The maximum size of a partial spread in $H(5, q^2)$ is $q^3 + 1$

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Abstract

In this paper, we show that the largest maximal partial spreads of the hermitian variety $H(5, q^2)$ consist of $q^3 + 1$ generators. Previously, it was only known that q^4 is an upper bound for the size of these partial spreads. We also show for $q \geq 7$ that every maximal partial spread of $H(5, q^2)$ contains at least $2q + 3$ planes. Previously, only the lower bound $q + 1$ was known.

Keywords: ovoid, polar space, Hermitian variety

1 Introduction

A *spread* of a polar space is a set of mutually skew generators that partition the space. In the case of finite hermitian polar spaces $H(d, q^2)$, a spread consists of $q^d + 1$ generators. J. Thas [9] showed that the finite hermitian variety $H(2n+1, q^2)$ over the field F_{q^2} posses no spread. Recently, D. Luyckx was able to considerably improve this result in the case $n = 2$. She showed in [5] that every partial spread (set of mutually skew generators) of $H(5, q^2)$ consists of at most q^4 planes. She also constructed examples for partial spreads of $H(4n + 1, q^2)$, $n \geq 1$, of size $q^{n+1} + 1$.

For $n = 2$ the construction is as follows. By choosing a suitable hermitian form to define $H(5, q^2)$ in $\mathrm{PG}(5, q^2)$, the restriction to field elements in F_q produced a symplectic polar space $W(5, q)$ in a Baer-subgeometry. It is known that this symplectic polar space has spreads, which have size $q^3 + 1$. These $q^3 + 1$ planes form a partial spread in $H(5, q^2)$. The main result of the present paper is to prove that this construction is optimal.

Theoren 1.1 *Every partial spread of $H(5, q^2)$ has at most $q^3 + 1$ elements.*

We checked for $q = 2$ that $H(5, q^2)$ has partial spreads of cardinality $q^3 + 1$ that do not arise from a symplectic spread as above. D. Luyckx in her paper also shows that a maximal partial spread of $H(2n + 1, q^2)$ must have size at least $q + 1$. It is likely that this bound is far away from the reality, but we can only make a slight improvement.

Theoren 1.2 *A maximal partial spread of $H(5, q^2)$ has at least $2q + 3$, if $q \geq 7$, at least $2q + 2$ generators for $q \in \{3, 4, 5\}$ and at least $2q + 1 = 5$ generators for $q = 2$.*

2 The proof

Consider a partial spread $\mathcal{S}$ of the Hermitian variety $H(5, q^2)$ embedded in $\mathrm{PG}(5, q^2)$. The points that are covered by the planes of $\mathcal{S}$ will be called covered points. The planes contained in $H(5, q^2)$ are called hermitian planes. Since the partial spread is maximal, every hermitian plane contains a covered point. On the other hand, a hermitian plane that is not in the partial spread can meet at most one of the planes of $\mathcal{S}$ in a line. The hermitian planes that are not in $\mathcal{S}$ and do not contain a line of a plane of $\mathcal{S}$ will be called *free planes*. Finally we put $x := q^4 + 1 - |\mathcal{S}|$.

Lemma 2.1 *Every covered point lies on x free planes. Every uncovered points of $H(5, q^2)$ lies on $q^3 + q + x$ free planes.*

Proof. Let P be an uncovered point. For every plane $\pi \in \mathcal{S}$ the subspace $\langle P, P^\perp \cap \pi \rangle$ is a hermitian plane on P meeting π in a line. Hence P lies on exactly $|\mathcal{S}|$ hermitian planes that meet a plane of $\mathcal{S}$ in a line. Then the number of free planes on P is $(q+1)(q^3+1) - |\mathcal{S}| = q^3 + q + x$. Now consider a covered point P in a plane π_0 of $\mathcal{S}$. The other planes π of S still give rise to the planes $\langle P, P^\perp \cap \pi \rangle$, but there are $(q^2 + 1)q$ hermitian planes on P that meet π_0 in a line, so now the number of free planes on P is $q^3 + q$ smaller than for the uncovered points. $\square$

This lemma shows that $x \geq 0$ and hence $|\mathcal{S}| \leq q^4 + 1$. This was noticed by D. Luyckx in [5]. The lemma has another interesting consequence. Consider the multiset $\mathcal{M}$ consisting of the free planes and q^3+q copies of each plane of $\mathcal{S}$. Then every hermitian point is covered exactly q^3+q+x times by planes of this multiset. This has powerful consequences. In order to prove these, we need the following remarkable property of hermitian varieties noticed by Thas [9].

