Partial Ovoids and Spreads in Generalized Quadrangles, and Related Combinatorial Structures

by

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Abstract

In this paper we overview what is known about partial ovoids and spreads of finite (classical) generalized quadrangles. In the first, respectively the second, part of the paper we will be mostly concerned with small, respectively large, maximal partial ovoids and spreads. Also connections with other interesting objects in finite geometry will be explained. Among the new results are new bounds on the smallest and largest maximal partial spreads of $\mathcal{Q}(5,q)$ and $\mathcal{H}(3,q^2)$, an improvement on a recent bound for small maximal partial ovoids of $W(q^3)$, and a new bound on small maximal partial spreads of $\mathcal{H}(4,q^2)$. We also classify transitive maximal partial ovoids of size (q^2-1) of $\mathcal{Q}(4,q)$, and discuss examples for small q. We introduce a theory for spreads and ovoids of affine generalized quadrangles, and apply earlier obtained results to this theory. Finally, we obtain new results in the study of maximal $(s+1)\times(s-1)$-grids in thick generalized quadrangles of order (s,t), and connect these objects to certain maximal partial spreads.

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Contents

1 Introduction

Generalized polygons were introduced in 1959 by Tits in the appendix of his celebrated paper on triality [36]. Ever since they have played a key role in algebraic and combinatorial geometry. Among the generalized polygons, the finite generalized quadrangles (GQs) occupy a special place, in particular because of the existence of several classical and non-classical examples which can be studied in a geometric, algebraic as well as combinatorial way, and are connected with a broad collection of other interesting objects from Galois geometry and algebra, such as polarities, (pseudo-)ovals, (pseudo-)ovoids, q-clans, flocks, fans, fibrations, herds, 4-gonal families, BN-pairs, ... Since their introduction several authors have been interested in the study of substructures of finite GQs, in particular in the existence of *spreads* and *ovoids* (see further for a formal introduction to these objects). As there already is an extensive literature on this subject containing some good surveys, we will only shortly overview the most important (recent) results on (partial) spreads and ovoids of finite GQs and refer the interested reader to the literature. More recently people have become interested in the existence, the construction and bounds on the size of maximal partial ovoids and spreads of finite GQs. In this paper we will give an overview of the most important results obtained so far on this subject, as well as contribute to this theory by providing some new results. The GQs encountered will be mostly the finite classical GQs, although if possible we will try to extend results to general finite GQs of order (s, t).

Let us start by revisiting some basic theory and combinatorics of finite GQs.

2 Preliminaries

We start with introducing several basic concepts.

2.1 Generalized quadrangles

A *finite GQ $\mathcal{Q}$ of order* (s, t) is a finite point-line incidence geometry such that

- two distinct lines intersect in at most one point;
- every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines;
- given any point p not on a given line L, there exists a unique line M containing p and intersecting L. (The unique point on L collinear with p will be denoted by $\text{proj}_L p$.)

Sometimes a GQ will be denoted by $\mathcal{Q} = (P, B, I)$, when we want to specify the point set P, the line set B and the incidence relation I.

If $s > 1$ and $t > 1$, then $\mathcal{Q}$ will be called *thick*, otherwise $\mathcal{Q}$ will be called *thin*. If $s = t$, then $\mathcal{Q}$ will be said to be of order s.

Clearly the definition of GQ is self-dual and the dual $\mathcal{Q}^D$ of a GQ $\mathcal{Q}$ is a GQ of order (t, s). Consequently every definition, theorem, ... for a GQ $\mathcal{Q}$ also results in a definition, theorem,... for its dual $\mathcal{Q}^D$; throughout this paper we will not always mention this explicitly.

An easy counting argument shows that a finite GQ of order (s, t) contains $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines. If $\mathcal{Q}'$ is a GQ with as point set a subset of the point set of the GQ $\mathcal{Q}$, with as line set a subset of the line set of $\mathcal{Q}$, and for which the incidence is the one inherited from $\mathcal{Q}$, then $\mathcal{Q}'$ will be called a *subquadrangle* or *subGQ* of $\mathcal{Q}$.

If two distinct points x and y are collinear, we will write $x \sim y$; dually we write $L \sim M$ for two distinct intersecting lines L and M. The perp $p^\perp$ of a point p is defined as the set of all points collinear with p. If A is a set of points of $\mathcal{Q}$, then we define $A^\perp := \bigcap\{p^\perp \parallel p \in A\}$. It is immediately clear that for two distinct points x and y, $\left|\{x, y\}^\perp\right| = s + 1$ or $t + 1$, according to $x \sim y$ or $x \not\sim y$. For $(A^\perp)^\perp$ we will simply write $A^{\perp\perp}$. Clearly $\left|\{x, y\}^{\perp\perp}\right| = s + 1$ or $\le t + 1$ according to $x \sim y$ or $x \not\sim y$, in the latter case the set $\{x, y\}^{\perp\perp}$ is called the *hyperbolic line* through x and y. A pair $\{x, y\}$ of two distinct non-collinear points for which the hyperbolic line $\{x, y\}^{\perp\perp}$ contains $t + 1$ points is called *regular*. By definition also every pair of two distinct collinear points is called *regular*. A point x is called *regular* if every pair of points $\{x, y\}$, $y \ne x$, is regular. A pair $\{x, y\}$ of non-collinear points is said to be *anti-regular* if $\left|\{x, y\}^\perp \cap z^\perp\right| \le 2$ for all points $z \notin \{x, y\}$. A point x is called *anti-regular* if $\{x, y\}$ is anti-regular for each point $y \not\sim x$. The *closure* of two non-collinear points x and y is defined as $cl(x, y) := \{z \parallel z^\perp \cap \{x, y\}^{\perp\perp} \ne \emptyset\}$. A *triad* of points is a set of three pairwise non-collinear points. Suppose that $\mathcal{Q}$ is a thick GQ of order (s, s^2). Then by a theorem of Bose and Shrikhande [5] $\left|\{x, y, z\}^\perp\right| = s + 1$ for each triad of points $\{x, y, z\}$. Consequently $\left|\{x, y, z\}^{\perp\perp}\right| \le s + 1$. We say that $\{x, y, z\}$ is *3-regular* if and only if $\left|\{x, y, z\}^{\perp\perp}\right| = s + 1$. The point x is called *3-regular* if each triad containing x is 3-regular. Let $x \ne y$ be two collinear points (still under the assumption that $\mathcal{Q}$ has order (s, s^2)). We say that the pair $\{x, y\}$ has *Property (G)* if every triad of points $\{x, u, v\}$ for which $y \in \{x, u, v\}^\perp$ is 3-regular. A flag (x, L) is said to have *Property (G)* if every pair $\{x, y\}$, with $y \in L \setminus \{x\}$, has Property (G). Finally, a line L is said to satisfy *Property (G)* if every pair of distinct points of L has Property (G).

Throughout this paper, we will work with thick GQs *unless mentioned explicitly otherwise.*

Also, usually we will use the notation "$\mathcal{Q}$" or "$\mathcal{S}$" for a GQ.

2.2 Classical quadrangles

We will now briefly overview the finite classical GQs and some of their basic combinatorial properties.

Let $\mathcal{Q}$ be a non-singular hyperbolic, parabolic, elliptic quadric in $\mathbf{PG}(3,q)$, $\mathbf{PG}(4,q)$, $\mathbf{PG}(5,q)$ respectively. Then the points and lines of $\mathcal{Q}$ form a GQ which will be denoted by $\mathcal{Q}(3,q)$, $\mathcal{Q}(4,q)$ and $\mathcal{Q}(5,q)$ respectively.

The GQ $\mathcal{Q}(3,q)$ has order $(q,1)$ and is a so called *grid* (see further).

The GQ $\mathcal{Q}(3,q)$ is contained as a subGQ in $\mathcal{Q}(4,q)$, which is a GQ of order q.

The GQ $\mathcal{Q}(4,q)$ is contained as a subGQ in the GQ $\mathcal{Q}(5,q)$, which has order (q,q^2).

Consider a symplectic polarity θ of $\mathbf{PG}(3,q)$. The point-line geometry with as point set the points of $\mathbf{PG}(3,q)$ and with as line set the set of totally isotropic lines with respect to θ is a GQ of order q, denoted by $W(q)$.

Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathbf{PG}(n,q^2)$, with $n \in \{3,4\}$. Then the points and lines of $\mathcal{H}(n,q^2)$ form a GQ of order (q^2,q) or (q^2,q^3) according as whether $n=3$ or $n=4$. These GQs will be denoted by $\mathcal{H}(n,q^2)$.

The following theorem provides some important relations between the classical GQs.

Theorem 2.1 ([15]) *The following isomorphisms hold:*

- $\mathcal{Q}(4,q) \cong W(q)^D$;
- $\mathcal{Q}(4,q) \cong W(q)$ *if and only if* q *is even;*
- $\mathcal{Q}(5,q) \cong \mathcal{H}(3,q^2)^D$.

We now provide the definition of a grid. A *grid* is an incidence structure $\Gamma = (P,B,I)$ with $P = \{x_{ij} \parallel i = 0,1,\ldots,s_1 \text{ and } j = 0,1,\ldots,s_2\}$, $s_1, s_2 > 0$ and with $B = \{L_0, L_1, \ldots, L_{s_1}, M_0, M_1, \ldots, M_{s_2}\}$, such that $x_{ij} I L_k$ if and only if $i = k$ and such that $x_{ij} I M_k$ if and only if $j = k$. The numbers s_1 and s_2 are called the *parameters* of the grid. We also say that Γ is an $(s_1+1) \times (s_2+1)$-*grid.*

In this paper we will occasionally encounter some specific types of non-classical GQs. As detailed knowledge of these GQs is not needed for a good understanding of the present paper we refer the reader to the standard literature. For the GQs $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ of Tits we refer to Chapter 3 of [15]; for translation GQs (TGQs) we refer to Chapter 8 of [15]; for the GQs $\mathcal{P}(\mathcal{S}, x)$ of Payne we refer to [14] or Chapter 3 of [15]; for flock GQs we refer to [26].

In the following section we will shortly discuss the existence of spreads and ovoids. In Section 4 we will give an overview of what is known about "small" maximal partial spreads and ovoids. We will also obtain some new results. In the following sections we will then discuss the other end of the spectrum and consider "large" maximal partial spreads and ovoids.

2.3 Combinatorial properties of the classical quadrangles

We recall some important combinatorial properties of classical generalized quadrangles.

PROPERTIES OF $\mathcal{Q}(4, q)$. All lines are regular; all points are regular if and only if q is even; all point are antiregular if and only if q is odd.

We also have the following important characterization theorem.

Theorem 2.2 ([15], 5.2.1) *A GQ of order s, $s > 1$, is isomorphic to $W(s)$ if and only if each point is regular.*

PROPERTIES OF $\mathcal{Q}(5, q)$. All lines are regular; all points are 3-regular.

Theorem 2.3 ([15], 5.3.3) *(i) Let $\mathcal{S}$ be a GQ of order (s, s^2), $s > 1$ and s odd. Then $\mathcal{S} \cong \mathcal{Q}(5, s)$ if and only if $\mathcal{S}$ has a 3-regular point.*

(ii) Let $\mathcal{S}$ be a GQ of order (s, s^2), s even. Then $\mathcal{S} \cong \mathcal{Q}(5, s)$ if and only if one of the following holds:

- *(a) all points of $\mathcal{S}$ are 3-regular;*
- *(b) $\mathcal{S}$ has at least one 3-regular point not incident with some regular line.*

PROPERTIES OF $\mathcal{H}(4, q^2)$. For each two distinct non-collinear points x, y we have that $|\{x, y\}^{\perp\perp}| = q + 1$; if L and M are non-concurrent lines, then $|\{L, M\}^{\perp\perp}| = 2$, but $\{L, M\}$ is not antiregular.

Theorem 2.4 ([15], 5.5.1) *A GQ of order (s^2, s^3), $s > 1$, is isomorphic to the classical GQ $\mathcal{H}(4, s^2)$ if and only if every hyperbolic line has at least $s+1$ points.*

2.4 Some notes on subquadrangles

Theorem 2.5 ([15], 2.2.1) *Let $\mathcal{S}'$ be a proper subquadrangle of order (s', t') of the GQ $\mathcal{S}$ of order (s, t). Then either $s = s'$ or $s \geq s't'$. If $s = s'$, then each external point of $\mathcal{S}'$ is collinear with $st' + 1$ mutually non-collinear points of $\mathcal{S}'$; if $s = s't'$, then each external point of $\mathcal{S}'$ is collinear with exactly $1 + s'$ points of $\mathcal{S}'$.*

Theorem 2.6 ([15], 2.3.1) *Let $\mathcal{S}' = (P', B', I')$ be a substructure of the GQ $\mathcal{S}$ of order (s, t) so that the following two conditions are satisfied:*

(i) if $x, y \in P'$ are distinct points of $\mathcal{S}'$ and L is a line of $\mathcal{S}$ such that $xILIy$, then $L \in B'$;

(ii) each element of B' is incident with $s + 1$ elements of P'.

