Algebraic inclusions of Moufang polygons

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Abstract

An inclusion of a Moufang polygon into another is called algebraic if the algebraic structures which describe them can be chosen in such a way that the one is a substructure of the other. We show that an inclusion of Moufang n-gons is always algebraic if $n \in \{3, 6, 8\}$, but that this is not always true when $n = 4$. We classify the algebraic inclusions of Moufang quadrangles in the case where none of the root groups is 2-torsion, which corresponds to the fact that the characteristic of the underlying (skew) field is different from 2. Finally, we show that all full and ideal inclusions of Moufang quadrangles without 2-torsion root groups are algebraic.

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1 Introduction

A generalized polygon is a rank-2 incidence geometry the incidence graph of which has diameter n and girth $2n$ for some $n \geq 3$ (and is then called a generalized n-gon). A generalized polygon is in fact the same as a rank-2 spherical building, and there is a vast literature on these objects. In many circumstances, one is interested in subpolygons of a given generalized polygon, for various reasons. To mention a few, they are used in characterizations of certain of these polygons, they can be used to discover or describe other interesting structures (such as spreads or ovoids), or they can be used in inductive arguments, for example to study embeddings of generalized polygons in projective spaces or other higher rank buildings.

A bit surprising, not too much has been written down on the study of subpolygons by itself. In the finite case, there are some results involving the order of the polygons; see, for example, [10, section 1.8]. The case of the classical compact connected polygons has been dealt with in [11].

In this paper, we will be interested in the case of the generalized polygons satisfying the Moufang condition. Although this condition looks rather restrictive, it is satisfied quite often, and in particular, all classical polygons belong to this class. Moreover, the Moufang polygons have been classified in [9] — but there is no hope to be able to classify all generalized polygons, since there exist free constructions, and even the finite case is still wide open. Two small pieces

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of the study of the inclusion of Moufang polygons have already been done before, namely the inclusion of Moufang octagons [5] and a study of the Moufang polygons which do not have full or ideal subpolygons [10, section 5.9]. It is also noteworthy that Moufang polygons (and spherical buildings of arbitrary rank in general) play an important role in algebraic group theory and related subjects, so our results might have consequences on the existence of subgroups of those groups.

We will start by recalling some definitions and notations, and prepare the setup for our algebraic approach to the problem. Then we will be dealing with the cases of Moufang triangles, hexagons and octagons, which can be completely settled by taking a closer look at the proof of the classification of Moufang polygons in [9]. It goes without saying that we will have to rely heavily on this book. The case of Moufang quadrangles is significantly harder, and it turns out that the inclusion of Moufang polygons does not always translate nicely into the inclusion of the corresponding algebraic structures. However, in many cases, it does, and in particular in the case that the characteristic of the underlying (skew) field is not 2, we show that all inclusions are either algebraic or dual (see below for the exact definitions of these expressions). In the section which follows, we then classify all algebraic inclusions, with the only restriction that we do not allow the characteristic to be equal to 2 — a case which seems to be much harder (although many of our results can be extended to this case as well). In the last section, we describe all full and ideal subquadrangles of a given Moufang quadrangle, and we show that our list is complete.

2 Preliminaries

We start with some definitions.

Definition 2.1. A *generalized n-gon* is a connected bipartite graph with diameter n and girth $2n$. A *generalized polygon* is a generalized n-gon for some finite $n \geq 2$. A generalized polygon Γ is called *thick* if $|\Gamma_x| \geq 3$ for all vertices x of Γ. A circuit of Γ of length $2n$ is called an *apartment* of Γ. A path of length n in Γ is called a *root* or a *half-apartment* of Γ.

Definition 2.2. If $\alpha = (v_0, \dots, v_n)$ is a root of a generalized n-gon Γ, then the group of all automorphisms of Γ which fix all the vertices of $\Gamma_{v_1} \cup \dots \cup \Gamma_{v_{n-1}}$ is called a *root group* of Γ (corresponding to the root α) and is denoted by U_α. If U_α acts regularly on the set of apartments through α, then α is called a *Moufang root*. If all roots of Γ are Moufang roots, then Γ is called a *Moufang n-gon*.

From now on, we assume that Γ is a thick Moufang n-gon for some $n \geq 3$, and we will fix an (arbitrary) apartment Σ which we label by the integers modulo $2n$ such that $i+1 \in \Gamma_i$ and $i+2 \neq i$ for all i. We define $U_i := U_{(i,i+1,\dots,i+n)}$ for all i, and we set $U_{[i,j]} = \langle U_i, U_{i+1}, \dots, U_j \rangle$ for all $i \leq j < i+n$ and $U_{[i,i-1]} = 1$ for all i.

The definitions 2.3–2.6 were introduced in [9]. We present them in a different but equivalent form.

Definition 2.3. Let $\hat{U}_{[1,n]}$ be a group generated by non-trivial subgroups $\hat{U}_1, \dots, \hat{U}_n$ for some $n \geq 3$. The $(n+1)$-tuple $(\hat{U}_{[1,n]}, \hat{U}_1, \dots, \hat{U}_n)$ is called a *root group sequence* if there exists a Moufang n-gon Γ and a labeled apartment $\Sigma =$

$(0, \ldots, 2n-1)$ in Γ such that there exists an isomorphism from $\hat{U}_{[1,n]}$ to $U_{[1,n]}$ mapping $\hat{U}_i$ to U_i for all $i \in \{1, \ldots, n\}$. We will denote this root group sequence by $\Theta(\Gamma, \Sigma)$. The number n will be called the *length* of the root group sequence.

Definition 2.4. If $\Theta = (U_{[1,n]}, U_1, \ldots, U_n)$ is a root group sequence, then $(U_{[1,n]}, U_n, \ldots, U_1)$ is also a root group sequence. It is called the *opposite of* Θ and is denoted by Θ^{op}.