Result 2.2 *Let π_1, π_2 and π be three distinct generators of $H(2n+1,q^2)$. Then the points of π that lie on a line of $H(2n+1,q^2)$ meeting π_1 and π_2 form a hermitian variety $H(n,q^2)$ in π.*

In the degenerate situation $n=1$, we mean by a hermitian variety $H(1,q^2)$ a set of $q+1$ collinear points. We remark that this property can be verified easily in the case $n=1$ by using the duality of $H(3,q^2)$ and $Q^-(5,q)$.

Lemma 2.3 *For two different planes π_1,π_2 of $\mathcal{S}$ the number of free planes intersecting both is equal to*

$$y := x(q^3+1)-(q^3+q)(q^2-q+1)(q-1).$$

Proof Let π_1 and π_2 be two different planes of $\mathcal{S}$. Then the union U of all hermitian lines meeting π_1 and π_2 has size $(q^4+q^2+1)(q^4+1)$. Now consider the multiset $\mathcal{M}$ constructed above whose planes cover every hermitian point q^3+q+x times. We count incident pairs $(P,\pi)\in U\times\mathcal{M}$. Each point of U occurs in q^3+q+x pairs.

The q^3+q copies of π_1 and π_2 in $\mathcal{M}$ occur each in q^4+q^2+1 pairs. A plane of $\mathcal{M}$ that is skew to π_1 and π_2 occurs q^3+1 times by the above result; this applies to the $(|S|-2)(q^3+q)$ of planes of $\mathcal{S}\setminus\{\pi_1,\pi_2\}$. For the free planes in $\mathcal{M}$ there are three possibilities. They can be skew to π_1 and π_2. Then they also meet U in q^3+1 points. They can meet π_1 and π_2 in one point. Then they meet U in a line, so these free planes occur in q^2+1 pairs. We denote by y the number of such free planes. Then the number of free planes that meet exactly one of π_1 and π_2 is $2(q^4+q^2+1)x-2y$ by Lemma 2.1. It follows from Result 2.2 that these free planes occur in $1+(q+1)q^2$ pairs.

Thus, each plane of $\mathcal{M}$ occurs in q^3+1 pairs, except that $2(q^3+q)$ occur $q^4-q^3+q^2$ extra times, $2(q^4+q^2+1)x-2y$ occur q^2 extra times, and y occur $q^3+1-(q^2+1)=q^3-q^2$ times less. Hence

$$\begin{aligned}|U|(q^3+q+x) &= |\mathcal{M}|(q^3+1)+2(q^3+q)(q^4-q^3+q^2)\\ &\quad+[2(q^4+q^2+1)x-2y]q^2-y(q^3-q^2)\end{aligned}$$

As the planes of $\mathcal{M}$ cover $H(5,q^2)$ exactly q^3+q+x times, we have $|\mathcal{M}| = (q^5+1)(q^3+q+x)$. Simplifying gives y as stated. $\square$

We have $|\mathcal{F}| = |\mathcal{M}| - |S|(q^3+q)$. Using the size for $|\mathcal{M}|$ from the above proof, we find

$$\sum_{F\in\mathcal{F}} 1 = |\mathcal{F}| = (q^5-q^4)(q^3+q) + x(q^5+q^3+q+1)$$

For $F \in \mathcal{F}$ denote by c_F the number of points of F that are covered by planes of the partial spread $\mathcal{S}$. Counting incident pairs (P,F) with points covered by $\mathcal{S}$ and free planes F, Lemma 2.1 gives

$$\sum_{F\in\mathcal{F}} c_F = |\mathcal{S}|(q^4+q^2+1)x.$$

Counting triples (P,P',F) of different points covered by $\mathcal{S}$ and free planes F with $P,P' \in F$, the preceding lemma gives

$$\sum_{F\in\mathcal{F}} c_F(c_F-1) = |\mathcal{S}|(|\mathcal{S}|-1)y.$$

Using these three equalities to evaluate the Cauchy-Schwarz-inequality

$$|\mathcal{F}|\sum_{F\in\mathcal{F}} c_F^2 \geq \left(\sum_{F\in\mathcal{F}} c_F\right)^2$$

using $x = q^4+1-|\mathcal{S}|$ and $s := |S|$, gives

$$\begin{aligned} 0 \;\leq\;& sq(q^2-q+1)(q^3+1-s)(q^{11}+q^{10}+q^9-sq^7+q^7+2q^6-2sq^6 \\ & -sq^4+q^4-sq^3+s^2q^3-sq^2+q^2+s^2q-2sq+q-s^2+2s-1). \end{aligned}$$

It follows that $|\mathcal{S}| \leq q^3+1$. Here we used that we have $|\mathcal{S}| \leq q^4+1$, see above.