Then there are four possibilities:

(1) *$\mathcal{S}'$ is a dual grid, so $s = 1$;*

(2) *the elements of B' are lines which are incident with a distinguished point of P, and P' consists of those points of P which are incident with these lines;*

(3) *$B' = \emptyset$ and P' is a set of pairwise non-collinear points of P;*

(4) *$\mathcal{S}'$ is a subquadrangle of order (s, t').*

Let $\{x, y, z\}$ be a 3-regular triad of the GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s \neq 1$ and s even. Let P' be the set of all points incident with lines of the form uv, with $u \in \{x, y, z\}^\perp = \mathbf{X}$ and $v \in \{x, y, z\}^{\perp\perp} = \mathbf{Y}$, and let B' be the set of lines L which are incident with at least two points of P'. Then J. A. Thas proves in [27] (see also [15, 2.6.2]) that, with I' the restriction of I to $(P' \times B') \cup (B' \times P')$, the geometry $\mathcal{S}' = (P', B', I')$ is a subGQ of $\mathcal{S}$ of order s. Moreover, $\{x, y\}$ is a regular pair of points of $\mathcal{S}'$, with $\{x, y\}^{\perp'} = \{x, y, z\}^\perp$ and $\{x, y\}^{\perp'\perp'} = \{x, y, z\}^{\perp\perp}$ (with the meaning of " $\perp'$ " being obvious).

The following theorem is crucial for this observation.

Theorem 2.7 ([15], 2.6.1) *Let $\{x, y, z\}$ be a 3-regular triad of the GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s \neq 1$, and let P' be the set of all points incident with lines of the form uv, with $u \in \{x, y, z\}^\perp = \mathbf{X}$ and $v \in \{x, y, z\}^{\perp\perp} = \mathbf{Y}$. If L is a line which is incident with no point of $\mathbf{X} \cup \mathbf{Y}$ and if k is the number of points in P' which are incident with L, then $k \in \{0, 2\}$ if s is odd and $k \in \{1, s + 1\}$ if s is even.*

2.5 Generalized quadrangles with small parameters

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s,t), $1 < s \leq t$. We consider the cases $s = 2, 3, 4$. Precise references can be found in [15].

THE CASE $s = 2$. If $s = 2$, then $t \in \{2, 4\}$. The GQ of order 2 is unique and is isomorphic to $\mathcal{Q}(4,2)$. The uniqueness of the GQ of order $(2,4)$ was proved independently at least five times, by S. Dixmier and F. Zara, J. J. Seidel, E. E. Shult, J. A. Thas and H. Freudenthal.

THE CASE $s = 3$. If $s = 3$, then $t \in \{3, 5, 6, 9\}$. The uniqueness of the GQ of order $(3,5)$ was proved by S. Dixmier and F. Zara. The uniqueness of the GQ of order $(3,9)$ was proved independently by S. Dixmier and F. Zara, and by P. J. Cameron in 1976 (see [15]). For $s = t = 3$ there are exactly two non-isomorphic GQs, due independently to S. Dixmier and F. Zara and S. E. Payne. Finally, S. Dixmier and F. Zara proved that no GQ of order $(3,6)$ exists.

THE CASE $s = 4$. If $s = 4$, then $t \in \{4, 6, 8, 11, 12, 16\}$. Nothing is known about the case $t = 11$ or $t = 12$. In the other cases, unique examples are known, but the uniqueness question is only settled for the case $t = 4$. The proof is due to S. E. Payne, with a gap filled by J. Tits in 1983, see also [15].

3 Spreads and Ovoids

In this short section we will briefly mention results concerning the (non-) existence of spreads and ovoids in finite classical GQs, and provide some references.

We first define the objects which will form the core of this article. A *spread* $\mathcal{S}$ of a GQ $\mathcal{Q}$ is a set of lines of $\mathcal{Q}$ partitioning the points of $\mathcal{Q}$. Dually an *ovoid* $\mathcal{O}$ of $\mathcal{Q}$ is a set of points such that each line of $\mathcal{Q}$ meets $\mathcal{O}$ in a unique point. A *partial spread* $\mathcal{M}$ of $\mathcal{Q}$ is a set of mutually disjoint lines of $\mathcal{Q}$. A partial spread is called *maximal* if it cannot be extended to a larger partial spread. Dually a *partial ovoid* is a set of mutually non-collinear points of $\mathcal{Q}$; it is called *maximal* if it cannot be extended to a larger partial ovoid.

A *k-arc* $\mathcal{K}$ of a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, is a set of k mutually non-collinear points, that is, a partial ovoid of size k. One easily observes that $k \leq st + 1$ (see, e.g., [15]), and if $k = st + 1$, then $\mathcal{K}$ is an ovoid of $\mathcal{S}$. A k-arc is *complete* if it is not contained in a k'-arc with $k' > k$. Dually, one defines *dual k-arcs* and *complete dual k-arcs*.

Finally we introduce a notation which will be used throughout this paper. If $\mathcal{M}$ is a set of lines of a GQ, we will denote by $\widetilde{\mathcal{M}}$ the set of all points covered by the lines of $\mathcal{M}$.

Theorem 3.1 ([15], 1.8.3) *A GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $t > s^2 - s$, has no ovoid. Dually, a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $s > t^2 - t$, has no spread.*

First note that a spread or an ovoid of a finite GQ of order (s,t) clearly contains $st + 1$ lines, respectively points.

The GQ $\mathcal{Q}(4,q)$ has spreads if and only if q is even [22] and has ovoids for all values of q (every $\mathcal{Q}^-(3,q) \subset \mathcal{Q}(4,q)$ is an ovoid of $\mathcal{Q}(4,q)$). Dualizing yields the corresponding results for the GQ $W(q)$.

The GQ $\mathcal{Q}(5,q)$ has spreads for all values of q [24], but never has ovoids [25]. Dualizing yields the corresponding results for the GQ $\mathcal{H}(3,q^2)$.

The (non-)existence of spreads of the GQ $\mathcal{H}(4,q^2)$ is a long-standing open problem. The only thing known so far is that $\mathcal{H}(4,4)$ does not admit a spread (by an unpublished computer result of A. Brouwer).
Finally the GQ $\mathcal{H}(4,q^2)$ does never admit an ovoid [25].

We synthesize some of these (and other) results in the following theorem.

Theorem 3.2 ([15], 3.4.1, 3.4.2 and 3.4.3; see also [31] for (v))

(*i*) *The GQ $\mathcal{Q}(4,q)$ always has ovoids. It has spreads if and only if q is even.*

(*ii*) *The GQ $T_2(\mathcal{O})$ of Tits always has ovoids.*

(*iii*) *The GQ $\mathcal{Q}(5,q)$ has spreads but no ovoids.*

(*iv*) *The GQ $T_3(\mathcal{O})$ of Tits has no ovoid but always has spreads.*

(*v*) *Each TGQ $T(\mathcal{O})$, where $\mathcal{O}$ is good at some element π, has spreads.*

(*vi*) *The GQ $\mathcal{H}(4,q^2)$ has no ovoid. For $q = 2$ it has no spread.*

(*vii*) *The GQ $\mathcal{P}(\mathcal{S},x)$ of S. E. Payne always has spreads. It has an ovoid if and only if $\mathcal{S}$ has an ovoid containing x.*

For a good reference on the existence question of spreads and ovoids in (not necessarily classical) finite GQs we refer to Thas and Payne [31].

4 Small Maximal Partial Spreads and Ovoids

Let $\mathcal{Q}$ be a finite GQ of order (s,t). We start by mentioning an absolute lower bound for the size of a maximal partial spread, respectively ovoid, of $\mathcal{Q}$.

Theorem 4.1 *A maximal partial ovoid of a finite GQ $\mathcal{Q}$ of order (s,t) contains at least $s+1$ points; dually a maximal partial spread of $\mathcal{Q}$ contains at least $t+1$ lines.*

Proof. Suppose that $\mathcal{K}$ is a maximal partial ovoid of $\mathcal{Q}$ with $|\mathcal{K}| < s+1$. Consider a line L exterior to $\mathcal{K}$ (such a line clearly exists). Then the set $P := \{\text{proj}_L p \parallel p \in \mathcal{K}\}$ contains less then $s+1$ points. Consequently there exists a point on L collinear with no point of $\mathcal{K}$. Such a point extends $\mathcal{K}$ to a larger partial ovoid, contradicting the assumed maximality of $\mathcal{K}$. $\square$

Theorem 4.2 *If a GQ $\mathcal{Q}$ of order (s,t) admits a maximal partial spread $\mathcal{M}$ of size $t+1$, then $\mathcal{M}$ is a spread of a subGQ $\mathcal{Q}'$ of $\mathcal{Q}$ of order $(s,t/s)$, so that consequently $t \geq s$ and $s \mid t$. Dually, if $\mathcal{Q}$ admits a maximal partial ovoid $\mathcal{K}$ of size $s+1$, then $\mathcal{K}$ is an ovoid of a subGQ $\mathcal{Q}'$ of $\mathcal{Q}$ of order $(s/t,t)$, $s \geq t$ and $t \mid s$.*

Proof. Consider any point p which is not incident with a line of $\mathcal{M}$. Then clearly every line through p intersects exactly one line of $\mathcal{M}$. Consequently every line which contains at least two distinct points of $\widetilde{\mathcal{M}}$ is completely contained in $\widetilde{\mathcal{M}}$. It is now easily seen that the point-line geometry with as point set $\widetilde{\mathcal{M}}$ and line set the set of lines having all their points in $\widetilde{\mathcal{M}}$ is a subGQ $\mathcal{Q}'$ of $\mathcal{Q}$ of order $(s,t/s)$. Of course $\mathcal{M}$ is a spread of $\mathcal{Q}'$. $\square$

For $s=t$, one can give a precise description of maximal partial ovoids, respectively spreads, of size $s+1$.

Theorem 4.3 *A GQ $\mathcal{Q}$ of order s admits a maximal partial ovoid (respectively spread) of size $s+1$ if and only if $\mathcal{Q}$ has a regular pair of non-collinear points (respectively lines).*

Proof. First suppose that $\mathcal{Q}$ has a regular pair $\{x,y\}$ of non-collinear points. An easy counting argument shows that every point of $\mathcal{Q} \setminus \{x,y\}^{\perp\perp}$ is collinear with some point of $\{x,y\}^{\perp\perp}$. As $\{x,y\}^{\perp\perp}$ is a set of $s+1$ mutually non-collinear points, $\{x,y\}^{\perp\perp}$ is a maximal partial ovoid of size $s+1$.

Conversely, suppose that $\mathcal{K}$ is a maximal partial ovoid of $\mathcal{Q}$ of size $s+1$. By Theorem 4.2 we know that $\mathcal{K}$ has to be an ovoid of a subGQ of order $(1,s)$ of $\mathcal{Q}$. The point set of this GQ is the set $\mathcal{K} \cup \mathcal{K}^{\perp}$ and hence one easily sees that $\mathcal{K}^{\perp\perp} = \mathcal{K}$, i.e. each pair of distinct points of $\mathcal{K}$ is regular. $\square$.

Remark 4.4 (i) Note that the previous theorem shows that if a GQ of order s admits a maximal partial ovoid $\mathcal{K}$ of size $s+1$, then $\mathcal{K}$ is necessarily a hyperbolic line.

(ii) When a thick GQ of order (s,t) admits a maximal partial ovoid of size $s+1$, we constructed a subGQ of order $(s/t,t)$. We will encounter similar behaviour when considering maximal partial ovoids of size $st-t/s$ in GQs of order (s,t); there, a subGQ of order $(s,t/s)$ will be constructed starting from such a hypothetical partial ovoid.

We will now discuss bounds on the size of small maximal partial ovoids and spreads for the classical GQs. Here we mean with "small", "relatively close" to the lower bound.

4.1 The GQs $\mathcal{Q}(4,q)$ and $W(q)$

Recall that these GQs are each others dual and that $W(q)$ is self-dual if and only if q is even. Most of the results here are taken from the recent paper [7], on which the first author of this paper reported at the Fifth Shanghai Conference on Combinatorics. The first result is an immediate corollary of Theorem 4.3.

Theorem 4.5 *The smallest maximal partial ovoids (respectively spreads) of $W(q)$ (respectively $\mathcal{Q}(4,q)$) have size $q+1$. If q is even, the smallest maximal partial spreads of $W(q)$ have size $q+1$.*
More generally, the same statement holds for translation generalized quadrangles *(cf. Chapter 8 of [15]) of order q, q even.*

Proof. The first part follows from the fact that all points of $W(q)$ are regular.
The second part follows from the fact that translation generalized quadrangles of order q, q even, have a regular translation point. □

Of course, once the smallest examples are known, one is interested in finding the second smallest example, or at least in finding a bound on its size. In order to obtain such results we need to explain a link between maximal partial ovoids of $W(q)$ and minimal *blocking sets* in $\mathbf{PG}(3,q)$.

Definition. A *blocking set* $\mathcal{B}$ in $\mathbf{PG}(n,q)$ is a set of points of $\mathbf{PG}(n,q)$ with the property that every hyperplane of $\mathbf{PG}(n,q)$ intersects it in at least 1 point. A blocking set is called *minimal* if it cannot be reduced to a smaller blocking set. A blocking set is called *non-trivial* if it does not contain the point set of a line.