Definition 2.5. Consider two root group sequences $\Theta = (U_{[1,n]}, U_1, \ldots, U_n)$ and $\Theta' = (U'_{[1,n]}, U'_1, \ldots, U'_n)$. An *isomorphism* from Θ to Θ' is an isomorphism from $U_{[1,n]}$ to $U'_{[1,n]}$ mapping U_i to U'_i for all $i \in \{1, \ldots, n\}$. An *anti-isomorphism* from Θ to Θ' is an isomorphism from Θ to $\Theta'^{\,\mathrm{op}}$.

Definition 2.6. Let $\Theta = (U_{[1,n]}, U_1, \ldots, U_n)$ be a root group sequence. For each $i \in \{1, \ldots, n\}$, let U'_i be a non-trivial subgroup of U_i, and let $U'_{[1,n]}$ denote the subgroup of $U_{[1,n]}$ generated by $U'_1, \ldots, U'_n$. If the n-tuple $\Theta' = (U'_{[1,n]}, U'_1, \ldots, U'_n)$ is again a root group sequence, then Θ' will be called a *subsequence* of Θ.

Recently, the classification of Moufang polygons has been completed by J. Tits and R. Weiss in [9]. The following theorem is essential.

Theorem 2.7. *Let Γ be an arbitrary Moufang n-gon. Then:*

(i) $n \in \{3, 4, 6, 8\}$.

(ii) *Let $\Sigma = (0, \ldots, 2n-1)$ be an arbitrary apartment of Γ. Then up to isomorphism, Γ is uniquely determined by the isomorphism class of its root group sequence $\Theta(\Gamma, \Sigma) = (U_{[1,n]}, U_1, \ldots, U_n)$. We denote this Moufang n-gon by by $\Gamma(\Theta)$.*

(iii) *If Θ_1 and Θ_2 are two root group sequences such that $\Gamma(\Theta_1) \cong \Gamma(\Theta_2)$, then Θ_1 and Θ_2 are isomorphic or anti-isomorphic.*

Proof. (i) See [9, (17.1)].

(ii) See [9, (7.6) and (7.7)].

(iii) Suppose that $\Theta_1 = \Theta(\Gamma_1, \Sigma_1)$ and $\Theta_2 = \Theta(\Gamma_2, \Sigma_2)$ for some Moufang n-gons Γ_1 and Γ_2 and some apartments $\Sigma_1 = (0, \ldots, 2n-1)$ and $\Sigma_2 = (0', \ldots, (2n-1)')$ of Γ_1 and Γ_2, respectively. It follows from (ii) that $\Gamma_1 \cong \Gamma(\Theta_1)$ and $\Gamma_2 \cong \Gamma(\Theta_2)$, and hence $\Gamma_1 \cong \Gamma_2$. Let ϕ be an isomorphism from Γ_1 to Γ_2, then ϕ maps Σ_1 to some apartment $\phi(\Sigma_1) = (\phi(0), \ldots, \phi(2n-1))$ of Γ_2. By [9, (4.12)], there exists an automorphism ψ of Γ_2 which maps $\phi(\Sigma_1)$ to Σ_2 and maps the edge $(\phi(n), \phi(n+1))$ to the edge $(n', (n+1)')$. So $\psi \circ \phi$ maps Σ_1 to Σ_2, and either it maps i to i for all i, in which case Θ_1 and Θ_2 are isomorphic, or it maps i to $2n+1-i$ for all i, in which case Θ_1 and Θ_2 are anti-isomorphic.

□

The following theorem defines the fundamental μ-maps, which play a very important role in the whole theory of Moufang polygons, in particular for the Moufang sets defined by opposite root groups in a Moufang polygon.

Theorem 2.8. *For each i and each $a_i \in U_i^*$, there exist a unique element $\mu(a_i) \in U_{i+n}^* a_i U_{i+n}^*$ such that $(i-1)^{\mu(a_i)} = i+1$ and $(i+1)^{\mu(a_i)} = i-1$. This element $\mu(a_i)$ fixes i and $i+n$ and reflects Σ, and $U_j^{\mu(a_i)} = U_{2i+n-j}$ for each $a_i \in U_i^*$ and each j.*

Proof. See [9, (6.1)]. □

By the following theorem, the study of subpolygons of Moufang polygons is equivalent to the study of subsequences of root group sequences. Nevertheless, we will still use the polygons, since the μ-maps which we get from Theorem 2.8 will turn out to be very useful.

Theorem 2.9. (i) *Let Γ_2 be a Moufang n-gon and let Γ_1 be a sub-n-gon of Γ_2. Then Γ_1 is also a Moufang n-gon. If α is an arbitrary root of Γ_1, with corresponding root groups $U_\alpha^{(1)}$ of Γ_1 and $U_\alpha^{(2)}$ of Γ_2, then $U_\alpha^{(1)}$ is a subgroup of $U_\alpha^{(2)}$.*

In particular, let Σ be an arbitrary labeled apartment of Γ_1, then $\Theta_1 := \Theta(\Gamma_1, \Sigma)$ is a subsequence of $\Theta_2 := \Theta(\Gamma_2, \Sigma)$.

(ii) *Let Θ_2 be a root group sequence and let Θ_1 be a subsequence of Θ_2. Then $\Gamma(\Theta_1)$ is isomorphic to a subpolygon of $\Gamma(\Theta_2)$.*

Proof. (i) The fact that Γ_1 is again Moufang is well known; see, for example, [10, Lemma 5.2.2].

