such that $\forall x, y : x < y \to F(x) < F(y)$ then $<$ is well founded.*

PROOF. The first assertion is obvious. Indeed the elements of the collection of elements of an infinite descending chain forms a non empty set without a minimal element.

For a proof of the second assertion assume that $X \neq \emptyset$. If $X \cap A = \emptyset$ then every $x \in X$ is a $<$-minimal element and the assertion follows.

Assume now that $X \cap A \neq \emptyset$ and define $\beta = \min\{F(x) \mid x \in X \cap A\}$. Let $x_0 \in X \cap A$ such that $F(x_0) = \beta$. Then $x_0$ is minimal. Inddeed, if there would exist an $x_1 \in X$ such that $x_1 < x_0$ $x_1 \in A$, then $F(x_1) < F(x_0)$ in contradiction with the assumption that $x_0$ is chosen minimally. □

## 3. Functions on ordinalen

**Definition 2.8.** *A function $F : \mathsf{On} \to \mathsf{On}$ is called order preserving if $\alpha < \beta$ yields $F(\alpha) < F(\beta)$.*

**Lemma 2.9.** *If $F : \mathsf{On} \to \mathsf{On}$ is order preserving, then $F(\alpha) \geq \alpha$ holds for all $\alpha \in \mathsf{On}$.*

PROOF. Assume that there exists an $\alpha \in \mathsf{On}$ such that $F(\alpha) < \alpha$. Define

$$C = \{\xi \in \mathsf{On} \mid F(\xi) < \xi\}$$

Then $C \neq \emptyset$ and $C \subseteq \mathsf{On}$, hence there exists $\alpha_0 = \min C$. Since $F$ is order preserving we see $F(\alpha_0) < \alpha_0$ so that $F(F(\alpha_0)) < F(\alpha_0)$ and thus $F(\alpha_0) \in C$. But we have $F(\alpha_0) < \alpha_0$ in contradiction with the minimality of $\alpha_0$. □

**Definition 2.10.** *A function $F : \mathsf{On} \to A$ is called ordering function for $A \subseteq \mathsf{On}$ if $F$ is order preserving and surjective on $A$.*

**Definition 2.11.** *$A \subseteq \mathsf{On}$ is unbounded if for all $\alpha \in \mathsf{On}$ there exists a $\beta \in A$ such that $\alpha < \beta$.*

**Definition 2.12.** *$A \subseteq \mathsf{On}$ is called closed if $\sup X \in A$ for all non empty sets $X \subseteq A$.*

**Definition 2.13.** *If $A \subseteq \mathsf{On}$ is closed and unbounded then $A$ is called club.*

**Lemma 2.14.** *Let $A$ be unbounded. Then there exists a uniquely determined ordering function* $\mathrm{Enum}_A$ *on $A$ such that*

$$\mathrm{Enum}_A(\alpha) = \min\{\beta \in A \mid \forall \xi < \alpha : \mathrm{Enum}_A(\xi) < \beta\}.$$

PROOF. We first prove the existence of an ordering function $F$ for $A$ using transfinite recursion. Assume inductively that $F(\xi)$ has been defined for all $\xi < \alpha$. Then $\{F(\xi) \mid \xi < \alpha\}$ is a set and hence there exists a minimal $\gamma \in \mathsf{On}$ such that $F(\xi) < \gamma$ for all $\xi < \alpha$. Because $A$ is unbounded there exists a minimal $\beta \in A$ such that $\gamma \leq \beta$. Hence $F(\alpha) = \beta$ and so $F$ is totally defined. We still have to show that $F$ is order preserving and onto with $\mathrm{rng}(F) = A$. That $F$ is order preserving is easy to see. To prove surjectivity let $\gamma \in A$. Then there exists $\alpha = \min\{\xi \mid \gamma \leq F(\xi)\}$. We have $\gamma \leq F(\alpha)$ and $\forall \xi < \alpha : F(\xi) < \gamma$ so that $F(\alpha) = \gamma$.

We now prove uniqueness. Let $F$ and $G$ both be ordering functions for $A$, thus $F, G : \mathsf{On} \to A$ and $F, G$ are surjective and order preserving. Assume by induction on $\alpha$ that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

- $F(\alpha) < G(\alpha)$. The surjectivity of $G$ yields that there exists a $\beta$ such that $F(\alpha) = G(\beta)$. Then $\beta > \alpha$ since $\beta = \alpha$ is impossible by assumption and if $\beta < \alpha$ the induction hypothesis yields that $F(\beta) = G(\beta) = F(\alpha)$. But $F$ is order preserving hence $\beta < \alpha$ yields $F(\beta) < F(\alpha)$, contradiction. Thus $\beta > \alpha$ and the order preservation of $G$ yields $G(\beta) > G(\alpha) > F(\alpha)$.
- $F(\alpha) > G(\alpha)$. Similarly.

We conclude $\forall \alpha \in \mathsf{On} : F(\alpha) = G(\alpha)$ hence $F = G$. □

### 3.1. Normal functions.

**Definition 2.15.** *A function $F$ is called continuous if* $\forall \lambda \in \mathsf{Lim} : F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$.

**Definition 2.16.** *A function $F$ is called normal if $F$ is continuous and order preserving.*

**Lemma 2.17.** *If $F : \mathsf{On} \to \mathsf{On}$ is continuous and if $\forall \alpha : F(\alpha) < F(\alpha')$. Then $F$ is normal.*

PROOF. By induction on $\alpha$ we show that $F$ is order preserving, i.e. $\forall \beta \in \mathsf{On} : \beta < \alpha \implies F(\beta) < F(\alpha)$. The $\alpha = 0$ is trivial. Assume that $\alpha = \gamma'$. Then the two cases follow from $\beta < \alpha = \gamma'$ in the following way:

- $\beta = \gamma$. Then $F(\beta) < F(\alpha)$ follows from the assumption $F(\beta) < F(\beta')$.
- $\beta < \gamma$. Then the induction hypothesis yields that $F(\beta) < F(\gamma)$ and hence $F(\beta) < F(\alpha)$ since $F(\gamma) < F(\gamma')$ is valid by assumption.

Assume now that $\alpha \in \mathsf{Lim}$. Since $\alpha$ is a limit we obtain from $\beta < \alpha$ also $\beta' < \alpha$. The continuity of $F$ implies $F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\}$. The induction hypothesis yields $F(\beta') \leq F(\alpha)$. By assumption we have $F(\beta) < F(\beta')$ hence $F(\beta) < F(\alpha)$. □

**Lemma 2.18.** *Let $F : \mathsf{On} \to \mathsf{On}$ be normal.*

(1) $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$ *if* $\alpha > 0$.
(2) *If* $\lambda \in \mathsf{Lim}$ *then* $F(\lambda) \in \mathsf{Lim}$.
(3) *For* $\gamma \geq F(0)$ *there exists a uniquely determined* $\alpha$ *such that* $F(\alpha) \leq \gamma \leq F(\alpha')$.
(4) *Let* $G$ *be normal. Then* $F \circ G$ *is normal, too.*
(5) *For a non empty set* $A$ *we have* $F(\sup A) = \sup F(A)$ *where* $F(A) = \{F(\alpha) \mid \alpha \in A\}$.

PROOF. (1) Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$.

Suppose first that $\alpha$ is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. $F$ is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$, hence $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$. Therefore $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.

Let $\alpha \in \mathsf{Lim}$. Then $F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup\{F(\xi') \mid \xi < \alpha\}$.

(2) Let $\lambda \in \mathsf{Lim}$. Then $F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$ because $F$ is continuous. Suppose $\gamma < F(\lambda)$. Then $\gamma < F(\xi)$ for some $\xi < \lambda$. Then $\gamma' \leq F(\xi) < F(\xi') < F(\lambda)$ because $F$ is order preserving.

(3) Suppose that $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. Suppose $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $F(\alpha) \leq \gamma < F(\alpha')$. Indeed, if $\alpha = 0$ then $\gamma \geq F(0)$ by assumption and the assertion follows. If $\alpha > 0$ then $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$. If $\xi < \alpha$ then the minimality of $\alpha$ yields $F(\xi') \leq \gamma$ so that $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\} \leq \gamma$.

(4) Is easy.

(5) Suppose that $A \subseteq \mathsf{On}$ is a non empty set with $\alpha = \sup A$. If $\alpha \in A$ then $F(\alpha) = F(\sup A) = \sup F(A)$ since $F$ is order preserving. If $\alpha \notin A$ then $\alpha \in \mathsf{Lim}$ and $F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup F(A)$.

□

**Lemma 2.19** (Fixed point lemma for normal functions)**.** *If* $F$ *is normal, then there exists a least* $\alpha$ *such that* $F(\alpha) = \alpha$.

PROOF. Let us define a sequence $\alpha_n$ as follows.

$$\alpha_0 = \alpha \qquad\qquad \alpha_{n+1} = F(\alpha_n)$$

Let $\beta := \sup\{\alpha_n \mid n < \omega\}$. Then $\alpha \leq \beta$. Moreover,

$$\begin{aligned} F(\beta) &= F(\sup\{\alpha_n \mid n < \omega\}) \\ &= \sup F(\{\alpha_n \mid n < \omega\}) \\ &= \sup\{F(\alpha_n) \mid n < \omega\} \\ &= \sup\{\alpha_{n+1} \mid n < \omega\} \\ &= \beta \end{aligned}$$

hence $\beta$ is a fixed point of $F$. Since there is one fixed point there is also a least one.

□

## 4. Ordinal arithmetic

### 4.1. The ordinal sum.

**Definition 2.20.** *The sum of two ordinals is defined as follows by transfinite recursion:*

$$\alpha + 0 = \alpha,$$
$$\alpha + \beta' = (\alpha + \beta)',$$
$$\alpha + \lambda = \sup\{\alpha + \xi \mid \xi < \lambda\} \text{ if } \lambda \text{ is a limit.}$$

**Lemma 2.21.**
(1) *The function $\beta \mapsto \alpha + \beta$ is normal.*
(2) $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.
(3) $\alpha, \beta \leq \alpha + \beta$.
(4) *For all $\gamma \geq \alpha$ existst a unique $\beta$ such that $\gamma = \alpha + \beta$.*
(5) $\alpha_0 \leq \alpha_1 \implies \alpha_0 + \beta \leq \alpha_1 + \beta$.
(6) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
(7) $\alpha, \beta < \omega \implies \alpha + \beta = \beta + \alpha$.
(8) $0 < k < \omega \implies k + \omega = \omega < \omega + k$.

PROOF. For a given $\alpha$ define $F(\beta) = \alpha + \beta$.

(1) $F$ continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that $F$ is normal.

(2) This assertion ollows from the normality of $F$.

(3) This assertion follows by Induction on $\beta$.

(4) Suppose $\gamma \geq \alpha$ and choose with Lemma 2.18 a $\beta$ such that $\alpha + \beta \leq \gamma < \alpha + \beta'$. Then $\gamma = \alpha + \beta$.

(5) By induction on $\beta$.

(6) By induction on $\gamma$.

If $\gamma = 0$, then $(\alpha + \beta) + \gamma = (\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0) = \alpha + (\beta + \gamma)$.

If $\gamma = \xi'$ then $(\alpha + \beta) + \xi' = ((\alpha + \beta) + \xi)'$. The induction hypothesis yields $((\alpha + \beta) + \xi)' = (\alpha + (\beta + \xi))' = \alpha + (\beta + \xi)' = \alpha + (\beta + \gamma)$.

If $\gamma \in \mathsf{Lim}$ then $(\alpha + \beta) + \gamma = \sup\{(\alpha + \beta) + \xi \mid \xi < \gamma\} = \alpha + \sup\{\beta + \xi \mid \xi < \gamma\} = \alpha + (\beta + \gamma)$.

(7) By induction on $\alpha$.

(8) If $0 < k < \omega$, then $k + \omega = \sup\{k + n \mid n < \omega\} = \sup\{m \mid m < \omega\} = \omega < \omega' \leq \omega + k$.

$\square$

Note that by now we may write $\alpha + 1$ for $\alpha'$.

### 4.2. The product of ordinals.

**Definition 2.22.** *The product of two ordinals is defined by transfinite recursion:*

$$\alpha \cdot 0 = 0,$$
$$\alpha \cdot \beta' = \alpha \cdot \beta + \alpha,$$
$$\alpha \cdot \lambda = \sup\{\alpha \cdot \xi \mid \xi < \lambda\} \text{ if } \lambda \text{ is a limit.}$$

**Lemma 2.23.**
(1) *If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.*
(2) $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.
(3) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
(4) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(5) $\alpha \cdot 0 = 0 = 0 \cdot \alpha$.
(6) $\alpha, \beta < \omega \implies \alpha \cdot \beta = \beta \cdot \alpha$.
(7) $1 < k < \omega \implies k \cdot \omega = \omega < \omega \cdot k$.

PROOF. All proofs are similar to the proofs we have seen before for the sum of ordinals. Let us check the distributivity property which is proved by induction on $\gamma$:

- $\gamma = 0$. Then $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0$.
- $\gamma = \xi'$. Then $\alpha \cdot (\beta + \xi') = \alpha \cdot (\beta + \xi)' = \alpha \cdot (\beta + \xi) + \alpha$. By induction hypthesis this is equal to $(\alpha \cdot \beta + \alpha \cdot \xi) + \alpha = \alpha \cdot \beta + (\alpha \cdot \xi + \alpha) = \alpha \cdot \beta + \alpha \cdot \xi' = \alpha \cdot \beta + \alpha \cdot \gamma$.
- $\gamma \in \mathsf{Lim}$. Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot \sup\{\beta + \xi \mid \xi < \gamma\} = \sup\{\alpha \cdot (\beta + \xi) \mid \xi < \gamma\}$. The induction hypothesis yields $\sup\{\alpha \cdot (\beta + \xi) \mid \xi < \gamma\} = \sup\{\alpha \cdot \beta + \alpha \cdot \xi) \mid \xi < \gamma\} = \alpha \cdot \beta + \sup\{\alpha \cdot \xi \mid \xi < \gamma\} = \alpha \cdot \beta + \alpha \cdot \gamma$.

□

### 4.3. Ordinal exponentiation.

**Definition 2.24.** *The exponentiation of two ordinals is defined by the following transfinite recursion:*

$$\alpha^0 = 1$$
$$\alpha^{\beta'} = \alpha^\beta \cdot \alpha$$
$$\alpha^\lambda = \sup\{\alpha^\xi \mid 0 < \xi < \lambda\} \text{ if } \lambda \text{ is a limit.}$$

**Lemma 2.25.**
(1) *The function $\beta \mapsto \alpha^\beta$ is normal for $\alpha \geq 2$.*
(2) $\alpha \leq \gamma \implies \alpha^\beta \leq \gamma^\beta$.
(3) $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.
(4) $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.
(5) *If $\beta > \beta_0 > \cdots > \beta_n$ and $\alpha > \delta_1, \ldots, \delta_n$ then $\alpha^\beta > \alpha^{\beta_0} \cdot \delta_0 + \cdots + \alpha^{\beta_n} \cdot \delta_n$.*

PROOF. All proofs are routine. Assertion 4 is proved by induction on $\gamma$ en assertion 5 is proved by induction on $n$. □

## 5. The Cantor normal form theorem

THEOREM 2.26 (Cantor's normal form theorem).

(1) *For all* $\alpha \geq 2$ *and* $\gamma \geq 1$ *there exist uniquely determined* $\beta, \delta, \gamma_0$ *so that* $0 < \delta < \alpha$ *and* $\gamma_0 < \alpha^\beta$ *and*
$$\gamma = \alpha^\beta \cdot \delta + \gamma_0.$$
(2) *For all* $\alpha \geq 2$ *and* $\gamma \geq 1$ *exists uniquely determined* $n$, $\beta_0 > \cdots > \beta_n$, $0 < \delta_0, \ldots, \delta_n < \alpha$ *such that*
$$\gamma = \alpha^{\beta_0} \cdot \delta_0 + \cdots + \alpha^{\beta_n} \cdot \delta_n.$$

PROOF.

(1) First we prove the existence of the decomposition as claimed. Since $\beta \mapsto \alpha^\beta$ is normal there exist a $\beta$ so that $\alpha^\beta \leq \gamma < \alpha^{\beta+1}$. Therefore there exists a $\delta$ with $0 < \delta < \alpha$ so that $\alpha^\beta \cdot \delta \leq \gamma < \alpha^\beta \cdot (\delta + 1)$. Moreover there exists a $\gamma_0$ with $\gamma_0 < \alpha^\beta$ so that $\alpha^\beta \cdot \delta + \gamma_0 \leq \gamma < \alpha^\beta \cdot \delta + \gamma_0 + 1$.

Let us now prove uniqueness. Suppose that $\gamma = \alpha^\beta \cdot \delta + \gamma_0 = \alpha^{\beta_1} \cdot \delta_1 + \gamma_1$ with $0 < \delta, \delta_1 < \alpha$ and $\gamma_0 < \alpha^\beta$, $\gamma_1 < \alpha^{\beta_1}$. Because $0 < \delta < \alpha$ and $\gamma_0 < \alpha^\beta$ we see $\alpha^\beta \leq \gamma < \alpha^{\beta+1}$. Moreover,
$$\alpha^{\beta+1} = \alpha^\beta \cdot \alpha \geq \alpha^\beta(\delta + 1) = \alpha^\beta \cdot \delta + \alpha^\beta > \alpha^\beta + \gamma_0$$
Similarly we find $\alpha^{\beta_1} \leq \gamma < \alpha^{\beta_1+1}$ and $\alpha^{\beta_1+1} > \alpha^{\beta_1} + \gamma_1$. Since exponentiation is normal we find $\beta = \beta_1$. We thus have $\gamma = \alpha^\beta \cdot \delta + \gamma_0 = \alpha^\beta \cdot \delta_1 + \gamma_1$. Since the ordinal product is normal, we see
$$\alpha^\beta \cdot \delta \leq \gamma < \alpha^\beta \cdot (\delta + 1)$$
$$\alpha^\beta \cdot \delta_1 \leq \gamma < \alpha^\beta \cdot (\delta_1 + 1)$$
hence $\delta = \delta_1$. Now we have $\gamma = \alpha^\beta \cdot \delta + \gamma_0 = \alpha^\beta \cdot \delta + \gamma_1$. The ordinal sum is right monotone, thence $\gamma_0 = \gamma_1$.

(2) By induction on $\gamma$. The previous assertion yields $\gamma = \alpha^\beta \cdot \delta + \gamma_0$ with $0 < \delta < \alpha$. Because $\gamma_0 < \gamma$ the induction hypothesis yields that $\gamma_0 = \alpha^{\beta_1} \cdot \delta_1 + \cdots + \alpha^{\beta_n} \cdot \delta_n$ and therefore $\gamma = \alpha^\beta \cdot \delta + \alpha^{\beta_1} \cdot \delta_1 + \cdots + \alpha^{\beta_n} \cdot \delta_n$. We conclude $\beta > \beta_1$ because $\gamma > \gamma_0$.

□

**Definition 2.27.** *We write* $\alpha =_{\mathrm{CNF}} \omega^{\alpha_0} k_0 + \cdots + \omega^{\alpha_n} k_n$ *if* $\alpha = \omega^{\alpha_0} k_0 + \cdots + \omega^{\alpha_n} k_n$ *where* $\alpha_0 > \cdots > \alpha_n$ *and* $k_0, \ldots, k_n < \omega$. *We call this representation the Cantor normal form of* $\alpha$.

Note that the CNF is uniquely defined by Cantor's theorem.

### 5.1. Additive principal ordinal numbers.

**Definition 2.28.** *The set* $\mathsf{AP}$ *of additive principal numbers is defined by*
$$\alpha \in \mathsf{AP} \iff \alpha > 0 \wedge \forall \xi, \eta < \alpha : \xi + \eta < \alpha.$$
*It is easy to see that* $1$ *is the first additive principal number. It is also easy to see that the other additive principal numbers are limit ordinals.*

**Lemma 2.29.**

(1) $\alpha \mapsto \omega^\alpha$ *is the ordering function of* $\mathsf{AP}$.
(2) $\alpha \in \mathsf{AP} \iff \forall \xi < \alpha : \xi + \alpha = \alpha$.

PROOF.