Lemma 4.6 ([7]) *With every maximal partial ovoid $\mathcal{K}$ of $W(q)$ there is associated a minimal blocking set of* $\mathbf{PG}(3, q)$.

Proof. Consider $W(q)$ in its natural representation in $\mathbf{PG}(3, q)$. As for every point p of $W(q)$ the set $\{p\} \cup p^{\perp}$ consists exactly of all points of the plane p^{θ}, where θ is the polarity defining $W(q)$, it is clear that $\mathcal{K}$ determines a blocking set of $\mathbf{PG}(3, q)$. It is easily shown that this blocking set is minimal (see [7]).□

This lemma now allows one to apply the theory of blocking sets, which has been done in [7].

Firstly, a nice bound for small maximal partial ovoids $W(p)$, p a prime, can be obtained using a result on blocking sets of $\mathbf{PG}(2, p)$ by Blokhuis [4].

Theorem 4.7 ([7]) *Let $\mathcal{K}$ be a second smallest maximal partial ovoid of $W(p)$, p prime. Then $|\mathcal{O}| \geq 3(p + 1)/2 + 1$.*

By results of Storme and Weiner [20] on blocking sets of $\mathbf{PG}(3, q^2)$ and $\mathbf{PG}(3, q^3)$, one can obtain the following results.

Theorem 4.8 ([7]) *The second smallest maximal partial ovoids $\mathcal{K}$ of $W(q^2)$, $q = p^h$, $p > 3$ prime, $h \geq 1$, contain at least $s(q^2) + 1$ points, where $s(q^2)$ denotes the cardinality of the second smallest non-trivial minimal blocking sets in $\mathbf{PG}(2, q^2)$. If $q = p > 2$, then $\mathcal{K}$ contains at least $3(p^2 + 1)/2 + 1$ points.*

Theorem 4.9 ([7]) *The second smallest maximal partial ovoids $\mathcal{K}$ of $W(q^3)$, $q = p^h$, $p \geq 7$ prime, $h \geq 1$, contain at least $q^3 + q^2 + q + 1$ points. If $|\mathcal{K}| = q^3 + q^2 + q + 1$, then $\mathcal{K}$ consists of the point set of a subgeometry $\mathbf{PG}(3, q)$ of $\mathbf{PG}(3, q^3)$.*

In [7] the existence of maximal partial ovoids of $W(q^3)$ of size q^3+q^2+q+1 was posed as an open problem. It was J. A. Thas who suggested to the authors to try to use the Klein correspondence in order to prove the (non)-existence of such a partial ovoid. We will now use this correspondence to prove its non-existing. We need the following lemma.

Lemma 4.10 *Consider a hyperplane π of $\mathbf{PG}(5, q^3)$ and a subspace $\Omega :=$ $\mathbf{PG}(5, q)$ of $\mathbf{PG}(5, q^3)$. Then π intersects Ω in at least a plane $\mathbf{PG}(2, q)$.*

Proof. Suppose $\phi := \mathbf{PG}(3, q^3) \subset \mathbf{PG}(5, q^3)$ is any $\mathbf{PG}(3, q^3)$ skew to Ω. (Note that if the lemma would be false such a ϕ exists.) Let π' be a $\mathbf{PG}(4, q^3)$ containing ϕ. First suppose that π' intersects Ω in some $\mathbf{PG}(3, q) =: \omega$. This subgeometry extends in a unique way to $\overline{\omega} := \mathbf{PG}(3, q^3) \subset \mathbf{PG}(5, q^3)$, and $\overline{\omega} \cap \phi$ is a plane $\mathbf{PG}(2, q^3)$. As ω determines a blocking set in $\overline{\omega}$ it follows that

$\overline{\omega} \cap \phi$ contains at least one point of ω, implying that ϕ is not skew to Ω, a contradiction.

For every point $p \in \Omega$ there is a $\mathbf{PG}(4, q^3)$, $\langle p, \phi \rangle$, containing ϕ. As in $\mathbf{PG}(5, q^3)$ there are $q^3 + 1$ $\mathbf{PG}(4, q^3)$-subspaces containing ϕ and as such a $\mathbf{PG}(4, q^3)$ contains at most $(q^3 - 1)/(q - 1)$ points of Ω it follows that each of these $q^3 + 1$ hyperplanes intersects Ω in exactly $(q^3 - 1)/(q - 1)$ points. This proves the lemma. □

Theorem 4.11 *The GQ $W(q^3)$, $q = p^h$, $p \geq 7$, does not admit a maximal partial ovoid of size $q^3 + q^2 + q + 1$.*

Proof. Assume that $W(q^3)$ admits a maximal partial ovoid $\mathcal{K}$ of size $q^3 + q^2 + q + 1$. We consider $W(q^3)$ in its natural representation in $\mathbf{PG}(3, q^3)$. Then $\mathcal{K}$ corresponds to some $\mathbf{PG}(3, q)$-subgeometry of $\mathbf{PG}(3, q^3)$. Using the Klein correspondence the lines of $\mathbf{PG}(3, q^3)$ are mapped onto the points of $\mathcal{Q}^+(5, q^3)$, the lines of $\mathbf{PG}(3, q)$ onto the points of some $\mathcal{Q}^+(5, q) \subset \mathcal{Q}^+(5, q^3)$ and the lines of $W(q^3)$ onto the points of some $\mathcal{Q}(4, q^3) \subset \mathcal{Q}^+(5, q^3)$. It is easily seen, since $\mathcal{K}$ is a partial ovoid, that $\mathcal{Q}(4, q^3) \cap \mathcal{Q}^+(5, q) = \emptyset$ (as point sets). However using the previous lemma, we see that the $\mathbf{PG}(4, q^3)$ determined by $\mathcal{Q}(4, q^3)$ and the $\mathbf{PG}(5, q)$ determined by $\mathcal{Q}^+(5, q)$ must intersect in at least a plane $\mathbf{PG}(2, q)$. As in $\mathbf{PG}(5, q)$ every plane has nonempty intersection with $\mathcal{Q}^+(5, q)$, $\mathcal{K}$ cannot be a partial ovoid. □

Concerning small maximal partial ovoids of $W(q)$, it is interesting to note that computer searches [7], exhaustive for $q \in \{2, 3, 4, 5\}$ and heuristic for $q \in \{7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27\}$, suggest that the second smallest maximal partial ovoids of $W(q)$ will probably have size $2q + 1$. A maximal partial ovoid of this size can easily be constructed by taking all points except one point r on a hyperbolic line H in $W(q)$, together with one arbitrary point (not collinear with one of the remaining points of H) from each of the $q + 1$ lines of $W(q)$ through r. In [7] also a theoretical construction of a maximal partial ovoid of size $3q - 1$ is given for every $q > 3$.

In view of the isomorphism relations between the GQs under consideration in this section, the only case not handled yet is the one of maximal partial spreads of $W(q)$, for odd q. In [7] a counting technique first introduced in [11] is used to prove the following bound. It is also explained how a fine tuning of this technique can in some cases slightly improve this result.

Theorem 4.12 ([7]) *Suppose that $\mathcal{M}$ is a maximal partial spread of $W(q)$, q odd. Then $|\mathcal{M}| \geq \lceil 1,419q \rceil$.*

Here as well, the theoretical obtained bound is rather small compared to the results obtained by computer for small q, which seem to point in the direction of a bound of order $q\sqrt{q}$ [7].

4.2 The GQs $\mathcal{Q}(5,q)$ and $\mathcal{H}(3,q^2)$

The most important results for these GQs were obtained by Ebert and Hirschfeld in [10] and by Aguglia, Ebert and Luyckx in [2]. After an overview we will provide an improvement of a bound which was obtained [2]. As the GQs under consideration are each others dual we will discuss all results in terms of $\mathcal{Q}(5,q)$. In [10] the following lower bound on the size of maximal partial ovoids of $\mathcal{Q}(5,q)$ was proved using a counting technique analogous to the one used to prove Theorem 4.12. Recall that the trivial lower bound equals $q+1$.

Theorem 4.13 ([10]) *Let $\mathcal{K}$ be a maximal partial ovoid of $\mathcal{Q}(5,q)$. Then $|\mathcal{K}| \geq 2q+1$. If $q \geq 4$, then $|\mathcal{K}| \geq 2q+2$.*

As far as it concerns constructions of small maximal partial ovoids of $\mathcal{Q}(5,q)$ for general q, the best known construction provides such a partial ovoid of size q^2+1. This goes as follows. Consider any elliptic quadric $\mathcal{Q}^-(3,q) \subset \mathcal{Q}(5,q)$. Then it is easily seen that the q^2+1 points of $\mathcal{Q}^-(3,q)$ form a maximal partial ovoid (see also [10]). Using a recent result of Ball [3] we can see that in fact every ovoid of $\mathcal{Q}(4,q) \subset \mathcal{Q}(5,q)$ determines a maximal partial ovoid of $\mathcal{Q}(5,q)$ (Ball shows that every $\mathcal{Q}^-(3,p^h) \subset \mathcal{Q}(4,p^h)$ intersects an ovoid of $\mathcal{Q}(4,p^h)$ in $1 \pmod p$ points). Also in [1] constructions of maximal partial ovoids of $\mathcal{Q}(5,q)$ of size q^2+1 are provided. There however the authors show, using the computer, the existence of maximal partial ovoids of $\mathcal{Q}(5,q)$ of size strictly less than q^2+1, for $q=7,8$. Very recently, in [8], Cimrakova and Fack, show by computer the existence of several maximal partial ovoids of size strictly less than q^2+1 for all $q \in \{4,5,7,8,9,11,13\}$. However, these maximal partial ovoids are still "much" larger than the bound from Theorem 4.13, so there still remains a lot of work to be done.

When it comes to the existence of small maximal partial spreads of $\mathcal{Q}(5,q)$, we know from the foregoing that the existence of spreads of $\mathcal{Q}(4,q)$ implies the existence of a maximal partial spread of size q^2+1 (the trivial lower bound) of $\mathcal{Q}(5,q)$, and this bound is reached if and only if q is even. So, if q is odd, a maximal partial spread contains at least q^2+2 lines. The size of small maximal partial spreads of $\mathcal{Q}(5,q)$ is discussed in [2]. Using an idea of J. A. Thas the authors show that $\mathcal{Q}(5,q)$ cannot admit a maximal partial spread of size q^2+2. We will now improve this result by showing that $\mathcal{Q}(5,q)$, $q \geq 5$, cannot admit a maximal partial spread of size q^2+3. We first prove the following lemma.

Lemma 4.14 *If a partial spread $\mathcal{M}$ of $\mathcal{Q}(5,q)$ which is not a spread of some $\mathcal{Q}(4,q) \subset \mathcal{Q}(5,q)$ covers all points of some $\mathcal{Q}(4,q) \subset \mathcal{Q}(5,q)$, then $|\mathcal{M}| \geq q^2+q+1$ if q is even, and $|\mathcal{M}| \geq 2q^2+q$ if q is odd.*

Proof. For q even this is trivial. For odd q this follows immediately from the fact that a maximal partial spread of $\mathcal{Q}(4,q)$ contains at most q^2-q+1 lines if q is odd ([21], see also the next section). $\square$

Theorem 4.15 *The GQ $\mathcal{Q}(5,q)$ does not admit a maximal partial spread of size q^2+3 if $q \geq 5$.*

Proof. Assume that $\mathcal{M}$ is a maximal partial spread of $\mathcal{Q}(5,q)$ of size q^2+3. Consider a line L of $\mathcal{Q}(5,q)$ that contains at least 4 points of $\widetilde{\mathcal{M}}$ and suppose that there is a point p on L not belonging to $\widetilde{\mathcal{M}}$. As $|\mathcal{M}| = q^2+3$ it follows that there exists a line through p not intersecting $\widetilde{\mathcal{M}}$, a contradiction. Hence a line of $\mathcal{Q}(5,q)$ intersects $\widetilde{\mathcal{M}}$ in either $1,2,3$ or $q+1$ points. Further, if a point p does not belong to $\widetilde{\mathcal{M}}$, then there is either a unique line through p intersecting $\widetilde{\mathcal{M}}$ in 3 points, while all other lines through p intersect $\widetilde{\mathcal{M}}$ in exactly 1 point, or there are exactly 2 lines through p intersecting $\widetilde{\mathcal{M}}$ in 2 points, and all other lines through p intersect $\widetilde{\mathcal{M}}$ in a unique point. We will denote the set of points of the former type by X_1 and the set of points of the latter type by X_2. Also define $x_i := |X_i|$, $i = 1,2$. Finally denote by $\mathcal{F}$ the set of lines of $\mathcal{Q}(5,q)$ not in $\mathcal{M}$ that are completely covered by $\widetilde{\mathcal{M}}$; here we put $f := |\mathcal{F}|$. An easy counting argument shows that there are $(q^2x_1 + (q^2-1)x_2)/q$ lines having exactly 1 point in $\widetilde{\mathcal{M}}$, that there are $x_1/(q-2)$ lines having exactly 3 points in $\widetilde{\mathcal{M}}$, and that there are $2x_2/(q-1)$ lines having exactly 2 points in $\widetilde{\mathcal{M}}$. Considering the points not covered by $\widetilde{\mathcal{M}}$ and the lines not in $\mathcal{M}$, we obtain:

$$\begin{cases} x_1 + x_2 = q^4 - q^2 - 2q - 2 \\ (q + \frac{1}{q-2})x_1 + (\frac{q^2-1}{q} + \frac{2}{q-1})x_2 + f = (q^2+1)(q^3+1) - q^2 - 3 \end{cases}$$