Consider an arbitrary root α of Γ_1, and its corresponding root groups $U_\alpha^{(1)}$ of Γ_1 and $U_\alpha^{(2)}$ of Γ_2. Let Σ_a and Σ_b be two apartments of Γ_1 through α. Then there is a unique element ϕ of $U_\alpha^{(2)}$ mapping Σ_a to Σ_b. Now consider the subgraph $\Delta := \Gamma_1 \cap \phi(\Gamma_1)$. Since Γ_1 and $\phi(\Gamma_1)$ have the apartment Σ_b in common, their intersection Δ is again a generalized n-gon (see, for example, [10, Proposition 1.8.4]). Since ϕ is a root elation, it fixes at least one pencil and at least one point row of Γ_1. It follows (see, for example, [10, Proposition 1.8.1]) that Δ is a full and ideal subpolygon of both Γ_1 and $\phi(\Gamma_1)$, and hence (see, for example, [10, Proposition 1.8.2]) Δ, Γ_1 and $\phi(\Gamma_1)$ coincide. We conclude that ϕ stabilizes Γ_1, and hence its restriction to Γ_1 must be the unique element of $U_\alpha^{(1)}$ mapping Σ_a to Σ_b. Since this holds for every pair of apartments Σ_a and Σ_b of Γ_1 through α, we have shown that every element of $U_\alpha^{(1)}$ is the restriction of a unique element of $U_\alpha^{(2)}$ to Γ_1. Hence $U_\alpha^{(1)}$ is a subgroup of $U_\alpha^{(2)}$, for all roots α of Γ_1.

(ii) It follows readily from the construction in [9, (7.1) and (7.2)] that the vertex set X_1 of $\Gamma(\Theta_1)$ can be canonically identified with a subset of the vertex set X_2 of $\Gamma(\Theta_2)$, and that any two elements $x, y \in X_1$ which are adjacent in $\Gamma(\Theta_1)$ are also adjacent in $\Gamma(\Theta_2)$. It follows that any two elements $x, y \in X_1$ which have distance i in $\Gamma(\Theta_1)$ also have distance i in $\Gamma(\Theta_2)$, for all $i \in \{0, \ldots, n\}$; in particular, two elements $x, y \in X_1$ which are non-adjacent in $\Gamma(\Theta_1)$ are also non-adjacent in $\Gamma(\Theta_2)$. We conclude that $\Gamma(\Theta_1)$ is isomorphic to a subpolygon of $\Gamma(\Theta_2)$.

□

Theorem 2.10. *If $\Theta = (U_{[1,n]}, U_1, \ldots, U_n)$ is a root group sequence, then*

(i) $[U_i, U_j] \le U_{[i+1,j-1]}$ *for all* $1 \le i < j \le n$*;*

(ii) *The product map from* $U_1 \times \cdots \times U_n$ *to* $U_{[1,n]}$ *is bijective.*

Proof. See [9, (5.5) and (5.6)]. $\square$

Definition 2.11. Let $a_i \in U_i$ and $a_j \in U_j$ with $i + 2 \le j < i + n$. For each k such that $i < k < j$, we set $[a_i, a_j]_k = a_k$, where a_k is the unique element of U_k appearing in the factorization of $[a_i, a_j] \in U_{[i+1,j-1]}$.

Lemma 2.12. *Let* Γ_2 *be a Moufang* n*-gon and let* Γ_1 *be a sub-*n*-gon of* Γ_2*; let* Σ *be an arbitrary labeled apartment of* Γ_1*. Then the* $\mu^{(1)}$*-maps defined by Theorem* 2.8 *with respect to* Γ_1 *are the restriction of the* $\mu^{(2)}$*-maps defined with respect to* Γ_2*, to the root groups of* Γ_1*.*

Proof. Note that, by Theorem 2.9.(i), the root groups of Γ_1 are indeed subgroups of the root groups of Γ_2, so the statement of this lemma makes sense.

But this same fact implies that every element $\mu^{(1)}(a_i) \in (U^{(1)}_{i+n})^* \cdot a_i \cdot (U^{(1)}_{i+n})^*$ with $a_i \in (U^{(1)}_i)^*$ is also an element of $(U^{(2)}_{i+n})^* \cdot a_i \cdot (U^{(2)}_{i+n})^*$, and by the uniqueness of the $\mu^{(2)}$-maps in Theorem 2.8, this element has to be equal to $\mu^{(2)}(a_i)$. $\square$

For each possible value of n, we will now use the appropriate algebraic structure to describe an arbitrary Moufang n-gon, and we will redo certain steps of the classification of Moufang n-gons, but for both the Moufang n-gon and its sub-n-gon simultaneously, and make some appropriate choices during the proof.

3 Subtriangles of Moufang triangles

We start with the study of all possible subtriangles of a given Moufang triangle. This is the easiest case, since Moufang triangles have a very simple description, as was already shown in 1933 (but in a slightly different form; see [1] or [3]) by R. Moufang (see [7]):

Definition 3.1. Let $(A, +, \cdot)$ be an arbitrary alternative division ring, and let U_1, U_2 and U_3 be three groups parametrized by $(A, +)$ via some (group) isomorphisms x_1, x_2 and x_3. Let U_+ be the group generated by U_1, U_2 and U_3 with respect to the commutator relations

$$[U_1, U_2] = [U_2, U_3] = 1 \ ,$$
$$[x_1(s), x_3(t)] = x_2(s \cdot t) \ ,$$

for all $s, t \in A$. Then $\Theta = (U_+, U_1, U_2, U_3)$ is a root group sequence of length 3; it is unique up to isomorphism (i.e., it does not depend on the choice of x_1, x_2 and x_3), and will be denoted by $\Theta_T(A)$. We also say that Θ is *parametrized* by A via the isomorphisms x_1, x_2 and x_3.

