Blocking all generators of $\mathrm{Q}^+(2n+1,3)$, $n \geqslant 4$

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Abstract

We determine the smallest minimal blocking sets of $\mathrm{Q}^+(2n+1,3)$, $n \geqslant 4$.

1 Introduction

Let $\mathrm{Q}^+(2n+1,q)$ denote the non-singular hyperbolic quadric of the finite projective space $\mathrm{PG}(2n+1,q)$. This quadric is an example of a classical polar space. The subspaces of $\mathrm{PG}(2n+1,q)$ of maximal dimension completely contained in $\mathrm{Q}^+(2n+1,q)$ are called *generators*; they have dimension n. An *ovoid* of $\mathrm{Q}^+(2n+1,q)$ is a set $\mathcal{O}$ of points of $\mathrm{Q}^+(2n+1,q)$ such that every generator meets $\mathcal{O}$ in exactly one point; it contains q^n+1 points. Any hyperplane of $\mathrm{PG}(2n+1,q)$ which is not a tangent hyperplane intersects $\mathrm{Q}^+(2n+1,q)$ in a non-singular parabolic quadric $\mathrm{Q}(2n,q)$. Also this quadric is an example of a classical polar space, its generators have dimension $n-1$, and an ovoid contains q^n+1 points.

It is known that $\mathrm{Q}^+(7,q)$ has ovoids when q is prime or $q \equiv 0$ or $2 \bmod 3$, we refer to [13] for a list of references. Furthermore, from [2], $\mathrm{Q}^+(2n+1,q)$, with $q=p^h$, $p>2$ prime, has no ovoids if

$$p^n > \binom{2n+p}{2n+1} - \binom{2n+p-2}{2n+1}.$$

By this formula, no ovoid of $\mathrm{Q}^+(7,q)$ is excluded, while ovoids of $\mathrm{Q}^+(9,q)$, for $q=3$, are excluded. Furthermore, since any ovoid $\mathcal{O}$ of $\mathrm{Q}^+(2n+1,q)$ induces an ovoid of $\mathrm{Q}^+(2n-1,q)$ via the projection of $\mathcal{O} \cap p^\perp$ onto the base $\mathrm{Q}^+(2n-1,q)$ of the cone $p^\perp \cap \mathrm{Q}^+(2n+1,q)$ from any point p with $p \in \mathrm{Q}^+(2n+1,q) \setminus \mathcal{O}$, $\mathrm{Q}^+(2n+1,q)$, $q=3$, has no ovoids for $n \geqslant 4$. We define a *blocking set* of $\mathrm{Q}^+(2n+1,q)$ as a set $\mathcal{K}$ of points of $\mathrm{Q}^+(2n+1,q)$ such that every generator of $\mathrm{Q}^+(2n+1,q)$ meets $\mathcal{K}$ in at least one point. A blocking set $\mathcal{K}$ is called minimal if $\mathcal{K} \setminus \{p\}$ is not a blocking set for any point $p \in \mathcal{K}$, or, equivalently, if for any point $p \in \mathcal{K}$, there exists a generator of $\mathrm{Q}^+(2n+1,q)$ meeting $\mathcal{K}$ only in the point p.

In this paper, we determine the smallest minimal blocking sets of $\mathrm{Q}^+(2n+1,3)$. In order to state the result, we need the notation of a truncated cone. Suppose that α is a subspace of $\mathrm{PG}(2n+1,q)$ and $\mathcal{O}$ an arbitrary geometric object lying in some subspace π such that $\alpha \cap \pi = \emptyset$. The *cone* $\alpha\mathcal{O}$ with vertex

*This author thanks the Fund for Scientific Research - Flanders (Belgium) for a research grant.

α and base $\mathcal{O}$ is the union of the spaces $\langle \alpha, p \rangle$, $p \in \mathcal{O}$. The *truncated cone* $\alpha^*\mathcal{O}$ is obtained by removing the points of the vertex α of the cone $\alpha\mathcal{O}$. When α is the empty subspace, $\alpha^*\mathcal{O}$ is by definition the set $\mathcal{O}$. We will prove the following theorem.

Theorem 1 *Suppose that $\mathcal{K}$ is a set of points of $\mathrm{Q}^+(2n+1,3)$, $n \geqslant 4$, $|\mathcal{K}| = 3^n + 1 + r$, $1 \leqslant r < 3^{n-3}$, and such that every generator of $\mathrm{Q}^+(2n+1,3)$ meets $\mathcal{K}$ in at least one point. Then $|\mathcal{K}| = 3^n + 3^{n-3}$ and $\mathcal{K}$ is a truncated cone $\pi^*_{n-4}\mathcal{O}$, π_{n-4} an $(n-4)$-dimensional subspace, $\pi_{n-4} \subset \mathrm{Q}^+(2n+1,3)$ and $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,3) \subset \mathrm{Q}^+(7,3)$, $\mathrm{Q}^+(7,3)$ the base of the cone $\pi^\perp_{n-4} \cap \mathrm{Q}^+(2n+1,3)$.*

In Section 3 we will prove this theorem for $n = 4$. In Section 4 we will use inductive arguments to generalize the result for $n > 4$. Since Theorem 1 describes an example $\pi^*_{n-4}\mathcal{O}$ of a minimal blocking set of size $3^n + 3^{n-3}$, we may assume that the smallest minimal blocking sets of $\mathrm{Q}^+(2n+1,3)$ have size smaller than or equal to $3^n + 3^{n-3}$. We will also use the following result.

Theorem 2 [9] *If $\mathcal{O}$ is an ovoid of $\mathrm{Q}^+(7,3)$ then $\langle \mathcal{O} \rangle$ is a hyperplane α of $\mathrm{PG}(7,3)$ and $\mathcal{O}$ constitutes also an ovoid of $\alpha \cap \mathrm{Q}^+(7,3) = \mathrm{Q}(6,3)$.*

Using the classification of ovoids of $\mathrm{Q}(4,p)$, p prime [1], similar results for small minimal blocking sets of $\mathrm{Q}(2n,p)$, $p > 3$ prime, $n \geq 3$, were obtained in [4], while results on small minimal blocking sets of $\mathrm{Q}(2n,3)$ were obtained in [6, 7]. General results on small minimal blocking sets of $\mathrm{Q}(6,q)$, $q \geqslant 32$, q even, were obtained in [5]. More general results on small minimal blocking sets of $\mathrm{Q}^-(2n+1,q), n \geqslant 2$, and $\mathrm{W}(2n+1,q), n \geqslant 2$, were obtained in [10, 11].

In Section 2 we will recall basic geometrical properties of ovoids of $\mathrm{Q}(6,q)$ that will be used in Section 3. Finally we define two notations. An i-dimensional subspace of $\mathrm{PG}(2n+1,q)$ will often be denoted by π_i, and we define $\theta_n := \frac{q^{n+1}-1}{q-1}$, i.e. the number of points in an n-dimensional projective space of order q.

2 On ovoids of $\mathrm{Q}(6,q)$

We mention that ovoids of $\mathrm{Q}(6,q)$ are rare. Presently, ovoids of $\mathrm{Q}(6,q)$ are only known when $q \equiv 0 \bmod 3$. Furthermore, all ovoids of $\mathrm{Q}(4,p)$, p prime, are elliptic quadrics $\mathrm{Q}^-(3,p)$ [1], which is a sufficient condition for the non-existence of ovoids of $\mathrm{Q}(6,p)$, $p > 3$ prime [12]. Finally we mention that $\mathrm{Q}(6,3)$ has, up to collineations, a unique ovoid [9].