(1) By induction on $\alpha$. Suppose $F(\alpha) = \omega^\alpha$. The we have to show that $F$ is a surjective and order preserving function from $\mathsf{On}$ into $\mathsf{AP}$ is. We know already that $F$ is order preserving. We still have to show that $\operatorname{rng} F = \mathsf{AP}$.

Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathsf{AP}$.

Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta \omega$. Then there exist $m, n < \omega$ such that $\xi < \omega^\beta n$ and $\eta < \omega^\beta m$. Then $\xi + \eta < \omega^\beta n + \omega^\beta m = \omega^\beta (n + m)$. Because $n + m < \omega$ we have $\omega^\beta (n + m) < \omega^\beta \omega = \omega^{\beta+1} = \omega^\alpha$. This yields $\omega^\alpha \in \mathsf{AP}$.

Assume now that $\alpha \in \mathsf{Lim}$ and let $\xi, \eta < \omega^\alpha$. Then there exist $\alpha_1, \alpha_2 < \alpha$ with $\xi < \omega^{\alpha_1}$ and $\eta < \omega^{\alpha_2}$. Then $\xi + \eta < \omega^{\alpha_1} + \omega^{\alpha_2} < \omega^{\max(\alpha_0, \alpha_1)+1} < \omega^\alpha$. Hence $\omega^\alpha \in \mathsf{AP}$.

Now suppose that $\alpha \in \mathsf{AP}$. Then we find a unique $\beta$ such that $\omega^\beta \leq \alpha < \omega^{\beta+1}$. We claim that $\omega^\beta = \alpha$. Otherwise $\alpha = \omega^\beta \cdot n + \delta$ for some $n, \delta$ with $\omega^\beta \cdot n, \delta < \alpha$ which is a contradiction.

(2) Suppose $\alpha \in \mathsf{AP}$. Then there are two cases:

- $\alpha = 1$. This case is trivial because the only $\xi < \alpha$ is the ordinal $0$ and in this case we have $\alpha + 0 = \alpha$.
- $\alpha \in \mathsf{Lim}$. Suppose $\xi < \alpha$. Then $\xi + \alpha = \sup\{\xi + \eta \mid \eta < \alpha\} \leq \alpha$. We have $\xi + \alpha \geq \alpha$ and so $\xi + \alpha = \alpha$.

For the other direction, suppose $\xi + \alpha = \alpha$ for all $\xi < \alpha$. Suppose $\xi, \eta < \alpha$. Then $\xi + \alpha, \eta + \alpha < \alpha$ and thence $\xi + \eta < \xi + \alpha = \alpha$ so that $\alpha \in \mathsf{AP}$.

□

**Definition 2.30.** *We write* $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$ *if* $\alpha = \alpha_0 + \cdots + \alpha_n$ *and* $\alpha_0 \geq \cdots \geq \alpha_n$ *and* $\alpha_0, \ldots, \alpha_n \in \mathsf{AP}$.

By reformulating Cantor's theorem we see easily that the following theorem holds.

**Lemma 2.31.** *For every* $\alpha > 0$ *there exist uniquely determined ordinals* $\alpha_0, \ldots, \alpha_n$ *such that* $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$.

### 5.2. The natural sum of ordinals.

**Definition 2.32.** *The natural sum* $\alpha \oplus \beta$ *is defined as follows.*

(1) $\alpha \oplus 0 = \alpha = 0 \oplus \alpha$
(2) *If* $\alpha =_{\mathrm{NF}} \alpha_1 + \cdots + \alpha_n$ *and* $\beta =_{\mathrm{NF}} \alpha_{n+1} + \cdots + \alpha_{n+m}$ *then* $\alpha \oplus \beta = \alpha_{p(0)} + \cdots + \alpha_{p(n+m)}$ *where* $p : \{1, \ldots, m+n\} \to \{1, \ldots, m+n\}$ *is a function with* $\alpha_{p(1)} \geq \cdots \geq \alpha_{p(n+m)}$.

**Exercise 2.33.**

(1) $\alpha \oplus \beta = \beta \oplus \alpha$.
(2) $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.
(3) *If* $\alpha_0, \ldots, \alpha_n \in \mathsf{AP}$ *met* $\alpha_0 \geq \cdots \geq \alpha_n$ *then* $\alpha_0 + \cdots + \alpha_n = \alpha_0 \oplus \cdots \oplus \alpha_n$.
(4) $\beta < \gamma \implies \alpha \oplus \beta < \alpha \oplus \gamma$.
(5) $\alpha, \beta < \omega^\gamma \implies \alpha \oplus \beta < \omega^\gamma$.
(6) $\alpha + \beta \leq \alpha \oplus \beta$.

**Exercise 2.34.** *For a set of ordinals let otype*$(M)$ *be the unique ordinal which is order isomorphic to* $M$. *Let* $M$ *and* $N$ *be to sets of ordinals. Then otype*$(M \cup N) \leq$ *otype*$(M) \oplus$ *otype*$(N)$.

Note: If one is familiar with the multiset ordering from term rewriting theory then one can interprete the natural sum of ordinals $\alpha$ en $\beta$ as union of the multisets of their exponents.

CHAPTER 3

# Proof-theoretic analysis of $Z$

In this chapter we treat the proof theoretic analysis of the formal system $Z$ which is a conservative extension of first order Peano arithmetic $PA$.

## 1. The system $Z$

We deal with a formal system which for convenience a priori includes enough machinery to deal with primitive recursive functions.

**Definition 3.1.** *The set of primitive recursive functions* $\mathsf{PRF}$ *is defined as the least set of number theoretic functions which is closed under the following formation rules. en is gedefinieerd als de kleinste verzameling die gesloten is onder volgende operaties:*

(1) $S : \mathbb{N} \to \mathbb{N} : m \mapsto m+1 \in \mathsf{PRF}$
(2) $0^n : \mathbb{N}^n \to \mathbb{N} : \boldsymbol{m} \mapsto 0 \in \mathsf{PRF}$
(3) $P^n_i : \mathbb{N}^n \to \mathbb{N} : \boldsymbol{m} \mapsto m_i \in \mathsf{PRF}$ *voor alle* $1 \leq i \leq n$
(4) *als* $h : \mathbb{N}^m \to \mathbb{N} \in \mathsf{PRF}$ *en* $g_1, \dots, g_m : \mathbb{N}^n \to \mathbb{N} \in \mathsf{PRF}$ *dan* $h \circ (g_1, \dots, g_m) : \mathbb{N}^n \to \mathbb{N} : \boldsymbol{m} \mapsto h(g_1(\boldsymbol{m}), \dots, g_m(\boldsymbol{m})) \in \mathsf{PRF}$
(5) *als* $g : \mathbb{N}^n \to \mathbb{N} \in \mathsf{PRF}$ *en* $h : \mathbb{N}^{n+2} \to \mathbb{N} \in \mathsf{PRF}$ *dan* $\mathrm{Rec}(g,h) : \mathbb{N}^{n+1} \to \mathbb{N} \in \mathsf{PRF}$ *met*

$$\mathrm{Rec}(g,h)(0,\boldsymbol{m}) = g(\boldsymbol{m})$$
$$\mathrm{Rec}(g,h)(n+1,\boldsymbol{m}) = h(n,\boldsymbol{m},\mathrm{Rec}(g,h)(n,\boldsymbol{m}))$$

We will work with these functions in a formal system and use a canonical signature for them.

**Definition 3.2.** *The set of* $\mathsf{PR}^n$ *of* $n$*-are primitive recursieve function symbols is defined by*

(1) $S \in \mathsf{PR}^1$
(2) $0^n, P^n_i \in \mathsf{PR}^n$ *for all* $1 \leq i \leq n$
(3) *if* $h \in \mathsf{PR}^m$ *and* $g_1, \dots, g_m \in \mathsf{PR}^n$ *then* $h \circ (g_1, \dots, g_m) \in \mathsf{PR}^n$
(4) *if* $g \in \mathsf{PR}^n$ *and* $h \in \mathsf{PR}^{n+2}$ *then* $\mathrm{Rec}(g,h) \in \mathsf{PR}^{n+1}$.

*Finally put* $\mathsf{PR} = \bigcup_n \mathsf{PR}^n$. *If* $f \in PR^n$ *then the arity* $\#f$ *of* $f$ *is* $n$.

**Definition 3.3.** *Let* $PR$ *be the set of primitive recursive function symbols and let* $X$ *be a countable infinite set of variables. The set of primitive recursive terms* $T(PR,X)$ *is the least set of terms such that*

(1) $X \subseteq T(PR,X)$,
(2) *if* $f \in PR$ *and* $\#f = n$ *and* $t_1, \dots, t_n \in T(PR,X)$ *then* $f(t_1, \dots, t_n) \in T(PR,X)$.

In this context we define the numerals as follows:

$$\underline{0} = 0$$
$$\underline{m+1} = S(\underline{m})$$
$$\underline{(m_1, \ldots, m_n)} = (\underline{m_1}, \ldots, \underline{m_n})$$

The we have, for example: $\underline{5} = S(\underline{4}) = S(S(\underline{3})) = \ldots = S(S(S(S(S(0)))))$. In the sequel we sometimes drop parentheses to shorten notation. Thus we write $\underline{(2,0,3)} = (SS0, 0, SSS0)$ instead of $(S(S(0)), 0, S(S(S(0))))$.

**Definition 3.4.** *Let us now define the formal system $Z$ as follows.*

- *The terms of $Z$ are the primitive recursive terms $T(PR, X)$.*

- *The formulas of $Z$ are the formulas for the language with the function symbols $PR$ and without additional constants or relation symbolds. So formulas are*
  (1) *$s = t$ if $s$ and $t$ are terms of $Z$;*
  (2) *$\perp$,*
  (3) *$\phi \to \psi$ if $\phi$ and $\psi$ are formulas of $Z$;*
  (4) *$\forall x : \phi(x)$ if $\phi(a)$ is a formula of $Z$ with $x$ is not a bound variable in $\phi(a)$.*
- *The axioms of $Z$ are all axioms of the sequent calculus together with the universal closures of the following mathematical axioms:*

$$\neg(Sx = 0) \qquad\qquad Sx = Sy \to x = y$$
$$0^n(\boldsymbol{x}) = 0 \qquad\qquad P_i^n(\boldsymbol{x}) = x_i$$

$$h \circ (g_1, \ldots, g_m)(\boldsymbol{x}) = h(g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x}))$$

$$\mathrm{Rec}(g, h)(0, \boldsymbol{x}) = g(\boldsymbol{x})$$
$$\mathrm{Rec}(g, h)(Sy, \boldsymbol{x}) = h(y, \boldsymbol{x}, \mathrm{Rec}(g, h)(y, \boldsymbol{x}))$$

$$F(0) \wedge (\forall x : F(x) \to F(Sx)) \to \forall x : F(x)$$

## 2. Provable instances of transfinite induction for $Z$

**Definition 3.5.** *Let $R$ be a formula with at most two free variables such that*

$$<= \{(n, m) \in \mathbb{N}^2 \mid \mathbb{N} \vDash R(n, m)\}$$

*is well founded. We then define the following formulas*

$$\mathrm{Prog}_<(F) \equiv \forall x : ((\forall y < x : F(y)) \to F(x))$$
$$\mathrm{TI}_<(F, t) \equiv \mathrm{Prog}_<(F) \to \forall x < t : F(x)$$
$$\mathrm{TI}_<(F) \equiv \mathrm{Prog}_<(F) \to \forall x : F(x)$$

The abbreviaten TI stands for "Transfinite induction" and Prog stands for "Progressive".

In the sequal we assume that we can code as usual sequences of natural numbers by primitive recursive operations. By $\langle a_0, \ldots, a_n \rangle$ we denote the natural number which denotes a sequence of length $n+1$ having $a_i$ at its $i$-th entry. We assume that the empty sequence is coded by $0$. Coding is a primitive recursive operation and as usual we assume that we have a corresponding machinery available in $Z$. For this machinery within $Z$ we use the same notation.

**Definition 3.6.** *Let* $a = (a_0, \ldots, a_n)$ *and* $b = (b_0, \ldots, b_m)$. *We define* $a <' b$ *if one of the following properties holds:*

- $a = (b_0, \ldots, b_k)$ *en* $k < m$;
- $a = (b_0, \ldots, b_{k-1}, a_k, \ldots, a_n)$ *where* $k \leq \min(n, m)$ *and* $a_k <' b_k$.

This is the lexicographic ordering also known in real life from a dictionary.

**Lemma 3.7.** $<'$ *is transistive, i.e.:* $Z \vdash x <' y \wedge y <' z \to x <' z$.

Proof as usual by a simple formal induction within the formal system.

**Definition 3.8.** *Definition of the set* $\mathsf{OT}$*:*

(1) $0 \in \mathsf{OT}$;

(2) *if* $a_0, \ldots, a_n \in \mathsf{OT}$ *and* $a_n \leq' \cdots \leq' a_0$ *the* $(a_0, \ldots, a_n) \in \mathsf{OT}$.

*We write* $a < b$ *als* $a <' b$ *en* $a, b \in \mathsf{OT}$.

**Definition 3.9.** *For* $a = (a_0, \ldots, a_n)$ *and* $b = (b_0, \ldots, b_m)$ *is the concatenation of* $a$ *and* $b$ *defined by*

$$a \circ b = (a_0, \ldots, a_n, b_0, \ldots, b_m)$$

*This is a primitive recursive operation and as usual we assume that we have a corresponding machinery available in* $Z$. *For this machinery within* $Z$ *we use the same notation.*

**Definition 3.10.** *For a formula* $F(y)$ *we define*

$$\bar{F}(y) \equiv \forall x : ((\forall z < x : F(z)) \to (\forall z < x \circ y : F(z)))$$

$\bar{F}(y)$ models in a certain sense the jump operation from recursion theory. In our context we use the notation just as a convenient abbreviation

**Lemma 3.11.** $Z \vdash \mathrm{Prog}_<(F) \to \mathrm{Prog}_<(\bar{F})$.

PROOF. (Informal in $Z$.)

- Assume (1) $\mathrm{Prog}_<(F)$. Then we have to show $\mathrm{Prog}_<(\bar{F})$.
- Assume (2) $\forall y < b : \bar{F}(y)$. Then we have to show $\bar{F}(b)$.
- Assume (3) $\forall z < a : F(z)$. Then we have to show $\forall z < a \circ b : F(z)$.

We first prove the follwing assertion (4):

$$\forall n : \forall y_1, \ldots, y_n < b : \forall z < a \circ (y_1, \ldots, y_n) : F(z)$$

Proof by induction on $n$. Assume $\forall z < a \circ (y_1, \ldots, y_n) : F(z)$ for $y_1, \ldots, y_n < b$. Assume $y_{n+1} < b$. Assumption (2) yields $\bar{F}((y_1, \ldots, y_{n+1}))$ hence with (3) we obtain $\forall z < a \circ (y_1, \ldots, y_{n+1}) : F(z)$. This proves Assertion (4).

Assume now $c < a \circ b$. Then we have $c < a$ or $c \leq a \circ (b_1, \ldots, b_n)$ with $b_n \leq \cdots \leq b_1 < b$. If $c < a$ then $F(c)$ follows from (3). If $c < a \circ (b_1, \ldots, b_n)$ then $\forall z < c : F(z)$ follows from (4). With (1) we obtain $F(c)$. $\square$

**Lemma 3.12.** $Z \vdash \mathrm{TI}_<(\bar{F}, y) \to \mathrm{TI}_<(F, (y))$.

PROOF. (Informal in $Z$.) Assume (1) $\mathrm{TI}_<(\bar{F}, y)$. We claim $\mathrm{TI}_<(F, y)$. To prove this, assume (2) $\mathrm{Prog}_<(F)$. lemma 1.11 yields (3) $\mathrm{Prog}_<(\bar{F})$. From (1) we obtain $\forall z < y : \bar{F}(z)$. With (3) we obtain $\bar{F}(y)$ and thus $\forall z < (y) : F(z)$ since $\forall z < 0 : F(z)$. $\square$

THEOREM 3.13. *Let $a \in \mathsf{OT}$. Then $Z \vdash \mathrm{TI}_<(F, a)$ for all $F$.*

PROOF. Define

$$c_0 = 0 \qquad\qquad c_{m+1} = (c_m)$$

Let $a \in \mathsf{OT}$. Then there exists an $m$ such that $a < c_m$. By induction on $m$ we prove $Z \vdash \mathrm{TI}_<(F, c_m)$:

- $m = 0$. In this case the assertion holds for trivial reasons.
- $m+1$. The induction hypothesis yields $Z \vdash \mathrm{TI}_<(\bar{F}, c_m)$. lemma 1.12 yields $Z \vdash \mathrm{TI}_<(F, c_{m+1})$.

We conclude $Z \vdash \mathrm{TI}_<(F, a)$. $\square$

**Definition 3.14.** *The ordinal $o(a)$ for $a \in \mathsf{OT}$ is defined by*

$$o(0) = 0$$
$$o((a_0, \ldots, a_n)) = \omega^{o(a_0)} + \cdots + \omega^{o(a_n)}$$

Recall that $\varepsilon_0$ is defined as the least fixed point of the function $\alpha \mapsto \omega^\alpha$.

Informally speaking $\varepsilon_0$ is gotten by:

$$\varepsilon_0 = \omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}}}}}$$

**Lemma 3.15.** *The function $o$ is bijective, order preserving and it fulfills $o(a) < \varepsilon_0$ for all $a \in \mathsf{OT}$.*

PROOF. By inductie on $a \in \mathsf{OT}$ on proves first that $<'$ is a linear ordering.

By induction on $a$ we prove $\forall a \in \mathsf{OT} : o(a) < \varepsilon_0$:

- $a = 0$. Then of course $o(a) = 0 < \varepsilon_0$.
- $a = (a_0, \ldots, a_n)$. The induction hypothesis yields $o(a_i) < \varepsilon_0$ for all $i$ and hence $\omega^{o(a_i)} < \omega^{\varepsilon_0} = \varepsilon_0$. Dus $o(a) = \omega^{o(a_0)} + \cdots + \omega^{o(a_n)} < \varepsilon_0$.

By induction on $a$ we thus proved $a < b \implies o(a) < o(b)$.

This yields that $o$ is one to one. By induction on $\alpha < \varepsilon_0$ one shows that $o$ is surjective and hence is a bijection. $\square$

lemma 1.15 yields that $Z$ the scheme of transfinite induction is provable for all ordinals below $\varepsilon_0$.

CHAPTER 4

# Upper bounds for the transfinite induction in $Z$

## 1. Definition of $Z^\infty$

**Definition 4.1.** *We define the infinitary system $Z^\infty$ as follows.*
*The system has the following symbols:*

- *bound number variables $x_1, x_2, x_3, \ldots$*
- *alle primitief recursieve function symbols*
- *free predicate variables $X_1, X_2, X_3, \ldots$*
- *logical symbols* \$ $\to$, $\bot$, $\forall$,

*The formulas of $Z^\infty$ are the closed formulas with respect to this languages. The atomic formulas are equations between closed primitive recursive terms and formulas of the form $X(t)$ where $X$ is a predicate variable and $t$ is a closed term. Every closed primitive recursive term $t$ has a standard interpretation provided by its value $val(t)$. Using this interpretation we can determine the truth or falsity of closed equations. When we speak about truth value of an atomic formula we assume that the formula is an equation and not a prime formula of the form $X(t)$.*
*The axioms of $Z^\infty$ are*

(1) $\Gamma : \Delta, \phi$ *if $\phi$ is a atomic formule,*
(2) $\Gamma, \phi : \Delta$ *if $\phi$ is a false atomic formule,*
(3) $\Gamma, X(s) : \Delta, X(t)$ *if* $\mathrm{val}(s) = \mathrm{val}(t)$.