Theorem 3.2. *Let* Γ *be an arbitrary Moufang triangle. Then there exists an alternative division ring* $(A, +, \cdot)$ *such that* $\Gamma \cong \Gamma(\Theta_T(A))$*.*

Proof. See [9, (17.2)]. $\square$

Theorem 3.3. *Let* Γ_1 *and* Γ_2 *be two Moufang triangles. Then* Γ_1 *is isomorphic to a subtriangle of* Γ_2 *if and only if there exists an alternative division ring* $\tilde{A}$ *and a subring* A *of* $\tilde{A}$ *such that* $\Gamma_1 \cong \Gamma(\Theta_T(A))$ *and* $\Gamma_2 \cong \Gamma(\Theta_T(\tilde{A}))$*.*

Proof. (i) Suppose that Γ_1 is a subtriangle of Γ_2; let $\Sigma = (0, \ldots, 5)$ be an arbitrary labeled apartment of Γ_1. We will write U_i and $\tilde{U}_i$ in place of $U_i^{(1)}$ and $U_i^{(2)}$, respectively, to denote the root groups of Γ_1 and Γ_2 with respect to the labeled apartment Σ. By Theorem 2.9.(i), $U_i \leq \tilde{U}_i$ for all i.

By [9, (19.4)], $\tilde{U}_1$ is abelian, and we choose an additive group $\tilde{A}$ isomorphic to $\tilde{U}_1$ and an isomorphism $t \mapsto x_1(t)$ from $\tilde{A}$ to $\tilde{U}_1$. Let $A := x_1^{-1}(U_1)$; then A is an additive group isomorphic to U_1. We now choose arbitrary elements $e_1 \in U_1^*$ and $e_3 \in U_3^*$, and for every $t \in \tilde{A}$, we let

$$x_2(t) := x_1(t)^{\mu(e_3)} \quad \text{and} \quad x_3(t) := x_2(t)^{\mu(e_1)} \ ,$$

where μ is defined by Theorem 2.8 with respect to Γ_2. By Lemma 2.12 however, $U_2 = U_1^{\mu(e_3)}$ and $U_3 = U_2^{\mu(e_1)}$ as well, and hence $U_i = x_i(A)$ for all $i \in \{1, 2, 3\}$.

Following [9, (19.6)], we now define a multiplication on $\tilde{A}$ by defining $uv = u \cdot v$ to be the unique element of $\tilde{A}$ such that

$$[x_1(u), x_3(v)] = x_2(uv) \ ,$$

for all $u, v \in \tilde{A}$. Since $U_i = x_i(A)$, it follows that A is also closed under this multiplication. By [9, (19.7)], the left and right distributive laws hold in $\tilde{A}$, and therefore also in A. Now let $1 \in \tilde{A}^*$ denote the element $x_1^{-1}(e_1)$; then $1 \in A^*$ as well. By [9, (19.9) and (19.13)], both $\tilde{A}$ and A are alternative division rings with unit 1. In particular, A is a subring of $\tilde{A}$, and $\Gamma_1 \cong \Gamma(\Theta_T(A))$ and $\Gamma_2 \cong \Gamma(\Theta_T(\tilde{A}))$.

(ii) Let $\Theta_2 = \Theta_T(\tilde{A})$ be the root group sequence parametrized by $\tilde{A}$ via some isomorphisms x_1, x_2 and x_3. Now let $\Theta_1 = \Theta_T(A)$ be the root group sequence parametrized by A via the restriction of these same isomorphisms x_1, x_2 and x_3 to A. Then Θ_1 is a subsequence of Θ_2, and hence, by Theorem 2.9.(ii), $\Gamma(\Theta_1)$ is isomorphic to a subtriangle of $\Gamma(\Theta_2)$.

□

4 Subhexagons of Moufang hexagons

We postpone the case of Moufang quadrangles for a while, and we now consider subhexagons of a given Moufang hexagon. All Moufang hexagons can be parametrized by an anisotropic cubic norm structure, which is more often called an hexagonal system is this context (see [9, (15.15)]):

Definition 4.1. Let $\Xi = (J, F, \sharp)$ be an arbitrary hexagonal system with norm N, trace T, (Freudenthal) cross product $\times$ and unit $1 \in J^*$. Let U_1, U_3 and U_5 be three groups parametrized by J via some isomorphisms x_1, x_3 and x_5, and let U_2, U_4 and U_6 be three groups parametrized by the additive group of F via some isomorphisms x_2, x_4 and x_6. Let U_+ be the group generated by $U_1, \ldots, U_6$

with respect to the commutator relations

$$\begin{aligned}
[U_1, U_2] = [U_1, U_4] = [U_2, U_3] = [U_2, U_4] = [U_2, U_5] = 1 \ , \\
[U_3, U_4] = [U_3, U_6] = [U_4, U_5] = [U_4, U_6] = [U_5, U_6] = 1 \ ,
\end{aligned}$$

$$\begin{aligned}
[x_1(a), x_3(b)] &= x_2(T(a,b)) \ , \\
[x_3(a), x_5(b)] &= x_4(T(a,b)) \ , \\
[x_1(a), x_5(b)] &= x_2(-T(a^\sharp, b)) x_3(a \times b) x_4(T(a, b^\sharp)) \ , \\
[x_2(t), x_6(u)] &= x_4(tu) \ , \\
[x_1(a), x_6(t)] &= x_2(-tN(a)) x_3(ta^\sharp) x_4(t^2 N(a)) x_5(-ta) \ ,
\end{aligned}$$

for all $a, b \in J$ and all $t, u \in F$. Then $\Theta = (U_+, U_1, U_2, U_3, U_4, U_5, U_6)$ is a root group sequence of length 6; it is unique up to isomorphism (i.e., it does not depend on the choice of the maps x_i), and will be denoted by $\Theta_H(\Xi)$. We also say that Θ is *parametrized* by Ξ via the isomorphisms x_i.

