Asymptotic Analysis of Coded Slotted ALOHA

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System model


• Bipartite graph, consisting of:
  – “AND” nodes
  – “OR” nodes

• Messages are exchanged along the edges of the graph:
  – Two values - \{0, 1\}
Message update rules

**OR nodes**
- Outgoing message is 1 if any of the incoming messages is 1

**AND nodes**
- Outgoing message is 1 if all incoming messages are 1
Message update probabilities

**OR nodes**
- \( p \) – probability that the value of the incoming message is zero
- \( q \) – probability that the value of the outgoing message is zero

\[
q = p^{k-1}
\]

**AND nodes**
- \( p \) – probability that the value of the outgoing message is zero
- \( q \) – probability that the value of the incoming message is zero

\[
p = 1 - (1 - q)^{j-1}
\]
Message update probabilities (cont’d)

OR nodes

• The expected (i.e., average) probability that the outgoing message is 0 is:

\[ q = \sum_k \lambda_k p^{k-1} = \lambda(p) \]

• where:
  – \( \lambda_k \) - probability that message is egressing a node of degree \( k \), \( \Sigma_k \lambda_k = 1 \)
  – \( \lambda(x) = \sum_k \lambda_k x^{k-1} \)

AND nodes

• The expected (i.e., average) probability that the outgoing message is 0 is:

\[ p = \sum_j \omega_j (1 - (1 - q)^{j-1}) \]

\[ = 1 - \omega (1 - q) \]

• where:
  – \( \omega_j \) - probability that message is egressing a node of degree \( j \), \( \Sigma_j \omega_j = 1 \)
  – \( \omega(x) = \sum_j \omega_j x^{j-1} \)

edge-oriented degree distributions
And-or tree evaluation

\[ q(i) = \lambda \left( 1 - \omega (1 - q(i - 1)) \right) \]

\[ p(i - 1) = 1 - \omega (1 - q(i - 1)) \]

\[ q(i - 1) = \lambda (p(i - 2)) \]

\[ p(i - 2) = 1 - \omega (1 - q(i - 2)) \]

\[ q(i - 2) \]
And-or tree evaluation (cont’d)

• Our graphs are not trees!
  – There are loops
    • i.e., interdependencies among messages
  – The results obtained by the and-or tree evaluation pose upper limits on the performance

• Probability of recovering a message:
  \[ P_R = 1 - \lim_{i \to \infty} q(i) \]
  – where \( q(0) = 1 \)

• And-or tree evaluation shows the expected asymptotic performance based on the statistical graph description expressed through \( \lambda(x), \omega(x) \)

• And-or tree evaluation is standardly used to assess the asymptotic performance of the erasure-correcting codes
Coded slotted ALOHA

- $N$ users
  - Equal length packets that “fit” into the slots

- $M$ slots

- Users contend for the access to the base station
  - Users repeat their transmission in several randomly chosen slots of the frame
  - Successive interference cancellation is used to remove already resolved transmissions
Generalized model

• Users and slots are divided into classes
  – We assume that the division of the users into classes is performed on the basis of the expected packet loss probability

• $L$ user classes
  – Fraction of $a_l$ users belongs to class $U_l$

• $J$ slot classes
  – Fraction of $b_j$ slots belongs to class $S_j$

• $e_l$ - expected packet-loss (erasure) probability of (the users belonging to) $U_l$
Generalized model (cont’d)

- $p_{lj}$ - expected fraction of edges egressing $U_l$ that ingress in $S_j$:
  
  $$p_{lj} = \frac{\alpha_{lj}}{a_l N}$$

- $\lambda_{lj}(x)$ – edge-oriented degree distribution of users from $U_l$ with respect to $S_j$

- $\omega_{jl}(x)$ – edge-oriented distribution of slots from $S_j$ with respect to $U_l$

- $\beta_{jl}$ – expected degree of a slot from $S_j$ with respect to $U_l$
  
  $$\beta_{jl} = \frac{1}{\int_0^1 \omega_{jl}(x)}$$

- $\beta_j$ - expected degree of a slot from $S_j$
  
  $$\beta_j = \sum_n \beta_{jn}$$
Theorem

- Probability of not recovering a message of a user belonging to class $U_l$ is after $i$-the iteration, $i \geq 1$, is:

$$q_l(i) = \prod_j \lambda_{lj} \left( 1 - \sum_m \frac{\beta_{jm}}{\beta_j} (1 - e_m) \prod_k \omega_{jk} (1 - q_k(i - 1)) \right)$$

$$q(i) = \lambda \left( 1 - \omega (1 - q(i - 1)) \right)$$

Performance parameters

• Asymptotic probability of recovering a message belonging to a user from $U_l$:
  \[ P_{R,l} = 1 - \lim_{i \to \infty} q_l(i) \]

• Average asymptotic probability of recovering a message:
  \[ P_R = \sum_l a_l P_{R,l} \]

• Expected throughput:
  \[ T = \frac{N \cdot P_R}{M} = \frac{P_R}{1 + \epsilon} \]
  
  where $1 + \epsilon = \frac{M}{N}$
Example

- Frameless ALOHA
  - Number of slots $M$ is not a priori fixed
  - Users perform access on a slot basis, using predefined slot access probability:
    \[ p_{U_l \rightarrow S_j} = p_l = \frac{\alpha_{lj}}{a_l N} \]

- It can be shown that:
  \[ \lambda_{lj}(x) = e^{-(1+\epsilon) \frac{b_j \alpha_{lj}}{a_l} (1-x)} \]
  \[ \omega_{jl}(x) = e^{-\alpha_{lj} (1-x)} \]
Example (cont’d)

- Assume three scenarios:
  1. single user class with packet-loss probability $e^{(1)} = 0$, single slot class
     - Slot access probability $p = \frac{\beta}{N}$
  2. single user class with packet loss probability $e^{(2)} = 0.375$, single slot class
     - Slot access probability $p = \frac{\beta}{N}$
  3. two user classes, equal fractions of user belonging to each class, $a_1^{(3)} = 0.5$ and $a_2^{(3)} = 0.5$, respective packet-loss probabilities are $e_1^{(3)} = 0.25$ and $e_2^{(3)} = 0.5$, single slot class
     - Slot access probabilities $p_1 = \frac{\alpha_1}{0.5N}$ and $p_2 = \frac{\alpha_2}{0.5N}$

- Goal: For given $M/N$, choose slot access probabilities (i.e., $\beta$, $\alpha_1$, $\alpha_2$) such that the throughput is maximized
Results:
Maximum throughput
Results:

Optimal slot access probabilities
Results:
Probability of recovering a message
Conclusions

• Generalization of the and-tree evaluation with the channel impairments (i.e., packet loss probability) are taken into account
  – Extendible further, by taking into account other criteria, e.g., message importance

• If the overall goal is to maximize the throughput, users with worse channels should not contend (i.e., access the channel)
  – If overall probability of user resolution is of interest as well, then all users should contend, but users with worse channels should access channel less frequently