For our purposes, the following theorem is an important result on ovoids. Let $\mathrm{Q}(6,q)$ denote the non-singular parabolic quadric of $\mathrm{PG}(6,q)$, $q = p^h$, p prime. Call a hyperplane α of $\mathrm{PG}(6,q)$ *elliptic, hyperbolic* or *tangent* respectively if $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(5,q)$, $\mathrm{Q}^+(5,q)$, or tangent to $\mathrm{Q}(6,q)$.

Theorem 3 [1] *Suppose that $\mathcal{O}$ is an ovoid of $\mathrm{Q}(6,q)$, $q = p^h$, p prime. Any elliptic hyperplane α intersects $\mathcal{O}$ in* 1 mod p *points.*

The property of Theorem 3 can easily be derived for hyperbolic and tangent hyperplanes.

We also mention the following theorem.

Theorem 4 [7] *Suppose that $\mathcal{O}$ is an ovoid of* $\mathrm{Q}(6,q)$*. Any hyperbolic hyperplane α has the property $\langle \alpha \cap \mathcal{O}\rangle = \alpha$.*

When $q = 3$, elliptic hyperplanes have the same property, which was checked using the computer [7].

Theorem 5 *Suppose that $\mathcal{O}$ is an ovoid of* $\mathrm{Q}(6,3)$*. Any elliptic hyperplane α has the property $\langle \alpha \cap \mathcal{O}\rangle = \alpha$.*

Consider now a singular quadric $p\mathrm{Q}(6,q)$ in $\mathrm{PG}(7,q)$, and suppose that $\mathcal{O}$ is an ovoid of the base $\mathrm{Q}(6,q)$. Consider the cone $p\mathcal{O}$. Any hyperplane of $\mathrm{PG}(7,q)$ not on p intersects the cone $p\mathcal{O}$ in an ovoid of $\mathrm{Q}(6,q)$. We will use the following lemma in Section 3.

Lemma 1 *Any two ovoids $\mathcal{O}_1$ and $\mathcal{O}_2$, which lie in the intersection of the hyperplanes π_1, π_2 respectively, (with $\pi_1 \neq \pi_2$, and $p \notin \pi_i$, $i = 1,2$) with the cone $p\mathcal{O}$, have at least one point in common.*

Proof. Consider $\mathcal{O}_1 = \pi_1 \cap p\mathcal{O}$. The hyperplane π_2 intersects π_1 in a 5-dimensional space not on p. Denote $\pi_1 \cap p\mathrm{Q}(6,q) = \mathrm{Q}_1(6,q)$, then $\pi_1 \cap \pi_2$ is either a hyperbolic, elliptic, or tangent hyperplane of $\mathrm{Q}_1(6,q)$. In any case, $|\pi_1 \cap \pi_2 \cap \mathcal{O}_1| \equiv 1 \bmod p$, hence, π_2 contains at least one point $s \in \mathcal{O}_1$, which necessarily lies on the cone $p\mathcal{O}$. Hence, $s \in \pi_2 \cap p\mathcal{O}$ and we conclude that $|\mathcal{O}_1 \cap \mathcal{O}_2| \geqslant 1$. $\square$

Finally, we describe a property of ovoids of $\mathrm{Q}(6,3)$ that will be very useful in the proof of Lemma 7.

Consider an ovoid $\mathcal{O}$ of $\mathrm{Q}(6,3)$. Consider all hyperplanes of $\mathrm{PG}(6,3)$. Denote the set of hyperbolic hyperplanes of $\mathrm{Q}(6,3)$ with $\mathcal{H}$, $|\mathcal{H}| = 378$. The following property was checked with a computer, using the software packages GAP [8] and pg [3].

Property 1

It is possible to find two elements $\beta_1, \beta_2 \in \mathcal{H}$, $\beta_1 \neq \beta_2$, such that $\langle \beta_1 \cap \beta_2 \cap \mathcal{O}\rangle$ is a 4-dimensional subspace of $\mathrm{PG}(6,3)$*. Define $\mathcal{B} := (\beta_1 \cup \beta_2) \cap \mathcal{O}$. Define $\mathcal{C} := \{\beta \in \mathcal{H}\setminus\{\beta_1,\beta_2\} \| \beta = \langle \mathcal{B}\cap\beta\rangle\}$; in other words, β is spanned by the points of $\beta\cap\mathcal{O}$ in $(\beta_1\cup\beta_2)\cap\mathcal{O}$. We find, using the computer, that $\mathcal{B}_1 := \mathcal{B}\cup(\bigcup_{\beta\in\mathcal{C}}(\beta\cap\mathcal{O}))$ contains 25 elements of $\mathcal{O}$. We fix β_1, and choose any $\beta_2' \in \mathcal{C}$. This gives rise to a new set $\mathcal{C}^{\beta_2'} := \{\beta \in \mathcal{H} \setminus \{\beta_1, \beta_2'\} \| \beta = \langle(\beta_1 \cup \beta_2') \cap \beta \cap \mathcal{O}\rangle\}$ and a new set $\mathcal{B}^{\beta_2'} := \bigcup_{\beta\in\mathcal{C}^{\beta_2'}}(\beta \cap \mathcal{O})$. It is possible to find two elements $\beta_2', \beta_2'' \in \mathcal{C}$ such that $\mathcal{O} = \mathcal{B}_1 \cup \mathcal{B}^{\beta_2'} \cup \mathcal{B}^{\beta_2''}$.*

3 The smallest minimal blocking sets of $\mathrm{Q}^+(9,3)$

From this section on we suppose that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}^+(2n+1,q)$, $|\mathcal{K}| = q^n + 1 + r$, $0 < r < q^{n-3}$. Only Lemma 2 will be proved for general n, afterwards we restrict to $n = 4$ for this section. For some lemmas, we will suppose that $q = 3$.

Lemma 2 *For any point $p \in \mathcal{K}$, $|p^\perp \cap \mathcal{K}| \leqslant 1 + r$.*

Proof. Since $\mathcal{K}$ is minimal, we can find a generator π_n of $\mathrm{Q}^+(2n+1,q)$ meeting $\mathcal{K}$ only in the point p. There are q^n $(n-1)$-dimensional subspaces of π_n not on p which lie in a second generator of $\mathrm{Q}^+(2n+1,q)$ that must be blocked by at least one point of $\mathcal{K}$. Hence, $|\mathcal{K} \setminus p^\perp| \geqslant q^n$; so $|p^\perp \cap \mathcal{K}| \leqslant 1 + r$. $\square$

Lemma 3 *Suppose that p is a point of $\mathrm{Q}^+(9,3) \setminus \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \geqslant 3^3 + 1$. If equality holds, then there exists a 7-dimensional space $\overline{\alpha}_p$ on p that meets $\mathrm{Q}^+(9,3)$ in a cone $p\mathrm{Q}(6,3)$. The set $p^\perp \cap \mathcal{K}$ is projected onto an ovoid $\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,3) \subset \mathrm{Q}^+(7,3)$, the base of the cone $p^\perp \cap \mathrm{Q}^+(9,3)$.*

Proof. Let $q = 3$. All $2(q^3+1)(q^2+1)(q+1)$ generators of $\mathrm{Q}^+(9,q)$ on p meet $\mathcal{K}$ in at least one point, but any point of $p^\perp \cap \mathcal{K}$ lies in exactly $2(q^2+1)(q+1)$ generators on p. Hence, at least q^3+1 points of $\mathcal{K}$ are needed to block all generators on p. Since $p^\perp \cap \mathrm{Q}^+(9,q) = p\mathrm{Q}^+(7,q)$, p projects the set $p^\perp \cap \mathcal{K}$ onto a blocking set $\mathcal{K}_p$ of $\mathrm{Q}^+(7,q)$. When $|p^\perp \cap \mathcal{K}| = q^3+1$, then $\mathcal{K}_p$ is necessarily an ovoid of $\mathrm{Q}^+(7,q)$. When $q = 3$, any ovoid of $\mathrm{Q}^+(7,q)$ lies in a hyperplane π of $\mathrm{PG}(7,q)$, and constitutes an ovoid of $\pi \cap \mathrm{Q}^+(7,q) = \mathrm{Q}(6,q)$ (Theorem 2). The 7-dimensional space $\overline{\alpha}_p$ is now the space $\langle p, \pi \rangle$, and the lemma follows. $\square$

Let $q = 3$. For any point $p \in \mathrm{Q}^+(9,q) \setminus \mathcal{K}$, we say that p is a *small point* if and only if $|p^\perp \cap \mathcal{K}| = q^3+1$. We will always denote the 7-dimensional space from the previous lemma by $\overline{\alpha}_p$.