*The inference rules are as follows:*

- $(\to S)$. *If* $\vdash^{\alpha_0}_{r} \Gamma, \phi : \psi, \Delta$ *and* $\alpha_0 < \alpha < \varepsilon_0$ *and* $\phi \to \psi \in \Delta$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta$.
- $(\to A)$. *If* $\vdash^{\alpha_0}_{r} \Gamma : \phi, \Delta$ *and* $\vdash^{\alpha_1}_{r} \Gamma, \psi : \Delta$ *and* $\phi \to \psi \in \Gamma$ *and* $\alpha_0, \alpha_1 < \alpha < \varepsilon_0$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta$.
- $(\forall A)$. *If* $\vdash^{\alpha_0}_{r} \Gamma, \phi(k) : \Delta$ *and* $\alpha_0 < \alpha < \varepsilon_0$ *and* $\forall x \phi(x) \in \Gamma$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta$.
- $(\forall S)$. *If* $\vdash^{\alpha_i}_{r} \Gamma : \phi(i), \Delta$ *for all* $i \in \mathbb{N}$ *with* $\alpha_i < \alpha < \varepsilon_0$ *and if* $\forall x \phi(x) \in \Delta$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta$.
- *(CUT). If* $\vdash^{\alpha_0}_{r} \Gamma, \phi : \Delta$ *and* $\vdash^{\alpha_1}_{r} \Gamma : \Delta, \phi$ *and* $\alpha_0, \alpha_1 < \alpha < \varepsilon_0$ *and* $|\phi| < r$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta$.

*Note that $Z^\infty$ has similar inference rules as $Z$. The new rule is the rule (often also called $\omega$-rule in the literature) $(\forall S)$ with infinitely many premises. Moreover the derivation lengths in $Z^\infty$ can be transfinite ordinals. $Z^\infty$ is an infinitary systeem: proofs in $Z^\infty$ can have transfinite depths and breadth in contrary to finitary systems like $Z$.*

The rationale behind $Z^\infty$ is that it is possible to derive all instances of complete induction by use of the $\omega$-rule. Since induction does no longer belong to the axioms one can carry out a Gentzen style cut elimination for $Z^\infty$ and therefore $Z^\infty$ is amenable for tracing back information from (cut free) proofs.

The free predicate variables allow to speak about second order universal quantification over sets of natural numbers. This allows a smooth modelling of well-foundedness which is basically of such a form. The moral is that cut free proofs in $Z^\infty$ of the well-orderedness of a definable well order of order type $\alpha$ necessarily needs $\alpha$ many steps.

## 2. Properties of $Z^\infty$

**Lemma 4.2** (Structural lemma). *If* $\vdash^{\alpha}_{r} \Gamma : \Delta$ *and* $\alpha < \alpha'$, $r \leq r'$, $\Gamma \subseteq \Gamma'$ *and* $\Delta \subseteq \Delta'$ *then* $\vdash^{\alpha'}_{r} \Gamma' : \Delta'$.

**Lemma 4.3** (Properties).

(1) *If* $\vdash^{\alpha}_{r} \Gamma, \phi : \Delta$ *and* $\phi$ *is a true atomic formula then* $\vdash^{\alpha}_{r} \Gamma : \Delta$.
(2) *If* $\vdash^{\alpha}_{r} \Gamma : \phi, \Delta$ *and* $\phi$ *is a false atomic formula then* $\vdash^{\alpha}_{r} \Gamma : \Delta$.
(3) *If* $\vdash^{\alpha}_{r} \Gamma : \Delta, X(s)$ *and* $\mathrm{val}(s) = \mathrm{val}(t)$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta, X(t)$.
(4) *If* $\vdash^{\alpha}_{r} \Gamma, X(s) : \Delta$ *and* $\mathrm{val}(s) = \mathrm{val}(t)$ *then* $\vdash^{\alpha}_{r} \Gamma, X(t) : \Delta$.

PROOF.

(1) By induction on $\alpha$. If $\Gamma, \phi : \Delta$ is an axiom, then $\Gamma : \Delta$ is an axiom, too and the assertion follows. If $\Gamma, \phi : \Delta$ is not an axiom then we distinguish cases according to the last applied inference rule. Suppose, for example, that this rule was $(\rightarrow A)$. Then $\psi \rightarrow \chi \in \Gamma, \phi$ and $\alpha_0, \alpha_1 < \alpha$ where $\vdash^{\alpha_0}_{r} \Gamma, \phi : \Delta, \psi$ and $\vdash^{\alpha_1}_{r} \Gamma, \phi, \chi : \Delta$. Since $\phi$ is atomic, we see$\psi \rightarrow \chi \in \Gamma$. The induction hypothesis yields $\vdash^{\alpha_0}_{r} \Gamma : \Delta, \psi$ and $\vdash^{\alpha_1}_{r} \Gamma, \chi : \Delta$. Another application of $(\rightarrow A)$ yields $\vdash^{\alpha}_{r} \Gamma : \Delta$. The other cases are similar.
(2) Similarly to the proof of assertion 1.
(3) By induction on $\alpha$. Suppose$\Gamma : \Delta, X(s)$ is an axiom. Then we distinguish the following cases:
   - There exists a true atomic formula $\psi \in \Delta$. Then $\Gamma : \Delta$ is an axiom.
   - There exists a false atomic formula $\psi \in \Gamma$. Then $\Gamma : \Delta$ is an axiom.
   - There exist terms $s'$ and $t'$ with $Y(s') \in \Gamma$ and $Y(t') \in \Delta, X(s)$ and $\mathrm{val}(s') = \mathrm{val}(t')$. If $Y(t') \in \Delta$ then $\Gamma : \Delta$ is an axiom. Otherwise $Y(t') = X(s)$. In this case $\mathrm{val}(t) = \mathrm{val}(s) = \mathrm{val}(t') = \mathrm{val}(s')$ and thus $\Gamma : \Delta, X(t)$ is an axiom.

   Now suppose that $\Gamma : \Delta, X(s)$ is the conclusion of $(\rightarrow A)$. Then $\phi \rightarrow \psi \in \Gamma$ and we have the premises $\vdash^{\alpha_0}_{r} \Gamma : \Delta, \phi, X(s)$ and $\vdash^{\alpha_1}_{r} \Gamma, \psi : \Delta, X(s)$. The induction hypothesis yields $\vdash^{\alpha_0}_{r} \Gamma : \Delta, \phi, X(t)$ and $\vdash^{\alpha_1}_{r} \Gamma, \psi : \Delta, X(t)$. An application of $(\rightarrow A)$ yields $\vdash^{\alpha}_{r} \Gamma : \Delta, X(t)$. The other cases are similar.
(4) Similarly to the proof of assertion 3.

□

**Lemma 4.4** (Inversion lemma).

(1) *If* $\vdash^{\alpha}_{r} \Gamma : \phi \rightarrow \psi, \Delta$ *then* $\vdash^{\alpha}_{r} \Gamma, \phi : \psi, \Delta$.
(2) *If* $\vdash^{\alpha}_{r} \Gamma : \Delta, \forall x \phi(x)$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta, \phi(k)$ *for all* $k \in \mathbb{N}$.
(3) *If* $\vdash^{\alpha}_{r} \Gamma, \phi \rightarrow \psi : \Delta$ *then* $\vdash^{\alpha}_{r} \Gamma : \Delta, \phi$ *and* $\vdash^{\alpha}_{r} \Gamma, \psi : \Delta$.

PROOF. By routine inductions on $\alpha$. One can mimic the proofs for the Gentzen calcululs without any problem. □

## 3. Cut-elimination in $Z^\infty$

We cannot prove cut eliminaton for $Z$ since the induction scheme causes a serious obstacle but for $Z^\infty$ we can prove cut elimination.

**Lemma 4.5** (Reduction lemma)**.** *If* $\vdash^{\alpha}_{r} \Gamma : \Delta, \phi$ *and* $\vdash^{\beta}_{r} \Gamma, \phi : \Delta$ *and* $|\phi| \leq r$ *then* $\vdash^{\alpha+\beta 2}_{r} \Gamma : \Delta$.

PROOF. By induction on $\beta$. Suppose $\beta = 0$. Then $\Gamma, \phi : \Delta$ is an axiom. We distinguish the following cases:

- $\Delta$ contains a true atomic formula. Then $\Gamma : \Delta$ is an axiom.
- $\Gamma$ contains a false atomic formula. Then $\Gamma : \Delta$ is an axiom.
- $\phi$ is false atomic formula. An application of lemma 1.3 to $\vdash^{\alpha}_{r} \Gamma : \Delta, \phi$ yields $\vdash^{\alpha}_{r} \Gamma : \Delta$ and thus the assertion.
- There exist terms $s$ and $t$ with $\mathrm{val}(s) = \mathrm{val}(t)$ and $X(s) \in \Gamma, \phi$ and $X(t) \in \Delta$. If $X(s) \in \Gamma$ then $\Gamma : \Delta$ is an axiom. If $X(s) = \phi$ then lemma 1.3 yields $\vdash^{\alpha}_{r} \Gamma : \Delta, X(t)$ which by implicit contraction is the same as $\vdash^{\alpha}_{r} \Gamma : \Delta$.

Assume now that the last applied rule was $(\rightarrow A)$. Then $\psi \rightarrow \chi \in \Gamma, \phi$ and $\vdash^{\beta_0}_{r} \Gamma, \phi : \Delta, \psi$ and $\vdash^{\beta_1}_{r} \Gamma, \phi, \chi : \Delta$ where $\beta_0, \beta_1 < \beta$. The induction hypothesis yields $\vdash^{\alpha+\beta_0 2}_{r} \Gamma : \Delta, \psi$ and $\vdash^{\alpha+\beta_1 2}_{r} \Gamma, \chi : \Delta$. There are two cases to consider:

- $\psi \rightarrow \chi \in \Gamma$. Then we again apply $(\rightarrow A)$ and obtain the assertion.
- $\psi \rightarrow \chi \equiv \phi$. The inversion lemma yields $\vdash^{\alpha}_{r} \Gamma, \psi : \Delta, \chi$. Because of $|\psi|, |\chi| < |\phi| \leq r$ we can apply now a cut. This yields $\vdash^{\alpha+\beta_0 2+1}_{r} \Gamma : \chi, \Delta$. Another cut yields $\vdash^{\alpha+\beta 2}_{r} \Gamma : \Delta$.

The other rules are treated analogously. □

THEOREM 4.6 (Cut elimination for $Z^\infty$). *If* $\vdash^{\alpha}_{r+1} \Gamma : \Delta$ *then* $\vdash^{3^\alpha}_{r} \Gamma : \Delta$.

PROOF. By induction on $\alpha$. If $\alpha = 0$ the the assertion is trivially true. We assume that $\alpha > 0$. We distinguish two cases:

- The last applied rule was not the cut rule. Then there are premises $\vdash^{\alpha_i}_{r+1} \Gamma_i : \Delta_i$ with $\alpha_i < \alpha$ for all $i$. The induction hypothesis yields $\vdash^{3^{\alpha_i}}_{r} \Gamma_i : \Delta_i$ for all $i$. Applying the same rule yields $\vdash^{3^\alpha}_{r} \Gamma : \Delta$.
- The last applied rule was a cut. Then there are premises of the form $\vdash^{\alpha_0}_{r+1} \Gamma, \phi : \Delta$ and $\vdash^{\alpha_1}_{r+1} \Gamma : \Delta, \phi$ met $|\phi| < r + 1$. The induction hypothesis yields $\vdash^{3^{\alpha_0}}_{r} \Gamma, \phi : \Delta$ and $\vdash^{3^{\alpha_1}}_{r} \Gamma : \Delta, \phi$ with $|\phi| \leq r$. The reduction lemma yields $\vdash^{3^\alpha}_{r} \Gamma : \Delta$.

□

THEOREM 4.7. *If* $\vdash^{\alpha}_{r} \Gamma : \Delta$ *then there exists a* $\beta$ *such that* $\vdash^{\beta}_{0} \Gamma : \Delta$. *The* $\beta$ *is an iterated tower of exponentials with base* $3$ *of hight* $r$ *with an* $\alpha$ *on top, thus of the form* $3_r(\alpha)$.

## 4. Embedding of $Z$ into $Z^\infty$

In this section we make the relation between the formal proof system $Z$ and the informal proof system $Z^\infty$ more explicit. We will prove in particular that $Z^\infty$ can prove every closed which is provable in $Z$. The proof yields an explicit construction how a $Z$-proof can be translated into a $Z^\infty$ proof. In doing so, we will show that $Z^\infty$ is able to prove the induction scheme $Z$ which is not an axiom of $Z^\infty$. The latter is precisely the point where infinite derivation lengths will enter the scene.

**Lemma 4.8.**

(1) *If* $\mathrm{val}(s) = \mathrm{val}(t)$ *and if* $\phi(x)$ *is a formula, then* $\vdash^{2|\phi|}_{0} \phi(s) : \phi(t)$.

(2) $\vdash^{\omega}_{0} \phi(0), \forall x(\phi(x) \to \phi(Sx)) : \forall x\phi(x)$.

PROOF.

(1) By inducion on the rank of $\phi$.
If $|\phi| = 0$ then $\phi$ is atomic and an easy case distinction yields that $\phi(s) : \phi(t)$ is an axiom, too.
If $|\phi| > 0$ then we have to consider two cases:

- $\phi \equiv \psi \to \chi$. The induction hypothesis yields $\vdash^{2|\psi|}_{0} \psi(t) : \psi(s)$ and $\vdash^{2|\chi|}_{0} \chi(s) : \chi(t)$. Then we continue as follows:
$$\vdash^{2\max(|\psi|,|\chi|)+1}_{0} \psi(s) \to \chi(s), \psi(t) : \chi(t),$$
$$\vdash^{2\max(|\psi|,|\chi|)+2}_{0} \phi(s) : \phi(t).$$

- $\phi(a) \equiv \forall y\psi(a, y)$. The induction hypothesis yields $\vdash^{2|\psi|}_{0} \psi(s, m) : \psi(t, m)$ for all $m \in \mathbb{N}$. This yields $\vdash^{2|\psi|+1}_{0} \forall y\psi(s, y) : \psi(t, m)$ for all $m$ and hence $\vdash^{2|\psi|+2}_{0} \forall y\psi(s, y) : \forall y\psi(t, y)$.

(2) Suppose that $k = 2|\phi|$ and $\xi \equiv \forall x(\phi(x) \to \phi(Sx))$. By induction on $n$ we prove
$$\vdash^{k+2n}_{0} \phi(0), \xi : \phi(n).$$
If $n = 0$ the the assertion follows from the first assertion. If $n > 0$ the induction hypothesis yields $\vdash^{k+2n}_{0} \phi(0), \xi : \phi(n)$. The first assertion yields $\vdash^{k}_{0} \phi(Sn) : \phi(Sn)$ and therefore by $(\to A)$ we obtain $\vdash^{k+2n+1}_{0} \phi(0), \xi, \phi(n) \to \phi(Sn) : \phi(Sn)$. This yields $\vdash^{k+2n+2}_{0} \phi(0), \xi : \phi(Sn)$ via $(\forall A)$ using the implicit contraction which is built into the calculus.

Finally, we apply $(\forall S)$ and obtain $\vdash^{\omega}_{0} \phi(0), \xi : \forall x\phi(x)$.

□

THEOREM 4.9 (Embedding). *If* $Z \vdash^{k}_{r} \Gamma : \Delta$ *and* $\mathrm{FV}(\Gamma : \Delta) \subseteq \{a_1, \ldots, a_n\}$ *then there exist* $k', r'$ *such that for all* $\boldsymbol{m} \in \mathbb{N}^n$ *we have* $\vdash^{\omega+k'}_{r'} \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$.

PROOF. By induction on $k$. If $k = 0$ then $\Gamma : \Delta$ is an axiom of the sequent calculus. Then there is an atomic formula $\phi$ with $\phi \in \Gamma \cap \Delta$ or we have $\bot \in \Gamma$, so that $\Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$ is an axioma of $Z^\infty$.

If $k > 0$, the we have to deal with the last applied derivation rule in the proof of $\Gamma : \Delta$:

- $(\rightarrow A)$. We then have

$$\frac{\Gamma : \phi, \Delta \qquad \Gamma, \psi : \Delta}{\Gamma, \phi \rightarrow \psi : \Delta} \rightarrow A$$

  The induction hypothesis yields

$$\left|\frac{\omega+k_1}{r_1}\right. \Gamma(\boldsymbol{m}) : \phi(\boldsymbol{m}), \Delta(\boldsymbol{m})$$
$$\left|\frac{\omega+k_2}{r_2}\right. \Gamma(\boldsymbol{m}), \psi(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$

  By applying $(\rightarrow A)$ [1] we obtain

$$\left|\frac{\omega+\max(k_1,k_2)+1}{\max(r_1,r_2)}\right. \Gamma(\boldsymbol{m}), \phi(\boldsymbol{m}) \rightarrow \psi(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$

- $(\rightarrow S)$. Similarly.
- $(\forall S)$. We have

$$\frac{\Gamma : \phi(a), \Delta}{\Gamma : \forall x \phi(x), \Delta} \forall S$$

  with $a \notin \mathrm{FV}(\Gamma : \forall x \phi(x), \Delta)$. The induction hypothesis yields

$$\left|\frac{\omega+k_1}{r_1}\right. \Gamma(\boldsymbol{m}) : \phi(k, \boldsymbol{m}), \Delta(\boldsymbol{m})$$

  for all $k, \boldsymbol{m}$ Applying $(\forall S)$ yields:

$$\left|\frac{\omega+k_1+1}{r_1}\right. \Gamma(\boldsymbol{m}) : \forall x \phi(x, \boldsymbol{m}), \Delta(\boldsymbol{m}).$$

- $(\forall A)$. We have

$$\frac{\Gamma, \phi(t) : \Delta}{\Gamma, \forall x \phi(x) : \Delta} \forall A$$

  It is possible that there are free variables $a_i$ which occur in $t$. For those extra variables we substitute simply zeros. The induction hypothesis yields

$$\left|\frac{\omega+k_1}{r_1}\right. \Gamma(\boldsymbol{m}), \phi(t)(\boldsymbol{m}, \boldsymbol{0}) : \Delta(\boldsymbol{m})$$

  Note that $\phi(t)(\boldsymbol{m}, \boldsymbol{0}) = (\phi(t(\boldsymbol{m}, \boldsymbol{0})))(\boldsymbol{m})$ and from lemma 1.8 we obtain

$$\left|\frac{2|\phi|}{0}\right. \phi(\mathrm{val}(t(\boldsymbol{m}, \boldsymbol{0})), \boldsymbol{m}) : \phi(t(\boldsymbol{m}, \boldsymbol{0}), \boldsymbol{m})$$

  We can apply a cut:

$$\dfrac{\dfrac{\Gamma(\boldsymbol{m}), \phi(t(\boldsymbol{m}, \boldsymbol{0}), \boldsymbol{m}), \phi(\mathrm{val}(t(\boldsymbol{m}, \boldsymbol{0})), \boldsymbol{m}) : \Delta(\boldsymbol{m}) \qquad \Gamma(\boldsymbol{m}), \phi(\mathrm{val}(t(\boldsymbol{m}, \boldsymbol{0})), \boldsymbol{m}) : \phi(t(\boldsymbol{m}, \boldsymbol{0}), \boldsymbol{m}), \Delta(\boldsymbol{m})}{\Gamma(\boldsymbol{m}), \phi(\mathrm{val}(t(\boldsymbol{m}, \boldsymbol{0})), \boldsymbol{m}) : \Delta(\boldsymbol{m})} \mathrm{CUT}}{\Gamma(\boldsymbol{m}), \forall x \phi(x, \boldsymbol{m}) : \Delta(\boldsymbol{m})} \forall A$$

[1] This is here the rule $(\rightarrow A)$ of $Z^\infty$

The assertion follows.

- $(=I)$. We have

$$\frac{\Gamma, t = t : \Delta}{\Gamma : \Delta} = I$$

The induction hypothesis yields

$$\vdash^{\omega+k_1}_{r_1} \Gamma(\boldsymbol{m}), t(\boldsymbol{m}, \boldsymbol{0}) = t(\boldsymbol{m}, \boldsymbol{0}) : \Delta(\boldsymbol{m})$$

Since $t(\boldsymbol{m}, \boldsymbol{0}) = t(\boldsymbol{m}, \boldsymbol{0})$ is a true atomic formula, we can conclude that

$$\vdash^{\omega+k_1}_{r_1} \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}).$$

- $(=F)$. We have

$$\frac{\Gamma, \boldsymbol{t} = \boldsymbol{s}, f(\boldsymbol{t}) = f(\boldsymbol{s}) : \Delta}{\Gamma, \boldsymbol{t} = \boldsymbol{s} : \Delta} = F$$

The induction hypothesis yields

$$\vdash^{\omega+k_1}_{r_1} \Gamma(\boldsymbol{m}), \boldsymbol{t}(\boldsymbol{m}) = \boldsymbol{s}(\boldsymbol{m}), f(\boldsymbol{t})(\boldsymbol{m}) = f(\boldsymbol{s})(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$

If there exists an $i$ so that $\mathbb{N} \not\models s_i(\boldsymbol{m}) = t_i(\boldsymbol{m})$ then $\Gamma(\boldsymbol{m}), \boldsymbol{s}(\boldsymbol{m}) = \boldsymbol{t}(\boldsymbol{m}) : \Delta(\boldsymbol{m})$ is an axiom and the assertion follows. Suppose that $\mathbb{N} \models s_i(\boldsymbol{m}) = t_i(\boldsymbol{m})$ for all $i$.