Now suppose that $|\mathcal{S}| = q^3+1$. Then we have equality and this implies that all planes of $\mathcal{F}$ have the same number f of covered points. The above equations for $\sum c_F$ and $|\mathcal{F}|$ show that this number is q^2-q+1. We also have $|\mathcal{F}| = q^6(q^3-1)$ and the number y of planes of $\mathcal{F}$ meeting two planes of $\mathcal{F}$ is

$$y = (q^4+q^2+1)(q-1)^2q.$$

This information shows that all spreads of size q^3+1 behave similar. However, we also mention that there might exist different spreads.

3 Small maximal partial spreads

In order to prove a lower bound for small maximal partial spreads of $H(5, q^2)$, we need to calculate some numbers. The crucial point of our counting argument is that the number of planes of $H(5, q^2)$ that meet three mutually skew planes of $H(5, q^2)$ is independent of the three planes chosen.

Lemma 3.1 (a) *Every plane of $H(5, q^2)$ meets $(q^4 + q^2 + 1)(q^4 + q)$ other planes of $H(5, q^2)$.*

(b) *If π_1 and π_2 are mutually skew planes of $H(5, q^2)$, then there exist exactly $(q^4 + q^2 + 1)(q^3 - q^2 + q + 1)$ planes of $H(5, q^2)$ meet π_1 and π_2.*

(b) *If π_1, π_2 and π are three mutually skew planes of $H(5, q^2)$, then $q^6 - 2q^5 + 3q^4 + q + 1$ planes of $H(5, q^2)$ meet π_1, π_2 and π.*

Proof (a) Each of the q^4+q^2+1 lines of a plane π of $H(5, q^2)$ lies in q further planes. A point of π lies in $(q + 1)(q^3 + 1)$ planes of $H(5, q^2)$, of which one is π and $(q^2 + 1)q$ other ones meet π in a line, so q^4 of which meet π only in this point. Thus there exist $(q^4 + q^2 + 1)q^4$ planes in $H(5, q^2)$ that meet π in a unique point.

(b) Consider a point $P \in \pi_1$. The number of planes on P that meet π_2 can be counted in the quotient geometry on P: Given two skew lines l_1 and l_2 in $H(3, q^2)$, there are exactly $1 + (q^2 + 1)q$ lines that meet l_2 and exactly $q^2 + 1$ of these meet also l_1. Thus, P lies in $q^3 + q + 1$ planes of $H(5, q^2)$ that meet π_2 and exactly $q^2 + 1$ of these meet π_1 in a line. It follows that there exists $q^4 + q^2 + 1$ planes of $H(5, q^2)$ that meet π_1 in a line and π_2 in a point, and there are $(q^4 + q^2 + 1)(q^3 - q^2 + q)$ planes in $H(5, q^2)$ that meet π_1 in a unique point and that meet also π_2.

(c) First we recall from Result 2.2 that we find a hermitian curve $H = H(2, q^2)$ in the plane π consisting of those points of π that lie on a line of $H(5, q^2)$ that meets π_1 and π_2. Alternatively, one can say that H consists of the points $P \in \pi$ such that the planes $\langle P, P^\perp \cap \pi_1\rangle$ and $\langle P, P^\perp \cap \pi_2\rangle$ meet in a line l (and this is the line of $H(5, q^2)$ on P that meets π_1 and π_2).