Lemma 4 *Suppose that L is a line of $\mathrm{Q}^+(9,3)$, $L \cap \mathcal{K} = \emptyset$ and $|L^\perp \cap \mathcal{K}| = 3^2+1$, then L contains at least two small points.*

Proof. Let $q = 3$. From Lemma 3, we have $|r^\perp \cap \mathcal{K}| \geqslant q^3+1$ for every $r \in L$. Define $n_r := |r^\perp \cap \mathcal{K}| - (q^3+1)$. Then $n_r = 0$ if and only if r is a small point. We find

$$\sum_{r \in L} |r^\perp \cap \mathcal{K}| = \sum_{r \in L} (q^3 + 1 + n_r) \leqslant q^4 + q + q(q^2+1),$$

which implies

$$\sum_{r \in L} n_r \leqslant q - 1.$$

Hence, at most $q-1$ points of L have $n_r > 0$, or, L contains at least two small points. $\square$

Lemma 5 *Suppose that π_4 is a generator of $\mathrm{Q}^+(9,q)$ meeting $\mathcal{K}$ in exactly one point p. Then π_4 contains at least θ_3 small points. Furthermore, every line of π_4 not on p, that contains a small point, contains a second small point.*

Proof. Count the number of pairs (r,s), $r \in \pi_4 \setminus \{p\}$, $s \in \mathcal{K} \setminus \{p\}$, $r \in s^\perp$. We find

$$\sum_{r \in \pi_4 \setminus \{p\}} |(r^\perp \cap \mathcal{K}) \setminus \{p\}| \leqslant (|\mathcal{K}| - 1)\theta_3.$$

The right hand side is at most $(q^4 + q - 1)\theta_3 = q^7 + q^6 + q^5 + 2q^4 - 1 < (\theta_4 - 1)(q^3+1)$. Since $p \in r^\perp \cap \mathcal{K}$, it follows that $|r^\perp \cap \mathcal{K}| - 1 < q^3 + 1$ for at least one point $r \in \pi_4 \setminus \{p\}$, hence π_4 contains a small point r.

Consider now a solid α of π_4, not on p, containing a small point r. Since $|r^\perp \cap \mathcal{K}| = q^3 + 1$, every generator on α meets $\mathcal{K}$ in exactly one point, hence $|\alpha^\perp \cap \mathcal{K}| = 2$. Count the number of pairs (t, s), $t \in \alpha$, $s \in \mathcal{K}$, $t \in s^\perp$. Then the two points of $\alpha^\perp \cap \mathcal{K}$ occur in θ_3 pairs; every other point of $\mathcal{K}$ occurs in exactly θ_2 pairs. We find

$$\sum_{t \in \alpha} |t^\perp \cap \mathcal{K}| \leqslant 2\theta_3 + (|\mathcal{K}| - 2)\theta_2 \leqslant \theta_3(q^3 + 1) + q^3 - 1.$$

Since θ_3 is the number of points of α, at least $\theta_3 - (q^3 - 1) = \theta_2 + 1$ points of α are small. Consider a fixed small point r in π_4. Counting the number of incident pairs (α, r'), α a solid of π_4 on r but not on p, and $r' \neq r$ a small point in π_4, we find that π_4 contains at least θ_3 small points.

When L is a line of π_4 containing a small point r but not p, $|r^\perp \cap \mathcal{K}| = q^3 + 1$ implies that $|L^\perp \cap \mathcal{K}| = q^2 + 1$. Applying Lemma 4 proves the last statement of this lemma. □

Lemma 6 *Suppose that* L *is a line of* $\mathrm{Q}^+(9, 3)$ *containing two small points* r *and* r'. *Let* $L \not\subset \overline{\alpha}_r$. *Then* $|L^\perp \cap \mathcal{K}| = 3^2 + 1$ *and* $\langle L^\perp \cap \mathcal{K} \rangle$ *is a* 5*-dimensional space.*

Proof. Let $q = 3$. Since $|r^\perp \cap \mathcal{K}| = q^3 + 1$, all generators on L meet $\mathcal{K}$ in exactly one point, hence $|L^\perp \cap \mathcal{K}| = q^2 + 1$. Consider the 7-dimensional space $\overline{\alpha}_r$, $\overline{\alpha}_r \cap \mathrm{Q}^+(9, q) = r\mathrm{Q}(6, q)$ and r projects the points of $r^\perp \cap \mathcal{K}$ onto an ovoid $\mathcal{O}$ of $\mathrm{Q}(6, q)$. It is clear that r projects the points of $L^\perp \cap \mathcal{K}$ onto an ovoid $\mathcal{O}'$ of $\mathcal{Q}_5 = \mathrm{Q}^+(5, q) \subseteq \mathrm{Q}(6, q)$. By Theorem 4, $\langle \mathcal{O}' \rangle$ is a 5-dimensional space.

Consider the second small point r' on L. Project $r'^\perp \cap \mathcal{K}$ from r' onto a hyperplane of $r'^\perp$, containing $r\mathcal{Q}_5$. Then $r'^\perp \cap \mathcal{K}$ is again projected onto an ovoid of a parabolic quadric $\mathrm{Q}(6, q)$. Again by Theorem 4, the projection of $L^\perp \cap \mathcal{K}$ from r' can only have dimension 5. Since this projection lies in $r\mathcal{Q}_5$, necessarily the points of $L^\perp \cap \mathcal{K}$ belong to a 5-dimensional hyperbolic quadric. □

Lemma 7 *Suppose that* $r \in \mathrm{Q}^+(9, 3) \setminus \mathcal{K}$ *is a small point, then* $r^\perp \cap \mathcal{K}$ *is an ovoid of* $\mathrm{Q}(6, 3)$ *and* $\langle r^\perp \cap \mathcal{K} \rangle$ *is a* 6*-dimensional space.*

Proof. Let $q = 3$. Consider the 7-dimensional space $\overline{\alpha}_r$ (Lemma 3). We can choose the base of the cone $\overline{\alpha}_r \cap \mathrm{Q}^+(9, q) = r\mathrm{Q}(6, q)$ such that $\mathrm{Q}(6, q) \subseteq \mathrm{Q}^+(7, q)$, the base of the cone $r^\perp \cap \mathrm{Q}^+(9, q)$. The set $r^\perp \cap \mathcal{K}$ is projected from r onto an ovoid $\mathcal{O}$ of $\mathrm{Q}(6, q)$. Denote by δ the 7-dimensional space containing $\mathrm{Q}^+(7, q)$ and by γ the hyperplane of δ containing $\mathrm{Q}(6, q)$. If β_1 is a hyperplane of γ intersecting $\mathrm{Q}(6, q)$ in a $\mathrm{Q}^+(5, q)$, and such that $\beta_1^{\perp_{\mathrm{Q}^+(7,q)}} \cap \mathrm{Q}^+(7, q) = \mathrm{Q}^+(1, q) = \{r', r''\}$, then there exists a line $L_{\beta_1} = \langle r, r' \rangle$, $L_{\beta_1} \cap \mathcal{K} = \emptyset$, such that r projects the set $L_{\beta_1}^\perp \cap \mathcal{K}$ exactly onto the set $\beta_1 \cap \mathcal{O}$. By Lemmas 4 and 6, the set $L_{\beta_1}^\perp \cap \mathcal{K}$ spans a 5-dimensional subspace β_1^r of $\overline{\alpha}_r$. Consider a hyperplane β_2 of γ, as described in Property 1, i.e. $\langle \beta_1 \cap \beta_2 \cap \mathcal{O} \rangle$ is a 4-dimensional space. We now find that the set $L_{\beta_2}^\perp \cap \mathcal{K}$ spans a 5-dimensional subspace β_2^r of $\overline{\alpha}_r$. Since $\langle \beta_1 \cap \beta_2 \cap \mathcal{O} \rangle$ is a 4-dimensional subspace of γ, $\zeta := \langle \beta_1^r, \beta_2^r \rangle$ is a 6-dimensional space of $\overline{\alpha}_r$. This means also that all points of $(\beta_1^r \cup \beta_2^r) \cap \mathcal{K}$ lie already in a 6-dimensional space.

