The (extended) rank weight enumerator and $q$-matroids

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Field extension $\mathbb{F}_{q^m}/\mathbb{F}_q$ gives $\mathbb{F}_q$-isomorphism

$$\mathbb{F}_{q^m}^n \to \mathbb{F}_q^{m\times n}, \quad x \mapsto m(x),$$

so vectors over $\mathbb{F}_{q^m}$ are mapped to $m \times n$ matrices over $\mathbb{F}_q$.

**Rank metric code** is subspace of $\mathbb{F}_{q^m} \leftrightarrow$ subspace of $\mathbb{F}_q^{m\times n}$. 
**q-Analogue**

<table>
<thead>
<tr>
<th>Finite Set</th>
<th>Subspace</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$\frac{q^n - 1}{q - 1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Subset</th>
<th>Subspace</th>
</tr>
</thead>
<tbody>
<tr>
<td>intersection</td>
<td>intersection</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Union</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>complement</td>
<td>orthogonal</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Size</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\binom{n}{k}$</td>
<td>$\left[ \begin{array}{c} n \ k \end{array} \right]_q$</td>
</tr>
</tbody>
</table>

From $q$-analogue to ‘normal’: let $q \to 1$. 
**C linear code**

\[ \text{supp}(\mathbf{c}) = \text{coordinates of } \mathbf{c} \text{ that are non-zero} \]
\[ \text{wt}_H(\mathbf{c}) = \text{size of support} \]

**Weight enumerator**

\[ W_C(X, Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w \]

with \( A_w = \text{number of words of weight } w. \)
C rank metric code

\( R^{\text{supp}}(c) = \) row space of \( m(c) \)

\( \text{wt}_R(c) = \) dimension of support

Rank weight enumerator

\[
W^R_C(X, Y) = \sum_{w=0}^{n} A^R_w X^{n-w} Y^w
\]

with \( A^R_w = \) number of words of weight \( w \).
$D \subseteq C$ subcode

$\text{supp}(D) =$ union of $\text{supp}(d)$ for all $d \in D$

$\text{wt}_H(D) =$ size of support

**Generalized weight enumerators**

For all $0 \leq r \leq \dim C$:

$$W_C^r(X, Y) = \sum_{w=0}^{n} A^r_w X^{n-w} Y^w$$

with $A^r_w =$ number of subcodes of dimension $r$ and weight $w$.

(Note: consistent with definition of generalized Hamming weights)
\[ D \subseteq C \text{ subcode} \]

\[ \text{Rsupp}(D) = \text{sum of Rsupp}(d) \text{ for all } d \in D \]

\[ \text{wt}_R(D) = \text{dimension of support} \]

**Generalized rank weight enumerators**

For all \( 0 \leq r \leq \dim C \):

\[ W^{R,r}_C(X, Y) = \sum_{w=0}^{n} A^{R,r}_w X^{n-w} Y^w \]

with \( A^{R,r}_w = \text{number of subcodes of dimension } r \text{ and weight } w \).

(Note: consistent with definition of generalized rank weights)
$\mathbb{F}_{q^e}/\mathbb{F}_q$ field extension

Extension code $C \otimes \mathbb{F}_{q^e}$: code over $\mathbb{F}_{q^e}$ generated by words of $C$.

Extended weight enumerator

$$W_C(X, Y, T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^n$$

with $A_w(T)$ polynomial such that $A_w(q^e) = \text{number of words of weight } w \text{ in } C \otimes \mathbb{F}_{q^e}$. 
Field extension $\mathbb{F}_{q^m}/\mathbb{F}_q$

Extension code $C \otimes \mathbb{F}_{q^m}$: code over $\mathbb{F}_{q^m}$ generated by words of $C$.