Theorem 4.2. *Let Γ be an arbitrary Moufang hexagon. Then there exists an hexagonal system $\Xi = (J, F, \sharp)$ such that $\Gamma \cong \Gamma(\Theta_H(\Xi))$.*

Proof. See [9, (17.5)]. □

Definition 4.3. Let $\tilde{\Xi} = (\tilde{J}, \tilde{F}, \tilde{\sharp})$ be an hexagonal system. Then we say that an hexagonal system $\Xi = (J, F, \sharp)$ is a *subsystem* of $\tilde{\Xi}$ if $F \subseteq \tilde{F}$, $J \subseteq \tilde{J}$, the scalar multiplication $F \times J \to J$ is the restriction of the scalar multiplication $\tilde{F} \times \tilde{J} \to \tilde{J}$, and $\sharp$ is the restriction of $\tilde{\sharp}$ to J.

Theorem 4.4. *Let Γ_1 and Γ_2 be two Moufang hexagons. Then Γ_1 is isomorphic to a subhexagon of Γ_2 if and only if there exists an hexagonal system $\tilde{\Xi}$ and a subsystem Ξ of $\tilde{\Xi}$ such that $\Gamma_1 \cong \Gamma(\Theta_H(\Xi))$ and $\Gamma_2 \cong \Gamma(\Theta_H(\tilde{\Xi}))$.*

Proof. (i) Suppose that Γ_1 is a subhexagon of Γ_2; let $\Sigma = (0, \dots, 9)$ be an arbitrary labeled apartment of Γ_1. We will write U_i and $\tilde{U}_i$ in place of $U_i^{(1)}$ and $U_i^{(2)}$, respectively, to denote the root groups of Γ_1 and Γ_2 with respect to the labeled apartment Σ. By Theorem 2.9.(i), $U_i \leq \tilde{U}_i$ for all i.

After relabeling the apartment Σ if necessary, we get, by [9, (29.11)], that $\Delta = (U_2 U_4 U_6, U_2, U_4, U_6)$ and $\tilde{\Delta} = (\tilde{U}_2 \tilde{U}_4 \tilde{U}_6, \tilde{U}_2, \tilde{U}_4, \tilde{U}_6)$ are root group sequences (of length 3); in particular, Δ is a subsequence of $\tilde{\Delta}$, and therefore $\Gamma(\Delta)$ is isomorphic to a subtriangle of $\Gamma(\tilde{\Delta})$, by Theorem 2.9.(ii). By Theorem 3.3, there exists an alternative division ring $\tilde{F}$ and a subring F such that $\Gamma(\Delta) \cong \Gamma(\Theta_T(F))$ and $\Gamma(\tilde{\Delta}) \cong \Gamma(\Theta_T(\tilde{F}))$. By the argument following [9, (29.14)] however, $\tilde{F}$ is a commutative field. Using the fact that $\Theta_T(F)^{\mathrm{op}} = \Theta_T(F)$ for every commutative field F, it follows from Theorem 2.7.(iii) that $\Delta \cong \Theta_T(F)$ and $\tilde{\Delta} \cong \Theta_T(\tilde{F})$.

We now choose arbitrary elements $e_1 \in U_1^*$ and $e_6 \in U_6^*$. By [9, (29.15)], there exist isomorphisms $t \mapsto x_i(t)$ from $\tilde{F}$ to $\tilde{U}_i$ for $i \in \{2, 4, 6\}$ such that $x_6(1) = e_6$, $x_6(t)^{\mu(e_1)} = x_2(t)$, $x_2(t)^{\mu(e_1)} = x_6(-t)$ and $[x_2(t), x_6(u)] = x_4(tu)$, for all $t, u \in \tilde{F}$. In particular, this last identity holds for all $t, u \in F$, and hence, by definition of the operator Θ_T, we have that $\Delta \cong \Theta_T(F) \cong (x_2(F) x_4(F) x_6(F), x_2(F), x_4(F), x_6(F))$. Hence we may assume that $U_i = x_i(F)$ for all $i \in \{2, 4, 6\}$.

As in [9, (29.16)], we choose an additive group $\tilde{J}$ isomorphic to $\tilde{U}_1$ and an isomorphism $a \mapsto x_1(a)$ from $\tilde{J}$ to $\tilde{U}_1$. Let $x_5(a) := x_1(-a)^{\mu(e_6)}$ and $x_3(a) := x_5(a)^{\mu(e_1)}$, for all $a \in \tilde{J}$. Let $J := x_1^{-1}(U_1)$. Then $U_i = x_i(J)$ for $i \in \{1, 3, 5\}$.

Now let $(t, a) \mapsto ta$ be the map from $\tilde{F} \times \tilde{J}$ to $\tilde{J}$ defined so that

$$[x_1(a), x_6(t)^{-1}]_5 = x_5(ta) \tag{1}$$

for all $t \in \tilde{F}$ and all $a \in \tilde{J}$, and let $\sharp : \tilde{J} \to \tilde{J}$ be the map defined by setting

$$[x_1(a), e_6]_3 = x_3(a^\sharp)\,, \tag{2}$$

for all $a, b \in \tilde{J}$. Then it is shown in [9, Chapter 29] that $\tilde{J}$ is a vector space over $\tilde{F}$ with scalar multiplication given by $(t, a) \mapsto ta$, that $\tilde{\Xi} = (\tilde{J}, \tilde{F}, \sharp)$ is an hexagonal system, and that $\Gamma_2 \cong \Gamma(\Theta_H(\tilde{\Xi}))$. Since $[x_1(J), x_6(F)]_5 = [U_1, U_6]_5 \subseteq U_5$, it follows from (1) that $F \cdot J = J$, and since $[x_1(J), e_6]_3 = [U_1, e_6]_3 \subseteq U_3$, it follows from (2) that $J^\sharp \subseteq J$. So by applying the same arguments, we can also conclude that $\Xi = (J, F, \sharp)$ is an hexagonal system, and that $\Gamma_1 \cong \Gamma(\Theta_H(\Xi))$. Clearly, Ξ is a subsystem of $\tilde{\Xi}$.