The goal is now to prove that $\langle \beta_1^r, \beta_2^r \rangle$ contains all points of $r^\perp \cap \mathcal{K}$. Therefore we will use Property 1.

Consider any 5-dimensional subspace $\beta \subset \gamma$ such that $\beta \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$ and such that $\beta = \langle (\beta_1 \cup \beta_2) \cap \beta \cap \mathcal{O} \rangle$, then β gives rise to a subspace $\beta^r \subseteq \langle \beta_1^r, \beta_2^r \rangle$ and all points of $\beta^r \cap \mathcal{K}$ are projected from r on the points of $\beta \cap \mathcal{O}$. This implies that all points of $\beta^r \cap \mathcal{K}$ lie in the space $\langle \beta_1^r, \beta_2^r \rangle$. Property 1 states actually that all points of $\mathcal{O}$ can be covered by a subspace like β, so considering all such subspaces β, we find that all points of $r^\perp \cap \mathcal{K}$ lie in the 6-dimensional space $\langle \beta_1^r, \beta_2^r \rangle$, and are projected onto $\mathcal{O}$. We conclude that $r^\perp \cap \mathcal{K}$ constitutes an ovoid of $\mathrm{Q}(6,q)$ and that $\langle r^\perp \cap \mathcal{K} \rangle$ is a 6-dimensional space. □

Lemma 8 *There exists a 7-dimensional subspace α such that $\alpha \cap \mathcal{K}$ contains at least $q+1$ ovoids $\mathcal{O}$ of $\mathrm{Q}(6,3)$, all containing a common point $p \in \mathcal{K}$ and sharing two by two q^2+1 points.*

Proof. Let $q = 3$. Consider a generator π_4 of $\mathrm{Q}^+(9,q)$ meeting $\mathcal{K}$ only in the point p. Lemma 5 implies that π_4 contains at least θ_3 small points r_i. Furthermore, $r_i^\perp \cap \mathcal{K} = \mathcal{O}_i$ is an ovoid of $\mathrm{Q}_i(6,q) \subset \mathrm{Q}_i^+(7,q) \subset r_i^\perp \cap \mathrm{Q}^+(9,q)$. Also, if $\langle r_i, r_j \rangle$, $i \neq j$, is a line of π_4 not on p, then $\mathcal{O}_i \cap \mathcal{O}_j$ contains q^2+1 points and constitutes an ovoid of $\mathrm{Q}_i(6,q) \cap \mathrm{Q}_j(6,q) = \mathrm{Q}^+(5,q)$. Consider a small point $r_1 \in \pi_4$, and a plane π through r_1 lying in π_4, but with $p \notin \pi$. Every line of π through r_1 contains a second small point r_2 (Lemma 5). So we find three non-collinear small points r_1, r_2 and r_3 in π.

The ovoids $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ share two by two an ovoid of some $\mathrm{Q}^+(5,q)$, but do all not contain a common ovoid of some $\mathrm{Q}^+(5,q)$, since that ovoid would lie in $\langle r_1, r_2, r_3 \rangle^\perp$. Hence, $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ span together a 7-dimensional subspace β. Lemma 5 implies that every line of $\pi \subset \pi_4$ on r_1 not containing r_2, r_3 contains a second small point r'. The points r_1, r_2 and r' are three non-collinear points spanning the plane π. Hence $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_{r'}$ span a 7-dimensional subspace which is necessarily β. Since there are $q+1$ choices for r', r_2 and r_3 included, we find that β contains $q+1$ ovoids $\mathcal{O}_i$, all containing p and sharing two by two q^2+1 points. □

Lemma 9 *The set $\mathcal{K}$ is a truncated cone $p^*\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,3) \subseteq \mathrm{Q}^+(7,3)$, the base of the cone $p^\perp \cap \mathrm{Q}^+(9,3)$.*

Proof. Consider the 7-dimensional subspace from Lemma 8 and call it β. The set $\beta \cap \mathcal{K}$ contains $q+1$ ovoids $\mathcal{O}_i$, sharing two by two q^2+1 points. Since $\beta \cap \mathcal{K}$ contains ovoids of $\mathrm{Q}(6,q)$, $\beta \cap \mathrm{Q}^+(9,q) = \mathrm{Q}^+(7,q)$, $\beta \cap \mathrm{Q}^+(9,q) = \mathrm{Q}^-(7,q)$ or $\beta \cap \mathrm{Q}^+(9,q) = s\mathrm{Q}(6,q)$.

Suppose that $\beta \cap \mathrm{Q}^+(9,q) = \mathrm{Q}^+(7,q)$. Consider two ovoids $\mathcal{O}_1$ and $\mathcal{O}_2$ contained in $\beta \cap \mathcal{K}$. Consider a point $p \in \mathcal{O}_1 \setminus \mathcal{O}_2$. All generators of $\mathrm{Q}^+(7,q)$ on p intersect $\mathcal{O}_2$ in exactly one point, hence, $|p^\perp \cap \mathcal{K}| > q+1$, a contradiction with Lemma 2.

Suppose that $\beta \cap \mathrm{Q}^+(9,q) = \mathrm{Q}^-(7,q)$. Consider again two ovoids $\mathcal{O}_1$ and $\mathcal{O}_2$ contained in $\beta \cap \mathcal{K}$, and consider a point $p \in \mathcal{O}_1 \setminus \mathcal{O}_2$. Since $p^\perp \cap \mathrm{Q}^-(7,q) = p\mathrm{Q}^-(5,q)$, $p^\perp$ intersects $\langle \mathcal{O}_2 \rangle$ in $\mathrm{Q}^-(5,q)$ and $\langle \mathrm{Q}^-(5,q) \rangle = \langle \mathcal{O}_2 \cap \mathrm{Q}^-(5,q) \rangle$, when $q = 3$ (Theorem 5). We find that $|\mathcal{O}_2 \cap \mathrm{Q}^-(5,q)| \geqslant 6 > q+1$, when $q = 3$, a contradiction with Lemma 2.

