Some properties of the Biggs-Smith geometry

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Abstract

We settle two problems posed by the first author in [2] concerning the Biggs-Smith geometry. The first is to determine the generating rank, the second is to prove that this geometry possesses exactly eighteen ovoids.

1 Introduction

The Biggs-Smith geometry is a bislim point-line geometry arising from the Biggs-Smith graph (for precise definitions, see below). It is introduced in [2] as one out of a family of 10 highly symmetric bislim geometries with nice properties, and connected with interesting small groups — usually (almost) simple ones. In the general study performed in [2], there were two questions related to the Biggs-Smith geometry that remained unsolved in [2]. The first one is the determination of the generating rank. It was shown in [2] that this rank was either 19 or 20. The conjecture was the first value, since the universal embedding rank is also 19. In the present paper, we show this conjecture to be true. This reinforces the general conjecture that these ranks are the same whenever the geometries have a rich enough automorphism group — but no-one knows how rich it precisely has to be, and no clue for a proof has been found yet, as far as we know.

The second problem concerns ovoids. In [2], eighteen ovoids were constructed — to which we refer as *classical* here — and these were used to construct a real embedding of the Biggs-Smith geometry. The existence of another class of ovoids would probably give rise to a different embedding. This would be quite interesting since such a situation would be new. In the present paper, however, we show that there are no non-classical ovoids. This negative result is, in our opinion, worth recording, also for its proof, which makes use of the interplay between two constructions of the Biggs-Smith graph: the original one as defined by Biggs and Smith [1], and the recent one by the first author [2].

2 Definitions

A point-line geometry is a system $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consisting of a point set $\mathcal{P}$, a line set $\mathcal{L}$ and a symmetric incidence relation $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$ expressing precisely when a point is incident with a line. Usually we think of a line as the set of points incident with it and we accordingly use phrases like "a point is on a line", "a line goes through a point", etc. If all lines of Γ carry the same number $s+1$ of points and all points are incident with the same number $t+1$ of lines, then we say that Γ has *order* (s,t). If $s=2$, then we call Γ *slim*. If also $t=2$, then we call Γ *bislim*.

An *isomorphism* (*monomorphism*) from the point-line geometry Γ to the point-line geometry Γ' is a bijection (injective map) from the point set of Γ to the point set of Γ' together with a bijection (injective map) from the line set of Γ to the line set of Γ' such that two elements of Γ are incident precisely when their images in Γ' are incident.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a point-line geometry and $\mathbf{PG}(d,\mathbb{K})$ the d-dimensional projective space over the field $\mathbb{K}$. An *embedding of* Γ *in* $\mathbf{PG}(d,\mathbb{K})$ is a monomorphism of Γ into the point-line geometry of $\mathbf{PG}(d,\mathbb{K})$. Usually, one identifies a point with its image in $\mathbf{PG}(d,\mathbb{K})$.

Let Γ be a slim geometry. If the positive integer d is maximal with respect to the property that Γ admits an embedding in $\mathbf{PG}(d,2)$, then $d+1$ is called the *universal embedding rank of* Γ. If $d>0$ does not exist, then we say that the universal embedding rank is zero. Also, we say that a subset of points of Γ *generates* Γ if the smallest slim subgeometry of Γ containing these points and with the property that, whenever two points of that subgeometry are collinear in Γ, they are also collinear in the subgeometry, coincides with Γ. The *generating rank of* Γ is the minimal number of points needed to generate Γ. Obviously the generating rank of Γ cannot be smaller than the universal embedding rank of Γ.

An *ovoid* of a point-line geometry Γ is a set of points with the property that each line is incident with exactly one point of the ovoid.

3 The Biggs-Smith geometry

The bislim point-line geometry **BS** is the *neighborhood geometry* arising from the non-bipartite Biggs-Smith graph on 102 vertices (see figure 1). This means that the points of **BS** are the vertices of the graph and the lines are a second copy of the same vertex set with the incidence relation induced by adjacency. The original construction of the Biggs-Smith graph uses 17 copies H_i, $i \in \mathbf{GF}(17)$ of an "H"-graph and tells one very explicitly how to join vertices of each of these copies (see figure 2).