(ii) Let $\Xi = (J, F, \sharp)$ and $\tilde{\Xi} = (\tilde{J}, \tilde{F}, \sharp)$. Let $\Theta_2 = \Theta_H(\tilde{\Xi})$ be the root group sequence parametrized by $\tilde{\Xi}$ via some isomorphisms x_i. Now let $\Theta_1 = \Theta_H(\Xi)$ be the root group sequence parametrized by Ξ via the restriction of these same isomorphisms x_1, x_3 and x_5 to J and the restriction of x_2, x_4 and x_6 to F. Then Θ_1 is a subsequence of Θ_2, and hence, by Theorem 2.9.(ii), $\Gamma(\Theta_1)$ is isomorphic to a subhexagon of $\Gamma(\Theta_2)$.

□

5 Suboctagons of Moufang octagons

We now consider suboctagons of a given Moufang octagon. Although this has already been solved in [5, Theorem B], we also give a complete proof of this result, to illustrate that our approach also works for the case of Moufang octagons.

All Moufang octagons can be parametrized by a so-called octagonal set (see [9, (10.11)]):

Definition 5.1. Let (K, σ) be an arbitrary octagonal set, let U_1, U_3, U_5 and U_7 be four groups parametrized by the additive group of K via some isomorphisms x_1, x_3, x_5 and x_7, and let U_2, U_4, U_6 and U_8 be four groups parametrized by $K_\sigma^{(2)}$ via some isomorphisms x_2, x_4, x_6 and x_8. Let U_+ be the group generated by $U_1, \ldots, U_8$ with respect to certain commutator relations which can be found in [9, (16.9)]. Then $\Theta = (U_+, U_1, \ldots, U_8)$ is a root group sequence of length 8; it is unique up to isomorphism (i.e., it does not depend on the choice of the maps x_i), and will be denoted by $\Theta_O(K, \sigma)$. We also say that Θ is *parametrized* by (K, σ) via the isomorphisms x_i.

Theorem 5.2. *Let Γ be an arbitrary Moufang octagon. Then there exists an octagonal set (K, σ) such that $\Gamma \cong \Gamma(\Theta_O(K, \sigma))$.*

Proof. See [9, (17.7)]. □

Definition 5.3. Let $(\tilde{K},\tilde{\sigma})$ be an octagonal set. Then we say that an octagonal set (K,σ) is a *subset* of $(\tilde{K},\tilde{\sigma})$ if $K \subseteq \tilde{K}$ and if σ is the restriction of $\tilde{\sigma}$ to K.

Theorem 5.4. *Let Γ_1 and Γ_2 be two Moufang octagons. Then Γ_1 is isomorphic to a suboctagon of Γ_2 if and only if there exists an octagonal set $(\tilde{K},\sigma)$ and a subset (K,σ) of $(\tilde{K},\sigma)$ such that $\Gamma_1 \cong \Gamma(\Theta_O(K,\sigma))$ and $\Gamma_2 \cong \Gamma(\Theta_O(\tilde{K},\sigma))$.*

Proof. (i) Suppose that Γ_1 is a suboctagon of Γ_2; let $\Sigma = (0,\ldots,11)$ be an arbitrary labeled apartment of Γ_1. We will write U_i and $\tilde{U}_i$ in place of $U_i^{(1)}$ and $U_i^{(2)}$, respectively, to denote the root groups of Γ_1 and Γ_2 with respect to the labeled apartment Σ. By Theorem 2.9.(i), $U_i \leq \tilde{U}_i$ for all i.

After relabeling the apartment Σ if necessary, we get, by [9, (31.8)], that $\Delta = (U_1U_3U_5U_7, U_1, U_3, U_5, U_7)$ is the root group sequences of an indifferent Moufang quadrangle.

Let $V_i := [U_{i-2}, U_{i+1}]$ and $\tilde{V}_i := [\tilde{U}_{i-2}, \tilde{U}_{i+1}]$ for all even i. By [9, (31.16) and (31.29)], this definition of V_i and $\tilde{V}_i$ coincides with the definition given in [9, (31.1)]. Clearly, $V_i \leq \tilde{V}_i$ for all even i.

We now choose arbitrary elements $e_1 \in U_1^*$ and $e_8 \in V_8^*$. By [9, (31.24)], there exists an octagonal set $(\tilde{K},\sigma)$, and isomorphisms $t \mapsto x_i(t)$ from $\tilde{K}$ to $\tilde{U}_i$ for $i \in \{1,3,5,7\}$ such that $x_1(1) = e_1$,

$$x_i(t)^{\mu(e_8)} = x_{8-i}(t) \tag{3}$$

for $i \in \{1,3,5,7\}$ and for all $t \in \tilde{K}$, and

$$[x_1(t), x_7(u)] = x_3(t^\sigma u)x_5(tu^\sigma) \tag{4}$$

for all $t,u \in \tilde{K}$. Moreover, we let

$$x_9(t) := x_1(t)^{\mu(e_1)} \tag{5}$$

for all $t \in \tilde{K}$.