Consider a point $P \in \pi$. Then every plane of $H(5, q^2)$ on P that meets π_1 and π_2, meets π_i in a point of the plane $E_i := \langle P, P^\perp \cap \pi_i\rangle$. Going into the quotient space $P^\perp/P$, in which we see a $H(3, q^2)$, the planes E_1, E_2 and π

become lines l_1, l_2, l. The number of planes of $H(5,q^2)$ on P that meet also π_1 and π_2 is equal to the number of lines in the $H(3,q^2)$ that meet l_1 and l_2. If $P \in H$, then the lines l_1 and l_2 meet in a point and l is disjoint to l_1 and l_2. In this case there are $q+1$ lines meeting l_1 and l_2 and one of these meets also l. If $P \notin H$, then l_1, l_2 and l are mutually skew, so there are q^2+1 lines in $H(3,q^2)$ that meet l_1 and l_2 and, by Result 2.2, exactly $q+1$ of these meet also l.

Thus the number of planes of $H(5,q^2)$ on P that meet π_1 and π_2 is $q+1$ in the first case and q^2+1 in the second case. Also in the first case one and in the second case $q+1$ of these planes meet π in line. Thus, from the planes of $H(5,q^2)$ that meet π_1 and π_2, exactly

$$(q^3+1)q + (q^4-q^3+q^2)(q^2-q) = q^6 - 2q^5 + 3q^4 - q^3 + q$$

meet π in a unique point, and

$$\frac{(q^3+1)\cdot 1 + (q^4-q^3+q^2)(q+1)}{q^2+1} = q^3+1$$

meet π in a line. □

Remark. In the previous proof, one can also show the following. A tangent line of the hermitian curve H in π lies on a unique plane of $H(5,q^2)$ meeting π_1 and π_2, whereas the other lines of π do not lie in planes that meet π_1 and π_2. This explains the term q^3+1 for the number of planes meeting π, π_1 and π_2.

To obtain from this information a lower bound we use a standard counting technique, see for example [7]. Suppose that $\mathcal{F}$ is a maximal partial spread of $H(5,q^2)$. Let n_i be the number of planes of $H(5,q^2)$ that are not in $\mathcal{F}$ and that meet exactly i planes of $\mathcal{F}$. Then $n_0 = 0$ as the spread is maximal. Also

$$\begin{aligned}
\sum_{i\geq 1} n_i &= (q+1)(q^3+1)(q^5+1) - |\mathcal{F}| &\quad (1)\\
\sum_{i\geq 1} n_i i &= |\mathcal{F}|(q^4+q^2+1)(q^4+q) &\quad (2)\\
\sum_{i\geq 1} n_i i(i-1) &= |\mathcal{F}|(|\mathcal{F}|-1)(q^4+q^2+1)(q^3-q^2+q+1) &\quad (3)
\end{aligned}$$

$$\sum_{i\geq 1} n_i i(i-1)(i-2) = |\mathcal{F}|(|\mathcal{F}|-1)(|\mathcal{F}|-2)(q^6-2q^5+3q^4+q+1). \quad (4)$$

The first equation holds, since $H(5,q^2)$ has $(q+1)(q^3+1)(q^5+1)$ generators. The second equation follows from a double counting argument, since each generator meets $(q^4+q^2+1)(q^4+q)$ other generators. Finally the third and forth equation follow from the lemma by counting suitable triple and 4-tuples. These equation enable us to calculate the sum

$$S := \sum_{i\geq 1} n_i(i-1)(i-3)(i-4).$$

Clearly $S \geq 0$. Simplifying using the above equation yields (here we put $|\mathcal{F}| = 2q+2+x$)

$$\begin{aligned} 0 \leq\ & (q^6 - 2\,q^5 + 3\,q^4 + q + 1)\,x^3 \\ & + (q^7 + 6\,q^2 + 9\,q^4 + 4\,q + 2\,q^5 - 10\,q^3 - 2 - 4\,q^6)\,x^2 \\ & + (-22\,q^4 - 1 - 7\,q^7 + 4\,q^2 + q^6 - 6\,q^3 + 14\,q^5 - 9\,q + 4\,q^8)\,x + 2 \\ & - 2\,q^8 - 16\,q^4 + 14\,q^6 - 12\,q^5 - 10\,q^2 - 8\,q^3 + 8\,q^7. \end{aligned}$$

For large q we immediately see that $x > 0$ and hence $|\mathcal{F}| \geq 2q+3$. In fact, this holds for $q \geq 7$. For $q = 5$ and $q = 3$ we still deduce $x > -1$, that is $|\mathcal{F}| \geq 2q+2$, and for $q = 2$ we find $|\mathcal{F}| \geq 2q+1 = 5$.