Hence, we conclude that $\beta \cap Q^+(9,q) = sQ(6,q)$, necessarily $s \notin \mathcal{K}$ by Lemma 2. Consider now an arbitrary ovoid $\mathcal{O}_i \subset \beta \cap \mathcal{K}$ and denote it by O_β; put $Q_\beta(6,q) := \langle \mathcal{O}_\beta \rangle \cap Q^+(9,q)$ and choose $Q^+_\beta(7,q)$ the base of the cone $s^\perp \cap Q^+(9,q)$ such that $Q_\beta(6,q) \subset Q^+_\beta(7,q)$. Denote $\langle \mathcal{O}_i \rangle \cap sQ(6,q)$ by $Q_i(6,q)$. Put $\mathcal{M} := \{t \in s\mathcal{O}_\beta \setminus \{s\} \| t \notin \mathcal{K}\}$, and suppose that $\mathcal{M} \neq \emptyset$. Consider a point $r \in \mathcal{M}$. By Lemma 3, we know that $|r^\perp \cap \mathcal{K}| \geqslant q^3 + 1$, so consider a point $r' \in r^\perp \cap \mathcal{K}$; and suppose that $r' \in s^\perp$. The line $\langle s, r' \rangle$ intersects $Q^+_\beta(7,q)$ in the point r'' (possibly $r' = r''$). Since $\mathcal{O}_\beta$ is an ovoid of $Q^+_\beta(7,q)$, $|(r''^\perp \cap Q^+_\beta(7,q)) \cap \mathcal{O}_\beta| = q^2+1$, implying that $|r'^\perp \cap \mathcal{K}| > q+1$, a contradiction with Lemma 2. Hence, $r' \notin s^\perp$ and $(sQ^+_\beta(7,q) \setminus s\mathcal{O}_\beta) \cap \mathcal{K} = \emptyset$.

Define $b := |s^*\mathcal{O}_\beta \cap \mathcal{K}|$ and $\mathcal{K}' := \mathcal{K} \setminus s\mathcal{O}_\beta$. The previous arguments show that $|r^\perp \cap \mathcal{K}'| \geqslant q^3$, for $r \in \mathcal{M}$. Furthermore, $b + |\mathcal{M}| = q(q^3+1)$ and $b + |\mathcal{K}'| = |\mathcal{K}| \leqslant q^4 + q = b + |\mathcal{M}|$, hence, $|\mathcal{K}'| \leqslant |\mathcal{M}|$.

Consider again the point $r \in \mathcal{M}$. Since no point $r' \in r^\perp \cap \mathcal{K}'$ lies in $s^\perp$, $\gamma := r'^\perp$ intersects $s\mathcal{O}_\beta$ in an ovoid $\mathcal{O}_\gamma$ of $Q_\gamma(6,q)$. Furthermore, Lemma 1 implies that $|\mathcal{O}_i \cap \mathcal{O}_\gamma| \geqslant 1$ for all ovoids $\mathcal{O}_i$. The 6-dimensional spaces $\langle Q_i(6,q) \rangle$ intersect $\langle Q_\gamma(6,q) \rangle$ in a 5-dimensional subspace ζ. Suppose that $\mathcal{O}_\gamma$ has with the union of all the $q+1$ ovoids $\mathcal{O}_i$ in $\beta \cap \mathcal{K}$ only one point p in common. Then $\langle \mathcal{O}_\gamma \rangle \cap \langle \mathcal{O}_i \rangle$ always must be the tangent hyperplane to $Q_\gamma(6,q)$ in p. So, two quadrics $Q_i(6,q)$ share a tangent hyperplane; this is a contradiction since they share q^2+1 points of $\mathcal{K}$. Hence, the $q+1$ ovoids $\mathcal{O}_i$ contain in total at least two different points of $\gamma \cap \mathcal{K}$, implying that $|\gamma \cap \mathcal{M}| = q^3 + 1 - |\gamma \cap \mathcal{K} \cap s\mathcal{O}_\beta| \leqslant q^3 - 1$. Count the number of pairs $(r, r') \in \mathcal{M} \times \mathcal{K}'$, with $r \in r'^\perp$, to obtain

$$|\mathcal{M}|q^3 \leqslant \sum_{r \in \mathcal{M}} |r^\perp \cap \mathcal{K}'| = \sum_{r' \in \mathcal{K}'} |r'^\perp \cap \mathcal{M}| \leqslant |\mathcal{K}'|(q^3 - 1).$$

Since $|\mathcal{K}'| \leqslant |\mathcal{M}|$, we find that $\mathcal{M} = \emptyset$. Hence, all points of $s\mathcal{O}_\beta \setminus \{s\}$ belong to $\mathcal{K}$. This proves the lemma. □

This result proves Theorem 1 for $n = 4$.

4 The smallest minimal blocking sets of $Q^+(2n+1,3)$

Throughout this section we assume that $n \geqslant 5$. As induction hypothesis we suppose that the smallest minimal blocking sets of $Q^+(2n_0+1,3)$, $4 \leqslant n_0 < n$, are truncated cones $\pi^*_{n_0-4}\mathcal{O}$, $\mathcal{O}$ an ovoid of $Q(6,3) \subset Q^+(7,3)$, the base of the cone $\pi^\perp_{n_0-4} \cap Q^+(2n_0+1,3)$. In the previous section exactly this hypothesis was proved for $n = 5$.

Lemma 10 *Suppose that p is a point of $Q^+(2n+1,3) \setminus \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \geqslant 3^{n-1} + 3^{n-4}$. If equality holds, then there exists an $(n+3)$-dimensional space $\overline{\alpha}_p$ on p that meets $Q^+(2n+1,3)$ in a cone $\pi_{n-4}Q(6,3)$. The set $p^\perp \cap \mathcal{K}$ is projected onto a truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $Q(6,3) \subset Q^+(7,3)$, the base of the cone $\pi^\perp_{n-4} \cap Q^+(2n+1,3)$.*

Proof. Let $q = 3$. All $2(q^{n-1}+1)\ldots(q^2+1)(q+1)$ generators of $Q^+(2n+1,q)$ on p meet $\mathcal{K}$ in at least one point, but any point of $p^\perp \cap \mathcal{K}$ lies in exactly

$2(q^{n-2}+1)\dots(q+1)$ generators on p. Hence, at least $q^{n-1}+1$ points of $\mathcal{K}$ are needed to block all generators on p. Since $p^\perp \cap \mathrm{Q}^+(2n+1,q) = p\mathrm{Q}^+(2n-1,q)$, p projects the set $p^\perp \cap \mathcal{K}$ onto a blocking set $\mathcal{K}_p$ of $\mathrm{Q}^+(2n-1,q)$. By the induction hypothesis, $\mathcal{K}_p$ contains at least $q^{n-1}+q^{n-4}$ points. If $|p^\perp \cap \mathcal{K}| = q^{n-1}+q^{n-4}$, then $\mathcal{K}_p$ is necessarily a truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q) \subset \mathrm{Q}^+(7,q)$, the base of the cone $\pi_{n-5}^\perp \cap \mathrm{Q}^+(2n-1,q)$, lying in an $(n+2)$-dimensional subspace. The $(n+3)$-dimensional subspace $\overline{\alpha}_p$ is now the space $\langle p, \pi_{n-5}, \mathcal{O}\rangle$, and the lemma follows. $\square$

Let $q=3$. For any point $p \in \mathrm{Q}^+(2n+1,q) \setminus \mathcal{K}$, we say that p is a *small point* if and only if $|p^\perp \cap \mathcal{K}| = q^{n-1}+q^{n-4}$. We will always denote the $(n+3)$-dimensional space from the previous lemma by $\overline{\alpha}_p$.