Solving for x_1 and x_2 in function of f, we obtain:

$$x_2 = \frac{1}{2}(q-1)q(2q^3 + q^2 - q^4 + 4q - 6 + (q-2)f).$$

As $x_2 \geq 0$ we obtain that

$$f \geq q^3 - q - 6 - \frac{6}{q-2}.$$

Next we count in two ways the ordered triples (K, L, M) with $K, M \in \mathcal{M}$, $K \neq M$, $L \in \mathcal{F}$ and $K \sim L \sim M$. We obtain

$$f(q+1)q = (q^2+3)(q^2+2)y,$$

where y is the average number of transversals in $\mathcal{F}$ of two distinct lines of $\mathcal{M}$. Using the above obtained bound for f, we deduce that $y > 2$ (recall that $q > 3$). This implies the existence of two distinct lines K and M of $\mathcal{M}$ with the property that at least 3 of their $q+1$ transversals are lines of $\mathcal{F}$. These lines determine in a unique way some $\mathcal{Q}(3,q) \subset \mathcal{Q}(5,q)$. Assume that not all points of this $\mathcal{Q}(3,q)$ would belong to $\widetilde{\mathcal{M}}$, say $p \in \mathcal{Q}(3,q) \setminus \widetilde{\mathcal{M}}$. Then the two lines through p in $\mathcal{Q}(3,q)$ intersect $\widetilde{\mathcal{M}}$ in respectively at least 2 and at least 3 points, a contradiction. Consequently all $q+1$ transversals of K and M belong to $\mathcal{F}$ and we have shown the existence of a $\mathcal{Q}(3,q) \subset \mathcal{Q}(5,q)$ which is completely covered by $\widetilde{\mathcal{M}}$. Consider such a fixed $\mathcal{Q}(3,q)$. The $(q^2+3)(q+1)-(q+1)^2$ remaining points of $\widetilde{\mathcal{M}}$ have to be partitioned by the $q+1$ $\mathcal{Q}(4,q)$ subGQs on $\mathcal{Q}(5,q)$ which contain $\mathcal{Q}(3,q)$. This implies that there exists a $\mathcal{Q}(4,q)$ containing $\mathcal{Q}(3,q)$ which is such that the set Z of points of $\widetilde{\mathcal{M}}$ in $\mathcal{Q}(4,q)$ not in $\mathcal{Q}(3,q)$, contains at least q^2-q+2 points. Consider such a (fixed) $\mathcal{Q}(4,q)$. We count in two ways the pairs (u,v) with $u \in \mathcal{Q}(4,q) \setminus (\mathcal{Q}(3,q) \cup Z)$, $v \in Z$ and $u \sim v$. We obtain

$$\left[q(q^2-1) - |Z|\right] h = |Z|\, z,$$

where h is the average number of points of Z collinear with a given point of $\mathcal{Q}(4,q) \setminus (\mathcal{Q}(3,q) \cup Z)$ and where z is the average number of points of $\mathcal{Q}(4,q) \setminus (\mathcal{Q}(3,q) \cup Z)$ collinear with a given point of Z. It is clear that $h \leq 2$, and since $|Z| \geq q^2-q+2$, we deduce that

$$z \leq 2\frac{q^3-q^2-2}{q^2-q+2}.$$

First suppose that $q > 5$. Then the above inequality implies the existence of a point v of Z such that at least $q-1$ of the lines of $\mathcal{Q}(4,q)$ through v belong to $\mathcal{F}$. Let C be $v^\perp \cap \mathcal{Q}(3,q)$. Then $C^\perp \cap \mathcal{Q}(4,q)$ either consists of 2 points or a single point, depending on whether q is odd or even. Let w be a point of $\mathcal{Q}(4,q) \setminus (\mathcal{Q}(3,q) \cup C^\perp)$ and suppose that w does not belong to Z. As $w^\perp \cap v^\perp \cap \mathcal{Q}(3,q)$ contains at most 2 points, there at least $q-3 > 2$ points of Z collinear with w and this on at least $q-3$ distinct lines through w (the $q-3$ points mentioned are points of $v^\perp$). This contradicts the fact that there are at most 2 lines through w which contain more than 1 point of $\widetilde{\mathcal{M}}$ (recall that the points of $\mathcal{Q}(3,q)$ are covered by $\widetilde{\mathcal{M}}$). Consequently $w \in Z$. One now easily shows that also all points of $C^\perp \cap \mathcal{Q}(4,q)$ must belong to Z, and finally it then follows that all points of $\mathcal{Q}(4,q)$ belong to Z, that is, $\mathcal{Q}(4,q)$ is covered by $\widetilde{\mathcal{M}}$. By the foregoing lemma it follows that $|\mathcal{M}| > q^2+3$, a contradiction.

Finally suppose that $q = 5$. Then the above inequality implies the existence of a point $v \in Z$, such that v is collinear with at most 8 points which do not belong to Z. Hence, at least 4 of the lines of $\mathcal{Q}(4,q)$ through v belong to $\mathcal{F}$. With the notation of the above paragraph $C^\perp \cap \mathcal{Q}(4,q)$ consists of two points v and w. Let $u \neq w$ be any point of $\mathcal{Q}(4,q) \setminus \mathcal{Q}(3,q)$ which is collinear with w. Then u is collinear with exactly 1 point of C. Consequently there are at least 3 points of Z collinear with u on 3 distinct lines incident with u, yielding at least 3 lines through u containing at least 2 points of $\widetilde{\mathcal{M}}$. This implies that $u \in Z$. Consequently all points of the cone wC belong to $\widetilde{\mathcal{M}}$. The proof can now be finished in a similar as in the first part of the proof. $\square$

Corollary 4.16 *Suppose that $q \geq 5$. Then the second smallest partial spread of $\mathcal{Q}(5,q)$, q even, and the smallest maximal partial spread of $\mathcal{Q}(5,q)$, q odd, contain at least q^2+4 points.*

It is however important to note that no constructions are known of maximal partial spreads of this size. In most cases the smallest known example (of size strictly bigger than q^2+1 if q is even) is not even close to this bound (an exception is given by $\mathcal{Q}(5,4)$ where computer searches showed the existence of a maximal partial spread of size 21 [8]). It is in the same paper [8] that the best known results on the size of maximal partial spreads can be found for $q \in \{3,4,5,7,8\}$ (computer results). The best theoretical construction (based on an idea of J. A. Thas) can be found in [2], where the authors prove, starting from a small maximal partial spread of $\mathbf{PG}(3,q)$, the existence for odd q of a maximal partial spread of size $(m+1)q^2+1$, with $m = \lceil 2\log_2(q) \rceil$. This is however still much larger than q^2+4. An other way to construct small maximal partial ovoids of $\mathcal{Q}(5,q)$, might be to start with a large partial spread (which is not a spread) of $\mathcal{Q}(4,q) \subset \mathcal{Q}(5,q)$ and then add lines. For a maximal partial spread of $\mathcal{Q}(4,q)$ of size $q^2-\epsilon$ this yields a maximal partial spread of $\mathcal{Q}(5,q)$ of size less than or equal to $q^2+(\epsilon+1)(q+1)$. Unfortunately not much is known about large maximal partial spreads of $\mathcal{Q}(4,q)$, and the largest known such partial spread is still much smaller than the theoretical bound q^2-q+1. Finally it is interesting to note that in [13] Hirschfeld and Korchmáros construct maximal partial spreads of $\mathcal{Q}(5,q)$ (in fact they construct maximal partial ovoids of $\mathcal{H}(3,q^2)$) from $\mathbf{GF}(q^2)$-maximal curves for even q. However, their examples have size $q^2+1+2gq$, where g is the genus of the curve used, which is also still much larger then our obtained lower bound (except for the trivial case of rational algebraic curves which have genus 0 and yield a maximal partial ovoid of size q^2+1).

4.3 The GQ $\mathcal{H}(4,q^2)$

Consider a fixed $\mathcal{H}(3,q^2) \subset \mathcal{H}(4,q^2)$ and a fixed $\mathcal{H}(2,q^2) \subset \mathcal{H}(3,q^2)$. Of course $\mathcal{H}(2,q^2)$ is an ovoid of $\mathcal{H}(3,q^2)$, and since every other $\mathcal{H}(2,q^2) \subset \mathcal{H}(3,q^2)$

contains at least 1 point of the fixed $\mathcal{H}(2,q^2)$ it is clear that $\mathcal{H}(2,q^2)$ is a maximal partial ovoid of $\mathcal{H}(4,q^2)$. It has size q^3+1, which is the trivial lower bound, and so it is the smallest maximal partial ovoid of $\mathcal{H}(4,q^2)$. To the authors' knowledge not much more is known about small maximal partial ovoids of $\mathcal{H}(4,q^2)$.

We now turn to small maximal partial spreads. Since $\mathcal{H}(3,q^2)$ is the only subGQ of order (q^2,q) of $\mathcal{H}(4,q^2)$ and $\mathcal{H}(3,q^2)$ does not admit a spread, it follows that the smallest maximal partial spread of $\mathcal{H}(4,q^2)$ contains at least q^3+2 lines. However using an analogous technique as was used in [2] to show the non-existence of a maximal partial spread of size q^2+2 of $\mathcal{Q}(5,q)$, we can exclude the existence of such a maximal partial spread.

Theorem 4.17 *The GQ $\mathcal{H}(4,q^2)$ does not admit a maximal partial spread of size q^3+2.*

Proof. Suppose that $\mathcal{M}$ is a maximal partial spread of size q^3+2, and let X be the set of points of $\mathcal{H}(4,q^2)$ not covered by $\mathcal{M}$. Then $|X| = q^7-q^3-q^2-1$. Further, through every point of X there is a unique line intersecting $\widetilde{\mathcal{M}}$ in exactly 2 points (while all other lines through such a point intersect $\widetilde{\mathcal{M}}$ in 1 point). Consequently the number of lines intersecting $\widetilde{\mathcal{M}}$ in 2 points equals $|X|/(q^2-1)$. This quantity can never be an integer, a contradiction. $\square$

Corollary 4.18 *The smallest maximal partial spread of $\mathcal{H}(4,q^2)$ contains at least q^3+3 lines.*

A maximal partial spread of this size is not known and does probably not exists. It could be that techniques analogous to the ones used in the proof of Theorem 4.15 can be used to prove the non-existence of maximal partial spread of size q^3+3. However this is not completely clear at the moment, and research by the first author and L. Storme on improvements of the bound of Corollary 4.18 is currently going on.

5 Large Maximal Partial Spreads of Generalized Quadrangles of Order (q^2,q)

We start this section with some results which are taken from J. A. Thas [29]. We then mention some results taken from K. Thas [32]. Recall that for any triad $\{L,M,N\}$ of lines of $\mathcal{H}(3,q^2)$, we have that $|\{L,M,N\}^\perp| = |\{L,M,N\}^{\perp\perp}| = q+1$.

Theorem 5.1 (J. A. Thas [29]) *Let $\mathcal{S}$ be isomorphic to the classical GQ $\mathcal{H}(3,q^2)$. If $\mathcal{S}$ has a partial spread $\mathbf{T}$, then*

$$|\mathbf{T}| \leq q^3 - q^2 + q + 1.$$

Corollary 5.2 (J. A. Thas [29]) *The classical GQ $\mathcal{H}(3,q^2)$ of order (q^2,q) has no spreads.*

Theorem 5.3 (J. A. Thas [29]) *Let $\mathcal{S}$ be isomorphic to the classical GQ $\mathcal{H}(3,q^2)$. Suppose $\mathbf{T}$ is a partial spread of $\mathcal{S}$ which contains three distinct lines L, M, N for which $\{L,M,N\}^{\perp\perp} \subseteq \mathbf{T}$. Then*

$$|\mathbf{T}| \leq \frac{q^3}{2} + \frac{q}{2} + 1.$$

Note that both Theorem 5.1 and Theorem 5.3 dualize to partial ovoids of $\mathcal{Q}(5,q)$.

Let us recall that if $\{U,V,W\}$ is a 3-regular triad of lines of a GQ $\mathcal{S}$ of order (s^2,s), $s > 1$, where $\{U,V,W\}^{\perp} = \{L_0, L_1, \ldots, L_s\}$ and $\{U,V,W\}^{\perp\perp} = \{M_0, M_1, \ldots, M_s\}$, then we have that each line of $\mathcal{S}$ which is not in $\{U,V,W\}^{\perp}$ $\cup \{U,V,W\}^{\perp\perp}$, either is incident with a point $L_i \cap M_j$ for some i and j in $\{0,1,\ldots,s\}$, or intersects exactly two lines of $\{U,V,W\}^{\perp}$ and no lines of $\{U,V,W\}^{\perp\perp}$, or intersects exactly two lines of $\{U,V,W\}^{\perp\perp}$ and no lines of $\{U,V,W\}^{\perp}$.