From lemma 1.3 we conclude

$$\vdash^{\omega+k_1}_{r_1} \Gamma(\boldsymbol{m}), \boldsymbol{t}(\boldsymbol{m}) = \boldsymbol{s}(\boldsymbol{m}) : \Delta(\boldsymbol{m}).$$

- $(=P)$. We have

$$\frac{\Gamma, R(\boldsymbol{t}), R(\boldsymbol{s}), \boldsymbol{s} = \boldsymbol{t} : \Delta}{\Gamma, R(\boldsymbol{s}), \boldsymbol{s} = \boldsymbol{t} : \Delta} = P$$

The induction hypothesis yields

$$\vdash^{\omega+k_1}_{r_1} \Gamma(\boldsymbol{m}), R(\boldsymbol{t})(\boldsymbol{m}), R(\boldsymbol{s})(\boldsymbol{m}), \boldsymbol{s}(\boldsymbol{m}) = \boldsymbol{t}(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$

If there exists an $i$ such that $\mathbb{N} \not\models s_i(\boldsymbol{m}) = t_i(\boldsymbol{m})$ then $\Gamma(\boldsymbol{m}), R(\boldsymbol{s})(\boldsymbol{m}), \boldsymbol{s}(\boldsymbol{m}) = \boldsymbol{t}(\boldsymbol{m}) : \Delta(\boldsymbol{m})$ is an axiom. So suppose $\mathbb{N} \models s_i(\boldsymbol{m}) = t_i(\boldsymbol{m})$ for all $i$. From lemma 1.3 we conclude

$$\vdash^{\omega+k_1}_{r_1} \Gamma(\boldsymbol{m}), R(\boldsymbol{s})(\boldsymbol{m}), \boldsymbol{s}(\boldsymbol{m}) = \boldsymbol{t}(\boldsymbol{m}) : \Delta(\boldsymbol{m}).$$

- $T$-regel. Here we have

$$\frac{\Gamma, \phi : \Delta}{\Gamma : \Delta} \mathrm{T}$$

where $\phi$ is an axiom of $Z$. The induction hypothesis yields [2]

$$\vdash^{\omega+k_1}_{r_1} \Gamma(\boldsymbol{m}), \phi : \Delta(\boldsymbol{m})$$

It is sufficient to prove

$$\vdash^{\omega+k_2}_{r_2} \phi$$

because we then could apply a cut:

[2] There are no free variables in the axioms of $Z$, thus $\phi(\boldsymbol{m}) \equiv \phi$.

$$\dfrac{\Gamma(\boldsymbol{m}), \phi : \Delta(\boldsymbol{m}) \qquad \Gamma(\boldsymbol{m}) : \phi, \Delta(\boldsymbol{m})}{\Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})}\ \text{CUT}$$

and the assertion would follow. We distinguish two cases:

- $\phi \equiv \forall x(Sx = 0 \to \bot)$. We prove this as follows:

$$\dfrac{\dfrac{Sm = 0 : \bot}{Sm = 0 \to \bot} \to S}{\forall x(Sx = 0 \to \bot)}\ \forall S$$

- $\phi \equiv \forall x \forall y(Sx = Sy \to x = y)$. We prove this as follows:

$$\dfrac{\dfrac{\dfrac{Sm = Sn : m = n}{Sm = Sn \to m = n} \to S}{\forall y(Sm = Sy \to m = y)}\ \forall S}{\forall x \forall y(Sx = Sy \to x = y)}\ \forall S$$

- $\phi \equiv F(0) \wedge \forall x(F(x) \to F(Sx)) \to \forall x F(x)$. From lemma 1.8 we conclude $\boldsymbol{m}$ for all $m$ and then,

$$\dfrac{\dfrac{\dfrac{F(0,\boldsymbol{m}), \forall x(F(x,\boldsymbol{m}) \to F(Sx,\boldsymbol{m})) : \forall x F(x,\boldsymbol{m})}{F(0,\boldsymbol{m}) : \forall x(F(x,\boldsymbol{m}) \to F(Sx,\boldsymbol{m})) \to \forall x F(x,\boldsymbol{m})} \to S}{F(0,\boldsymbol{m}) \to \forall x(F(x,\boldsymbol{m}) \to F(Sx,\boldsymbol{m})) \to \forall x F(x,\boldsymbol{m})} \to S}{\forall \boldsymbol{y}(F(0,\boldsymbol{y}) \to \forall x(F(x,\boldsymbol{y}) \to F(Sx,\boldsymbol{y})) \to \forall x F(x,\boldsymbol{y}))}\ \forall S$$

  In fact we have to prove a slightly different formula but this does not cause any problem and the details are left to the reader.

- The other cases are similar

□

THEOREM 4.10. *If $Z \vdash \Gamma : \Delta$ where $\Gamma : \Delta$ is a closed sequent then there is an $\alpha < \varepsilon_0$ so that* $\left|\frac{\alpha}{0}\right. \Gamma : \Delta$.

PROOF. The embedding theorem yields $Z^\infty \left|\frac{\omega+k}{r}\right. \Gamma : \Delta$. Cut elimination yields $Z^\infty \left|\frac{\alpha}{0}\right. \Gamma : \Delta$ with $\alpha < \varepsilon_0$. □

## 5. Transfiniete inductie in $Z^\infty$

**Definition 4.11.** *Zij $\prec$ een gefundeerde binaire relatie op $\mathbb{N}$. We definiëren:*

$$|m|_\prec = \sup\{|n|_\prec + 1 \mid n \prec m\}$$
$$\|{\prec}\| = \sup\{|m|_\prec + 1 \mid m \in \mathbb{N}\}$$
$$\|Z\|_{\sup} = \sup\{\|{\prec}\| \mid Z \vdash \mathrm{TI}_\prec(X) \textit{ met } \prec L_0\textit{-definieerbaar}\}$$

The prominent goal of this section is to prove $\|Z\|_{\text{sup}} = \varepsilon_0$.

**Definition 4.12.** *We define the notion of $X$-positive and $X$-negative formulas as followst:*
(1) *If $X$does not occur in $\phi$, then $\phi$ is $X$-positive and $X$-negative,*
(2) *the formula $X(t)$ is $X$-positive,*
(3) *if $\phi$ is $X$-positive then $\psi \to \phi$ is $X$-positive and $\phi \to \psi$ is $X$-negative,*
(4) *if $\phi$ is $X$-negative then $\psi \to \phi$ is $X$-negative and $\phi \to \psi$ is $X$-positive,*
(5) *if $\phi$ is $X$-positive then $\forall x \phi$ is $X$-positive;*
(6) *if $\phi$ is $X$-negative then $\forall x \phi$ is $X$-negative.*

**Definition 4.13.** *Zij $U \subseteq \mathbb{N}$ en $\phi$ een formule die alleen $X$ als vrije predikaatvariabele mag bevatten. We define the validity of $\phi(X)$ in the structure $(\mathbb{N}, U)$ as follows.*

$$(\mathbb{N}, U) \models \phi(X)$$

*by stipulating the interpretation of $X$ via $\mathbb{N} \models X(t)$ iff* $\mathrm{val}(t) \in U$.

**Lemma 4.14** (Monotonicity). (1) *Suppose that $\phi(X)$ is an $X$-positive formula where $X$ is the only predicate variable in $\phi$. If $U \subseteq V \subseteq \mathbb{N}$ and $(\mathbb{N}, U) \models \phi(X)$ the $(\mathbb{N}, V) \models \phi(X)$.*
(2) *Suppose that $\phi(X)$ is an $X$-negative formula where $X$ is the only predicate variable in $\phi$. If $U \subseteq V \subseteq \mathbb{N}$ en $(\mathbb{N}, V) \models \phi(X)$ dan $(\mathbb{N}, U) \models \phi(X)$.*

PROOF. By induction on $|\phi|$.

Suppose that $|\phi| = 0$. Then there are two cases to consider:

- $X$ does not occur in $\phi$. In this case we have$\phi \equiv P(\boldsymbol{t})$ and the assertion is trivially true.
- $X$ does occur$\phi$. Then $\phi \equiv X(t)$ for some $t$. We have by assumption $(\mathbb{N}, U) \models X(t)$ and so $\mathrm{val}(t) \in U$. Then by assumption $\mathrm{val}(t) \in V$ and $(\mathbb{N}, V) \models X(t)$.

If $|\phi| > 0$ then there are two cases:

- $\phi(X) \equiv \psi(X) \to \chi(X)$. The $\chi(X)$ and $\psi(X)$ are $X$-positive and $X$-negative respectively. By assumption we have $(\mathbb{N}, U) \models \psi(X) \to \chi(X)$ hence $(\mathbb{N}, U) \not\models \psi(X)$ or $(\mathbb{N}, U) \models \chi(X)$.
  - Suppose that $(\mathbb{N}, U) \models \chi(X)$. The induction hypothesis yields $(\mathbb{N}, V) \models \chi(X)$ and thus $(\mathbb{N}, V) \models \psi(X) \to \chi(X)$.
  - Suppose that $(\mathbb{N}, U) \not\models \chi(X)$. Then $(\mathbb{N}, U) \not\models \psi(X)$. If $(\mathbb{N}, V) \models \psi(X)$, then by induction hypothesis for the second assertion $(\mathbb{N}, U) \models \psi(X)$ contradiction. Thus $(\mathbb{N}, V) \not\models \psi(X)$ and $(\mathbb{N}, V) \models \psi(X) \to \chi(X)$.
- $\phi(X) \equiv \forall y \psi(y, X)$. The assumption yields $(\mathbb{N}, U) \models \forall y \psi(y, X)$ and therefore $(\mathbb{N}, U) \models \psi(n, X)$ for all $n \in \mathbb{N}$. The induction hypothesis yields $(\mathbb{N}, V) \models \psi(n, X)$ for all $n$ hence $(\mathbb{N}, V) \models \forall y \psi(y, X)$.

The second assertion is shown similarly. $\square$

**Lemma 4.15.** *Suppose* $\vdash^{\alpha}_{0} X(t_1), \ldots, X(t_n), \mathrm{Prog}_{\prec}(X), \Gamma : \Delta$ *and that* $\prec$ *is well founded. Define*

$$\delta = \max\{|t_1|_{\prec}, \ldots, |t_n|_{\prec}\}$$
$$M_\alpha = \{m \in \mathbb{N} \mid |m|_{\prec} < \delta + 2^{\alpha}\}$$

*Suppose that* $X$ *occurs only negatively in* $\Gamma$ *and only positively in* $\Delta$*. Then*

$$(\mathbb{N}, M_\alpha) \models \bigwedge \Gamma \to \bigvee \Delta$$

PROOF. By induction on $\alpha$.

Suppose $\alpha = 0$. Then $X(t_1), \ldots, X(t_n), \mathrm{Prog}_{\prec}(X), \Gamma : \Delta$ is an axiom and there are two cases:

- $\Gamma : \Delta$ is an axiom. The the assertion follows.
- We have $X(t) \in \Delta$ with $\mathrm{val}(t) = \mathrm{val}(t_i)$ for some $i$. Because $|t_i|_{\prec} < \delta + 1$ we see $\mathrm{val}(t_i) = \mathrm{val}(t) \in M_0$ and the assertion follows.

Now suppose $\alpha > 0$. We have to deal with different cases according the the last applied inference rule:

- $(\to A)$. In this case we have

$$\vdash^{\alpha_0}_{0} X(t_1), \ldots, X(t_n), \mathrm{Prog}_{\prec}(X), \Gamma : \Delta, \phi$$
$$\vdash^{\alpha_1}_{0} X(t_1), \ldots, X(t_n), \mathrm{Prog}_{\prec}(X), \Gamma, \psi : \Delta$$

  where $\phi \to \psi \in \Gamma$. The induction hypothesis yields

$$(\mathbb{N}, M_{\alpha_0}) \models \bigwedge \Gamma \to \bigvee \Delta \vee \phi$$
$$(\mathbb{N}, M_{\alpha_1}) \models \bigwedge \Gamma \wedge \psi \to \bigvee \Delta$$

  We have $M_{\alpha_0}, M_{\alpha_1} \subseteq M_\alpha$ and with monotony we conclude

$$(\mathbb{N}, M_\alpha) \models \bigwedge \Gamma \to \bigvee \Delta \vee \phi \tag{1}$$
$$(\mathbb{N}, M_\alpha) \models \bigwedge \Gamma \wedge \psi \to \bigvee \Delta \tag{2}$$

  We have to prove that

$$(\mathbb{N}, M_\alpha) \models \bigwedge \Gamma \to \bigvee \Delta$$

  Assume that $(\mathbb{N}, M_\alpha) \models \bigwedge \Gamma$. If $(\mathbb{N}, M_\alpha) \models \bigvee \Delta$ then the assertion follows. Thus suppose $(\mathbb{N}, M_\alpha) \not\models \bigvee \Delta$. From (1) we find $(\mathbb{N}, M_\alpha) \models \phi$ and since $\phi \to \psi \in \Gamma$ we conclude $(\mathbb{N}, M_\alpha) \models \psi$. Then $(\mathbb{N}, M_\alpha) \models \bigwedge \Gamma \wedge \psi$ and from (2) we see $(\mathbb{N}, M_\alpha) \models \bigvee \Delta$. This is a contradiction.

- $(\forall A)$. This rule can be applied to $\mathrm{Prog}_{\prec}(X)$ or to $\Gamma$. If it has been applied to $\Gamma$ then the proof is as before. So assume that this rule has been applied to $\mathrm{Prog}_{\prec}(X)$. Then we had the premises

$$\vdash^{\alpha_0}_{0} X(t_1), \ldots, X(t_n), \mathrm{Prog}_{\prec}(X), (\forall y \prec k : X(y)) \to X(k), \Gamma : \Delta$$

  with $\alpha_0 < \alpha$ and $k \in \mathbb{N}$. We can not yet apply the induction hypothesis to this sequent because the is antecedent $X$-positive. By inversion applied to $(\to A)$ we nevertheless find

$$\vdash^{\alpha_0}_{0} X(t_1), \ldots, X(t_n), \mathrm{Prog}_{\prec}(X), \Gamma : \Delta, (\forall y \prec k : X(y)) \tag{3}$$
$$\vdash^{\alpha_0}_{0} X(t_1), \ldots, X(t_n), \mathrm{Prog}_{\prec}(X), \Gamma, X(k) : \Delta \tag{4}$$

The induction hypothesis applied to (3) yields:

(5)
$$(\mathbb{N}, M_{\alpha_0}) \models \bigwedge \Gamma \to \bigvee \Delta \vee (\forall y \prec k : X(y))$$

We have to show that

$$(\mathbb{N}, M_\alpha) \models \bigwedge \Gamma \to \bigvee \Delta$$

If $(\mathbb{N}, M_{\alpha_0}) \models \bigwedge \Gamma \to \bigvee \Delta$ then the assertion follows by monotony. Thus suppose $(\mathbb{N}, M_{\alpha_0}) \not\models \bigwedge \Gamma \to \bigvee \Delta$. Then $(\mathbb{N}, M_{\alpha_0}) \models \bigwedge \Gamma$ and $(\mathbb{N}, M_{\alpha_0}) \not\models \bigvee \Delta$. From (5) we conclude $(\mathbb{N}, M_{\alpha_0}) \models \forall y \prec k : X(y)$ and thus $|k|_\prec \leq \delta + 2^{\alpha_0}$. Define

$$\delta' = \max\{\delta, |k|_\prec\}$$

The induction hypothesis yields

$$(\mathbb{N}, \{m \in \mathbb{N} \mid |m|_\prec < \delta' + 2^{\alpha_0}\}) \models \bigwedge \Gamma \to \bigvee \Delta$$

We have $\{m \in \mathbb{N} \mid |m|_\prec < \delta' + 2^{\alpha_0}\} \subset M_\alpha$. The assertion follows by monotonicity.

- The other rules can be treated similarly.

$\square$

**Lemma 4.16.** *Suppose that* $\left|\frac{\alpha}{0}\right. \mathrm{Prog}_\prec(X) \to \forall x.Xx$ *and that* $\prec$ *is well founded. Then* $\|\prec\| \leq 2^\alpha$.

PROOF. The inversion lemma yields $\left|\frac{\alpha}{0}\right. \mathrm{Prog}_\prec(X) : \forall x.X(x)$. The previous lemma yields

$$(\mathbb{N}, M_\alpha) \models \forall x.Xx$$

Since $\forall x.Xx$, we see $\forall k \in \mathbb{N}.k \in M_\alpha$ en dus $\mathbb{N} \subseteq M_\alpha$ so that:

$$\|\prec\| = \sup\{|m|_\prec + 1 \mid m \in \mathbb{N}\} \leq \sup\{|m|_\prec + 1 \mid m \in M_\alpha\}$$

We have in this context $M_\alpha = \{m \in \mathbb{N} \mid |m|_\prec < 2^\alpha\}$ and thus $\sup\{|m|_\prec + 1 \mid m \in M_\alpha\} \leq 2^\alpha$. $\square$

THEOREM 4.17. *Suppose* $Z \vdash \mathrm{TI}_\prec(X)$ *and that* $\prec$ *is well founded. Then* $\|\prec\| < \varepsilon_0$.

PROOF. The embedding theorem (theorem 1.9) yields the existence of $k'$ and $r'$ so that $\left|\frac{\omega+k'}{r'}\right. \mathrm{TI}_\prec(X)$. Using the cut elimination theorem (theorem 1.7) we obtain: $\left|\frac{3_{r'}(\omega+k')}{0}\right. \mathrm{TI}_\prec(X)$ where $3_0(\alpha) := \alpha$ and $3_{r+1}(\alpha) := 3^{3_r(\alpha)}$.

The boundedness lemma lemma 1.16 yields:

$$\|\prec\| \leq 2^{(3_{r'}(\omega+k')} < \varepsilon_0.$$

$\square$

A binary relation $\prec$ over $\mathbb{N}$ is well founded if there does not exist an infinite $\prec$-descending chain of elements in $\mathbb{N}$. The rank of a wellfounded relation $\prec \subseteq \mathbb{N} \times \mathbb{N}$ is the least $\alpha$ such that there exists a function $f : \mathbb{N} \to \alpha$ such that $m \prec n$ implies $f(m) < f(n)$. The rank $rk(m)$ of $m$ is the rank of the restriction of $\prec$ to $\{n \in Nat : n \prec m\}$.

**Exercise 4.18.** *Let* $\prec$ *be an arithmetical well founded order (of rank less than* $\varepsilon_0$*).*
*Show* $Z \left|\frac{3\cdot(rk(n)+1)}{0}\right. \mathrm{Prog}_\prec(X), n \in X$ *for all* $n$.

CHAPTER 5

# Provably recursive funtions of $Z$

The goal of this section is to investigate provable and and unprovable (with regard to the formal system $Z$) instances of sentences of the form $\forall x \exists y \phi(x, y)$ with $\phi$ quantifier free. In fact our method will work with some minor modifications also for sentences of the form $\forall x \exists y \phi(x, y)$ with $\phi$ being an existantial sentence.

Note that $\forall x \exists y \phi(x, y)$ describes the totality of an algorithm: "for every input $x$ there is an output $y$ fulfilling a certain specification".