Extended rank weight enumerator

$$W_C^R(X, Y, T) = \sum_{w=0}^{n} A_w^R(T)X^{n-w}Y^n$$

with $A_w^R(T)$ polynomial such that $A_w^R(q^m) =$ number of words of weight $w$ in $C \otimes \mathbb{F}_{q^m}$. 
$J$ subset of $[n]$

$$C(J) = \{ c \in C : \text{supp}(c) \subseteq J^c \}$$

**Lemma**

$C(J)$ is a subspace of $\mathbb{F}_q^n$ and $l(J) = \dim_{\mathbb{F}_q} C(J)$
\( J \) subspace of \( \mathbb{F}_q^n \)

\[
C(J) = \{ c \in C : \text{Rsupp}(c) \subseteq J^\perp \}
\]

Lemma

\( C(J) \) is a subspace of \( \mathbb{F}_{q^m}^n \)

\[
l(J) = \dim_{\mathbb{F}_{q^m}} C(J)
\]
Determining extended weight enumerator

\[ \leftrightarrow \]

Determining generalized weight enumerators

\[ \leftrightarrow \]

Determining \( l(J) \) for all \( J \subseteq [n] \)
Determining extended \textit{rank} weight enumerator

\[\iff\]

Determining generalized \textit{rank} weight enumerators

\[\iff\]

Determining \(I(J)\) for all \(J \subseteq \mathbb{F}_q^n\)
Matroid

$E$ finite subset

Independent sets $\mathcal{I} \subseteq 2^E$

- $\emptyset \in \mathcal{I}$
- If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.
- If $A, B \in \mathcal{I}$ and $|A| > |B|$ then there is an $a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{I}$.

Rank function $r : 2^E \to \mathbb{N}$

- $0 \leq r(A) \leq |A|$
- If $A \subseteq B$ then $r(A) \leq r(B)$.
- $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)
Fact: a linear code gives a matroid with

\[ E = \text{columns of generator matrix} \]

\[ r(J) = \text{dimension of subspace spanned by vectors of } J \]

Theorem

\[ r(J) = \dim C - l(J) \]
Rank generating function

\[
R_M(X, Y) = \sum_{J \subseteq E} X^{r(E) - r(J)} Y^{|J| - r(J)}
\]

(Tutte polynomial: replace \(X\) by \(X - 1\) and \(Y\) by \(Y - 1\).)

Theorem (Greene, 1976)

The Tutte polynomial determines the weight enumerator.

Theorem

The extended weight enumerator determines the Tutte polynomial and vice versa.
$q$-Matroid

$E = \mathbb{F}_q^n$

$q$-independent sets $\mathcal{I} \subseteq \{\text{subspaces of } E\}$

- $0 \in \mathcal{I}$
- If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.
- If $A, B \in \mathcal{I}$ and $\dim A > \dim B$ then there is a 1-dimensional subspace $a \subseteq A$, $a \not\subseteq B$ such that $B + a \in \mathcal{I}$.

$q$-Rank function $r : \{\text{subspaces of } E\} \rightarrow \mathbb{N}$

- $0 \leq r(A) \leq \dim A$
- If $A \subseteq B$ then $r(A) \leq r(B)$.
- $r(A + B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)
Theorem

Let \( r(J) = \dim C - l(J) \) for a rank metric code \( C \). Then \( r(J) \) is the rank function of a \( q \)-matroid.

Lemma

\( l(A + B) + l(A \cap B) \geq l(A) + l(B) \)
\textit{q-Rank generating function}

\[ R^q_M(X, Y) = \sum_{J \subseteq \mathbb{F}_q^n} X^{r(E) - r(J)} Y^{\dim J - r(J)} \]

\textbf{Question}: Are the extended rank weight enumerator and the \textit{q}-rank generating function equivalent?

\textbf{Answer}: Not sure, but probably “yes”.
Why study $q$-matroids?

Matroids generalize:
- codes
- graphs
- some designs

$q$-Matroids generalize:
- rank metric codes
- $q$-graphs?
- $q$-designs?
Further work

- Equivalence between polynomials
- Various definitions of $q$-matroids
- “Representable” $q$-matroids
- Deletion and contraction
Thank you for your attention.
\( e \) element of finite set \( E \)

\[
\{ \text{subsets containing } e \} \cup \{ \text{subsets of } e^c \} = 2^E
\]

\( e \) 1-dimensional subspace of \( \mathbb{F}_q^n \)

\[
\{ \text{subspaces containing } e \} \cup \{ \text{subspaces of } e^\perp \} \neq \{ \text{all subspaces of } \mathbb{F}_q^n \}
\]