We now generalize Theorem 5.1 to the following result.

Theorem 5.4 *Let $\mathcal{S}$ be isomorphic to the classical GQ $\mathcal{H}(3,q^2)$. Suppose that $\mathbf{T}$ is a partial spread of $\mathcal{S}$ which contains three distinct lines L, M, N so that $|\{L,M,N\}^{\perp\perp} \cap \mathbf{T}| = r + 3$, $r \in \mathbb{N}$, $r \leq q - 2$. Then*

$$|\mathbf{T}| \leq q^3 - \frac{q}{2}(q+1)(r+1) + 1.$$

Proof. Let $\{L,M,N\}^{\perp\perp} = L_0, L_1, \ldots, L_q$, set $\{L,M,N\}^{\perp} = M_0, M_1, \ldots, M_q$ and put $\{L,M,N\}^{\perp\perp} \cap \mathbf{T} = X$. Then $|X| = r + 3$. For each line U of the $q - r - 2$ lines of $\{L,M,N\}^{\perp\perp}$ which are not contained in $\mathbf{T}$, there are at most $q + 1$ lines of $\mathbf{T}$ which intersect U (in one point) and no other line of $\{L,M,N\}^{\perp\perp}$. The lines of $\mathbf{T}$ which do not intersect exactly one of these lines U and which do not belong to $\{L,M,N\}^{\perp\perp}$, either intersect $\{L_0, L_1, \ldots, L_q\}$ in 2 lines and no line of $\{M_0, M_1, \ldots, M_q\}$, or they intersect $\{M_0, M_1, \ldots, M_q\}$ in 2 lines and no line of $\{L_0, L_1, \ldots, L_q\}$. We obtain that

$$|\mathbf{T}| \leq (r+3) + (q-r-2)(q+1) + \frac{(q-2-r)(q^2-q)}{2} + \frac{(q+1)(q^2-q)}{2}.$$

Hence the result. $\square$

Note that, if $r < r' \leq q-2$, we have that

$$q^3 - \frac{q}{2}(q+1)(r'+1) + 1 \leq q^3 - \frac{q}{2}(q+1)(r+1) + 1.$$

Remark 5.5 For $r = 0$, we obtain a weaker result than Theorem 5.1. For $r = 1$, however, the bound is already (slightly) stronger. For $r = q-2$, we obtain Theorem 5.3.

One also notes that Theorem 5.4 dualizes to a result on partial ovoids of $\mathcal{Q}(5, q)$.

As a direct corollary of the proof of Theorem 5.4, we have the next result.

Theorem 5.6 *Let $\mathcal{S}$ be a GQ of order (s^2, s), $s > 1$, which has a 3-regular triad of lines $\{L, M, N\}$. Suppose that $\mathbf{T}$ is a partial spread of $\mathcal{S}$ so that $|\{L, M, N\}^{\perp\perp} \cap \mathbf{T}| = r + 3$, $r \in \mathbb{N}$, $r \leq s - 2$. Then*

$$|\mathbf{T}| \leq s^3 - \frac{s}{2}(s+1)(r+1) + 1.$$

Proof. Immediate. $\square$

Corollary 5.7 *Theorem 5.6 can be stated for each flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q), $q > 1$, and for the point-line dual $T_3(\mathcal{O})^D$ of $T_3(\mathcal{O})$, $\mathcal{O}$ any ovoid $\mathcal{O}$ of $\mathbf{PG}(3, q)$.*

Proof. If $\mathcal{F}$ is a flock of the quadratic cone of $\mathbf{PG}(3, q)$, then $\mathcal{S}(\mathcal{F})^D$ satisfies Property (G) at its line $[\infty]$ (which corresponds to the point (∞) of the flock GQ), and hence $\mathcal{S}(\mathcal{F})$ has 3-regular triads of lines. The point (∞) of $T_3(\mathcal{O})$ is 3-regular. $\square$

6 Complete $(st - t/s)$-Arcs in Generalized Quadrangles

In this section (and the following ones), "complete (k)-arc" means "maximal (k)-arc".

6.1 Complete $(st - t/s)$-arcs

The following theorem is an important observation.

Theorem 6.1 ([15], 2.7.1) *An $(st - m)$-arc in a GQ of order (s, t), where $-1 \leq m < t/s$ and $s \neq 1 \neq t$, is always contained in a uniquely defined ovoid.*

Considering Theorem 6.1, it is a natural question to ask whether or not complete $(st - t/s)$-arcs exist.
Let us first recall some notions and results concerning complete $(st-t/s)$-arcs. Let $\mathcal{K}$ be a complete $(st - t/s)$-arc in the GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$. If B' is the set of lines incident with no point of $\mathcal{K}$, P' the set of points on the lines of B', and I' the restriction of I to points of P' and lines of B', then $\mathcal{S}' = (P', B', I')$ is a subquadrangle of $\mathcal{S}$ of order $(s, t/s)$ (see [15, 2.7.2]). We denote this subGQ by $\mathcal{S}(\mathcal{K})$.

The following result is taken from J. A. Thas [23], see also 1.4.2 (ii) of [15].

Theorem 6.2 (J. A. Thas [23]; see also 1.4.2 (ii) of [15]) *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s, t), $s, t \neq 1$ and $s \neq t$, and let $\{x, y\}^{\perp\perp}$ be a hyperbolic line of size $p + 1$, where $pt = s^2$. Then every point of $\mathcal{S}$ not in $cl(x, y)$ is collinear with $t/s + 1 = s/p + 1$ points of $\{x, y\}^{\perp}$.*

Remark 6.3 If $s = t$ and the hyperbolic line $\{x, y\}^{\perp\perp}$ has size $s + 1$, that is, $\{x, y\}$ is regular, then $cl(x, y)$ coincides with the point set of $\mathcal{S}$. If $|\{x, y\}^{\perp\perp}| = p + 1$, $x \not\sim y$, with $pt = s^2$, then $\mathcal{S} \setminus cl(x, y) \neq \emptyset$ if and only if $s \neq t$.

Lemma 6.4 *Let $\mathcal{S}$ be a GQ of order (s, t), where $s \neq 1 \neq t$. Assume that $\mathcal{S}$ contains a complete $(st - t/s)$-arc $\mathcal{K}$ and that $\mathcal{S}(\mathcal{K})$ is the corresponding GQ of order $(s, t/s)$.*

(1) *Then every point off $\mathcal{K}$ is collinear with either $t - t/s$ points of $\mathcal{K}$ or $t+1$ points of $\mathcal{K}$, according to whether this point is contained in $\mathcal{S}(\mathcal{K})$ or not.*

(2) *If x and y are distinct non-collinear points of $\mathcal{S}$ and $\{x, y\}^{\perp\perp} \cap \mathcal{S}(\mathcal{K}) \neq \emptyset$, then $|\{x, y\}^{\perp} \cap \mathcal{S}(\mathcal{K})| = t/s + 1$.*

Proof. Easy. □

6.2 Nonexistence of complete $(st - t/s)$-arcs

The results of this section are taken from [34].

Theorem 6.5 *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose $\mathcal{K}$ is a complete $(st - t/s)$-arc of $\mathcal{S}$. If $\{x,y\}^{\perp\perp} = \mathbf{H}$ is a hyperbolic line of $\mathcal{S}$ of size $p+1$ with $pt = s^2$, then either $|\mathbf{H} \cap \mathcal{K}| \in \{0,1\}$, or $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,2)$ or $\mathcal{Q}(5,2)$.*

Theorem 6.6 *The classical generalized quadrangle $\mathcal{S} = \mathcal{H}(4,q^2)$ has no complete $(q^5 - q)$-arcs.*

Proof. Every hyperbolic line of $\mathcal{H}(4,q^2)$ has size $q+1$, see Section 2.3. Now suppose that $\mathcal{K}$ is a complete $(q^5 - q)$-arc of $\mathcal{S}$. Then for every two distinct points x and y on $\mathcal{K}$ there holds that $|\{x,y\}^{\perp\perp}| = q+1$, in contradiction with Theorem 6.5. □

As a nice corollary, we have the following upper bound for partial ovoids of the Hermitian quadrangle $\mathcal{H}(4,q^2)$.

Theorem 6.7 *If $\mathcal{K}$ is a partial ovoid of $\mathcal{H}(4,q^2)$, then we have that $|\mathcal{K}| \leq q^5 - q - 1$.*

Proof. By Theorem 3.2, $\mathcal{H}(4,q^2)$ has no ovoids. The theorem then follows from Theorem 6.6 and Theorem 6.1. □

Remark 6.8 In P. Govaerts, L. Storme and H. Van Maldeghem [12], an improvement of Theorem 6.7 is obtained; there it is shown that $|\mathcal{K}| \leq q^5 - \frac{4}{3}q + 2$ if $q > 2$.

The following theorem completely solves the problem under consideration for all GQs of order (s,s^2), $s > 1$.

Theorem 6.9 *Let $\mathcal{S}$ be a GQ of order (s,s^2), $s \neq 1$. Then $\mathcal{S}$ has no complete $(s^3 - s)$-arcs unless $s = 2$, i.e. $\mathcal{S} \cong \mathcal{Q}(5,2)$. In that case there is a unique example up to isomorphism.*

Proof. (1) Every hyperbolic line in a GQ of order (s,s^2) (with $s \neq 1$) has size 2, hence we can apply Theorem 6.5 to conclude that $\mathcal{S} \cong \mathcal{Q}(5,2)$.

(2) Fix a subGQ $\mathcal{S}'$ of order 2 of $\mathcal{S} \cong \mathcal{Q}(5,2)$ (which is isomorphic to the classical GQ $\mathcal{Q}(4,2)$), and let v be a point of $\mathcal{S}$ outside $\mathcal{S}'$. Then $v^\perp$ intersects $\mathcal{S}'$ in the points of an ovoid $\mathcal{O}$ of $\mathcal{S}'$, see Theorem 2.5, and it is well-known that this ovoid is subtended by v and exactly one other point of $\mathcal{S} \setminus \mathcal{S}'$, say v'. It is clear that $\mathcal{K} = \{v\} \cup ((v')^\perp \setminus (\{v'\} \cup \mathcal{O}))$ and $\mathcal{K}' = \{v'\} \cup (v^\perp \setminus (\{v\} \cup \mathcal{O}))$ are two disjoint 6-arcs in $\mathcal{S} \setminus \mathcal{S}'$ (remark that $\mathcal{K} \cup \mathcal{K}'$ is the point set of $\mathcal{S} \setminus \mathcal{S}'$).

As $\mathcal{S}$ has no ovoid (see, e.g., the appendix of this chapter or Theorem 3.2) the 6-arcs $\mathcal{K}$ and $\mathcal{K}'$ are complete, and moreover $\mathcal{S}(\mathcal{K}) = \mathcal{S}(\mathcal{K}') = \mathcal{S}'$. It is easy to show that for given $\mathcal{S}'$ and v in $\mathcal{S} \setminus \mathcal{S}'$, $\mathcal{K}$ is the unique 6-arc $\overline{\mathcal{K}}$ containing v for which $\mathcal{S}' = \mathcal{S}(\overline{\mathcal{K}})$. It is well-known that the stabilizer of $\mathcal{S}(\mathcal{K}) \cong \mathcal{Q}(4,2)$ in the automorphism group G of $\mathcal{S} \cong \mathcal{Q}(5,2)$ acts transitively on the points of $\mathcal{S} \setminus \mathcal{S}'$, and hence G acts transitively on the complete 6-arcs of $\mathcal{S}$ since the automorphism group of $\mathcal{S}$ acts transitively on the subGQs of order 2. $\square$

Remark 6.10 It is clear that the number of distinct complete 6-arcs in $\mathcal{Q}(5,2)$ equals two times the number of distinct $\mathcal{Q}(4,2)$'s in $\mathcal{Q}(5,2)$, and hence this number equals 72.