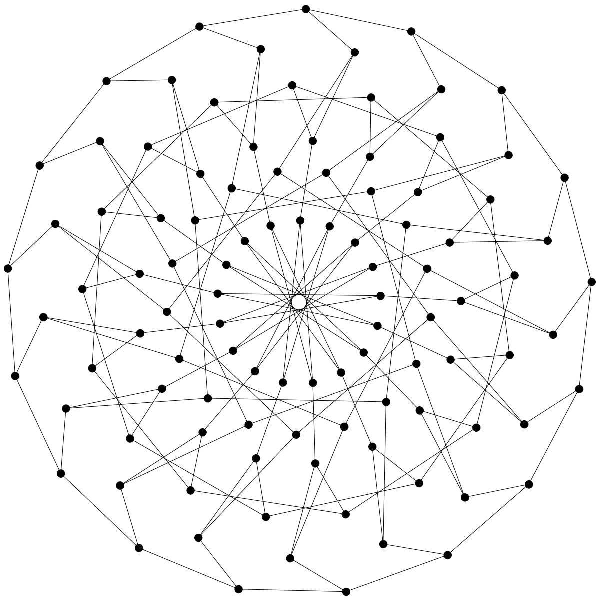

Figure 1: Biggs-Smith graph

So the vertices are $p_0, \ldots, p_{16}$, $r_0, \ldots, r_{16}$, etc. The notation 17/4 at the q-vertex of the "H"-graph means that we join q_i with q_{i+4}. Similarly for the others.

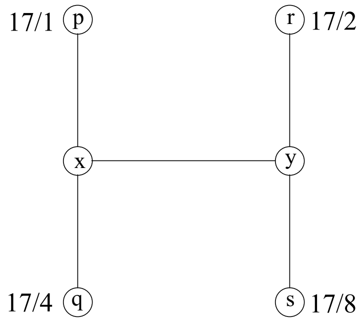

Figure 2: "H"-graph

We now repeat the construction of the Biggs-Smith graph and the geometry **BS** as given in [2].

Consider the projective line $\mathbf{PG}(1, 17)$ over the field $\mathbf{GF}(17)$. A *harmonic triplet* consists of three pairs of points $\{a, b\}, \{c, d\}, \{e, f\}$, with $a, b, \ldots, f$ all distinct, such that we have the following equality of cross ratios: $(a, b; c, d) = (a, b; e, f) = (c, d; e, f) = -1$. Three mutually disjoint harmonic triples will be called a *trisection*. The group $\mathbf{PSL}_2(17)$ acts in two orbits of size 102 on the set of harmonic triplets. Let $\mathcal{P}$ be one of these orbits. Then $\mathcal{P}$ is the vertex set of the Biggs-Smith graph, and two vertices are adjacent if the corresponding harmonic triplets have a pair of points in common, e.g. $\{\{\infty, 0\}, \{1, -1\}, \{4, -4\}\}$ is adjacent to $\{\{\infty, 0\}, \{2, -2\}, \{8, -8\}\}$. The point set of **BS** is also $\mathcal{P}$, and the trisections within $\mathcal{P}$ form the line set.

It is trivial to see that the harmonic triplets for which some pair contains a fixed point of $\mathbf{PG}(1, 17)$ form an ovoid of **BS**. Such an ovoid will be called *classical.* Since there are eighteen points in $\mathbf{PG}(1, 17)$, there are eighteen classical ovoids.

Our Main Result reads:

Main Result *The geometry* **BS** *has generating rank equal to* 19*. Consequently, the generating rank is equal to the universal embedding rank. Also,* **BS** *does not contain ovoids different from the classical ones.*

4 Generating rank

Theorem 4.1 *The generating rank of* **BS** *is equal to* 19*.*

Proof. Since the universal embedding rank of **BS** is equal to 19, the generating rank is at least 19. We consider the construction of the Biggs-Smith graph with the "H"-graph.

We assert that $\mathcal{G} := \{r_i, p_0, p_1, p_2 | i \in \mathbf{GF}(17) \setminus \{-5\}\}$ is a generating set for **BS**. It suffices to prove that r_{-5} belongs to the slim subgeometry $\mathcal{S}$ of **BS** generated by the points of $\mathcal{G}$. Then this subgeometry coincides with **BS**, as is easy to see (and which, in fact, will also follow from our proof).

A vertex r_i, $i \in \mathbf{GF}(17)$ is adjacent to two other vertices r_{i-2} and r_{i+2} and one vertex y_i. Hence, the vertices y_i with $i \in \mathbf{GF}(17) \setminus \{-3, -7\}$ belong to $\mathcal{S}$.