**Definition 5.1.** *Let $\alpha = \omega^{\alpha_1} m_1 + \cdots + \omega^{\alpha_n} m_n$ be in Cantor normal form. Then*

$$\max(\alpha) = \max\{\max(\alpha_1), \ldots, \max(\alpha_n), m_1, \ldots, m_n\}$$

*and*

$$N(\alpha) = n + N(\alpha_1) + \cdots + N(\alpha_n)$$

## 1. The Hardy-hierarchy

**Definition 5.2.** *The Hardy hierarchy $(H_\alpha)_{\alpha \leq \varepsilon_0}$ is defined as follows:*

$$\begin{aligned} H_0(n) &= n, \\ H_{\alpha+1}(n) &= H_\alpha(n+1), \\ H_\lambda(n) &= H_{\lambda[n]}(n) && \textit{where} \lambda \in \mathsf{Lim}. \end{aligned}$$

*Here $\alpha[n]$ is the $n$-th element of the fundamental sequence for $\alpha < \varepsilon_0$:*

$$\begin{aligned} \alpha[n] &= \begin{cases} 0 & \textit{als } \alpha = 0 \\ \alpha_0 + \cdots + \alpha_{m-1} + \alpha_m[n] & \textit{als } \alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_m \end{cases} \\ \omega^{\alpha+1}[n] &= \omega^\alpha(n+1) \\ \omega^\lambda[n] &= \omega^{\lambda[n]} \end{aligned}$$

*Moreover for $\alpha = \varepsilon_0$ we put $\alpha[n] := \omega_n$ where $\omega_0 := 1$ and $\omega_{l+1} := \omega^{\omega_l}$ for $l < \omega$.*

We use this hierarchy for scaling the provably total functions of $Z$. The idea is that the computational complexity of $H_\alpha$ grows when $\alpha$ becomes larger and larger. We will see shortly that the functions $H_\alpha$ grow rather quickly even for rather small ordinals like $\omega^\omega$.

**Definition 5.3.** *We say* $\mathrm{NF}(\alpha, \beta)$ *if one the following conditions hold:*

(1) $\alpha = 0$;
(2) $\beta = 0$;
(3) $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ *and* $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$ *and* $\alpha_1 \geq \beta_1$.

The idea behind this definition is that $\mathrm{NF}(\alpha, \beta)$ implies $\alpha \oplus \beta = \alpha + \beta$ which means that no subterms of $\alpha$ disappear after performing the sum with $\beta$.

**Lemma 5.4.**

(1) $\alpha \in \mathsf{Lim} \implies \alpha[n] < \alpha[n+1]$ *and* $\sup_{n<\omega} \alpha[n] = \alpha$ *if* $n \to \omega$
(2) $\alpha > 0 \implies N(\alpha[0]) + 1 = N(\alpha)$.

PROOF. Both assertions can be proved by easy inductions on $\alpha$. □

**Lemma 5.5** (The Bachmann property). $\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0]$.

PROOF. Assume that $\beta =_{\mathrm{NF}} \beta_0 + \cdots + \beta_k$ with $k \geq 0$. There are the following three cases

(1) $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_m$ with $m > 0$. Then
$$\alpha[n] = \alpha_0 + \cdots + \alpha_m[n] < \beta_0 + \cdots + \beta_k < \alpha_0 + \cdots + \alpha_m$$
This yields $k \geq m$ en $\alpha_i = \beta_i$ for all $i < m$ so that
$$\alpha_m[n] < \beta_m + \cdots + \beta_k < \alpha_m.$$
This yields $\alpha_m[n] \leq \beta_m < \alpha_m$. If $k = m$ then $\alpha_m[n] < \beta_m < \alpha_m$. Then the induction hypothesis yields $\alpha_m[n] \leq \beta_m[0] \leq \alpha_m$. If $k > m$ then $\beta[0] =\geq \beta_0 + \cdots + \beta_m \geq \alpha[n]$.

(2) Suppose now $\alpha = \omega^{\gamma+1}$. Then
$$\alpha[n] = \omega^{\gamma}(n+1) < \beta < \omega^{\gamma+1}$$
This yields $\beta_0 = \cdots = \beta_n = \omega^{\gamma}$ and $\omega^{\gamma} < \beta_{n+1} + \ldots + \beta_k < \omega^{\gamma+1}$. The case $k = n+1$ is impossible. Hence $k > n+1$ and
$$\omega^{\gamma}(n+1) \leq \beta_0 + \cdots + \beta_n + \cdots + \beta_k[0]$$

(3) Suppose $\alpha = \omega^{\lambda}$. Then
$$\alpha[n] = \omega^{\lambda}[n] = \omega^{\lambda[n]} < \beta < \omega^{\lambda}$$
If $k > 0$ then $\beta[0] \geq \beta_0 \geq \omega^{\lambda[n]}$. If $k = 0$ and $\beta_0 = \omega^{\gamma}$ then $\lambda[n] < \gamma < \lambda$. The induction hypothesis yields $\lambda[n] \leq \gamma[0]$. Thence
$$\omega^{\lambda[n]} \leq \omega^{\gamma[0]} = \beta[0].$$

□

**Lemma 5.6.** $\alpha[n] < \beta < \alpha \implies N(\alpha[n]) < N(\beta)$.

PROOF. This follows from assertion 2 of Lemma lemma 1.4 and from Lemma lemma 1.5. □

**Lemma 5.7.** $\alpha < \beta \implies \alpha \leq \beta[N(\alpha)]$.

PROOF. From lemma 1.6 we obtain for $\beta \in \mathsf{Lim}$ that

$$N(\beta[n]) < N(\beta[n+1])$$

hence

$$\implies N(\alpha) \leq N(\beta[N(\alpha)]).$$

Suppose $\beta[N(\alpha)] < \alpha < \beta$. Then lemma 1.6 yields a contradiction. □

**Lemma 5.8.**
(1) $H_\alpha(n) < H_\alpha(n+1)$,
(2) $\beta[m] < \alpha < \beta \implies H_{\beta[m]}(n+1) \leq H_\alpha(n)$,
(3) $\beta < \alpha \wedge N(\beta) \leq n \implies H_\beta(n) < H_\alpha(n)$,
(4) $\alpha > 0 \implies H_\alpha(n) = \min\{k \geq n \mid \alpha[n]\ldots[k{-}1] = 0\} = n{+}\min\{k \mid \alpha[n]\ldots\alpha[n{+}k{-}1] > 0\}$.

PROOF. Let $k \geq n$ be minimal such that $\alpha[n]\ldots[k-1] = 0$. Then

$$H_\alpha(n) = H_{\alpha[n]}(n+1) = H_{\alpha[n][n+1]}(n+2) = H_{\alpha[n]\ldots[k-1]}(k) = k$$

The first two assertions are proved simultaneously by induction on $\alpha$. The third assertion follows by induction on $\alpha$ noting that $\beta < \alpha$ and $N\beta \leq n$ yields $\beta \leq \alpha[n]$ by Lemma lemma 1.5. □

**Lemma 5.9.**
(1) $\mathrm{NF}(\alpha, \beta)$ *yields* $H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$.
(2) $H_{\omega^{\alpha+1}}(n) = H^{n+1}_{\omega^\alpha}(n+1)$ *en* $H_{\omega^\lambda}(n) = H_{\omega^{\lambda[n]}}(n+1)$.
(3) *For all primitive recursive functions* $f : \mathbb{N}^d \to \mathbb{N}$ *there exists a* $k$ *such that for all* $\boldsymbol{x} \in \mathbb{N}^d$ *we have* $f(\boldsymbol{x}) < H_{\omega^k}(\max \boldsymbol{x})$.

PROOF.

(1) By induction on $\beta$.
(2) $H_{\omega^{\alpha+1}}(n) = H_{\omega^{\alpha+1}[n]}(n+1) = H_{\omega^\alpha(n+1)}(n+1) = H^{n+1}_{\omega^\alpha}(n+1)$.
(3) This assertion follows as usual from the second assertion by an induction along the generation history of $f$.

□

## 2. Operator-controlled derivations

In the sequel we consider formal and semiformal systems without set variables. These set variables have been used to model well foundedness properties but are not relevant in the context of provably recursive functions. Nevertheless all statements and proofs will go through for the versions with set variables. To obtain a better control about the computational content of existential statements we introduce a modified infinitary system in which derivations are controlled by a operator $F : \mathbb{N} \to \mathbb{N}$ which satisfies

$$\forall m, n \in \mathbb{N} : (m \leq n \implies F(m) \leq F(n))$$

and

$$\forall m \in \mathbb{N} : F(m) \geq m$$

We call such operators nice. Those operators are nice in the sense that they preserve information which is coded in arguments or function values for smaller arguments.

For such operators we write $F \leq G$ if for all $x \in \mathbb{N}$ we have $F(x) \leq G(x)$.

**Definition 5.10.** *Let $F$ be a nice operator and $i < \omega$. Then we define the operator $F[i]$ as follows.*

$$F[i](x) = F(\max\{i, x\}).$$

Then $F[i]$ is of course a nice operator, too.

**Definition 5.11.** *Let $F$ be a nice operator. $F \left|\frac{\alpha}{r}\right. \Gamma : \Delta$ holds if $N(\alpha) \leq F(0)$ and one of the following cases holds:*

- *Axiom: $\Gamma$ contains a false atomic formula or $\Delta$ contains a true atomaic formula;*
- *$(\to A)$: $F \left|\frac{\alpha_0}{r}\right. \Gamma : \Delta, \phi$ and $F \left|\frac{\alpha_1}{r}\right. \Gamma, \psi : \Delta$ and $\alpha_0, \alpha_1 < \alpha$ and $\phi \to \psi \in \Gamma$.*
- *$(\to S)$: $F \left|\frac{\alpha_0}{r}\right. \Gamma, \phi : \psi, \Delta$ and $\alpha_0 < \alpha$ and $\phi \to \psi \in \Delta$.*
- *$(\forall A)$: $F \left|\frac{\alpha_0}{r}\right. \Gamma, \phi(k) : \Delta$ and $k \leq F(0)$ and $\alpha_0 < \alpha$ and $\forall x \phi \in \Gamma$.*
- *$(\forall S)$: $F[i] \left|\frac{\alpha_i}{r}\right. \Gamma : \Delta, \phi(i)$ and $\alpha_i < \alpha$ for all $i \in \mathbb{N}$ and $\forall x \phi \in \Delta$.*
- *(CUT): $F \left|\frac{\alpha_0}{r}\right. \Gamma, \phi : \Delta$ and $F \left|\frac{\alpha_1}{r}\right. \Gamma : \phi, \Delta$ and $\alpha_0, \alpha_1 < \alpha$ and $|\phi| < r$.*

Note that this system is very similar to $Z^\infty$. The major difference is due to the rules for the quantifiers where the operators play a role. The idea is that existantial witnesses are saved into the controlling operator in case of an application of $(\forall A)$. The role of the extra condition $N\alpha \leq F(0)$ becomes transparent in the proof of the cut reduction theorem.

**Lemma 5.12** (Structural lemma)**.** *Suppose $F \left|\frac{\alpha}{r}\right. \Gamma : \Delta$ and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ and $r \leq r'$ and $\alpha \leq \alpha'$ and $\forall l : F(l) \leq G(l)$. Then $G \left|\frac{\alpha'}{r'}\right. \Gamma' : \Delta'$.*

PROOF. Proof by induction on $\alpha$. □

**Lemma 5.13** (Inversie)**.**

(1) *If $F \left|\frac{\alpha}{r}\right. \Gamma : \Delta, \phi$ and $\phi$ is a false atomic formula then $F \left|\frac{\alpha}{r}\right. \Gamma : \Delta$.*
(2) *If $F \left|\frac{\alpha}{r}\right. \Gamma, \phi : \Delta$ and $\phi$ is a true atomic formula then $F \left|\frac{\alpha}{r}\right. \Gamma : \Delta$.*
(3) *If $F \left|\frac{\alpha}{r}\right. \Gamma : \Delta, \phi \to \psi$ then $F \left|\frac{\alpha}{r}\right. \Gamma, \phi : \psi, \Delta$.*
(4) *If $F \left|\frac{\alpha}{r}\right. \Gamma, \phi \to \psi : \Delta$ then $F \left|\frac{\alpha}{r}\right. \Gamma : \phi, \Delta$ and $F \left|\frac{\alpha}{r}\right. \Gamma, \psi : \Delta$.*
(5) *If $F \left|\frac{\alpha}{r}\right. \Gamma : \Delta, \forall x \phi$ then $F[i] \left|\frac{\alpha}{r}\right. \Gamma : \Delta, \phi(i)$ for all $i$.*

PROOF. All proofs can be carried out by routine inductions on $\alpha$. Only the proof of the last assertion needs some extra care. The proof is still by induction on $\alpha$. Suppose $\alpha = 0$. Then $\Gamma : \Delta, \forall x \phi$ is an axiom. Since $\forall x \phi$ is not an atomic formule, we see that $\Gamma : \Delta$ is an axiom and the assertion follows. Now suppose $\alpha > 0$. We may assume that a last inference rule has been applied:

- $(\forall A)$. Then $F \overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, \psi(k) : \Delta, \forall x\phi$ and $k \leq F(0)$ and $\alpha_0 < \alpha$ and $\forall y\psi \in \Gamma$. The induction hypothesis yields $F[i] \overset{\alpha_0}{\underset{r}{\vdash}} \Gamma, \psi(k) : \Delta, \phi(i)$ for all $i$. An application of $(\forall A)$ yields the assertion since $k \leq F(0) \leq F[i](0)$.
- $(\forall S)$. Then $F[i] \overset{\alpha_i}{\underset{r}{\vdash}} \Gamma : \Delta, \psi(i), \forall x\phi$ and $\alpha_i < \alpha$ for all $i \in \mathbb{N}$ and $\forall y\psi \in \Delta, \forall x\phi$. The induction hypothesis yields $F[i][j] \overset{\alpha_i}{\underset{r}{\vdash}} \Gamma : \Delta, \psi(i), \phi(j)$ for all $j$. There are two cases to consider:
  - $\forall y\psi \not\equiv \forall x\phi$. Then we apply $(\forall S)$ and the assertion follows.
  - $\forall y\psi \equiv \forall x\phi$. By implicit contraction we obtain:
  $$F[i][i] \overset{\alpha_i}{\underset{r}{\vdash}} \Gamma : \Delta, \phi(i)$$
  Since $F[i][i] = F[i]$ we thus obtain:
  $$F[i] \overset{\alpha_i}{\underset{r}{\vdash}} \Gamma : \Delta, \phi(i)$$
  The assertion follows.
- The other cases are easy.

□

## 3. Cut-elimination

Now we are going to prove the cut elimination. This will have a subtle effect on the underlying operators.

**Definition 5.14.** *Suppose that $F$ and $G$ are nice operators. Define*

$$C(F, G) = F \circ G + F + G$$

The idea is that $C(F, G)$ models the composition of $F$ and $G$. The extra two summands are built in for conve nience and have to do with technical constraints regarding controlling the norms of ordinals in later applications and are in fact superfluous.

**Lemma 5.15** (Reduction lemma)**.** *Suppose that $G \overset{\beta}{\underset{r}{\vdash}} \Gamma, \phi : \Delta$ and $F \overset{\alpha}{\underset{r}{\vdash}} \Gamma : \Delta, \phi$ and $|\phi| \leq r$. Then*

$$C(F, G) \overset{\alpha+\beta 2}{\underset{r}{\vdash}} \Gamma : \Delta$$

PROOF. The proof is by induction on $\beta$. Note first that $N(\alpha + \beta \cdot 2) \leq F(0) + G(0) + G(0) \leq C(F, G)(0)$. There are two cases to consider:

- $\phi$ was not the principal formula of the last inference. If $\Gamma, \phi : \Delta$ is an axiom then the assertion follows immediately. Otherwise we distinguish cases according the last applied inference rule. Assume, for example, that the rule $(\forall S)$ has been applied. Then we see for all $i$,
$$G[i] \overset{\beta_i}{\underset{r}{\vdash}} \Gamma, \phi : \Delta, \psi(i)$$
with $\forall x\psi \in \Delta$. The induction hypothesis yields
$$C(F, G[i]) \overset{\alpha+\beta_i 2}{\underset{r}{\vdash}} \Gamma : \Delta, \psi(i)$$
We have $C(F, G[i]) \leq C(F, G)[i]$. Indeed, we have for all $l < \omega$

$$C(F, G[i])(l) = F(G[i](l)) + F(l) + G[i](l) \leq F(G(\max\{i, l\})) + F(\max\{i, l\}) + G(\max\{i, l\}) = C(F, G)[i](l)$$

and thus we can apply $(\forall S)$ and the assertion follows.

- $\phi$ was the principal formula of the last inference.
  - If $\Gamma, \phi : \Delta$ is an axiom then $\phi$ is a false atomic formula and the assertion follows by assertion two of Lemma lemma 1.13 applied to $F \overset{\alpha}{\underset{r}{\vdash}} \Gamma : \Delta, \phi$.
  - If $\phi \equiv \psi \to \chi$ then inversion yields
$$G \overset{\beta_0}{\underset{r}{\vdash}} \Gamma, \phi : \psi, \Delta$$
$$G \overset{\beta_1}{\underset{r}{\vdash}} \Gamma, \phi, \chi : \Delta$$
The induction hypothesis yields
$$C(F,G) \overset{\alpha+\beta_0 2}{\underset{r}{\vdash}} \Gamma : \psi, \Delta$$
$$C(F,G) \overset{\alpha+\beta_1 2}{\underset{r}{\vdash}} \Gamma, \chi : \Delta$$
Applying inversion (Lemma 5.15) on $F \overset{\alpha}{\underset{r}{\vdash}} \Gamma : \Delta, \phi$ yields
$$F \overset{\alpha}{\underset{r}{\vdash}} \Gamma, \psi : \chi, \Delta$$
By applying cut we obtain:
$$\dfrac{\dfrac{\Gamma : \psi, \chi, \Delta \qquad \Gamma, \psi : \chi, \Delta}{\Gamma : \chi, \Delta}\text{CUT} \qquad \Gamma, \chi : \Delta}{C(F,G) \overset{\alpha+\beta 2}{\underset{r}{\vdash}} \Gamma : \Delta}\text{CUT}$$
Applying cut is allowed since $|\psi|, |\chi| < |\phi| \leq r$. Note that $\alpha + \max\{\beta_0, \beta_1\} + 2 \leq \alpha + \beta \cdot 2$.
  - $\phi \equiv \forall x \psi$. We have the premises
$$G \overset{\beta_0}{\underset{r}{\vdash}} \Gamma, \phi, \psi(k) : \Delta$$
with $k \leq G(0)$. The induction hypothesis yields
$$C(F,G) \overset{\alpha+\beta_0 2}{\underset{r}{\vdash}} \Gamma, \psi(k) : \Delta$$
Inversion applied to $F \overset{\alpha}{\underset{r}{\vdash}} \Gamma : \Delta, \phi$ yields
$$F[k] \overset{\alpha}{\underset{r}{\vdash}} \Gamma : \Delta, \psi(k)$$
Now we have $k \leq G(0)$ so that $F[k] \leq F[G(0)] \leq C(F,G)$ and we can again apply a cut to obtain:
$$C(F,G) \overset{\alpha+\beta 2}{\underset{r}{\vdash}} \Gamma : \Delta$$

□

**Definition 5.16.** *For a nice operator $F$ we define*
$$F^\alpha(x) = \max\left(\{F(x)+1\} \cup \{C(F^\gamma, F^\delta)(x) \mid \gamma, \delta < \alpha \wedge N(\gamma), N(\delta) \leq F(x)\}\right)$$

**Lemma 5.17.** *Let $F$ be a nice operator. Then we have for all $i, m \in \mathbb{N}$ and $\alpha \in \mathsf{On}$ that*

$$(F[i])^\alpha(m) \leq F^\alpha[i](m).$$

PROOF. By indcution on $\alpha$. If $\alpha = 0$ then the assertion is clear. So suppose $\alpha > 0$. For some $\gamma, \delta < \alpha$ and $N(\gamma), N(\delta) \leq F[i](m)$ we obtain

$$\begin{aligned}
F[i]^\alpha(m) &= C(F[i]^\gamma, F[i]^\delta)(m) \\
&= F[i]^\gamma(F[i]^\delta(m)) + F[i]^\gamma(m) + F[i]^\delta(m) \\
&\leq F[i]^\gamma(F^\delta[i](m)) + F^\gamma[i](m) + F^\delta[i](m) \\
&\leq F^\gamma[i](F^\delta[i](m)) + F^\gamma[i](m) + F^\delta[i](m) \\
&= F^\gamma(F^\delta(\max\{i, m\})) + F^\gamma(\max\{i, m\}) + F^\delta(\max\{i, m\}) \\
&\leq F^\alpha(\max\{i, m\}) = F^\alpha[i](m).
\end{aligned}$$

□

THEOREM 5.18 (Cut-elimination). *If $F \vdash^{\alpha}_{r+1} \Gamma : \Delta$ then $F^\alpha \vdash^{\omega^\alpha}_{r} \Gamma : \Delta$.*

PROOF. By induction on $\alpha$. If $\alpha = 0$ then the assertion is obvious since no cut is involved. Suppose that $\alpha > 0$. Note that $N\alpha \leq F(0)$ yields $N\omega^\alpha) \leq F^\alpha(0)$. We have to consider two cases:

- The last inference rule was a cut. Then we have the premises:

  $$F \vdash^{\alpha_0}_{r+1} \Gamma, \phi : \Delta$$
  $$F \vdash^{\alpha_1}_{r+1} \Gamma : \Delta, \phi$$

  with $\alpha_0, \alpha_1 < \alpha$, $N(\alpha_0), N(\alpha_1) \leq F(0)$ and $|\phi| < r + 1$. The induction hypothesis yields

  $$F^{\alpha_0} \vdash^{\omega^{\alpha_0}}_{r} \Gamma, \phi : \Delta$$
  $$F^{\alpha_1} \vdash^{\omega^{\alpha_1}}_{r} \Gamma : \Delta, \phi$$

  with $|\phi| \leq r$. The reduction lemma yields

  $$C(F^{\alpha_0}, F^{\alpha_1}) \vdash^{\omega^{\alpha_0} + \omega^{\alpha_1} 2}_{r} \Gamma : \Delta$$

  We have $C(F^{\alpha_0}, F^{\alpha_1}) \leq F^\alpha$ since $N(\alpha_0), N(\alpha_1) \leq F(0)$.