Lemma 11 *Suppose that L is a line of $\mathrm{Q}^+(2n+1,3)$, $L \cap \mathcal{K} = \emptyset$ and $|L^\perp \cap \mathcal{K}| = 3^{n-2}+3^{n-5}$, then L contains 4 small points.*

Proof. Let $q=3$. By Lemma 10, $|r_i^\perp \cap \mathcal{K}| \geqslant q^{n-1}+q^{n-4}$ for all points $r_i \in L$. The sets $r_i^\perp \cap \mathcal{K}$ have exactly $q^{n-2}+q^{n-5}$ points in common, which implies that $|\mathcal{K}| \geqslant (q+1)(q^{n-1}+q^{n-4}-q^{n-2}-q^{n-5})+q^{n-2}+q^{n-5} = q^n+q^{n-3} \geqslant |\mathcal{K}|$. Hence, $|r_i^\perp \cap \mathcal{K}| = q^{n-1}+q^{n-4}$ for all points $r_i \in L$ and $|\mathcal{K}| = q^n+q^{n-3}$. $\square$

Lemma 12 *Suppose that π_n is a generator of $\mathrm{Q}^+(2n+1,q)$ meeting $\mathcal{K}$ in exactly one point p. Then π_n contains at least one small point.*

Proof. Count the number of pairs (r,s), $r \in \pi_n \setminus \{p\}$, $s \in \mathcal{K} \setminus \{p\}$, $r \in s^\perp$. We find

$$\sum_{r \in \pi_n \setminus \{p\}} |(r^\perp \cap \mathcal{K}) \setminus \{p\}| \leqslant (|\mathcal{K}|-1)\theta_{n-1}.$$

The right hand side is at most $(q^n+q^{n-3}-1)\theta_{n-1} < (\theta_n-1)(q^{n-1}+q^{n-4})$ (using $q\theta_{n-1} = \theta_n - 1$). Since $p \in r^\perp \cap \mathcal{K}$, it follows that $|r^\perp \cap \mathcal{K}| - 1 < q^{n-1}+q^{n-4}$ for at least one point $r \in \pi_n \setminus \{p\}$, hence π_n contains a small point r. $\square$

Lemma 13 *Suppose that $r \in \mathrm{Q}^+(2n+1,3) \setminus \mathcal{K}$ is a small point. If $\overline{\beta}$ is a hyperplane of $\overline{\alpha}_r$ on r, not containing the vertex π^r_{n-4} of the cone $\overline{\alpha}_r \cap \mathrm{Q}^+(2n+1,3)$, then the points of $\overline{\beta} \cap \mathcal{K}$ lie in an $(n+1)$-dimensional subspace β of $\overline{\beta}$, $r \notin \beta$.*

Proof. Let $q=3$. Since $\overline{\beta}$ is a hyperplane of $\overline{\alpha}_r$ on r not containing the vertex π^r_{n-4} of the cone $\overline{\alpha}_r \cap \mathrm{Q}^+(2n+1,q) = \pi^r_{n-4}\mathrm{Q}(6,q)$, $\overline{\beta} \cap \mathrm{Q}^+(2n+1,q)$ is a cone with base $\mathrm{Q}^{\overline{\beta}}(6,q)$ and vertex $\pi^{\overline{\beta}}_{n-5}$, an $(n-5)$-dimensional subspace on r. When $n=5$, this subspace is the point r itself. It is clear that $\overline{\beta}^\perp \cap \mathrm{Q}^+(2n+1,q) = \pi^{\overline{\beta}}_{n-5}\mathrm{Q}^{\overline{\beta}}(2,q)$, and this cone meets the cone $\overline{\alpha}_r \cap \mathrm{Q}^+(2n+1,q)$ in the space π^r_{n-4}. Thus there must exist a line L of $\mathrm{Q}^+(2n+1,q)$ contained in $\overline{\beta}^\perp$ such that $L \cap \overline{\alpha}_r = \{r\}$ and such that $L \not\subset \overline{\alpha}_r^\perp$. Since $L \subset \overline{\beta}^\perp$, we find $\overline{\beta} = L^\perp \cap \overline{\alpha}_r$. By Lemma 10, L does not meet $\mathcal{K}$.

Since $L^\perp \cap \mathcal{K} \subseteq r^\perp \cap \mathcal{K} \subseteq \overline{\alpha}_r$, it is clear that $L^\perp \cap \mathcal{K} = \overline{\beta} \cap \mathcal{K}$. Since $L \not\subset \overline{\alpha}_r^\perp$, Lemma 10 implies that $|L^\perp \cap \mathcal{K}| = q^{n-2}+q^{n-5}$. Suppose that p is a point of

$L \setminus \{r\}$. Lemma 11 implies that $|p^{\perp} \cap \mathcal{K}| = q^{n-1} + q^{n-4}$. By Lemma 10, there exists an $(n+3)$-dimensional subspace $\overline{\alpha}_p$ that meets $\mathrm{Q}^+(2n+1, q)$ in the cone $\pi^p_{n-4}\mathrm{Q}_p(6, q)$ and $p^{\perp} \cap \mathcal{K} \subset \overline{\alpha}_p$. Furthermore, $\overline{\alpha}_p$ contains $q^{n-1} + q^{n-4}$ points of $\mathcal{K}$, while $L^{\perp}$ contains $q^{n-2} + q^{n-5}$ points of $\mathcal{K}$, hence $L^{\perp}$ intersects $\overline{\alpha}_p$ in a hyperplane $\overline{\beta}'$ of $\overline{\alpha}_p$, with $p \in \overline{\beta}'$. We conclude that $L^{\perp} \cap \mathcal{K}$ is a subset of $\overline{\beta}$ and $\overline{\beta}'$. The spaces $\overline{\beta}$ and $\overline{\beta}'$ are different since $\overline{\beta}$ does not contain the line L, and so $p \notin \overline{\beta}$. Hence, $L^{\perp} \cap \mathcal{K}$ lies in the $(n+1)$-dimensional subspace $\beta = \overline{\beta} \cap \overline{\beta}'$; it cannot lie in a subspace of lower dimension by Lemma 10. It is impossible that $r \in \beta = \overline{\beta} \cap \overline{\beta}'$; or else r projects the points of $\beta \cap \mathcal{K}$ onto an n-dimensional subspace, but the projected points form a truncated cone $\pi^*_{n-6}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6, q)$, which lies in a space of dimension $n+1$. The subspace $\beta = \overline{\beta} \cap \overline{\beta}'$ intersects $\mathrm{Q}^+(2n+1, q)$ in a cone $\pi^{\beta}_{n-6}\mathrm{Q}(6, q)$, since $\langle \beta, r \rangle = \overline{\beta} \subseteq r^{\perp}$ and $\overline{\beta}$ intersects $\mathrm{Q}^+(2n+1, q)$ in $\pi^{\overline{\beta}}_{n-5}\mathrm{Q}^{\overline{\beta}}(6, q)$. □

Lemma 14 *Suppose that $r \in \mathrm{Q}^+(2n+1, 3) \setminus \mathcal{K}$ is a small point. Then there exists an $(n+2)$-dimensional subspace α_r, $r \notin \alpha_r$, such that $\alpha_r \cap \mathrm{Q}^+(2n+1, 3) = \pi_{n-5}\mathrm{Q}^r(6, 3)$, and such that the truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}^r(6, 3)$, is equal to the set $r^{\perp} \cap \mathcal{K}$.*