We remark that the same technique can be applied to $H(2n+1,q^2)$ for any n, only one has to calculate the numbers as in Lemma 3.1. For example, if $n = 1$, the numbers are easy to calculate, where again for the last number Result 2.2 is needed. Thus, if $\mathcal{F}$ is a partial spread of $H(3,q^2)$, then

$$\begin{aligned} \sum_{i\geq 1} n_i &= (q+1)(q^3+1) - |\mathcal{F}| \\ \sum_{i\geq 1} n_i i &= |\mathcal{F}|(q^2+1)q \\ \sum_{i\geq 1} n_i i(i-1) &= |\mathcal{F}|(|\mathcal{F}|-1)(q^2+1) \\ \sum_{i\geq 1} n_i i(i-1)(i-2) &= |\mathcal{F}|(|\mathcal{F}|-1)(|\mathcal{F}|-2)(q+1). \end{aligned}$$

The calculating the same sum as above gives $|\mathcal{F}| \geq 2q+1$ for $q = 2, 3$, and $|\mathcal{F}| \geq 2q+2$ for $q \geq 4$

4 Particular results for small values of q

Using the computer algebra system GAP [6] and the package pg [3], we constructed all maximal partial spreads of $H(5,4)$ and $H(3,q^2)$, $q=2,3$. For $H(5,4)$ we were interested in the different kind of maximal partial spreads that exist, and some of their geometric properties. For $H(3,q^2)$, $q=2,3$ we were more interested in the possible sizes that maximal partial spreads can have.

maximal partial spreads of $H(5,4)$

The following table summarizes the information on the different maximal partial spreads of $H(5,4)$.

class	size	stabilizer group	order of the group
symplectic	9	$L_2(8) : C_3$	$9\cdot 8\cdot 7\cdot 3 = 1512$
derivation 1	9	$C_3 \wr S_3$	$3^3\cdot 2\cdot 3 = 162$
derivation 2	9	$(C_2^3 : C_7) : C_3$	$2^3\cdot 7\cdot 3 = 168$
derivation 3	7	C_2	2
derivation 4	7	$D_{10} \cong C_5 : C_2$	10

There are, up to collineation, 3 examples of maximal partial spreads of size 9 and two examples of maximal partial spreads of size 7. The (up to collinetion unique) spread of $W(5,2)$ (embedded in $H(5,4)$) is called the symplectic example. Its stabilizer group, i.e. the subgroup of $\mathrm{P\Gamma U}(6,4)$ stabilizing the spread planewise, has actually order 9072, but its action on the planes of the maximal partial spread S is isomorphic with $L_2(8) : C_3$, which is a group of order 1512. The reason is that a collineation of $W(5,2)$ embedded in $H(5,4)$ can be extended in several ways to a collineation of $H(5,4)$. The group acts 3-transitively on the planes, the normal subgroup $L_2(8)$ acts sharply 3-transitively on the planes and is simple.

Suppose that S is a maximal partial spread of size 9. Consider a triple T of planes of S. Denote with $F(T)$ the set of free planes of $H(5,4)$ intersecting every plane of T. Suppose that S is the symplectic example. Then for any triple T the set $F(T)$ contains at least one triple of mutually skew planes intersecting no other planes of S than the three planes of the chosen triple.

We say that the set $F(T)$ staisfies condition D. Hence replacing the planes from the triple with such a triple of mutually skew planes yield a new spread of size 9. We call this procedure *derivation*. Since the stabilizer group of S acts 3-transitively on the planes of S, it is clear that a derivation from any chosen triple yields the same spread. We call this example a derivation 1 example. Its stabilizer group has order 162 and acts transitively on the planes of the spread. The action is imprimitive. A non-trivial block system exists where each block contains exactly three planes, the action of the group on the blocks is isomorphic to S_3 (the symmetric group on three elements). The subgroup stabilizing each block is elementary abelian and isomorphic with C_3^3. The derivation process can be executed on this example, but not all chosen triples will yield the same spread now. For some triples T, the set $F(T)$ will even not satisfy condition D, but if it does, the derivation can be a symplectic example, a derviation 1 or a derivation 2 example.