  Moreover, we have $\omega^{\alpha_0} + \omega^{\alpha_1} 2 \leq \omega^\alpha$ and the assertion follows.

- The last inference rule was not a cut. We distinguish cases according to the last applied derivation rule. Consider, for example, the case of $(\forall S)$. (The other cases can be treated similarly.) We have the premises

  $$F[i] \vdash^{\alpha_i}_{r+1} \Gamma : \Delta, \phi(i)$$

  and

  $$N(\alpha_i) \leq F[i](0)$$

for all $i$ with $\forall x\phi \in \Delta$. The induction hypothesis yields

$$F[i]^{\alpha_i} \left|\frac{\omega^{\alpha_i}}{r}\right. \Gamma : \Delta, \phi(i)$$

From lemma 1.17 and $N(\alpha_i) \leq F[i](0)$ we obtain $F[i]^{\alpha_i} \leq F^{\alpha_i}[i] \leq F^{\alpha}[i]$ and hence

$$F^{\alpha}[i] \left|\frac{\omega^{\alpha_i}}{r}\right. \Gamma : \Delta, \phi(i)$$

An application of $(\forall S)$ yields

$$F^{\alpha} \left|\frac{\omega^{\alpha}}{r}\right. \Gamma : \Delta$$

□

## 4. Witness bounds for existential sentences

**Lemma 5.19.** *Suppose that $F \left|\frac{\alpha}{0}\right. \exists x\phi(x)$ with $\phi$ atomic. Then there exists a $k \leq F(0)$ so that $\mathbb{N} \models \phi(k)$.*

PROOF. Suppose $F \left|\frac{\alpha}{0}\right. \exists x\phi(x)$. This yields

$$F \left|\frac{\alpha}{0}\right. \forall x(\phi(x) \to \bot) : \bot$$

Now we perform an induction on $\alpha$.

Since the proof is cut free the last applied rule is $(\forall A)$. We thus had premises

$$F \left|\frac{\alpha_0}{0}\right. \forall x(\phi(x) \to \bot), \phi(k) \to \bot : \bot$$

where $k \leq F(0)$. If $\mathbb{N} \models \phi(k)$ the the assertion follows. So assume $\mathbb{N} \not\models \phi(k)$ for all $k \leq F(0)$. By inversion (Lemma lemma 1.13 we obtain

$$F \left|\frac{\alpha_0}{0}\right. \forall x(\phi(x) \to \bot) : \phi(k), \bot$$

where $\phi(k)$ is a false atomic formula. Another application of Inversion yields

$$F \left|\frac{\alpha_0}{0}\right. \forall x(\phi(x) \to \bot) : \bot$$

for some $\alpha_0 < \alpha$. By induction hypothesis we obtain a $k \leq F(0)$ so that $\mathbb{N} \models \phi(k)$. Contradiction.

□

THEOREM 5.20. *If $F \left|\frac{\alpha}{0}\right. \forall x\exists y\phi(x, y)$ with $\phi$ atomic, then we have for all $m \in \mathbb{N}$ that there exists an $n \leq F(m)$ with $\mathbb{N} \models \phi(m, n)$.*

PROOF. Suppose $F \left|\frac{\alpha}{0}\right. \forall x\exists y\phi(x, y)$ with $\phi$ atomic. Inversion yields

$$F[m] \left|\frac{\alpha}{0}\right. \exists y\phi(i, y)$$

for a given $m$. From lemma 1.19 we obtain that $\mathbb{N} \models \phi(m, n)$ for some $n \leq F[m](0) = F(m)$. □

## 5. Embedding of $Z$

We can prove the embedding of $Z$ into $Z^\infty$ also in the context of operator-controlled derivations.

**Lemma 5.21.**
(1) *If* $\mathrm{val}(s) = \mathrm{val}(t)$ *then* $2|\phi| + \mathrm{id} \vdash_0^{2|\phi|} \phi(s) : \phi(t)$.
(2) $2|\phi| + 2\,\mathrm{id} + 2 \vdash_0^{\omega} \phi(0), \forall x(\phi(x) \to \phi(Sx)) : \forall x \phi(x)$.

PROOF.

(1) By induction on $|\phi|$. Most cases are routine. The interesting case is $\phi \equiv \forall y \psi(y)$. The induction hypothesis yields

$$2|\psi| + \mathrm{id} \vdash_0^{2|\psi|} \psi(s, m) : \psi(t, m)$$

for all $m$. We find

$$(2|\psi| + \mathrm{id} + 1)[m] \vdash_0^{2|\psi|+1} \forall y \psi(s, y) : \psi(t, m)$$

since $m \leq (22|\psi| + id + 1[m](0)$. Using $(\forall S)$ we obtain

$$(2|\psi| + \mathrm{id} + 2) \vdash_0^{2|\psi|+1} \forall y \psi(s, y) : \forall y \psi(t, y).$$

(2) Note that $(2|\phi| + 2\,\mathrm{id})[n](0) \geq N(2|\phi| + 2n)$. By induction on $n$ we prove that

$$(2|\phi| + 2\,\mathrm{id})[n] \vdash_0^{2|\phi|+2n} \phi(0), \xi : \phi(n)$$

where $\xi \equiv \forall x(\phi(x) \to \phi(Sx))$. If $n = 0$, then the first assertion yields

$$(2|\phi| + 2\,\mathrm{id})[0] \vdash_0^{2|\phi|} \phi(0), \xi : \phi(0)$$

Assume now $n > 0$. The induction hypothesis yields

$$(2|\phi| + 2\,\mathrm{id})[n] \vdash_0^{2|\phi|+2n} \phi(0), \xi : \phi(n)$$

The first assertion yields

$$(2|\phi| + \mathrm{id})[n] \vdash_0^{2|\phi|} \phi(Sn) : \phi(Sn)$$

So we find

$$(2|\phi| + 2\,\mathrm{id} + 1)[n] \vdash_0^{2|\phi|+2n+1} \phi(0), \xi, \phi(n) \to \phi(Sn) : \phi(Sn).$$

This yields

$$(2|\phi| + 2\,\mathrm{id} + 2)[n] \vdash_0^{2|\phi|+2n+1} \phi(0), \xi : \phi(Sn)$$

and thus

$$(2|\phi| + 2\,\mathrm{id})[n+1] \vdash_0^{2|\phi|+2n+2} \phi(0), \xi : \phi(Sn)$$

The assertion follows.

$\square$

THEOREM 5.22 (Embedding). *Suppose that* $\mathrm{FV}(\Gamma : \Delta) \subseteq \{a_1, \ldots, a_n\}$ *and* $Z \vdash \Gamma : \Delta$. *Then there exist* $k, r < \omega$ *and a primitive recursive nice operator* $F$ *so that for all* $\boldsymbol{m} = (m_1, \ldots, m_n)$ *we have* $F[\boldsymbol{m}] \mid\!\frac{\omega+k}{r}\, \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$.

PROOF. By induction of the length of the derivation of $\Gamma : \Delta$ in $Z$. If $\Gamma : \Delta$ is an axioma of $Z$, then $\Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$ is an axiom, too. Suppose that a last inference rule has been applied:

- The rules $(\to A)$ and $(\to S)$ can be treated as for $Z^\infty$.
- $(\forall A)$. Then we have $\forall x \phi(x) \in \Gamma$ and we have a premis $\vdash \Gamma, \phi(t) : \Delta$. The induction hypothesis yields
$$G[\boldsymbol{m}] \mid\!\frac{\omega+k_0}{r_0}\, \Gamma(\boldsymbol{m}), \phi(t(\boldsymbol{m})) : \Delta(\boldsymbol{m})$$
with $r_0, k_0 < \omega$. From lemma 1.21 we obtain
$$2|\phi| + \mathrm{id} \mid\!\frac{2|\phi|}{0}\, \phi(\mathrm{val}(t(\boldsymbol{m}))) : \phi(t(\boldsymbol{m}))$$
We may assume that $G[\boldsymbol{m}] \geq 2|\phi| + \mathrm{id}$ and $G[\boldsymbol{m}](0) \geq N(\omega + k_0 + 1)$ and can apply a cut:
$$G[\boldsymbol{m}] \mid\!\frac{\omega+k_0+1}{\max(r_0, |\phi|+1)}\, \Gamma(\boldsymbol{m}), \phi(\mathrm{val}(t(\boldsymbol{m}))) : \Delta(\boldsymbol{m})$$
An application of $(\forall A)$ is not yet immediately allowed since we have to guarantee that $\mathrm{val}(t(\boldsymbol{m})) \leq G(0)$. This is not necessarily true. But we can majorize $m \mapsto \mathrm{val}(t(\boldsymbol{m}))$ by a sufficiently large branch $A_d$ of the Ackermann-function and we can obtain:
$$(G + A_d)[\boldsymbol{m}] \mid\!\frac{\omega+k_0+2}{\max(r_0, |\phi|+1)}\, \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$
Here $d$ has to fulfill $A_d(\max(\boldsymbol{m})) \geq \mathrm{val}(t(\boldsymbol{m})) + 1$.
- $(\forall S)$. Then $\forall y \psi(y) \in \Delta$ and we have a premis $\vdash \Gamma : \psi(a), \Delta$ with $a \notin \mathrm{FV}(\Gamma : \Delta)$. The induction hypothesis yields for all $l \in \mathbb{N}$
$$G[\boldsymbol{m}, l] \mid\!\frac{\omega+k_0}{r_0}\, \Gamma(\boldsymbol{m}) : \psi(l, \boldsymbol{m}), \Delta(\boldsymbol{m})$$
We may assume $G[\boldsymbol{m}](0) \geq N(\omega + k_0 + 1)$ Applying $(\forall S)$ yields:
$$G[\boldsymbol{m}] \mid\!\frac{\omega+k_0+1}{r_0}\, \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$
This works because $G[\boldsymbol{m}, l] = G[\boldsymbol{m}][l]$.
- We can deal with CUT en the equality rules as $Z^\infty$.
- $T$-rule. Suppose we have a premis $\vdash \Gamma, \phi : \Delta$ where $\phi$ is an axiom $Z$. The induction hypothesis yields
$$G[\boldsymbol{m}] \mid\!\frac{\omega+k_0}{r_0}\, \Gamma(\boldsymbol{m}), \phi : \Delta(\boldsymbol{m})$$
It is enough to show
$$H \mid\!\frac{\omega+k_1}{r_1}\, \phi$$
The we can write down the following proof:
$$\frac{G[\boldsymbol{m}] \mid\!\frac{\omega+k_0}{r_0}\, \Gamma(\boldsymbol{m}), \phi : \Delta(\boldsymbol{m}) \qquad H \mid\!\frac{\omega+k_1}{r_1}\, \Gamma(\boldsymbol{m}) : \phi, \Delta(\boldsymbol{m})}{F[\boldsymbol{m}] \mid\!\frac{\omega+\max(k_0,k_1)+1}{\max(r_0, r_1, |\phi|+1)}\, \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})}\ \mathrm{CUT}$$

for some suitable $F$ majorizing $G$ and $H$. That is precisely what we want to show. For this we distinguish cases according to the shape of $\phi$. We consider the case $\phi \equiv \forall x(Sx = 0 \to \bot)$, since the other cases are similarly:

$$\begin{aligned}
&\mathrm{id} \,\Big|\frac{0}{0}\, Sm = 0 : \bot \\
&\mathrm{id}+1 \,\Big|\frac{1}{0}\, Sm = 0 \to \bot \\
&\mathrm{id}+2 \,\Big|\frac{2}{0}\, \forall x(Sx = 0 \to \bot)
\end{aligned}$$

- The induction rule can be treated by the induction lemma.

□

## 6. Gödel incompleteness

**Definition 5.23.** *Let $\alpha < \varepsilon_0$. A function $F$ is called $\alpha$-recursive if it can be generated from the zero function and the projection functions by substitution, primitive recursion and the closure under formations rule: If $F$ is $\alpha$ recursive and $\beta \le \alpha$ then $F^\alpha$ is $\alpha$ recursive.*

THEOREM 5.24. *Suppose $Z \vdash \forall x \exists y \phi(x, y)$ with $\phi$ atomic. Then there exists an $\alpha < \varepsilon_0$ and an $\alpha$-recursive function $F$ so that for all $m$ there exists an $n \le F(m)$ such that $\mathbb{N} \models \phi(m, n)$.*

PROOF. Suppose $Z \vdash \forall x \exists y \phi(x, y)$. The embedding theorem yields the existence of a primitive recursive function $F$ and of numbers $k, r < \omega$ so that

$$F \,\Big|\frac{\omega+k}{r}\, \forall x \exists y \phi(x, y)$$

Iterated applications of cut elimination yields:

$$\begin{aligned}
&F \,\Big|\frac{\omega+k}{r}\, \forall x \exists y \phi(x, y) \\
&F^{\omega+k} \,\Big|\frac{\omega^{\omega+k}}{r-1}\, \forall x \exists y \phi(x, y) \\
&\left(F^{\omega+k}\right)^{\omega^{\omega+k}} \,\Big|\frac{\omega^{\omega^{\omega+k}}}{r-2}\, \forall x \exists y \phi(x, y) \\
&\vdots \\
&G \,\Big|\frac{\alpha}{0}\, \forall x \exists y \phi(x, y)
\end{aligned}$$

for some suitable $\alpha < \varepsilon_0$. The function $G$ will be $\alpha$-recursive. By applying lemma 1.19 the assertion follows. □

We are now going to compare $\alpha$ recursive functions with functions from the Hardy hierarchy. The crucial ingredient for the proof is the property $\beta < \alpha \wedge N(\beta) \le n \implies H_\beta(n) < H_\alpha(n)$ which we have shown in Lemma lemma 1.8.

Some natural majorization properties of the Hardy hierarchy are collected in the following lemma.

**Lemma 5.25.**

(1) $H_\alpha(n) \le H_{\omega^\alpha}(n)$
(2) $H_\alpha(n) \le H_{\alpha \oplus \beta}(n)$
(3) $N(\alpha) \le H_\alpha(n)$

PROOF.

(1) By induction on $\alpha$. If $\alpha = 0$ then $H_0(n) = n < H_1(n)$. For $\alpha = \beta + 1$ we find

$$H_{\omega^\alpha}(n) = H_{\omega^\beta(n+1)}(n) = H^{n+1}_{\omega^\beta}(n+1) \overset{\text{IH}}{\geq} H_\beta(n+1) = H_\alpha(n).$$

If $\alpha$ is a limit, then:

$$H_\alpha(n) = H_{\alpha[n]}(n+1) \leq H_{\omega^{\alpha[n]}}(n+1) = H_{\omega^\alpha}(n)$$

(2) Suppose $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and $\beta = \omega^{\alpha_{m+1}} + \cdots + \omega^{\alpha_{m+n}}$. Then

$$H_\alpha(n) = H_{\omega^{\alpha_1}}(\ldots H_{\omega^{\alpha_m}}(n) \ldots)$$
$$H_{\alpha \oplus \beta} = H_{\omega^{\alpha_{\pi(1)}}}(\ldots H_{\omega^{\alpha_{\pi(n+m)}}}(n) \ldots)$$

where$\pi$ is an permutation so that $\alpha_{\pi(1)} \geq \cdots \geq \alpha_{\pi(n+m)}$. The assertion follows from the composition lemma 5.11 for Hardy hierarchies.

(3) If $\alpha = 0$ then the assertion is clear. For $\alpha + 1$ geldt

$$N(\alpha+1) = 1 + N((\alpha+1)[0]) \leq H_{(\alpha+1)[0]}(0) \leq H_{(\alpha+1)[0]}(1) \leq H_{\alpha+1}(0)$$

If $\alpha$ is a limit:

$$N(\alpha) = 1 + N(\alpha[0]) \leq 1 + H_{\alpha[0]}(0) \leq H_{\alpha[0]}(1) = H_\alpha(0) \leq H_\alpha(n)$$

□

Recall that for a nice operator we defined

$$F^\alpha(x) = \max\big(\{F(x)+1\} \cup \{C(F^\gamma, F^\delta)(x) \mid \gamma, \delta < \alpha \wedge N(\gamma), N(\delta) \leq F(x)\}\big)$$

**Lemma 5.26.** *Suppose that $F$ is a nice operator.*
(1) $\alpha < \beta \implies F^\alpha(x) \leq F^\beta(x)$.
(2) $4k \leq H_{\omega 2}(k)$ *and* $8k \leq H_{\omega 3}(k)$ *or more generally* $2^i k \leq H_{\omega i}(k)$
(3) $H_{\omega 2}(k) \leq H_{\omega^{\alpha \oplus \beta + 1}}(k)$ *if* $\alpha > 0$ *and* $x > 0$.
(4) $F \leq H_\alpha \implies F^\beta(x) \leq H_{\omega^{\alpha \oplus \beta + 1} + 8}(x)$.

PROOF.

(1) By indcution on $\alpha$. For $\alpha = 0$ we obtain

$$F^\alpha(x) = F(x) + 1 \leq F^\beta(x).$$

For $\alpha > 0$ we find

$$F^\alpha(x) = F(x) + 1$$

or

$$F^\alpha(x) = F^\gamma(F^\delta(x)) + F^\gamma(x) + F^\delta(x)$$

for $\gamma, \delta < \alpha$ with $N(\gamma), N(\delta) \leq F(x)$. Then the definition of $F^\beta(x)$ yields $F^\gamma(F^\delta(x)) + F^\gamma(x) + F^\delta(x) \leq F^\beta(x)$ since $\gamma, \delta < \beta$ with $N(\gamma), N(\delta) \leq F(x)$.

(2) We compute:

$$\begin{aligned} H_\omega(k) &= H_{\omega[k]}(k+1) = H_{k+1}(k+1) = H_k(k+2) = H_{k-1}(k+3) = 2k+2 \\ H_{\omega 2}(k) &= H_{(\omega 2)[k]}(k+1) = H_{\omega+k+1}(k+1) = H_\omega(2k+2) = H_{\omega[2k+2]}(2k+3) \\ &= H_{2k+3}(2k+3) = 4k+6 \\ H_{\omega 3}(k) &= H_{\omega 2+k+1}(k+1) = H_{\omega 2}(2k+2) \geq 4(2k+2) \geq 8k \end{aligned}$$

(3) This is easy.

(4) By induction on $\beta$ we prove for all $x \geq 8$,

$$F^\beta(x) \leq H_{\omega^{\alpha\oplus\beta+1}}(x)$$

and this yields the assertion since $H_{\omega^{\alpha\oplus\beta+1}}(x+8) \leq H_{\omega^{\alpha\oplus\beta+1}+8}(x)$.