Theorem 6.11 *Suppose $\mathcal{S}$ is a GQ of order s, $s > 2$, with a regular point p. Then $\mathcal{S}$ contains no complete (s^2-1)-arcs.*

Proof. Suppose $\mathcal{K}$ is a complete (s^2-1)-arc of the GQ $\mathcal{S}$ of order s, $s > 2$. Then $\mathcal{S}(\mathcal{K})$ is a grid with parameters $s+1, s+1$. If the regular point p is a point of either $\mathcal{S}(\mathcal{K})$ or $\mathcal{K}$, then we can take x on $\mathcal{K}$ and y in $\mathcal{S}(\mathcal{K})$ ($p \in \{x,y\}$), and so the assertion clearly follows from the previous observation. Suppose $p \in \mathcal{S} \setminus (\mathcal{K} \cup \mathcal{S}(\mathcal{K}))$. Let $x \in p^\perp \cap \mathcal{K}$ and $y \in p^\perp \cap \mathcal{S}(\mathcal{K})$. Then $\{x,y\}$ is a regular pair of points, and $|\{x,y\}^{\perp\perp} \cap \mathcal{K}| = 1$ and $\{x,y\}^{\perp\perp} \cap \mathcal{S}(\mathcal{K}) \neq \emptyset$. The result now follows from the observation. $\square$

Theorem 6.12 *The dual of the GQ $T_2(\mathcal{O})$ of order q, $q > 2$, has no complete (q^2-1)-arcs. In particular, the classical GQ $W(q)$ has no complete (q^2-1)-arcs if $q \neq 2$. The $T_2(\mathcal{O})$ of Tits of order q has no complete (q^2-1)-arcs if q is even and $q > 2$.*

Proof. The dual of $T_2(\mathcal{O})$ always has a regular point as $T_2(\mathcal{O})$; see [15, 3.3.2]. The GQ $T_2(\mathcal{O})$ of Tits of order q, q even, always has a regular point; see [15, 3.3.2]. $\square$

6.3 Complete $(st - t/s)$-arcs in the known GQs of order (s,t), $s \neq 1 \neq t$

Suppose $\mathcal{S}$ is a known GQ of order (s,t), $s \neq 1 \neq t$. Then we have that $t \in \{s-2, s^{2/3}, \sqrt{s}, s, s+2, s^{3/2}, s^2\}$. If $\mathcal{S}$ has a complete $(st - t/s)$-arc, then necessarily s is a divisor of t, and hence $t \geq s$, i.e. $t \in \{s, s+2, s^{3/2}, s^2\}$. If $\mathcal{S}$ is of order $(s, s+2)$, only the case $s = 2$ is allowed, and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,2)$. The only known GQ of order $(s, s^{3/2})$, $s > 1$, is isomorphic to the classical GQ $\mathcal{H}(4,s)$ (where s is the square of some prime power). Now suppose $\mathcal{K}$ is a complete (s^2-1)-arc of a GQ $\mathcal{S}$ of order s, $s > 2$. Then $\mathcal{S}(\mathcal{K})$ is a grid with parameters $s+1, s+1$, and hence $\mathcal{S}$ contains a regular pair of lines. The only known GQs of order s which have a regular pair of lines are the $T_2(\mathcal{O})$ of Tits and the dual of $T_2(\mathcal{O})$ for s even, and by Theorem 6.12 we only have to consider the case where s is odd. In that case, by the theorem of B. Segre [18], the oval $\mathcal{O}$ is a conic, and hence $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$, q odd.

The following theorem is a direct corollary of the preceding considerations and provides a first version of a classification result for GQs of order (s,t) with complete $(st - t/s)$-arcs, $s \neq 1 \neq t$.

Theorem 6.13 ([34]) *Let $\mathcal{S}$ be a known GQ of order (s,t) with $s \neq 1 \neq t$, and suppose that $\mathcal{S}$ has a complete $(st-t/s)$-arc $\mathcal{K}$. Then we necessarily have one of the following possibilities.*

(1) *$\mathcal{S} \cong \mathcal{Q}(4,2)$ and up to isomorphism there is a unique example.*

(2) *$\mathcal{S} \cong \mathcal{Q}(5,2)$ and up to isomorphism $\mathcal{K}$ is unique.*

(3) *$\mathcal{S} \cong \mathcal{Q}(4,q)$ with q odd.*

In fact, there is an example of a complete 8-arc in $\mathcal{Q}(4,3)$. This example will be obtained in the next section, and will be revisited later on.

7 Complete Dual $(s-1)\times(s+1)$-Grids and Complete (s^2-1)-Arcs in GQs of Order s, $s > 1$

This section is taken from [33].

We start this section with an easy observation. Consider an $s\times(s+1)$-grid Γ in a GQ $\mathcal{S}$ of order (s,t), $s,t > 1$. Suppose $\{L_1, L_2, \ldots, L_s, M_1, M_2, \ldots, M_{s+1}\}$ is the line set of Γ, where $M_i \not\sim M_j$ if $i \neq j$, $1 \leq i,j \leq s+1$ and $L_k \not\sim L_r$ if $1 \leq k \neq r \leq s$. Suppose $M, N, O \in \{M_1, M_2, \ldots, M_{s+1}\}$ are arbitrary but distinct, and suppose $m \mathrm{I} M$ so that $m \not{\mathrm{I}}\, L_i$ for $i = 1, 2, \ldots, s$. Then one observes that $m, \mathrm{proj}_N m, \mathrm{proj}_O m$ are on the same line, since otherwise $m, \mathrm{proj}_N m, \mathrm{proj}_O m$ are the points of a triangle of $\mathcal{S}$ (as clearly $\mathrm{proj}_N m = \mathrm{proj}_N(\mathrm{proj}_O m)$), contradiction. Hence

Theorem 7.1 *If Γ is an $s\times(s+1)$-grid in a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, then Γ is contained in an $(s+1)\times(s+1)$-grid. Hence $\mathcal{S}$ contains a regular pair of lines, and $s \leq t$.* □

We call a $k \times (s+1)$-grid Γ of the GQ $\mathcal{S}$ *complete* if there is no $k' > k$ so that Γ is contained in a $k' \times (s+1)$-grid. Dually, we define *complete dual $k \times (t+1)$-grids*. Also, if Γ is an $(s-1) \times (s+1)$-grid, then by $\mathcal{R}$, respectively $\mathcal{R}'$, we denote the set of $s-1$, respectively $s+1$, mutually non-concurrent lines of Γ. Observe the following.

Theorem 7.2 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t > 1$, which has a complete $(s-1) \times (s+1)$-grid. Then $s \leq t$.*

Proof. Let X be the set of points which are on the lines of $\mathcal{R}'$, and let Y be the set of points which are incident with a line of $\mathcal{R}'$ but not with a line of $\mathcal{R}$. Let $z \in P \setminus X$. It is clear that z is incident with $s-1$ lines which are incident with a point of $X \setminus Y$, and that at most two lines incident with z intersect Y. Hence we have that $t \geq s$ or $t = s-1$. The latter cannot occur as it contradicts the standard divisibility condition for GQs. $\square$

Theorem 7.3 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t > 1$, and let Γ be a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. If $s = t$, then $\mathcal{S}$ contains a complete dual (s^2-1)-arc. Also, in that case $\mathcal{S}$ contains a spread.*

Proof. Suppose that X is the set of points which are on the lines of $\mathcal{R}'$, and let Y be the set of points which are incident with a line of $\mathcal{R}'$ but not with a line of $\mathcal{R}$. Consider $z \in P \setminus X$. By projecting z onto the lines of $\mathcal{R}$, z is incident with $s-1$ lines which are incident with a point of $X \setminus Y$, and that at most two lines incident with z intersect Y. As a direct corollary of this argument, we have that, as Γ is a complete $(s-1) \times (s+1)$-grid, each line of $B \setminus (\mathcal{R} \cup \mathcal{R}')$ contains at most two distinct points of X, respectively Y (hence each such line hits at most two distinct lines of $\mathcal{R}'$). Suppose L is a line of $B \setminus (\mathcal{R} \cup \mathcal{R}')$ which is incident with $y \in Y$. Put $\mathcal{R}' = \{L_0, L_1, \ldots, L_s\}$, and suppose that yIL_0. By projecting y onto $L_1, L_2, \ldots, L_s$, the fact that $s = t$ infers that $L \in \{\mathrm{proj}_y L_1, \mathrm{proj}_y L_2, \ldots, \mathrm{proj}_y L_s\}$. Hence each such line L hits *precisely* two distinct lines of $\mathcal{R}'$. A direct corollary is that each point of $P \setminus X$ is incident with precisely one line of $\mathcal{S}$ which does not contain a point of X. Hence, the set $\mathbf{T}'$ of all lines of $\mathcal{S}$ which have empty intersection with X is a partial spread of $\mathcal{S}$ of size $s^2 - s$. It is now clear that $\mathbf{T}' \cup \mathcal{R}$ is a complete dual (s^2-1)-arc, and that $\mathbf{T}' \cup \mathcal{R}'$ is a spread of $\mathcal{S}$. $\square$

Corollary 7.4 *The GQ $\mathcal{Q}(4,3)$ has complete 8-arcs.*

Proof. Suppose x and y are non-collinear points of $\mathcal{Q}(4,3)$. Then $\{x, y\}^{\perp} \cup \{x, y\}^{\perp\perp}$ is a complete dual 2×4-grid. Now apply the dual of Theorem 7.3. $\square$

Of course, the last corollary dualizes to $W(3)$.

Remark 7.5 Suppose that $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t > 1$, and let Γ be a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. Use the notation of the proof of Theorem 7.3. By the proof of that theorem, the geometry $\mathcal{S}' = (P', B', I')$, where $P' = Y$, B' is the set of lines of $\mathcal{S}$ which intersect Y in two points (note that $\mathcal{R}' \subseteq B'$), and where $I' = I \cap ((P' \times B') \cup (B' \times P'))$, is a dual grid with parameters $s+1, s+1$ ('non-collinearity' in $P' = Y$ is an equivalence relation).

Assume that $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t > 1$, and that Γ is a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. If L is a line of $\mathcal{S}$ which is not concurrent with a line of Γ, then we call L an *exterior line of* Γ. Note the following corollary of Theorem 7.3.

Corollary 7.6 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t > 1$, and let Γ be a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. Suppose X is as before. If each point of $P \setminus X$ is incident with a constant number of exterior lines of Γ, then $s = t$ and the conclusion of Theorem 7.3 holds.*

Proof. Each point of Y is on $t - s$ lines which intersect X in exactly one point. So if $t > s$ there are points of $P \setminus X$ on exactly $t - s$ exterior lines, and for any $t \geq s$ there are points of $P \setminus X$ on precisely $t - s + 1$ exterior lines. Hence if each point of $P \setminus X$ is incident with a constant number of exterior lines of Γ, then $s = t$. $\square$

Corollary 7.7 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t > 1$, and let Γ be a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. Let X be as before. If there is a group of automorphisms of $\mathcal{S}$ which fixes Γ and which acts transitively on $P \setminus X$, then $s = t$, $\mathcal{S}$ contains a complete dual $(s^2 - 1)$-arc and $\mathcal{S}$ contains a spread.*

Proof. Immediate. $\square$

Remark 7.8 Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t > 1$, which has a complete $(s-1) \times (s+1)$-grid Γ, and suppose that $s \neq t$. Assume that X is as before. In view of the proof of Theorem 7.3, it could be interesting to investigate the following point-line geometry Γ':

(a) POINTS are the points of $\mathcal{S} \setminus X$;

(b) LINES are those lines of $\mathcal{S}$ which are not incident with a point of X;

(c) INCIDENCE is inherited from $\mathcal{S}$.

For $s = t$, Γ' is just a partial spread of $\mathcal{S}$ with $s^2 - s$ elements.

For GQs of order (s, s^2), $s > 1$ and s odd, there is a complete solution to the problem:

Theorem 7.9 *A GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s > 1$ and s odd, with a complete $(s-1) \times (s+1)$-grid does not exist.*

Proof. Adopt the notation $P' = Y, B', I', \mathcal{R}, \mathcal{R}'$ from Remark 7.5. Then $\mathcal{S}' = (P', B', I')$ is a dual $(s+1) \times (s+1)$-grid. By Theorem 2.7, each line of $\mathcal{S}$ which is not incident with a point of Y must hit 0 or 2 lines of B' as s is odd, a contradiction since each line of $\mathcal{R}'$ is concurrent with each line of $\mathcal{R}$, and $\mathcal{R} \subseteq B'$. $\square$

For s even, the situation is slightly different. Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, s^2), $s > 1$ and s even, and assume that Γ is a complete $(s-1)\times(s+1)$-grid of $\mathcal{S}$. Suppose $\mathcal{R}, \mathcal{R}', X, Y, \mathcal{S}' = (P', B', I')$ are as before. As $\mathcal{S}'$ is a dual grid with parameters $s+1, s+1$ and s is even, the geometry $\mathcal{S}''$ which consists of the lines of $\mathcal{S}$ which hit at least two distinct lines of B', and the points of $\mathcal{S}$ on these lines (and the natural incidence), is a subGQ of $\mathcal{S}$ of order s, see Section 2.4. As $\mathcal{S}''$ clearly contains $\mathcal{R}$ and $\mathcal{R}'$, there follows that $\mathcal{S}''$ also contains the complete $(s-1) \times (s+1)$-grid Γ. Whence by Theorem 7.3, $\mathcal{S}''$ contains a complete dual (s^2-1)-arc.
Now suppose that $\mathcal{S}''$ is a known GQ. Then by Theorem 7.3, $s = 2$, $\mathcal{S}'' \cong W(2)$, and $\mathcal{S}'' \cong \mathcal{Q}(5, 2)$ by Section 2.5.
Note that $\mathcal{R}'$ is just a unicentric triad of lines of $\mathcal{S}'$, and if $\mathcal{R} = \{L\}$, then L is the unique center of $\mathcal{R}'$.

8 Partial Ovoids and Spreads in Affine Generalized Quadrangles

This section is based on a part of [33].