Proof. Let $q = 3$. Consider the $(n+3)$-dimensional space $\overline{\alpha}_r$ with $\overline{\alpha}_r \cap \mathrm{Q}^+(2n+1, q) = \pi_{n-4}\mathrm{Q}(6, q)$. Suppose that $\overline{\beta}_1$ is a hyperplane of $\overline{\alpha}_r$, not containing π_{n-4} and containing the point r. By Lemma 13, $\overline{\beta}_1$ contains an $(n+1)$-dimensional subspace β_1, $r \notin \beta_1$, such that $\beta_1 \cap \mathrm{Q}^+(2n+1, q) = \pi^{\beta_1}_{n-6}\mathrm{Q}^{\beta_1}(6, q)$ and $\overline{\beta}_1 \cap \mathcal{K} = \beta_1 \cap \mathcal{K} = \pi^{\beta_1 *}_{n-6}\mathcal{O}^{\beta_1}$, $\mathcal{O}^{\beta_1}$ an ovoid of $\mathrm{Q}^{\beta_1}(6, q)$. Define $\pi_1 := \langle \mathcal{O}^{\beta_1} \rangle$. Choose a hyperbolic hyperplane $\alpha \subseteq \pi_1$, $\alpha \cap \mathrm{Q}^{\beta_1}(6, q) = \mathrm{Q}^+_{\alpha}(5, q)$. We can find a hyperplane $\overline{\beta}_2$ of $\overline{\alpha}_r$, $\overline{\beta}_2 \neq \overline{\beta}_1$, $r \in \overline{\beta}_2$, $\beta_1 \not\subseteq \overline{\beta}_2$, $\pi_{n-4} \not\subseteq \overline{\beta}_2$, but $\pi^{\beta_1}_{n-6}\mathrm{Q}^+_{\alpha}(5, q) \subseteq \overline{\beta}_2$. Again, by Lemma 13, we find an $(n+1)$-dimensional subspace β_2, $r \notin \beta_2$, $\beta_2 \cap \mathrm{Q}^+(2n+1, q) = \pi^{\beta_2}_{n-6}\mathrm{Q}^{\beta_2}(6, q)$, $\overline{\beta}_2 \cap \mathcal{K} = \beta_2 \cap \mathcal{K} = \pi^{\beta_2 *}_{n-6}\mathcal{O}^{\beta_2}$, $\mathcal{O}^{\beta_2}$ an ovoid of $\mathrm{Q}^{\beta_2}(6, q)$. Necessarily, $\pi^{\beta_1}_{n-6} = \pi^{\beta_2}_{n-6}$, and $\mathrm{Q}^+_{\alpha}(5, q) \subset \mathrm{Q}^{\beta_2}(6, q) \neq \mathrm{Q}^{\beta_1}(6, q)$. Define now $\pi_2 := \langle \mathcal{O}^{\beta_2} \rangle$.

Consider the $(n+2)$-dimensional space $\gamma = \langle \pi^{\beta_1}_{n-6}, \pi_1, \pi_2 \rangle$. The two 6-dimensional spaces π_1 and π_2 are skew to π_{n-4}, hence, $\pi_{n-4} \not\subseteq \gamma$. Furthermore, $r \notin \gamma$, since then γ would be an $(n+2)$-dimensional subspace on r, not containing π_{n-4}, spanned by points of $r^{\perp} \cap \mathcal{K}$, a contradiction with Lemma 13. We conclude that $\gamma \cap \mathrm{Q}^+(2n+1, q) = \pi^{\gamma}_{n-5}\mathrm{Q}^{\gamma}(6, q)$.

Choose now an arbitrary hyperplane α', $\alpha' \neq \alpha$, of π_1, such that $\langle \alpha' \cap \mathcal{O}^{\beta_1} \rangle = \alpha'$. Since $q = 3$, both hyperbolic and elliptic hyperplanes have this property (Theorems 4 and 5). Consider the $q+1$ $(n+1)$-dimensional spaces $\delta_i \subset \gamma$ through the n-dimensional space $\langle \alpha', \pi^{\beta_1}_{n-6} \rangle$. One of them, say δ_1, is the space $\langle \alpha', \pi^{\gamma}_{n-5} \rangle$. Consider now a space δ_i, $i \neq 1$. This space δ_i intersects π_2 in a 5-dimensional space through the 4-dimensional space $\epsilon := \alpha \cap \alpha'$. At most two 5-dimensional spaces through ϵ are tangent hyperplanes to $\mathrm{Q}^{\beta_2}(6, q)$, hence, at least $q-2$ elliptic and hyperbolic hyperplanes of $\mathrm{Q}^{\beta_2}(6, q)$ on ϵ remain, hence, at least $q - 2 \geq 1$ spaces δ_i are spanned by points of $\mathcal{K}$ (Since $q = 3$, we can use both the elliptic and hyperbolic hyperplanes). Consider such a δ_i, spanned by points of $\mathcal{K}$. The space $\langle \delta_i, r \rangle$ is a hyperplane of $\overline{\alpha}_r$ not containing π_{n-4}; so it contains an $(n+1)$-dimensional space spanned by $\langle \delta_i, r \rangle \cap \mathcal{K}$. This must be

δ_i since δ_i is spanned by its intersection with $\mathcal{K}$. We conclude that every point $p \in \pi_{n-5}^{\gamma*}\mathcal{O}^{\beta_1}$ lies in $\mathcal{K}$, provided p lies in some subspace δ_i (which depends on the choice of α'), spanned by points of $\mathcal{K}$.

We complete the proof by showing that every point $p \in \pi_{n-5}^{\gamma*}\mathcal{O}^{\beta_1}$ lies in such an $(n+1)$-dimensional space δ_i of γ, not containing π_{n-5}, spanned by points of $\mathcal{K}$.

Consider $p \in (\pi_{n-5}^{\gamma*}\mathcal{O}^{\beta_1}) \setminus (\beta_1 \cup \beta_2)$. The $(n-4)$-dimensional subspace $\langle \pi_{n-5}^{\gamma}, p\rangle \subseteq \gamma$ intersects the $(n+1)$-dimensional space β_2 in an $(n-5)$-dimensional space ζ. If $n = 5$, then this is a point u belonging to π_2. If $n > 5$, then ζ intersects π_2 in exactly one point u.

Choose a point $x \in (\pi_2 \cap \mathcal{K}) \setminus \zeta$, $x \notin \beta_1$. This is possible since we excluded at most one point of $\mathcal{O}^{\beta_2}$, namely the point $u \in \zeta \cap \pi_2$. It is impossible that $\mathcal{O}^{\beta_2} = \{u\} \cup (\mathcal{O}^{\beta_1} \cap \mathcal{O}^{\beta_2})$ since $\langle \mathcal{O}^{\beta_1} \cap \mathcal{O}^{\beta_2}\rangle$ intersects $\mathrm{Q}^{\beta_1}(6, q)$ in a hyperbolic quadric, and an ovoid of a hyperbolic quadric contains $q^2 + 1$ points. Hence, $x \in (\pi_2 \cap \mathcal{K}) \setminus \zeta$, $x \notin \beta_1$, exists.

The line $\langle p, x\rangle$ intersects β_1 in exactly one point $y \notin \pi_{n-6}^{\beta_1}$, else $\langle p, y\rangle \subseteq \zeta$, but $x \notin \zeta$.

The space $\langle y, \pi_{n-6}^{\beta_1}\rangle$ intersects π_1 in exactly one point z. If $z \in \alpha$ and $z = y$, then $\langle x, y\rangle = \langle x, z\rangle \subseteq \pi_2$, so $p \in \beta_2$, which is false. If $z \in \alpha$ and $z \neq y$, then $y \in \beta_2$ and hence, $p \in \beta_2$. We conclude that $z \notin \alpha$. Choose one 5-dimensional space $\alpha' \subseteq \pi_1$, $\alpha \neq \alpha'$, through z such that $\langle \alpha' \cap \mathcal{O}^{\beta_1}\rangle = \alpha'$. Then $\langle \pi_{n-6}^{\beta_1}, z, \alpha', x\rangle = \langle \pi_{n-6}^{\beta_1}, \alpha', x\rangle$ is an $(n+1)$-dimensional subspace of γ not containing π_{n-5}^{γ}. For, suppose that $\pi_{n-5}^{\gamma} \subseteq \Omega := \langle \pi_{n-6}^{\beta_1}, \alpha', x\rangle$, then since $z \in \alpha'$, $z \in \Omega$ and $\pi_{n-6}^{\beta_1} \subseteq \Omega$, also $y \in \Omega$. Furthermore, $x \in \Omega$ and $y \in \Omega$, which implies $p \in \Omega$. Finally, $\pi_{n-5}^{\gamma} \subseteq \Omega$, $p \in \Omega$, which implies $u \in \Omega$. Hence, selecting α' in such a way that $u \notin \langle x, \alpha'\rangle$ will imply that $\pi_{n-5}^{\gamma} \not\subseteq \langle \pi_{n-6}^{\beta_1}, \alpha', x\rangle$. This is possible. For, $\langle \pi_1, \pi_2\rangle$ is a 7-dimensional space, while $\langle x, \alpha'\rangle$ is a 6-dimensional space intersecting π_2 in a hyperplane. All hyperbolic 5-spaces of π_1 on z intersect only in z, hence, all spaces $\langle x, \alpha'\rangle$ only intersect in the line $\langle x, z\rangle$. So we can find an α' through z, such that $\langle x, \alpha'\rangle$ does not contain the point u. $\square$