For $\beta = 0$ we obtain:

$$\begin{aligned} F^0(x) &= F(x) + 1 \leq H_\alpha(x) + 1 \leq H_{\alpha+1}(x) \leq H_{\alpha\oplus\beta+1}(x) \\ &\leq H_{\omega^{\alpha\oplus\beta+1}}(x) \end{aligned}$$

For $\beta > 0$ we have

$$F^\beta(x) = F^\gamma(F^\delta(x)) + F^\gamma(x) + F^\delta(x)$$

for $\gamma, \delta < \beta$ with $N(\gamma), N(\delta) \leq F(x)$. Let $\xi = \max(\gamma, \delta)$. Then we obtain

$$F^\beta(x) \leq F^\xi(F^\xi(x)) \cdot 3 \leq F^\xi(F^\xi(x)) \cdot 4$$

The induction hypothesis and (2) yield

$$\begin{aligned} F^\xi(F^\xi(x)) \cdot 4 &\leq H_{\omega 2}(H_{\omega^{\alpha\oplus\xi+1}}(H_{\omega^{\alpha\oplus\xi+1}}(x))) \\ &\leq H_{\omega^{\alpha\oplus\xi+1}}(H_{\omega^{\alpha\oplus\xi+1}}(H_{\omega^{\alpha\oplus\xi+1}}(H_{\omega^{\alpha\oplus\xi+1}}(x)))) \end{aligned}$$

Because of the composition law $H_\alpha(H_\beta(x)) = H_{\alpha+\beta}(x)$ for $NF(\alpha, \beta)$ we see

$$F^\xi(F^\xi(x)) \cdot 4 \leq H_{\omega^{\alpha\oplus\xi+1}4}$$

Thus $F^\beta(x) \leq H_{\omega^{\alpha\oplus\xi+1}4}(H_{\omega^{\alpha\oplus\beta}5}(x))$. Moreover we find

$$\begin{aligned} N(\omega^{\alpha\oplus\xi+1} \cdot 4) &\leq 4(1 + N(\alpha) \oplus N(\xi) + 1) \leq 8(2H_\alpha(x)) = 16H_\alpha(x) \\ &\leq H_{\omega 4}(H_\alpha(x)) \leq H_{\omega^{\alpha\oplus\beta}5}(x) \end{aligned}$$

Now we can show that $F^\beta(x) \leq H_{\omega^{\alpha\oplus\beta}4}(H_{\omega^{\alpha\oplus\beta}5}(x))$.

For $\xi < \beta$ we have $\xi + 1 \leq \beta$ and there are two options:

- If $\xi + 1 = \beta$ then $H_{\omega^{\alpha\oplus\xi+1}4}(H_{\omega^{\alpha\oplus\beta}5}(x)) = H_{\omega^{\alpha\oplus\beta}4}(H_{\omega^{\alpha\oplus\beta}5}(x))$.
- If $\xi + 1 < \beta$ then we apply assertion 3 of Lemma lemma 1.8 (our crucial majorization property) and obtain

$$F^\beta(x) \leq H_{\omega^{\alpha\oplus\xi+1}4}(H_{\omega^{\alpha\oplus\beta}5}(x)) \leq H_{\omega^{\alpha\oplus\beta}4}(H_{\omega^{\alpha\oplus\beta}5}(x)) \leq H_{\omega^{\alpha\oplus\beta}9}(x) \leq H_{\omega^{\alpha\oplus\beta+1}}(x)$$

Now we can conclude

$$\begin{aligned} &H_{\omega^{\alpha\oplus\beta+1}}(x) \\ &\leq H_{\omega^{\alpha\oplus\beta+1}}(x+8) \\ &\leq H_{\omega^{\alpha\oplus\beta+1}+1}(x+7) \leq \cdots \leq H_{\omega^{\alpha\oplus\beta+1}+8}(x) \end{aligned}$$

The assertion follows.

$\square$

**Lemma 5.27.** *Let $F$ be a nice operator.*

(1) *Suppose that $3^{F(m)+1} \cdot 2 + 1 \le H_{\omega^k}(m)$ for all $m < \omega$. Then $F^{\omega \cdot l + p}(m) \le H^{3^{p+1}}_{\omega^l \oplus k+1}(m)$ for all $m < \omega$.*

(2) *Suppose that $F^{\omega \cdot l+p}(m) \le H^{3^{p+1}}_{\omega^l \oplus k+1}(m)$ for all $m < \omega$. Then $F^{3^{\omega \cdot l+p}}(m) \le H_{\omega^{\omega^l \cdot 3^p \oplus \omega^{l \oplus +k+1} \cdot 3^{p+1}}}(m)$ for all $m < \omega$.*

PROOF. The first assertion is proved by induction on $\omega \cdot l + p$. The second assertion follows from $3^{\omega \cdot l+p} = \omega^l \cdot 3^p$ and Lemma lemma 1.26 $\square$

THEOREM 5.28. *If $Z \vdash \forall x \exists y \phi(x,y)$ with $\phi$ atomic then there exists an $\alpha < \varepsilon_0$ so that for all $m$ there exists an $n \le H_\alpha(m)$ such that $\mathbb{N} \models \phi(m,n)$.*

PROOF. From theorem 1.24 we conclude that there exists an $\alpha$-recursive function $F$ such that $\mathbb{N} \models \phi(m,n)$ for some $n < F(m)$. For this $\alpha$-recursive function we have a primitive recursive function $G$ with:

$$F := \left(\dots\left((G^{\omega+k})^{\omega^{\omega+k}}\right)\dots\right)^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega+k}}}}}}$$

Without loss of generality we may assume that $G$ is nice: Otherwise we replace $G$ by a nice majorant. Since this function is primitive recursive we find an $l$ such that $\forall m : G(m) < H_{\omega^l}(m)$. By an iterated application of the third assertion of the last lemma the assertion follows. $\square$

THEOREM 5.29. *Assume that $Z \vdash \forall x \exists y \phi(x,y)$ with $\phi$ an existential formula. Then there exists an $\alpha < \varepsilon_0$ so that for all $m$ there exists an $n \le H_\alpha(m)$ such that $\mathbb{N} \models \phi(m,n)$.*

PROOF. Let $\phi(x,y) \equiv \exists z \psi(x,y,z)$ with $\psi$ atomic. If $Z \vdash \forall x \exists y \phi(x,y)$ then $Z \vdash \forall x \exists u \exists y \le u \exists z \le u \psi(x,y)$. Since primitive recursive predicates are closed under bounded quantification we find an atomic formula $\chi(x,u) \equiv \exists y \le u \exists z \le u \psi(x,y)$. Morever this equivalence is provable in $Z$. and we find $Z \vdash \forall x \exists u \chi(x,u)$.

From theorem 1.24 we conclude that there exists an $\alpha$-recursive function $F$ such that $\mathbb{N} \models \chi(m,n)$ for some $n < F(m)$. Then also $\mathbb{N} \models \phi(m,n)$. For the $\alpha$-recursive function $F$ we find in fact a primitive recursive nice function $G$ with:

$$F := \left(\dots\left((G^{\omega+k})^{\omega^{\omega+k}}\right)\dots\right)^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega+k}}}}}}$$

Since $G$ is primitive recursive we find an $l$ such that $\forall m : G(m) < H_{\omega^l}(m)$. By an iterated application of the third assertion of the last lemma the assertion follows. $\square$

THEOREM 5.30. *Let $\alpha < \varepsilon_0$. Then $Z \vdash \forall x \exists y H_\alpha(x) = y$.*

PROOF. Let $a$ be the natural number which codes an element in OT of order type $\beta$. Let us define a formule $H(a,x,y,z)$ which formalizes that there exists a compuation tree $z$ for $H_\beta(x)$ with result $y$.

We assume that there is a primitive recursive function $f$ such that $f(a, x)$ is $a[x]$ if $a \in OT$. Let $H(a, x, y, z)$ be the formula

$$a \in OT \wedge z \in Seq \wedge \forall i < lh(y)[lh((y)_i) = 3] \wedge [i = 0 \to \exists v \leq y(y)_i = \langle 0, v, v\rangle] \tag{6}$$

$$\wedge[i = lh(y) - 1 \to (y)_i = \langle a, x, y\rangle] \tag{7}$$

$$\wedge[i + 1 < lh(y) - 1 \to \forall u, v, w((y)_{i+1} = \langle u, v, w\rangle \to (y)_i = \langle u[v], v + 1, w, x, y\rangle)] \tag{8}$$

(9)

Then $Z \vdash a \in OT \to (H(a[x], x + 1, z, y) \leftrightarrow H(a, x + 1, z \star \langle a, x, y\rangle, y)$

Let $\alpha$ be represented by $k$. By transfinite induction up to $k$ within $Z$ we can prove $Z \vdash b \preceq k \to \forall x\exists z\exists y H(b, x, z, y)$. This yields the assertion.

$\square$

Therefore $Z \not\vdash \forall x\exists y H_{\varepsilon_0}(x) = y$. Assume otherwise that $Z \vdash \forall x\exists y H_{\varepsilon_0}(x) = y$. Assume that the graph of $H_{\varepsilon_0}$ is formalized in $Z$ by some canonical existantial formula. Then by the preceding theorem there exists an $\alpha < \varepsilon_0$ such that for all $m$ there exists an $n \leq H_\alpha(m)$ with $H_{\varepsilon_0}(m) = n \leq H_\alpha(m)$. This is a contradiction. So there exist formulas which are true over the structure $\mathbb{N}$ but which are unprovable in $Z$ and we obtain among other things a version of the first Gödel incompleteness theorem.

**Exercise 5.31.** *In this exercise we extend our methods to larger segments of ordinals so that the provably recursive functions of stronger theories than $Z$ can be treated once those systems have been reduced to $Z + TI_{\restriction\prec}$ where $\prec$ is an arithmetical well ordering of order type $\tau$ and where $TI_{\restriction\prec}$ formailizes the transfinite induction along all strict initial segments of $\prec$. We assume that we can extend the norm function $N$ as follows to the larger domain $\tau$. Assume that for every $k$ the set $\{\alpha < \tau : N\alpha\}$ has finite cardinality. $N0 := 0.$, $N(\alpha \oplus \beta) = N\alpha + N\beta$, $N(\overline{\omega}^\alpha) = N\alpha + 1$. Here $\overline{\omega}^\alpha := \omega^{\alpha+1}$ if there exists an $\alpha_0$ and an $n$ such that $\alpha_0 + n = \alpha$ and $\omega^{\alpha_0} = \alpha_0$. Otherwise $\overline{\omega}^\alpha := \omega^\alpha$. Assume that $\prec$ is a primitive recursive well order and $o : \mathbb{N} \to \tau$ is a function such that for all $m, n \in \mathbb{N}$ we have $m \prec n \Rightarrow o(m) < o(n)$. Moreover assume that there exists a primitive recursive function $G$ such that $N(o(n)) \leq G(n)$ and $N(o(n)) \leq G(n)$.*

(1) *Show that there exist a primitive recursive operator $F$ such that for all $n$ we have $F \left|\frac{2rk(A)\oplus 3\cdot(rk(n)+1)}{0}\right. Prog_\prec(A) : \forall <\prec nA(y)$.*
(2) *If $Z + TI_{\restriction\prec}\forall x\exists y\phi(x, y)$ then there exists a $\beta < \tau$ and a $\beta$ recursive function $H$ such that forall $m$ there exists an $n \leq H(m)$ such that $\mathbb{N} \models \phi(m, n)$. (Here the definition of $\beta$ recursive function has to be adapted to $\tau$ in a straight forward way.) (Hint: Adapt the machinery from this section.)*

**Exercise 5.32.** *Assume that $d$ is the Gödel number of a proof for $Z \vdash \exists y H_{\varepsilon_0}(H_{\varepsilon_0}(100)) = y$ Show that $d \geq H_{\varepsilon_0}(50)$. (Hint: Assume that $d < H_{\varepsilon_0}(50)$ and exploit the constructive content of the embeddind procedure together with the bounding lemma. Analyze the resulting term using the theory about the Hardy functions.)*

CHAPTER 6

# Provably recursive functions of $\mathsf{I}\Sigma_n$

## 1. The formal system $\mathsf{I}\Sigma_n$

In this section we refine our previous results to obtain a classification of the provably recursive functions of systems with $n$-quantifier induction. In these systems the number of allowed quantifiers in induction formulas does not exceed $n$. This enterprise requires a good bookkeeping regarding the involved complexities. In a first step we redefine the notion of rank of a formula so that the rank of a formula $\phi$ equals the rank of the formula $\neg\phi$. This has the desired effect that the rank of a formula $\forall x\phi$ coincides with the rank of the formula $\exists x\phi$. This has the effect that a formula with at most $n$ unbounded quantifiers can equivalently be written by a formula of rank not exceeding $n$.

**Definition 6.1.** *From the rest of this section we change the definition of complexity $|\phi|$ of a formula $\phi$ so that*

$$|\phi \to \psi| = \begin{cases} \max(|\phi|, |\psi|) + 1 & \textit{if } \psi \not\equiv \bot \\ |\phi| & \textit{otherwise} \end{cases}$$

**Definition 6.2.** *We define the system $\mathsf{I}\Sigma_n$ for $n \in \mathbb{N}$ as follows. $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma : \Delta$ if one of the following cases holds*

(1) *Axiom: there exists a prime formula $\phi \in \Gamma \cap \Delta$.*
(2) $(\to A)$*: $\mathsf{I}\Sigma_n \vdash^{a_1}_{r} \Gamma : \phi, \Delta$ and $\mathsf{I}\Sigma_n \vdash^{a_2}_{r} \Gamma, \psi : \Delta$ and $\phi \to \psi \in \Gamma$ and $a_1, a_2 < a$.*
(3) $(\to S)$*: $\mathsf{I}\Sigma_n \vdash^{a_1}_{r} \Gamma, \phi : \psi, \Delta$ and $a_1 < a$ and $\phi \to \psi \in \Delta$.*
(4) $(\forall S)$*: $\mathsf{I}\Sigma_n \vdash^{a_1}_{r} \Gamma : \Delta, \phi(b)$ and $\forall x\phi \in \Delta$ and $b \notin \mathrm{FV}(\Gamma : \Delta)$*
(5) $(\forall A)$*: $\mathsf{I}\Sigma_n \vdash^{a_1}_{r} \Gamma, \phi(t) : \Delta$ and $a_1 < r$ and $\forall x\phi \in \Gamma$.*
(6) $(= I)$*, $(= F)$ en $(= P)$ as defined for $Z$.*
(7) *CUT: $\mathsf{I}\Sigma_n \vdash^{a_1}_{r} \Gamma, \phi : \Delta$ en $\mathsf{I}\Sigma_n \vdash^{a_2}_{r} \Gamma : \phi, \Delta$ and $|\phi| < r$.*
(8) *If $\phi$ is an axiom for a primitive recursive function then (as for $Z$, see chapter 1) $\phi \in \Delta$. To be explicit such an axiom is an element of the set $\{\neg Sa = 0, Sa = Sb \to a = b, 0^n(\boldsymbol{a}) = 0, P^n_i(\boldsymbol{a}) = a_i, h \circ (g_1, \ldots, g_m)(\boldsymbol{a}) = h(g_1(\boldsymbol{a}), \ldots, g_m(\boldsymbol{a})), \mathrm{Rec}(g, h)(0, \boldsymbol{a}) = g(\boldsymbol{a}) \,\mathrm{Rec}(g, h)(Sy, \boldsymbol{a}) = h(y, \boldsymbol{a}, \mathrm{Rec}(g, h)(y, \boldsymbol{a}))\}$ where $g$ and $h$ range over $PRF$.*
(9) *IND: $\mathsf{I}\Sigma_n \vdash^{a_1}_{r} \Gamma : \phi(0), \Delta$ and $\mathsf{I}\Sigma_n \vdash^{a_2}_{r} \Gamma, \phi(b) : \Delta, \phi(Sb)$ and $a_1, a_2 < a$ and $\phi(s) \in \Delta$ and $|\phi| \leq n$ and $b \notin \mathrm{FV}(\Gamma, \Delta, \forall x\phi)$.*

The intention to define $\mathsf{I}\Sigma_n$ is to analyse the induction scheme more profoundly: the parameter $n$ controls the complexity of the formulas over which we perform induction. $\mathsf{I}\Sigma_0$ allows only induction over atomic formulas, $\mathsf{I}\Sigma_1$ allows only induction on formules of complexity at most 1, etc.

The crucial feature of the calculus for $\mathsf{I}\Sigma_n$ is that the formula in the induction rule IND is of lowest possible complexity. This allows for a Gentzen style cut elimination of cuts of ranks bigger than $n$. Moreover the induction rule IND implies the scheme of inductions for all formulas of rank not exceeding $n$.

**Lemma 6.3.**

(1) *If* $a \leq a'$, $r \leq r'$, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ *and* $\vdash^{a}_{r} \Gamma : \Delta$ *then* $\vdash^{a'}_{r'} \Gamma' : \Delta'$.
(2) *For a free variable* $b$ *we have that* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma(b) : \Delta(b)$ *implies* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma(t) : \Delta(t)$
(3) $\mathsf{I}\Sigma_n \vdash^{2|\phi|}_{0} \phi : \phi$
(4) *If* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma : \forall x\phi, \Delta$ *then* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma : \Delta, \phi(t)$
(5) *If* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma : \Delta, \phi \to \psi$ *then* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma, \phi : \psi, \Delta$
(6) *If* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma, \phi \to \psi : \Delta$ *then* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma : \Delta, \phi$ *and* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma, \psi : \Delta$
(7) *If* $|\phi| \leq n$ *then* $\mathsf{I}\Sigma_n \vdash^{a}_{0} \phi(0) \to (\forall x(\phi(x) \to \phi(Sx)) \to \forall x\phi(x))$ *for some* $a$.

PROOF. All assertions can be proved routinely. Only the last assertion requires some extra care:

(7) We know that $\vdash^{2|\phi|}_{0} \phi(0) : \phi(0)$. Assume $k = 2|\phi|$ and $k' \geq k$, then

$$\mathsf{I}\Sigma_n \vdash^{k'}_{0} \underbrace{\forall x(\phi(x) \to \phi(Sx))}_{G} : \forall x(\phi(x) \to \phi(Sx))$$

Inversion yields

$$\mathsf{I}\Sigma_n \vdash^{k'}_{0} G : \phi(b) \to \phi(Sb)$$
$$\mathsf{I}\Sigma_n \vdash^{k'}_{0} G, \phi(b) : \phi(Sb)$$

Now suppose $l = \max(k, k') + 1$ and that $c$ is a new free variable. Then noting that $|\phi| \leq n$ we are allowed to apply IND and obtain the following derivation:

$$\mathsf{I}\Sigma_n \vdash^{l} G, \phi(0) : \phi(c),$$
$$\mathsf{I}\Sigma_n \vdash^{l+1} G, \phi(0) : \forall x\phi(x),$$
$$\mathsf{I}\Sigma_n \vdash^{l+2} \phi(0) : G \to (\forall x\phi(x)),$$
$$\mathsf{I}\Sigma_n \vdash^{l+3} \phi(0) \to (G \to (\forall x\phi(x)).$$

□

## 2. Cut-elimination for $\mathsf{I}\Sigma_n$

For $\mathsf{I}\Sigma_n$ we can in contrast to $Z$ prove a special sort of cut-elimination, the so called *partial cut-elimination*. We can eliminate all cuts of rank strictly greater than $n$. This is due to the fact that the complexity of the formulas which are allowed for the induction scheme is bounded by $n$. For formulas of larger complexity we basically can perform the original Gentzen style argument.

**Lemma 6.4** (Reduction lemma)**.** *Suppose that* $\mathsf{I}\Sigma_n \vdash^{a}_{r} \Gamma : \Delta, \phi$ *and* $\mathsf{I}\Sigma_n \vdash^{b}_{r} \Gamma, \phi : \Delta$ *and* $r > n$ *and* $|\phi| \leq r$. *Then* $\mathsf{I}\Sigma_n \vdash^{a+2b}_{r} \Gamma : \Delta$.