8.1 Affine generalized quadrangles

Motivated by the results on complete $(st - t/s)$-arcs in generalized quadrangles of order (s, t), $s \neq 1 \neq t$, we now will introduce the notion of *(partial) spreads* and *(partial) ovoids* in *affine generalized quadrangles.*

A *geometrical hyperplane* $\mathcal{H}$ of a generalized quadrangle $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, is a subgeometry of $\mathcal{S}$ in the usual sense (this means that $P' \subseteq P$, $B' \subseteq B$ and that I' is the induced incidence), so that each line of $\mathcal{S}$ intersects $\mathcal{H}$ in 1 point, or is completely contained in $\mathcal{H}$. Geometrical hyperplanes of generalized quadrangles are characterized by the following theorem, which is merely a direct corollary of Theorem 2.6.

Theorem 8.1 *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, and suppose $\mathcal{H}$ is a geometrical hyperplane of $\mathcal{S}$. Then $\mathcal{H}$ is given by one of the following.*

(1) *$\mathcal{H}$ is an ovoid of $\mathcal{S}$.*

(2) *$\mathcal{H}$ is a point, together with all the lines incident with that point and all the points on those lines.*

(3) *$\mathcal{H}$ is a (not necessarily thick) subGQ of $\mathcal{S}$ of order $(s, t/s)$ (and hence s divides t).*

Proof. Use Theorem 2.6 and easy counting. □

An *affine generalized quadrangle (AGQ)* is the complement of a geometrical hyperplane $\mathcal{H}$ of a thick generalized quadrangle $\mathcal{S}$, that is, the natural geometry formed by the points and lines of $\mathcal{S}$ which are not in $\mathcal{H}$. In the following, we will speak of *AGQs of Type (1),(2),(3)*, according to whether the corresponding geometrical hyperplane is given by respectively (1), (2) or (3) of Theorem 8.1. H. Pralle [16] has derived a set of axioms (AGQ1)–(AGQ7) so that each point-line geometry satisfying these axioms arises as the complement of a geometrical hyperplane in some thick generalized quadrangle. Such a point-line geometry will be called an *abstract affine generalized quadrangle*. ¿From the work of H. Pralle follows that an abstract affine generalized quadrangle Γ corresponds — up to isomorphism — to a unique generalized quadrangle from which it arises. We call this generalized quadrangle the *generalized quadrangle spanned by* Γ, denoted $\mathcal{S} = \mathcal{S}(\Gamma)$. If Γ is an AGQ, then we say that Γ is of *order* (s,t) if the corresponding generalized quadrangle is of order $(s+1,t)$. As in the case of GQs, we will sometimes speak of an 'affine quadrangle' instead of 'affine generalized quadrangle'.

8.2 Partial ovoids and partial spreads in affine generalized quadrangles

A *partial ovoid* of an AGQ is a set of two by two non-collinear points. An *ovoid* of an AGQ is a partial ovoid such that any line has non-empty intersection with it. Dually, one defines *partial spreads* and *spreads* of AGQs.

Theorem 8.2 *Suppose Γ is an affine generalized quadrangle of order $(s-1,t)$, $s,t > 1$. An ovoid of Γ has $st+1$, st or $st - t/s$ points according as Γ is respectively of Type (1), (2) or (3). A spread also contains $st+1$, st or $st - t/s$ lines according as Γ is respectively of Type (1), (2) or (3).*

Proof. In each of the cases, we count the number of flags $(\mathcal{X},\mathcal{Y})$, where $\mathcal{X}$ is an element of the ovoid or spread, and $\mathcal{Y}$ is an element of Γ for which $\mathcal{X} I \mathcal{Y}$ (where 'I' denotes incidence in Γ), in two ways. By $\mathbf{S}$, we denote the ovoid or spread of Γ; by k, we denote $|\mathbf{S}|$. We will look at the three distinct cases as defined by Theorem 8.1.

CASE 1: AGQS OF TYPE (1). If $\mathbf{S}$ is an ovoid of Γ, then $k(t+1) = (t+1)(st+1)$, and if $\mathbf{S}$ is a spread, we have that $ks = s(st+1)$.

CASE 2: AGQS OF TYPE (2). If $\mathbf{S}$ is an ovoid of Γ, then $k(t+1) = (t+1)st$, and if $\mathbf{S}$ is a spread, then $ks = (s+1)(st+1) - (st+s+1)$.

CASE 3: AGQS OF TYPE (3). If $\mathbf{S}$ is an ovoid of Γ, then $k(t+1) = (t+1)(st+1) - (t/s+1)(t+1)$, and if $\mathbf{S}$ is a spread, then $ks = (s+1)(st+1) - (s+1)(t+1)$. □

8.3 Existence of spreads and ovoids of AGQs

Theorem 8.3 *An AGQ of Type (2) or (3) always has spreads.*

Proof. Suppose Γ is an affine generalized quadrangle of Type (2) or (3) of order $(s-1,t)$, $s,t > 1$, and suppose $\mathcal{H}$ is the corresponding geometrical hyperplane of $\mathcal{S}(\Gamma)$. Consider a line L of $\mathcal{H}$ (note that as Γ is not of Type (1), $\mathcal{H}$ has lines), and suppose $\mathbf{T}$ is the set of all the lines of Γ which intersect L in $\mathcal{S}(\Gamma)$. Then clearly, if Γ is of Type (2), we have that $\mathbf{T}$ contains st elements, and if Γ is of Type (3), then $\mathbf{T}$ has $st - t/s$ lines. Hence the result. □

Remark 8.4 One could make the spread problem for AGQs of Type (2) and (3) (of order $(s-1,t)$, $s,t > 1$) more interesting by demanding special properties for such a spread. For instance, one could ask the spread to induce a partial spread in $\mathcal{S}(\Gamma)$. In that case, if the AGQ is of Type (2), this would imply that $\mathcal{S}(\Gamma)$ also has spreads (cf. the dual of Theorem 6.1), and if the AGQ is of Type (3), $\mathcal{S}(\Gamma)$ would have a partial spread of large size, namely of size $st - t/s$.

Theorem 8.5 *Suppose Γ is an AGQ of Type (1), with spanned GQ $\mathcal{S}(\Gamma)$, and where Γ arises from the ovoid $\mathbf{O}$ of $\mathcal{S}(\Gamma)$. Then Γ has a spread $\mathbf{T}$ if and only if it induces a spread of $\mathcal{S}(\Gamma)$.*
So, the study of spreads of AGQs of Type (1) is equivalent to the study of GQs which have both spreads and ovoids.

Proof. Suppose Γ is of order $(s-1,t)$, $s,t > 1$. We identify lines of Γ with the corresponding lines of $\mathcal{S}(\Gamma)$. Suppose p is a point of $\mathbf{O}$, and suppose p is incident with r lines of $\mathbf{T}$ in $\mathcal{S}(\Gamma)$. Note that in $\mathcal{S}(\Gamma)$, every line of $\mathbf{T}$ intersects $\mathbf{O}$ in precisely one point, and distinct lines of $\mathbf{T}$ can only intersect on $\mathbf{O}$. If $L_0, L_1, \ldots, L_t$ are the lines through p in $\mathcal{S}(\Gamma)$ and if $L_0, L_1, \ldots, L_{r-1}$ are the lines of $\mathbf{T}$ through p in $\mathcal{S}(\Gamma)$, then every line of $\mathbf{T}$ which is not incident with p in $\mathcal{S}(\Gamma)$ intersects a line of $L_r, L_{r+1}, \ldots, L_t$. We obtain that:

$$st + 1 - r \leq (t + 1 - r)s,$$

hence $r = 1$. The result follows. $\square$

Remark 8.6 (i) Not a lot is known about GQs having both spreads and ovoids. If such a GQ $\mathcal{S}$ has order (s, t), $s, t > 1$, then by E. E. Shult [19] and J. A. Thas [23], see Theorem 3.1, we have that $t \leq s^2 - s$ and $s \leq t^2 - t$, and hence the GQs of order (k^i, k^j), with $k \neq 1$ and $(i, j) \in \{(1, 2), (2, 1)\}$, are trivially excluded. Hence, of the known orders of GQs, we have that, up to duality, $(s, t) \in \{(s, s), (s, s+2), (s, \sqrt{s^3})\}$. If $\mathcal{S}$ is of order $(s, \sqrt{s^3})$, $s > 1$, then the only known GQ is the classical GQ $\mathcal{H}(4, s)$ and this generalized quadrangle does not allow ovoids, see Theorem 3.2. Of the other classical GQs, the only one with the desired property is $\mathcal{Q}(4, s)$ with s even, see Theorem 3.2. The $T_2(\mathcal{O})$ of Tits always has ovoids, see Theorem 3.2, and several of them also have spreads [6]. The GQ $\mathcal{P}(\mathcal{S}, x)$ has spreads and ovoids if and only if $\mathcal{S}$ has an ovoid containing x, see Theorem 3.2.

(ii) A geometrical hyperplane $\mathcal{H}$ of a generalized quadrangle has the property that if any two points are collinear in the GQ, they are also collinear in $\mathcal{H}$, and hence the corresponding affine quadrangle will also have this property (actually, that is exactly why the existence problem of ovoids in AGQs is much harder to investigate than the existence problem of spreads: the line set of an AGQ does not have this property). Hence, if $\mathbf{O}$ is an ovoid of an AGQ Γ, then it will be a (large) partial ovoid of $\mathcal{S}(\Gamma)$.

Suppose that Γ is an AGQ, that $\mathcal{S}(\Gamma)$ is the GQ spanned by Γ and that Γ is of order $(s-1, t)$, $s, t > 1$. Also, $\mathcal{H}$ denotes the geometrical hyperplane of $\mathcal{S}(\Gamma)$ corresponding to Γ. Suppose that Γ has an ovoid $\mathbf{O}$. If Γ is of Type (1), then as $\mathcal{H}$ is an ovoid of $\mathcal{S}(\Gamma)$, it follows that $t \leq s^2 - s$. It is clear that $\mathcal{H}$ and $\mathbf{O}$ are disjoint ovoids of $\mathcal{S}(\Gamma)$, and $O = \mathcal{H} \cup \mathbf{O}$ has the property that any line of $\mathcal{S}(\Gamma)$ meets O in exactly two points. In the sense of J. A. Thas [28], this means that O is a 2-*ovoid*.

Remark 8.7 (i) It is not difficult to show that, conversely, if θ is an automorphism of the AGQ Γ, then θ induces an automorphism of $\mathcal{S}(\Gamma)$ which fixes $\mathcal{H}$. Hence $Aut(\Gamma) \cong Aut(\mathcal{S})_{\mathcal{H}}$.
(ii) Note that, if Γ is the AGQ of Type (3) which arises from $\mathcal{Q}(4, 3)$, then Γ also contains ovoids, see Corollary 7.4.

9 Partial Ovoids of $\mathcal{Q}(4,q)$ of Size q^2-1 for $q \in \{2,3,5,7,11\}$, and Beyond

9.1 Maximal partial ovoids of $\mathcal{Q}(4,q)$ of size q^2-1 for $q \in \{2,3,5,7,11\}$

Partial ovoids of $\mathcal{Q}(4,q)$ of size q^2-1 are only known for $q \in \{2,3,5,7,11\}$. For $q=2$ there is, up to isomorphism, a unique 3-arc of $\mathcal{Q}(4,2)$. This is an immediate consequence of the results of Section 3, Theorem 4.3, see also [34] (note that for $q=2$, $q^2-1=q+1$). For $q=3$ an 8-arc can easily be constructed as follows. Consider any elliptic quadric $\mathcal{Q}^-(3,3) \subset \mathcal{Q}(4,3)$ and any point $p \notin \mathcal{Q}^-(3,3)$. Then p is collinear with exactly 4 points of $\mathcal{Q}^-(3,3)$, and these points determine a conic C. The set $C^\perp$ contains two points (among which p). It is easily seen that $(\mathcal{Q}^-(3,3) \setminus C) \cup C^\perp$ is a complete 8-arc of $\mathcal{Q}(4,3)$. It has longtime been thought that these were the only values of q for which (q^2-1)-arcs of $\mathcal{Q}(4,q)$ existed. However in 2003 Penttila disproved this by showing with the computer the existence of (q^2-1)-arcs of $\mathcal{Q}(4,q)$ for $q \in \{5,7,11\}$ [17]. In [7] Cimrakova and Fack confirmed these results, also with the use of the computer. Their searches were heuristic and so by no means exclude the existence of (q^2-1)-arcs of $\mathcal{Q}(4,q)$ for other values of odd q. It is important to note that no computer free constructions are known of (q^2-1)-arcs for $q \in \{7,11\}$. It is however possible to describe the above construction of an 8-arc of $\mathcal{Q}(4,3)$ in a slightly different way and then generalize this construction for $q=5$. We first provide the "alternative" construction of the 8-arc (which is essentially the same as in Section 7).