Suppose that S is a derivation 2 example. Its stabilizer group fixes exactly one plane, the action of the group is 2-transitive on the 8 remaining planes and is isomorphic to $(C_2^3 : C_7) : C_3$. The normal subgroup $C_2^3 : C_7$ acts sharply 2-transitive on the 8 remaining planes. Again the process of derivation can be executed and if, for a chosen triple T, the set $F(T)$ satisfies condition D, then the derivation of S is always a derivation 1 example.

Suppose now that S is a derivation 1 or a derivation 2 example. It is always possible to find a triple T of planes of S such that $F(T)$ does not satisfy condition D, such that $F(T)$ contains a plane π that intersects two planes of $S \setminus T$, such that $F(T)$ contains two more planes π' and π'', not intersecting the planes of $S \setminus T$ and such that π, π' and π'' are mutually skew. Removing the two planes of $S \setminus T$ that intersect π, and the three planes of T, and adding the three planes π, π' and π'', yields a maximal partial spread of size 7. Starting from a derivation 1 or derivation 2 example, we can always construct derivation 3 and derivation 4 examples if we choose a suitable triple T.

If S is a derivation 3 example, then its stabilizer group fixes one plane of S. Its action on the remaining six planes is involutory, the stabilizer group is isomorphic with C_2. If S is a derivation 4 example, then its stabilizer group has order 10, fixes no plane, but does not act transitively on the planes of S. There are two orbits, one has length 7, the second one has length 2. The stabilizer group is isomorphic with $D_{10} \cong C_5 : C_2$, the dihedral group of order 10.

We recall that in the case $H(5,4)$ the theoretic lower bound from the previous section was 5. We see that in reality the smallest maximal partial spread has size 7.

Maximal partial spreads of $H(3,q^2)$, $q=2,3$

From the previous section we known that the a maximal partial spread of $H(3,q^2)$, $q=2,3$ contains at least $2q+1$ planes. That $H(3,q^2$ has a maximal partial spread of size q^2+1 is observed in [4], as in the $H(5,q^2)$, we can embed the symplectic polar space (now of rank 2) into $H(3,q^2)$. Alternatively, one says that $W(3,q)$ is a subquadrangle of the generalized quadrangle $H(3,q^2)$ [8]. It is known that $W(3,q)$ has spreads. Suppose that S is a spread of $W(3,q)$, than one can show that the extension in $H(3,q^2)$ is a maximal partial spread as follows. Consider the dual situation, i.e. we interchange the role of the points and the lines in the generalized quadrangles $W(3,q)$ and $H(3,q^2)$. The dual of $W(3,q)$ is isomorphic with $Q(4,q)$ and the dual of $H(3,q^2)$ is isomorphic with $Q^-(5,q)$. Hence in the dual situation we consider an *ovoid* of $Q(4,q)$ embedded in $Q^(5,q)$. An ovoid of $Q(4,q)$ is a set $\mathcal{O}$ of points of $Q(4,q)$ such that every line of $Q(4,q)$ meets the ovoid in exactly one point. (remark that this is exactly the dual of a spread of $W(3,q)$. Considering the ovoid $\mathcal{O}$ of $Q(4,q)$, we have to show that any point of $Q^-(5,q)$ is collinear on $Q^-(5,q)$ with at least one point of $\mathcal{O}$. Consider a point $p \in Q^-(5,q) \setminus Q(4,q)$, then all points collinear with p lie in a hyperplane π_4 of $\mathrm{PG}(5,q)$. Consider the 4-dimensional space π_4 containing $Q(4,q)$. Then π intersects π_4 in a 3-dimensional space, and it is known that each 3-dimensional space contains exactly $1 \bmod p$ points of $\mathcal{O}$, with $q=p^h$ [1, 2]. This shows that any spread of $W(3,q)$ constitutes a maximal partial spread of $H(3,q^2)$, of size q^2+1.

Using an exhaustive search, we found that $H(3,q^2)$, $q=2$ has, up to collineation, 1 maximal partial spread of size 5 and 1 maximal partial spread of size 6. Hence the lower bound $2q+1$ is reached in this case. For $q=3$, we found that $H(3,q^2)$ has maximal partial spreads of size 10, 11, 12, 13 and 16. In this case the lower bound $2q+1$ is not reached. We observe that the size of the symplectic spread (q^2+1) is the real lower bound for $q=2,3$, so the question arises whether this is true for general q. We remark that more results on maximal partial spreads of $H(3,q^2)$, $q=2,3$ can be found in [4].

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