Each vertex p_i, $i \in \mathbf{GF}(17)$ is adjacent with two vertices p_{i+1} and p_{i-1} and with one vertex x_i. The latter has next to p_i still two other neighbors: q_i and y_i. Consequently, for $i = 0, 1, 2$, the point q_i belongs to $\mathcal{S}$. Also, we have that x_1 belongs to the subgeometry $\mathcal{S}$.

A vertex s_i, $i \in \mathbf{GF}(17)$ is adjacent to two other vertices s_{i+8} and s_{i-8} and with one vertex y_i. The latter has still two neighbors: r_i and x_i. It's easy to see that the point s_1 is a point of $\mathcal{S}$.

The vertex s_{-8} is adjacent to the vertices s_1 and y_{-8}, representing points which belong to $\mathcal{S}$. Hence, s_0 belongs to $\mathcal{S}$. Inductively, by reducing each value of i in the previous case with 1, we conclude that the vertices s_i with $i \in \mathbf{GF}(17) \setminus \{1, 2, 3, 4, 5\}$ belong to the subgeometry $\mathcal{S}$.

For i being different from $-5, 1, 2, 3, 4, 5$ we have that the vertices s_i and r_i are in $\mathcal{S}$. Hence, the twelve vertices x_i with $i \in \mathbf{GF}(17) \setminus \{-5, 2, 3, 4, 5\}$ belong to $\mathcal{S}$. (Remark that for $i = 1$ we already have shown that x_1 belongs to $\mathcal{S}$.)

Given a vertex q_i. This vertex is adjacent to two other vertices q_{i+4} and q_{i-4}. Also, q_i is adjacent to the vertex x_i. Hence, we have that $q_i \in \mathcal{S}$ for $i = -8, -7, -6$.

The vertex p_i is adjacent to two vertices p_{i-1} and p_{i+1} and with x_i. Starting with considering the vertex with i-coordinate equal to 0 and letting i successively diminish with one, we get that $p_i \in \mathcal{S}$ for $i = -1, -2, -3, -4, -5$. Because of the fact that p_{-6} is adjacent to p_{-5} and to x_{-6} we have $p_{-7} \in \mathcal{S}$.

The vertex x_{-7} is adjacent to p_{-7} and to q_{-7}, hence y_{-7} belongs to $\mathcal{S}$.

The vertex r_{-7} is adjacent to y_{-7} and to r_8, hence the vertex $r_{-5} \in \mathcal{S}$. Since this vertex is the missing vertex, we have that the generating rank of **BS** is equal to 19. □

5 Ovoids

Theorem 5.1 *The point-line geometry* **BS** *has* 18 *ovoids, each containing* 34 *points.*

Proof. Let $\mathcal{O}$ be any ovoid of **BS**. We will show that $\mathcal{O}$ is classical. Consider any point v of $\mathcal{O}$. Then, conceiving v as a vertex of the Biggs-Smith graph, and then interpreting v as a line of **BS**, we see that v is adjacent to exactly one other vertex w that, viewed as a point, belongs to $\mathcal{O}$. Now, the harmonic triplets corresponding to v and w have a pair of points of $\mathbf{PG}(1, 17)$ in common, and hence they are contained in exactly two classical ovoids. Hence, in order to prove the theorem, it suffices to show that two adjacent vertices v and w are contained in exactly two ovoids of **BS**. By transitivity of the automorphism group on pairs of adjacent vertices, we may take $v = p_0$ and $w = x_0$.

We identify the points of $\mathcal{O}$ with the corresponding vertices of the Biggs-Smith graph.

So suppose $p_0, x_0 \in \mathcal{O}$. It's easy to see that each copy of the "H"-graph contains exactly two vertices of $\mathcal{O}$: one vertex corresponding with $\{p, q, y\}$ together with one vertex corresponding with $\{r, s, x\}$. Hence, r_0, y_0, q_0, s_0 are not in $\mathcal{O}$. Also, the vertices $x_1, p_2, x_{-1}, p_{-2}, p_1, p_{-1}, q_4, q_{-4}$ do not belong to $\mathcal{O}$. Next, we distinguish two cases based on H_1: either $y_1 \in \mathcal{O}$ or $q_1 \in \mathcal{O}$.