Consider a fixed $\mathcal{Q}(3,3) \subset \mathcal{Q}(4,3)$ and consider any point $p \in \mathcal{Q}(4,3) \setminus \mathcal{Q}(3,3)$. Then the points collinear with p in $\mathcal{Q}(3,3)$ form a conic C. Clearly $C^\perp$ consists of two points, say p and p'. Now let $\mathcal{K}'$ be the set of all points of $\mathcal{Q}(4,3) \setminus \mathcal{Q}(3,3)$ not contained in one of the cones pC and $p'C$. It is easily seen that $\mathcal{K} := \mathcal{K}' \cup \{p,p'\}$ is an 8-arc of $\mathcal{Q}(4,3)$ (isomorphic to the higher described 8-arc). We now generalize this construction for $q=5$. A conic C of $\mathcal{Q}(3,q) \subset \mathcal{Q}(4,q)$ will be called *doubly subtended* if $\left|C^\perp\right| = 2$. Again consider a fixed $\mathcal{Q}(3,5) \subset \mathcal{Q}(4,5)$. Let C_i, $i=1,2,3,4$, be a collection of four doubly subtended conics of $\mathcal{Q}(3,5)$, with the property that $|C_i \cap C_j| = 2$ if $i \neq j$. Let $\{p_i,p_i'\} := C_i^\perp$, $i=1,2,3,4$.

Theorem 9.1 *With the above notation* $\mathcal{K} := \mathcal{Q}(4,5) \setminus (\mathcal{Q}(3,5) \cup_i (p_iC_i \cup p_i'C_i)) \cup_i \{p_i,p_i'\}$ *is a 24-arc of* $\mathcal{Q}(4,5)$.

Proof. We first show that $|\mathcal{K}| = 24$. Consider a cone $p_i^{(\prime)}C_i$ (here $p_i^{(\prime)}$ means p_i or p_i'). Then there are exactly 6 cones p_jC_j, $p_j'C_j$ intersecting $p_i^{(\prime)}C_i \setminus$

$(\{p_i^{(\prime)}\} \cup C_i)$ in exactly 4 points. As $\left|p_i^{(\prime)}C_i \setminus (\{p_i^{(\prime)}\} \cup C_i)\right| = 24$ this implies that every point of $p_i^{(\prime)}C_i \setminus (\{p_i^{(\prime)}\} \cup C_i)$ is contained in an average of 2 of the 8 cones p_jC_j, p'_jC_j, $j = 1,2,3,4$. Assume that there would exist such a point x, contained in at least 3 of the considered cones. First notice that the four conics C_1, C_2, C_3, C_4 cover exactly 12 points of $\mathcal{Q}(3,5)$, and that each of these points is contained in precisely 2 conics C_i. Further, because of our assumption, the point x is collinear with at least 3 distinct points of $\cup_i C_i$. This implies the existence of a conic C_k, $k \in \{1,2,3,4\}$, such that x is collinear with 2 points of C_k, while x is contained in either p_kC_k or p'_kC_k, a contradiction. Consequently every point of $p_i^{(\prime)}C_i \setminus (\{p_i^{(\prime)}\} \cup C_i)$, $i = 1,2,3,4$, is contained in exactly 2 of the 8 considered cones. From this we now easily deduce that $\left|\cup_i(p_i^{\perp} \cup p_i'^{\perp}) \setminus \mathcal{Q}(3,5)\right| = 104$. It follows that $|\mathcal{K}| = 24$. We now proceed by proving that the set $\mathcal{K}$ is a partial ovoid. We already know from the construction that the 8 points p_i, p'_i, $i = 1,2,3,4$, are two by two non-collinear, while for every other point $p \in \mathcal{K}$ we have that $p \not\sim p_i$ and $p \not\sim p'_i$, $i = 1,2,3,4$. Assume that $p \sim p'$ with $p, p' \in \mathcal{K}$ and $p \neq p'$. Then the line pp' intersects $\mathcal{Q}(3,5)$ in a point $z \notin \cup_i C_i$. Consequently every p_i and p'_i is collinear with one of the points of $pp' \setminus \{p, p', z\}$. As $|pp' \setminus \{p,p',z\}| = 3$ we find a point of $pp' \setminus \{p,p',z\}$ that is contained in at least 3 of the cones p_iC_i, p'_iC_i, a contradiction in view of the first part of the proof. Finally it is easily seen that $\mathcal{K}$ is maximal. □

The previous theorem implies that in order to construct a 24-arc in $\mathcal{Q}(4,5)$ it is sufficient to find 4 doubly subtended conics in some $\mathcal{Q}(3,5) \subset \mathcal{Q}(4,5)$ which intersect two by two in exactly 2 points. In order to do so consider $\mathcal{Q}(4,5)$ in $\mathbf{PG}(4,5)$ determined by $X_0^2 + X_1X_2 + X_3X_4 = 0$ and the $\mathcal{Q}(3,5) \subset \mathcal{Q}(4,5)$ in the hyperplane $X_0 = 0$. Then the conics in the hyperplane $X_0 = 0$ with equations

$$C_1 : \begin{cases} X_1 = X_2 \\ X_1^2 + X_3X_4 = 0 \end{cases} \qquad C_2 : \begin{cases} X_1 = -X_2 \\ -X_1^2 + X_3X_4 = 0 \end{cases}$$

$$C_3 : \begin{cases} X_3 = X_4 \\ X_3^2 + X_1X_2 = 0 \end{cases} \qquad C_4 : \begin{cases} X_3 = -X_4 \\ -X_3^2 + X_1X_2 = 0 \end{cases}$$

are quickly seen to be doubly subtended and satisfy the desired property ($|C_i \cap C_j| = 2$ if $i \neq j$). Consequently we have constructed a 24-arc in $\mathcal{Q}(4,5)$.

One might wonder whether it is possible to generalize this construction for other values of q, and at least in theory this seems to be possible for $q \in \{7, 11\}$. Suppose that for $q = 7$ we find a set of 12 doubly subtended conics $C_1, \ldots, C_{12}$ in $\mathcal{Q}(3,7) \subset \mathcal{Q}(4,7)$ such that $|C_i \cap C_j| \in \{0,2\}$, $C_1, \ldots, C_{12}$ cover exactly 32 points of $\mathcal{Q}(3,7)$ and such that each of these 32 points is contained in exactly 3 conics C_i. Then a construction analogous to the one for $q = 5$ would yield a 48-arc of $\mathcal{Q}(4,7)$. However it is not clear whether such a set of conics exists. Finally, also for $q = 11$ one could obtain a generalization (60 conics covering completely $\mathcal{Q}(3,11) \subset \mathcal{Q}(4,11)$), but here as well the existence of such a set is not known. It may be interesting to study the link between such sets of conics and (partial) flocks of $\mathcal{Q}(3,q)$ (cf. [30]).

Tim Penttila has noted to us in a private communication [17] that the examples $\mathcal{K}$ of complete (q^2-1)-arcs of $\mathcal{Q}(4,q)$ which were constructed by him for $q = 5, 7, 11$ all satisfy the following property:

- $(q^2-1)^2$ *divides the size of* $Aut(\mathcal{Q}(4,q))_{\mathcal{K}}$. (*)

We end our paper by showing that, conversely, if such an arc satisfies (*), we necessarily have $q \in \{5, 7, 11\}$. We do not consider the case $q = 3$, as then all maximal 8-arcs are known.

9.2 Classification of complete (q^2-1)-arcs of $\mathcal{Q}(4,q)$, q odd, satisfying Property (*)

Recall Dickson's classification of the subgroups of $\mathbf{PSL}(2,q)$, with $q = p^h$, p a prime (see [9, Hauptsatz 8.27, p. 213]); we list the possible subgroups $H \leq \mathbf{PSL}(2,q)$, as follows:

(i) H is an elementary abelian p-group;

(ii) H is a cyclic group of order k, where k divides $\frac{q\pm1}{r}$, where $r = gcd(q-1, 2)$;

(iii) H is a dihedral group of order $2k$, where k is as in (ii);

(iv) H is the alternating group A_4, where $p > 2$ or $p = 2$ and $h \equiv 0 \mod 2$;

(v) H is the symmetric group S_4, where $p^{2h} - 1 \equiv 0 \mod 16$;

(vi) H is the alternating group A_5, where $p = 5$ or $p^{2h} - 1 \equiv 0 \mod 5$;

(vii) H is a semidirect product of an elementary abelian group of order p^m with a cyclic group of order k, where k divides $p^m - 1$ and $p^h - 1$;

(viii) H is a $\mathbf{PSL}(2,p^m)$, where m divides h, or a $\mathbf{PGL}(2,p^n)$, where $2n$ divides h.

Let $\mathcal{K}$ be a complete (q^2-1)-arc of $\mathcal{Q}(4,q)$, q odd and $q > 3$. Suppose the size of $G = Aut(\mathcal{Q}(4,q))_{\mathcal{K}}$ is divisible by $(q^2-1)^2$. Note that G fixes the grid $\mathcal{S}(\mathcal{K})$ which consists of the lines skew from $\mathcal{K}$ and the points incident with these lines. Put

$$|G| = (q^2-1)^2 r,$$

where r is natural. We remark that in $\mathbf{PGL}(5,q)_{\mathcal{Q}(4,q)}$, the stabilizer of $\mathcal{S}(\mathcal{K})$ has size $2(q^3-q)^2$, and inside $\mathbf{PSL}(5,q)_{\mathcal{Q}(4,q)}$, this stabilizer *restricted to its action on* $\mathcal{S}(\mathcal{K})$ (so after moding out the kernel) has size $\frac{(q^3-q)^2}{4}$; it is isomorphic to the direct product

$$\mathbf{PSL}(2,q) \times \mathbf{PSL}(2,q).$$

There is a unique involution fixing $\mathcal{S}(\mathcal{K})$ pointwise, and we denote the group it generates by N. Suppose H is the subgroup of GN/N inside $\mathbf{PSL}(5,q)$; then

$$|H| = (q^2-1)^2 r',$$

where $r' \geq \frac{r}{8h}$. Then H can be written as

$$H = H_1 \times H_2 \leq \mathbf{PSL}(2,q) \times \mathbf{PSL}(2,q),$$

where H_1 is the linewise stabilizer of one regulus ($\mathcal{L}_1$) of $\mathcal{S}(\mathcal{K})$ in H, and H_2 the linewise stabilizer of the other regulus ($\mathcal{L}_2$).

Assume that p (the odd prime divisor of q) divides $|H|$; then the subgroup of $\mathbf{PSL}(5,q)_{\mathcal{Q}(4,q)}$ that induced H on $\mathcal{S}(\mathcal{K})$ has a p-element θ, and this necessarily is a symmetry about some line M of $\mathcal{S}(\mathcal{K})$ by work of W. M. Kantor and K. Thas (see Chapter 7 of [35]). Suppose $U \sim M$ is a line not in $\mathcal{S}(\mathcal{K})$; then $U \cap \mathcal{K}$ is a point u. As θ fixes U, $u^\theta \neq u$ is a point collinear with u while being in $\mathcal{K}$, a contradiction. So there are no p-elements in H, and this means Cases (i), (vii) and (viii) of the subgroup list of $\mathbf{PSL}(2,q)$ are ruled out.

For now, we want to suppose not to be in the Cases (iv), (v), (vi), so (ii) and (iii) remain to be handled. Suppose we are in one of these cases. Clearly there is an $i \in \{1,2\}$ such that, if N_i is the elementwise stabilizer of $\mathcal{L}_i$ in H, we have

$$|H/N_i| \geq \frac{(q^2-1)\sqrt{r}}{2\sqrt{2h}},$$

while

$$2(q+1) \geq |H/N_i|.$$

So $p = 5$ and $h = 1$. Now we look at the Cases (iv), (v), (vi). In the same way as for the Cases (ii) and (iii), we obtain

$$\frac{(q^2-1)\sqrt{r}}{2\sqrt{2h}} \leq |H/N_i|,$$

where $H/N_i \in \{A_4, A_5, S_4\}$, so $|H/N_i| \in \{12, 24, 60\}$. When $H/N_i = A_4$, we easily obtain $h = 1$ and $p = 5$. When $H/N_i = S_4$, we obtain $h = 1$ and $p \in \{5, 7\}$. When $H/N_i = A_5$, we obtain $p \in \{5, 7, 11, 13\}$ and $h = 1$. Now suppose $p = 13$. Then

$$60 \text{ divides } (13^2 - 1)^2 r' = 168^2 r',$$

so r' is a multiple of 5.[1] This provides us with the desired contradiction, and ends the proof of the result.

Remark 9.2 (Application to Affine Quadrangles) Let Γ be an AGQ of order $(s-1, t)$, with $s-1, t \geq 2$ of Type (3). Suppose Γ has an ovoid $\mathbf{O}$. Then clearly $\mathbf{O}$ yields a maximal $(st - t/s)$-arc of the GQ $\mathcal{S}$ from which Γ arises by taking away the subGQ $\mathcal{S}(\Gamma)$ of order $(s, t/s)$. If $\mathcal{S}$ is $\mathcal{Q}(4, q)$, q odd, and Γ is an $(s+1) \times (s+1)$-grid of Γ, then we obtain a maximal $(q^2 - 1)$-arc of $\mathcal{Q}(4, q)$. As the automorphisms of Γ are induced by automorphisms of $\mathcal{S}$, the observations of this section can now be applied to this situation.

Final Note. The authors are presently working on the classification of transitive complete $(q^2 - 1)$-arcs of $\mathcal{Q}(4, q)$, q odd.

[1] One could also note that $13^2 - 1 = 168$ is not a multiple of 5, an observation which contradicts the description of Case (vi).

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