Lemma 15 *The set $\mathcal{K}$ is a truncated cone $\pi_{n-4}^{*}\mathcal{O}$, $\pi_{n-4} \subset \mathrm{Q}^+(2n+1, 3)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6, 3) \subset \mathrm{Q}^+(7, 3)$, the base of the cone $\pi_{n-4}^{\perp} \cap \mathrm{Q}^+(2n+1, 3)$.*

Proof. From Lemma 12, we find a point $r \in \mathrm{Q}^+(2n+1, q) \setminus \mathcal{K}$ satisfying $|r^{\perp} \cap \mathcal{K}| = q^{n-1} + q^{n-4}$. The $(n+2)$-dimensional subspace α_r from Lemma 14 meets $\mathrm{Q}^+(2n+1, q)$ in a cone $\pi_{n-5}^{r}\mathrm{Q}^{r}(6, q)$. Choose $Q = \mathrm{Q}^+(2n-1, q)$ as the base of the cone $r^{\perp} \cap \mathrm{Q}^+(2n+1, q)$ in such a way that $\langle Q\rangle$ contains the cone $\pi_{n-5}^{r}\mathrm{Q}^{r}(6, q)$. Let L be a line of $\mathrm{Q}^+(2n+1, q)$ on r such that $L \not\subseteq \pi_{n-5}^{r\perp}$, which implies that $L^{\perp}$ does not contain the vertex π_{n-5}^{r} of α_r. Thus $L^{\perp}$ meets α_r in a hyperplane of α_r, and this hyperplane of α_r meets $\mathrm{Q}^+(2n+1, q)$ in a cone $\pi_{n-6}^{L}\mathrm{Q}^{L}(6, q)$. Note that $n \geqslant 5$. If $n = 5$, then this hyperplane of α_r meets $\mathrm{Q}^+(2n+1, q)$ in a quadric $\mathrm{Q}^{L}(6, q)$.

As $L^{\perp} \cap \mathcal{K}$ is contained in $r^{\perp} \cap \mathcal{K} = \alpha_r \cap \mathcal{K}$, it follows that $L^{\perp} \cap \mathcal{K}$ is a truncated cone $\pi_{n-6}^{L*}\mathcal{O}^{L}$, $\mathcal{O}^{L}$ an ovoid of $\mathrm{Q}^{L}(6, q)$. Hence, $|L^{\perp} \cap \mathcal{K}| = q^{n-2} + q^{n-5}$. By Lemma 11, $|s^{\perp} \cap \mathcal{K}| = q^{n-1} + q^{n-4}$ for all points $s \in L$. Every point s gives rise to a truncated cone $s^{\perp} \cap \mathcal{K} = \pi_{n-5}^{s*}\mathcal{O}^{s}$, $\mathcal{O}^{s}$ an ovoid of $\mathrm{Q}^{s}(6, q)$, and all these truncated cones share the truncated cone $L^\perp \cap \mathcal{K} = \pi_{n-6}^{L*}\mathcal{O}^L$. Denote the subspace spanned by $L^\perp \cap \mathcal{K}$ by β_L.

Every point of $\mathcal{K}$ is collinear with a point of L, which implies that $\mathcal{K}$ is the union of these $q+1$ cones $\pi_{n-5}^{s*}\mathcal{O}_s$, $s \in L$. It follows that $|\mathcal{K}| = q^n + q^{n-3}$, and that $\mathcal{K}$ is contained in the union of the $q+1$ $(n+2)$-dimensional subspaces α_s, $s \in L$, that share the $(n+1)$-dimensional subspace β_L.

Consider now a second line L' of $\mathrm{Q}^+(2n+1,q)$ on r such that $L' \not\subseteq \pi_{n-5}^{r\perp} \cap \mathrm{Q}^+(2n+1,q)$ and choose it in such a way that $\beta_L \not\subseteq L'^\perp$. This is possible since $\langle \beta_L, r\rangle^\perp$ has only dimension $n-2$. Then, as for L, the subspace $\beta_{L'} := \langle L'^\perp \cap \mathcal{K}\rangle$ has dimension $n+1$ and is contained in α_s for all $s \in L'$. We have $\beta_L \neq \beta_{L'}$. Let p be a point of L' with $p \neq r$. Then α_p has dimension $n+2$ and meets α_r in $\beta_{L'}$. Furthermore, $\beta_{L'} \cap \mathrm{Q}^+(2n+1,q) = \pi_{n-6}^{L'*}\mathrm{Q}^{L'}(6,q)$, $\beta_{L'} \cap \mathcal{K} = \pi_{n-6}^{L'*}\mathcal{O}^{L'}$, $\mathcal{O}^{L'}$ an ovoid of $\mathrm{Q}^{L'}(6,q)$ and $|\mathcal{O}^{L'} \cap \mathcal{O}^L| \geqslant 1$, since, by Theorem 3, $\mathcal{O}^L$ intersects every hyperplane of $\langle \mathcal{O}^L\rangle$.

Varying the point $p \in L'$, the tangent hyperplanes $p^\perp$ vary over the hyperplanes through $L'^\perp$, hence, every point of the $(n-5)$-dimensional spaces π_{n-5}^s, $s \in L$, lies in some $p^\perp$, $p \in L'$. For every point $x \in \pi_{n-5}^s$, $s \in L$, the line $\langle x, y\rangle$, $y \in \mathcal{O}^L \cap \mathcal{O}^{L'}$, contains q points of $\mathcal{K}$. Hence, x belongs to one of the vertices π_{n-5}^p, $p \in L'$.

Consider a fixed point $s \in L \setminus \{r\}$, fixed points $p_1 \in \pi_{n-5}^r$, $p_2 \in \pi_{n-5}^s$, $p_1, p_2 \notin \pi_{n-5}^r \cap \pi_{n-5}^s = \pi_{n-6}^L$. Consider a fixed point $u \in \pi_{n-5}^{r*}\mathcal{O}^r$, then it is possible to select a line L'', satisfying the conditions of L', for which $u \in L''^\perp$. Then the preceding arguments show that the set $\langle u, p_2\rangle \setminus \{p_2\}$ is contained in $\mathcal{K}$.

Consider an arbitrary line M of $\pi_{n-5}^{r*}\mathcal{O}^r$ passing through p_1 and containing q points of $\mathcal{K}$. The q^2 points of $\langle M, p_2\rangle \setminus \langle p_1, p_2\rangle$ all lie in $\mathcal{K}$; this implies that the truncated cone $\langle \pi_{n-5}^r, \pi_{n-5}^s\rangle^*\mathcal{O}^r$ lies in $\mathcal{K}$. Since $|\mathcal{K}| = |\langle \pi_{n-5}^r, \pi_{n-5}^s\rangle^*\mathcal{O}^r| = q^n + q^{n-3}$, this truncated cone must be equal to $\mathcal{K}$. □

This result proves Theorem 1 for $n \geqslant 5$.

Acknowledgement

The authors thank the two referees for their useful comments and suggestions.

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