If y_1 belongs to $\mathcal{O}$ then q_1 does not. The vertices r_3, r_{-1}, s_{-8} and s_{-7} are not in $\mathcal{O}$. Since y_0 and s_{-8} are not in $\mathcal{O}$ it follows that s_8 is an element of the ovoid. Consequently r_8, x_8, s_7, y_{-1} are not in $\mathcal{O}$. The two vertices of H_{-1} belonging to the ovoid are q_{-1} and s_{-1}. Hence, $x_3, q_7, x_{-5}, q_8, s_{-2}, y_7, y_8$ are not in $\mathcal{O}$. Since q_0 and q_8 do not belong to $\mathcal{O}$, we have that x_4 is an element of $\mathcal{O}$ which implies that r_4, s_4, p_3 and p_5 are not in $\mathcal{O}$. Since x_3 and p_2 are not in the ovoid, the vertex p_4 is element of $\mathcal{O}$. The vertices y_4, x_5 and p_6 are not in $\mathcal{O}$. But the vertex s_3 is element of the ovoid. This means that s_2, y_{-6} and y_{-5} are not contained in the ovoid. Since r_0 and r_4 do not belong to $\mathcal{O}$ the vertex y_2 is element of the ovoid. Hence, q_2, s_{-6} do not belong to $\mathcal{O}$. Also, since x_8 and q_4

are not in $\mathcal{O}$, the vertex q_{-5} is in $\mathcal{O}$. Consequently, p_{-5} and q_3 are not elements of the ovoid. The second vertex of H_3, next to s_3, belonging to the ovoid, is y_3, implying that r_1, r_5 and s_{-5} are not contained in $\mathcal{O}$. It is easy to see that r_{-5} belongs to $\mathcal{O}$. Also x_2 is element of the ovoid since p_1 and p_3 are not. Hence, $y_{-3}, y_{-7}, r_2, q_{-2}$ and q_6 are not vertices of $\mathcal{O}$. We deduce that y_{-2} is in $\mathcal{O}$ and consequently, s_6 and r_{-4} are not in the ovoid. Since y_4 and s_{-5} are not in $\mathcal{O}$ the vertex s_{-4} is an element of the ovoid. We thus have that x_{-4}, y_5 and s_{-3} are not contained in the ovoid. Since x_{-4} and p_{-5} are not in $\mathcal{O}$ the vertex p_{-3} is an element of the ovoid. It is also easy to see that q_5 and s_5 are contained in $\mathcal{O}$. It follows that q_{-3}, x_{-2}, x_{-8} and y_{-4} are not contained in the ovoid. Since x_{-2} and s_{-2} are not in $\mathcal{O}$ it follows that r_{-2} belongs to $\mathcal{O}$. The vertices y_{-4} and q_{-4} are not contained in the ovoid, hence p_{-4} is a vertex in $\mathcal{O}$. So, $\{r_{-6}, x_{-3}, p_{-6}\}$ is not a subset of $\mathcal{O}$. It is clear that r_{-3} is a vertex of $\mathcal{O}$, but r_{-7} is not. We already found four elements of H_{-6} not in $\mathcal{O}$, hence the two remaining vertices, x_{-6} and q_{-6} do belong to the ovoid. Thus, x_7 and p_{-7} are not in $\mathcal{O}$. An analogous reasoning holds for H_{-7}: x_{-7} and q_{-7} belong to $\mathcal{O}$, but x_6 and p_{-8} not. In the same way we deduce that r_{-8}, r_6 and y_6 are in $\mathcal{O}$. Also, we have that p_7 and r_7 are vertices contained in the ovoid, inducing that y_{-8} is not in $\mathcal{O}$ and hence q_{-8} is element of $\mathcal{O}$. Since y_8 and q_8 are both not in $\mathcal{O}$, it follows that p_8 is a vertex of $\mathcal{O}$. The copy H_1 has already one vertex contained in $\mathcal{O}$ and four not contained in $\mathcal{O}$, thus the remaining vertex s_1 is an element of $\mathcal{O}$. We now have examined all vertices of the Biggs-Smith graph and may conclude that the ovoid $\mathcal{O}$ which contains the vertices p_0, x_0 and y_1 is unique.

Now let us consider the second case where next to p_0 and x_0, also q_1 is element of $\mathcal{O}$. The 4-arc transitivity of the Biggs-Smith graph implies that there is an automorphism taking the 4-arc $(x_0, p_0, p_1, x_1, y_1)$ onto the 4-arc $(x_0, p_0, p_1, x_1, q_1)$. Hence there is a unique ovoid containing the vertices x_0, p_0 and q_1.

The theorem is proved. □

References

[1] N. L. Biggs and D. H. Smith, On trivalent graphs, *Bull. Lond. Math. Soc.* **3** (1971), 127 – 131.

[2] H. Van Maldeghem, Ten exceptional geometries from trivalent distance regular graphs, *Ann. Combin.* **6** (2002), 209 – 228.