PROOF. By induction on $b$ with a subsidiary proof on the length of $\phi$. The whole proof is by routine. Note that if $\phi$ has been introduced by an induction rule then we can apply a cut to $\phi$ since $n < r$. The other cases are similar to Gentzen style cut elimination for predicate calculus. Since the definition of rank has been modified let us consider the case that $\phi \equiv \psi \to \bot$. Inversion yields

$$\mathsf{I}\Sigma_n \left|\frac{b}{r}\right. \Gamma : \psi, \Delta$$

By inversion we find moreover, $\mathsf{I}\Sigma_n \left|\frac{a}{r}\right. \Gamma, \psi : \bot, \Delta$. But this yields $\mathsf{I}\Sigma_n \left|\frac{a}{r}\right. \Gamma, \psi : \Delta$. Now we can apply the induction hypothesis on the length of $\phi$ to obtain the assertion.

$$\mathsf{I}\Sigma_n \left|\frac{a+2b}{r}\right. \Gamma : \Delta.$$

□

THEOREM 6.5 (Partial cut-elimination). *If* $\mathsf{I}\Sigma_n \left|\frac{a}{r+1}\right. \Gamma : \Delta$ *and* $r > n$ *then* $\mathsf{I}\Sigma_n \left|\frac{3^a}{r}\right. \Gamma : \Delta$.

**Lemma 6.6.** *If* $\mathsf{I}\Sigma_n \vdash \phi$ *then there exists a* $k$ *so that* $\mathsf{I}\Sigma_n \left|\frac{k}{n+1}\right. \phi$.

## 3. Embedding with operator-controlled derivations

In this section we show that we can embed the systeem $\mathsf{I}\Sigma_n$ into the infinitary systeem for operator-controlled derivations. Special care is needed to keep the derivation heights small so that we can extract relative tight bounds on provable instances of existential formulas.

**Lemma 6.7.** *If* $F \left|\frac{\alpha}{r}\right. \Gamma(t) : \Delta(t)$ *and* $\mathrm{val}(t) = k$ *then* $F \left|\frac{\alpha}{r}\right. \Gamma(k) : \Delta(k)$.

THEOREM 6.8 (Embedding). *Suppose* $\mathsf{I}\Sigma_n \left|\frac{a}{n+1}\right. \Gamma : \Delta$ *where* $\mathrm{FV}(\Gamma : \Delta) \subseteq \{a_1, \ldots, a_m\}$.
*Then there exists a primitive recursive operator* $F$ *with* $F[\boldsymbol{m}] \left|\frac{\omega d+a}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$ *where* $d$ *denotes the number of iteration of the scheme IND.*

PROOF. We only deal with the cases which are not routine:

- $(\forall S)$. In this case we have $\forall x\phi \in \Delta$ and $\mathsf{I}\Sigma_n \left|\frac{a'}{n+1}\right. \Gamma : \Delta, \phi(a)$ with $a$ a variable which is new for the context. The induction hypothesis yields the existence of a primitive recursive $F$ such that
$$F[\boldsymbol{m}, p] \left|\frac{\omega d+a'}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(p, \boldsymbol{m})$$
for all $\boldsymbol{m}, p$. This yields
$$(1+F)[\boldsymbol{m}] \left|\frac{\omega d+1+a'}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$
- $(\forall A)$. In this case we have $\mathsf{I}\Sigma_n \left|\frac{a'}{n+1}\right. \Gamma, \phi(t) : \Delta$ met $\forall x\phi \in \Gamma$. The induction hypothesis yields
$$F[\boldsymbol{m}, \boldsymbol{p}] \left|\frac{\omega d+a'}{n+1}\right. \Gamma(\boldsymbol{m}), \phi(\boldsymbol{m}, t(\boldsymbol{m}, \boldsymbol{p})) : \Delta(\boldsymbol{m})$$

From lemma 6.7 we obtain

$$F[\boldsymbol{m},\boldsymbol{p}] \left|\frac{\omega d+a'}{n+1}\right. \Gamma(\boldsymbol{m}), \phi(\boldsymbol{m}, \mathrm{val}\, t(\boldsymbol{m},\boldsymbol{p})) : \Delta(\boldsymbol{m})$$

This holds for all $p$, in particular for $p = 0$

$$F[\boldsymbol{m},0] \left|\frac{\omega d+a'}{n+1}\right. \Gamma(\boldsymbol{m}), \phi(\boldsymbol{m}, \mathrm{val}\, t(\boldsymbol{m},0)) : \Delta(\boldsymbol{m})$$

For some $k$ we have $\mathrm{val}\, t(\boldsymbol{m}, 0) \leq A_k(\max \boldsymbol{m})$ since $t$ is primitive recursive, hence

$$(F + A_k)[\boldsymbol{m}] \left|\frac{\omega d+1+a'}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m})$$

- IND. Suppose $\mathsf{I}\Sigma_n \left|\frac{a_1}{n+1}\right. \Gamma : \Delta, \phi(0)$ and $\mathsf{I}\Sigma_n \left|\frac{a_2}{n+1}\right. \Gamma, \phi(b) : \phi(Sb), \Delta$ where $\mathsf{I}\Sigma_n \left|\frac{a}{n+1}\right. \Gamma : \Delta, \phi(s)$. The induction hypothesis yields

$$F_1[\boldsymbol{m}] \left|\frac{\omega d+a_1}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(0, \boldsymbol{m}) \tag{10}$$

$$F_2[\boldsymbol{m}, p] \left|\frac{\omega d+a_2}{n+1}\right. \Gamma(\boldsymbol{m}), \phi(p, \boldsymbol{m}) : \phi(Sp, \boldsymbol{m}), \Delta(\boldsymbol{m}) \tag{11}$$

Choose $k$ such that $A_k(\max(\boldsymbol{m}, p)) \geq \max(F_1(\boldsymbol{m}), F_2(\boldsymbol{m}, p)) + p$. By induction on $p$ we prove that

$$A_k[\boldsymbol{m}, p] \left|\frac{\omega d+p+a_2}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(p, \boldsymbol{m})$$

For $p = 0$ this is immediate from Lemma lemma 6.3. For $p + 1$ the induction hypothesis yields

$$A_k[\boldsymbol{m}, p] \left|\frac{\omega d+p+a_2}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(p, \boldsymbol{m})$$

In connection with (11) we can apply a cut:

$$A_k[\boldsymbol{m}, p+1] \left|\frac{\omega d+p+1+a_2}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(Sp, \boldsymbol{m}).$$

So we arrive at

$$A_k[\boldsymbol{m}, p] \left|\frac{\omega(d+1)}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(p, \boldsymbol{m})$$

and this yields the assertion. For $p := \mathrm{val}\, s(\boldsymbol{m}, \boldsymbol{n})$ we obtain form lemma 6.7 that

$$A_k[\boldsymbol{m}, \mathrm{val}\, s(\boldsymbol{m}, \boldsymbol{n})] \left|\frac{\omega(d+1)}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(\mathrm{val}\, s(\boldsymbol{m}, \boldsymbol{n}), \boldsymbol{m})$$

Now we have $A_l(\max \boldsymbol{m}) \geq A_k(\max(\boldsymbol{m}, \mathrm{val}\, s(\boldsymbol{m}, \boldsymbol{n})))$ for some $l$, hence

$$A_l[\boldsymbol{m}] \left|\frac{\omega(d+1)}{n+1}\right. \Gamma(\boldsymbol{m}) : \Delta(\boldsymbol{m}), \phi(s(\boldsymbol{m}, \boldsymbol{n}), \boldsymbol{m})$$

□

## 4. Bounds on the lengths of proofs of existantial statements

In the last section we come to a highlight of all the investigations done before.

**Definition 6.9.** *Let us recall the definition of the arithmetical hierarchy: $\Sigma^0_n$, $\Pi^0_n$ by recursion on $n$. The sets $\Sigma^0_0$ en $\Pi^0_0$ consist of the quantifier free formulas. A formula $\phi$ belongs to $\Sigma^0_{n+1}$ if $\phi \equiv \exists \boldsymbol{x}\psi$ where $\psi \in \Pi^0_n$. A formula $\phi$ belongs to $\Pi^0_{n+1}$ if $\phi \equiv \forall \boldsymbol{x}\psi$ where $\psi \in \Sigma^0_n$.*
*Note that if $\phi \in \Sigma^0_n$ (resp. $\Pi^0_n$) then also $\phi \in \Sigma^0_m$ (resp. $\Pi^0_m$) for all $m > n$. So the set $\Pi^0_1$ consists of all uit all quantifier free formulas and formulas of the form $\forall x\phi$ where $\phi$ is quantifier free.*

**Definition 6.10.** *If $\Delta = \forall x_1.\phi_1, \dots, \forall x_n.\phi_n, \Delta'$ met voor alle i, where $\phi_i$ (for all $i$) and $\Delta'$ are quantifier free.*
*Then $\mathbb{N} \models \Delta^{(a_1,\dots,a_n)}$ iff $\mathbb{N} \models \forall x_1 < a_1.\phi_1, \dots, \forall x_n < a_n.\phi_n, \Delta'$.*

THEOREM 6.11. *Suppose $F \left|\frac{\alpha}{2}\right. \Gamma : \Delta$ with $\Gamma, \Delta \subseteq \Pi^0_1$ where $\Delta = \{\forall x_1\phi_1, \dots, \forall x_n\phi_n\} \cup \Delta'$ with $\Delta'$ quantifier. The for all $\boldsymbol{m} \in \mathbb{N}^n$ we have $\mathbb{N} \models \bigwedge \Gamma^{F^\alpha(\max \boldsymbol{m})} \to \bigvee \Delta^{\boldsymbol{m}}$.*

PROOF. By induction on $\alpha$. If $\alpha = 0$ then the assertion holds trivially since axioms do not involve quantifiers.

So assume $\alpha > 0$. We distinguish cases according to the last applied inference rule.

- Axiom case. This case is fine.
- $(\to S)$. We necessariliy have quantifier free formulas $A \to B \in \Delta$. The premis is: $F \left|\frac{\alpha_0}{2}\right. \Gamma.A : B, \Delta$. Together with the induction hypothesis we obtain:
$$\mathbb{N} \models \bigwedge \Gamma^{F^{\alpha_0}(\max \boldsymbol{m})} \wedge A \to \bigvee \Delta^{\boldsymbol{m}} \vee B$$
Suppose that $\mathbb{N} \models \bigwedge \Gamma^{F^\alpha(\max \boldsymbol{m})}$. Assume $\mathbb{N} \not\models \bigvee \Delta^{\boldsymbol{m}}$. Then we have $\mathbb{N} \not\models A \to B$. Thus $\mathbb{N} \models A$ and $\mathbb{N} \not\models B$. Because of $F^\alpha > F^{\alpha_0}$ we find $\mathbb{N} \models \bigwedge \Gamma^{F^\alpha(\max \boldsymbol{m})} \wedge A \to \bigvee \Delta^{\boldsymbol{m}} \vee B$. Thence $\mathbb{N} \models \bigvee \Delta^{\boldsymbol{m}} \vee B$. Moreover since $B$ is false we obtain: $\mathbb{N} \models \bigvee \Delta^{\boldsymbol{m}}$ contradiction.
- $(\to A)$. This case is similar to the case $(\to S)$
- $(\forall S)$. Without loss of generality we may assume that $\forall x_1.A(x_1) \in \Delta$ is principal formula. The premises yield for all $n$:
$$F[n] \left|\frac{\alpha_n}{2}\right. \Gamma : \Delta.A_1(n)$$
Using the induction hypothesis we find for all $n < m_1$:
$$\mathbb{N} \models \bigwedge \Gamma^{F[n]^{\alpha_n}(\boldsymbol{m})} \to \bigvee \Delta^{\boldsymbol{m}} \vee A_1(n)$$
By diagonalization we obtain:
$$F[n]^{\alpha_n}(\boldsymbol{m}) \le F^{\alpha_n}(\max(n, \boldsymbol{m})) \le F^{\alpha_n}(\boldsymbol{m}) \le F^\alpha(\boldsymbol{m})$$
Suppose that $\mathbb{N} \models \bigwedge \Gamma^{F^\alpha(\boldsymbol{m})}$. The we also find $\mathbb{N} \models \bigwedge \Gamma^{F^{\alpha_n}(\boldsymbol{m})}$, hence:
$$\mathbb{N} \models \bigvee \Delta^{\boldsymbol{m}} \vee A_1(n) \text{ for all } n \le m_1$$
If $\mathbb{N} \models \Delta^{\boldsymbol{m}}$ then the assertion is clear. Otherwise we see $\mathbb{N} \not\models \Delta^{\boldsymbol{m}}$ and therefore $\mathbb{N} \models A_1(n)$ holds for all $n \le m_1$. By definition 6.10 we obtain: $\mathbb{N} \models (\forall x_1.A_1(x_1))^{m_1}$ and therefore $\mathbb{N} \models \Delta^{\boldsymbol{m}}$ which is a contradiction.
- $(\forall A)$. In this case we have $\forall x A \in \Gamma$ and $F \left|\frac{\alpha_0}{2}\right. \Gamma, A(k) : \Delta$ where $k \le F[0]$. The indction hypothesis yields:
$$\mathbb{N} \models \bigwedge \Gamma^{F^{\alpha_0}(\boldsymbol{m})} \wedge A(k) \to \bigvee \Delta^{\boldsymbol{m}}$$
Suppose that $\mathbb{N} \models \bigwedge \Gamma^{F^\alpha(\boldsymbol{m})}$. If in addition $\mathbb{N} \models \bigvee \Delta^{\boldsymbol{m}}$ then the assertion is clear. Suppose that $\mathbb{N} \not\models \bigvee \Delta^{\boldsymbol{m}}$.

  Because of $\mathbb{N} \models \bigwedge \Gamma^{F^\alpha(\boldsymbol{m})}$ we see that $\mathbb{N} \not\models A(k)$. But we have $\forall x A \in \Gamma$ and therefore $\mathbb{N} \models A(k')$ for all $k'$ which yields a contradiction.

- CUT. Suppose that we have the following premises:
$$F \vdash^{\alpha_0}_{2} \Gamma, D : \Delta \qquad F \vdash^{\alpha_1}_{2} \Gamma : \Delta, D$$
We have $|D| < 2$, and therefore $D$ contains at most one quantifier or $D \equiv \neg D'$ for $D' \in \Pi^0_1$. We may assume without loss of generality that $D \in \Pi^0_1$. So assume $D \equiv \forall x_{n+1} A$. We apply the induction hypothesis to the right side of the sequent and obtain:
$$\mathbb{N} \models \bigwedge \Gamma^{F^{\alpha_1}(\boldsymbol{m}, m')} \to \bigvee \Delta^{\boldsymbol{m}} \vee (\forall x_{n+1} A_{m+1})^{m'}$$
for an arbitrary $m'$. This thus holds for $m' := F^{\alpha_0}(\boldsymbol{m})$. Jointly with $F^{\alpha_1}(\boldsymbol{m}, F^{\alpha_0}(\boldsymbol{m})) \leq F^{\alpha}(\boldsymbol{m})$ we see:
$$\mathbb{N} \models \bigwedge \Gamma^{F^{\alpha}(\boldsymbol{m})} \to \bigvee \Delta^{\boldsymbol{m}} \vee (\forall x_{n+1} A_{m+1})^{F^{\alpha_0}(\boldsymbol{m})}$$
Suppose that $\mathbb{N} \models \bigwedge \Gamma^{F^{\alpha}(\boldsymbol{m})}$. If in addition $\mathbb{N} \models \bigvee \Delta^{\boldsymbol{m}}$ then the assertion follows. So assume $\mathbb{N} \not\models \bigvee \Delta^{\boldsymbol{m}}$. Then we see that $\mathbb{N} \models (\forall x_{n+1} A_{m+1})^{F^{\alpha_0}(\boldsymbol{m})}$. Together with the already obtained results we see: $\mathbb{N} \models \bigwedge \Gamma^{F^{\alpha}(\boldsymbol{m})} \wedge (\forall x_{n+1} A_{m+1})^{F^{\alpha_0}(\boldsymbol{m})}$. By applying the induction hypothesis to the left side of the sequent we obtain $\mathbb{N} \models \bigvee \Delta^{\boldsymbol{m}}$.

□

**Lemma 6.12.** *Suppose that $F \vdash^{\alpha}_{n+1} \Gamma : \Delta$ and $F(x) \geq 3^x$. Then $F \vdash^{3^\alpha}_{n} \Gamma : \Delta$.*

Recall that
$$\alpha_k(x) = \alpha^{\alpha^{\cdot^{\cdot^{\cdot^{\alpha^x}}}}}$$
where the tower of exponents has hight $k$.

THEOREM 6.13. *Suppose $\mathsf{I}\Sigma_n \vdash \forall x \exists y \phi(x, y)$ where $\phi$ is prime. Then there exists an $l < \omega$ such that for all $p$ there exists a $q \leq H_{\omega_n(l)}(p)$ so that $\mathbb{N} \models \phi(p, q)$.*

PROOF. Suppose that $\mathsf{I}\Sigma_n \vdash \forall x \exists y \phi(x, y)$. Then there exist $l, r$ with
$$\vdash^{l}_{r} \forall x \exists y \phi(x, y).$$
We apply now cut elimination to obtain:
$$\vdash^{l'}_{n+1} \forall x \exists y \phi(x, y).$$
The embedding theorem yields a primitive recursive $F$ so that
$$F \vdash^{\omega d + l'}_{n+1} \forall x \exists y \phi(x, y).$$
If $n = 1$ then we are done since $F^{\omega d + l}$ is primitive recursive. Assume $n > 1$. Cut elimination in the system with operator control yields
$$\vdash^{\omega^\alpha}_{2} \forall x \exists y \phi(x, y)$$
Cut elimination in the system with operator control yields
$$F^{\omega d + l} \vdash^{3^{\omega d + l}}_{2} \forall x \exists y \phi(x, y).$$

This yields the assertion if $n = 2$. If $n > 2$ we perform further cut elimination to obtain

$$G^{\omega_{n_2}(\omega^d \cdot 3^l} \Big|\frac{3^{\omega d+l}}{2} \forall x \exists y \phi(x, y).$$

for some function $G$ of the right complexity. Inversion yields

$$G^{\omega_{n_2}(\omega^d \cdot 3^l}[p] \Big|\frac{\omega^\alpha}{2} \exists y \phi(p, y)$$
$$G^{\omega_{n_2}(\omega^d \cdot 3^l}[p] \Big|\frac{\omega^\alpha}{2} \forall y(\phi(p, y) \to \bot) \to \bot$$
$$G[^{\omega_{n_2}(\omega^d \cdot 3^l} p] \Big|\frac{\omega^\alpha}{2} \forall y(\phi(p, y) \to \bot) : \bot$$

Lemma theorem 6.11 yields the existence of some $q \leq G^{\omega_{n-1}(\omega^d \cdot 3^l}(p)$ with $\mathbb{N} \models \phi(p, q)$. The assertion follows by bounding $G$ in terms of the Hardy hierarchy:

$$G^{\omega_{n-1}(\omega^d \cdot 3^l}(p) \leq H_{\omega_n(l)}(p).$$

□

THEOREM 6.14. *Suppose that* $\mathsf{I}\Sigma_1 \vdash \exists y H_{\omega^{d100}}(100) = y$ *for a proof* $D$ *in which the number of symbols in* $D$ *is bounded by* $H_{\omega^{d50}}$. *Then* $\ulcorner D \urcorner > H_{\omega^{d50}}(50)$.

PROOF. Suppose $D \Big|\frac{l}{r} \exists y H_{\omega^{d100}}(100) = y$. Let $d = \ulcorner D \urcorner$. Then $D' \Big|\frac{3_{r-2}(l)}{2} \exists y H_{\omega^{d100}}(100) = y$ with a modified proof $D'$ stil in the finitary system. The term lengths of terms in $D'$ are bounded by the Gödel number $d$ of $D$. This yields

$$H_{\omega^d} \Big|\frac{\omega d + 3_{r-2}(l)}{2} \exists y H_{\omega^{d100}}(100) = y.$$

So there exists a $y \leq (H_{\omega^d})^{\omega d + 3_{d-2}(l)}$ such that $H_{\omega^{d100}}(100) = y$. Suppose that $d \leq H_{\omega^{d50}}(50)$. Then

$$H_{\omega^{d100}}(100) \leq (H_{\omega^{d50}})^{\omega d + 3_{d-2}(l)} < H_{\omega^{d100}}(100).$$

This is a contradiction and therefore we arrive at $d > H_{\omega^{d50}}(50)$. □