Now let $K := x_1^{-1}(U_1)$; then $(K,+)$ is an additive subgroup of $(\tilde{K},+)$ isomorphic to U_1. From (3) for $i = 1$, we get that $x_7(K) = x_1(K)^{\mu(e_8)} = U_1^{\mu(e_8)} = U_7$, and similarly, it follows from (5) that $x_9(K) = U_9$. It follows from (4) that $x_3(K) = [e_1, x_7(K)]_3 = [e_1, U_7]_3 = U_3$, and by (3) with $i = 3$, we get that $x_5(K) = U_5$ as well.

By [9, (31.9.ii) and (31.32)], we have that $\mu(x_1(t)) = x_9(t^{-1})x_1(t)x_9(t^{-1})$ for all $t \in \tilde{K}$. If we restrict this identity to K, then it follows from the fact that $\mu(a_1) \in U_9^* a_1 U_9^*$ for all $a_1 \in U_1^*$ and from Lemma 2.12 that K is closed under inverses.

Let $x_6(t) := [x_3(t), e_8]$ for all $t \in \tilde{K}$. By the argument following [9, (31.25)], the map $a_3 \mapsto [a_3, e_8]$ is an isomorphism from $\tilde{U}_3$ to $\tilde{V}_6$, and hence x_6 is an isomorphism from $\tilde{K}$ to $\tilde{V}_6$. By the same argument, the map $a_3 \mapsto [a_3, e_8]$ restricted to U_3 is an isomorphism from U_3 to V_6, and since $U_3 = x_3(K)$, the restriction of x_6 to K is an isomorphism from K to V_6. If we now set $x_4(t) := x_6(t)^{\mu(e_1)}$, $x_2(t) := x_6(t)^{\mu(e_8)}$ and $x_8(t) := x_2(t)^{\mu(e_1)}$ for all $t \in \tilde{K}$, then $\tilde{V}_i = x_i(\tilde{K})$ and $V_i = x_i(K)$ for all $i \in \{2,4,6,8\}$.

By [9, (31.26.ii)], we have that $[x_1(t), x_6(u)] = x_4(tu)$ for all $t,u \in \tilde{K}$. If we restrict this identity to $t,u \in K$, then it follows from the fact that

$[U_1, V_6] = V_4$ that K is closed under multiplication, and hence K is a subfield of $\tilde{K}$. Moreover, it follows from (4) that $K^\sigma \subseteq K$, and hence (K, σ) is a subset of $(\tilde{K}, \sigma)$.

By the remainder of the classification result in [9, Chapter 31], we get that $\Gamma_1 \cong \Gamma(\Theta_O(K, \sigma))$ and $\Gamma_2 \cong \Gamma(\Theta_O(\tilde{K}, \sigma))$.

(ii) Let $\Theta_2 = \Theta_O(\tilde{K}, \sigma)$ be the root group sequence parametrized by $(\tilde{K}, \sigma)$ via some isomorphisms x_i. Now let $\Theta_1 = \Theta_O(K, \sigma)$ be the root group sequence parametrized by (K, σ) via the restriction of these same isomorphisms x_1, x_3, x_5 and x_7 to K and the restriction of x_2, x_4, x_6 and x_8 to $K_\sigma^{(2)}$. Then Θ_1 is a subsequence of Θ_2, and hence, by Theorem 2.9.(ii), $\Gamma(\Theta_1)$ is isomorphic to a suboctagon of $\Gamma(\Theta_2)$.

□

6 Subquadrangles of Moufang quadrangles

We now consider subquadrangles of a given Moufang quadrangle. In this case, the result is not as nice as in the other cases; in particular, it is not true that every inclusion of Moufang quadrangles can be described by the inclusion of the corresponding algebraic structures, as we will see in Theorem 6.6. The situation is not too bad, however, as will be illustrated by Lemma 6.15.

All Moufang quadrangles can be parametrized by a so-called quadrangular system (see [2]):

Definition 6.1. Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be an arbitrary quadrangular system with corresponding biadditive maps F and H; let U_1 and U_3 be two groups parametrized by W via some isomorphisms x_1 and x_3, and let U_2 and U_4 be two groups parametrized by V via some isomorphisms x_2 and x_4. Let U_+ be the group generated by U_1, U_2, U_3 and U_4 with respect to the commutator relations

$$\begin{aligned} [U_1, U_2] = [U_2, U_3] &= [U_3, U_4] = 1 \ , \\ [x_1(w_1), x_3(w_2)] &= x_2(H(w_1, w_2)) \ , \\ [x_2(v_1), x_4(v_2)] &= x_3(F(v_1, v_2)) \ , \\ [x_1(w), x_4(v)] &= x_2(vw)x_3(wv) \ , \end{aligned}$$

for all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$, where we have denoted the maps τ_V and τ_W by juxtaposition, i.e. $vw := \tau_V(v, w)$ and $wv := \tau_W(w, v)$ for all $v \in V$ and all $w \in W$. Then $\Theta = (U_+, U_1, U_2, U_3, U_4)$ is a root group sequence of length 4; it is unique up to isomorphism (i.e., it does not depend on the choice of the maps x_i), and will be denoted by $\Theta_Q(\Omega)$. We also say that Θ is *parametrized* by Ω via the isomorphisms x_i.

Theorem 6.2. *Let Γ be an arbitrary Moufang quadrangle. Then there exists a quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ such that $\Gamma \cong \Gamma(\Theta_Q(\Omega))$.*

Proof. See [2, Section 5]. □

Definition 6.3. Let Γ be a Moufang quadrangle, and let $\Sigma = (0, \ldots, 7)$ be a labeled apartment of Γ. Let $U_0, \ldots, U_7$ be the root groups associated to Σ. Then we write $V_i := [U_{i-1}, U_{i+1}] \le U_i$ for all i, and we let $Y_i := C_{U_i}(U_{i-2}) \le U_i$ for each i. By [9, (21.20.i)], we have $Y_i = C_{U_i}(U_{i+2})$ as well.

Theorem 6.4. *By relabeling the vertices of Σ by the transformation $i \mapsto 5-i$ if necessary, we can assume that*

(i) $Y_i \neq 1$, $[U_i, U_i] \leq V_i \leq Y_i \leq Z(U_i)$ *for all odd* i *;*

(ii) U_i *is abelian for all even* i *.*

Proof. See [9, (21.28)]. □

Definition 6.5. Let $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \tilde{\epsilon}, \tilde{\delta})$ be a quadrangular system. Then we say that a quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a *subsystem* of $\tilde{\Omega}$ if $V \subseteq \tilde{V}$, $W \subseteq \tilde{W}$, $\epsilon = \tilde{\epsilon}$, $\delta = \tilde{\delta}$, and if τ_V and τ_W are the restriction of $\tau_{\tilde{V}}$ and $\tau_{\tilde{W}}$ to $V \times W$ and $W \times V$, respectively.

Theorem 6.6. *Let Γ_1 and Γ_2 be two Moufang quadrangles.*

(i) *Suppose that Γ_1 is a subquadrangle of Γ_2. Let Σ be an apartment of Γ_1, labeled in such a way that the statements of Theorem 6.4 hold for the root groups $\tilde{U}_i$ of Γ_2. If $Y_1 \cap \tilde{Y}_1 \neq 1$, and if one of the conditions*

  (a) $\tilde{Y}_4 = 1$,

  (b) $\tilde{Y}_4 \neq 1$, $Y_4 = 1$ *and* $U_4 \cap \tilde{Y}_4 \neq 1$,

  (c) $\tilde{Y}_4 \neq 1$, $Y_4 \neq 1$ *and* $Y_4 \cap \tilde{Y}_4 \neq 1$,

is satisfied, then there exists a quadrangular system $\tilde{\Omega}$ and a subsystem Ω of $\tilde{\Omega}$ such that $\Gamma_1 \cong \Gamma(\Theta_Q(\Omega))$ and $\Gamma_2 \cong \Gamma(\Theta_Q(\tilde{\Omega}))$.

(ii) *If $\Gamma_1 \cong \Gamma(\Theta_Q(\Omega))$ and $\Gamma_2 \cong \Gamma(\Theta_Q(\tilde{\Omega}))$ for some quadrangular system $\tilde{\Omega}$ and some subsystem Ω of $\tilde{\Omega}$, then Γ_1 is isomorphic to a subquadrangle of Γ_2.*

Proof. (i) Suppose that Γ_1 is a subquadrangle of Γ_2. Let $\Sigma = (0, \ldots, 7)$ be an apartment of Γ_1, labeled in such a way that the statements of Theorem 6.4 hold for the root groups of Γ_2. As before, we will write U_i and $\tilde{U}_i$ in place of $U_i^{(1)}$ and $U_i^{(2)}$, respectively, to denote the root groups of Γ_1 and Γ_2 with respect to the labeled apartment Σ, and by Theorem 2.9.(i), $U_i \leq \tilde{U}_i$ for all i.

We first show that the statements of Theorem 6.4 also hold for the root groups U_i of Γ_1. By applying this theorem on Γ_1, we see that either the statements hold for the given labeling, or they hold for the labeling transformed by the map $i \mapsto 5-i$. We may assume the latter. Then U_i is abelian for all odd i, and it is then obvious that $[U_i, U_i] \leq V_i$ and $Y_i \leq Z(U_i)$. By Definition 6.3, the statement $V_i \leq Y_i$ is equivalent to $[[U_{i-1}, U_{i+1}], U_{i-2}] = 1$; hence it follows from $\tilde{V}_i \leq \tilde{Y}_i$ that $V_i \leq Y_i$ for all odd i. Finally, it follows from the assumption $Y_1 \cap \tilde{Y}_1 \neq 1$ that $Y_i \neq 1$ for all odd i, and it follows from the fact that $\tilde{U}_i$ is abelian for all even i that U_i is abelian for all even i. So the statements of Theorem 6.4 hold for the given labeling of the root groups U_i, after all.

By Theorem 6.4.(ii), $\tilde{U}_4$ is abelian, so choose a group $(\tilde{V}, +)$ isomorphic to U_4 and an isomorphism $v \mapsto x_4(v)$ from $\tilde{V}$ to $\tilde{U}_4$, and choose a (possibly non-abelian) group $(\tilde{W}, \boxplus)$ isomorphic to U_1 and an isomorphism $w \mapsto x_1(w)$ from $\tilde{W}$ to $\tilde{U}_1$. Let $V := x_4^{-1}(U_4)$ and let $W := x_1^{-1}(U_1)$.

Since $Y_1 \cap \tilde{Y}_1 \neq 1$, we can choose an element $e_1 \in Y_1^* \cap \tilde{Y}_1^*$; let $\delta := x_1^{-1}(e_1)$. If we are in case (a) and $Y_4 = 1$, then choose $e_4 \in U_4^*$ arbitrarily; if $Y_4 \neq 1$, then choose $e_4 \in Y_4^*$ arbitrarily. In both cases, $e_4 \in \tilde{U}_4^*$ as well. If we are in case (b), then choose $e_4 \in U_4^* \cap \tilde{Y}_4^*$. Finally, if we are in case (c), then choose $e_4 \in Y_4^* \cap \tilde{Y}_4^*$. Let $\epsilon := x_1^{-1}(e_4)$. Then δ and ϵ satisfy the assumptions which are required in [2, Section 5], for both Γ_1 and Γ_2.

We now set $x_3(w) := [x_1(w), e_4^{-1}]_3$ and $x_5(w) := x_1(w)^{\mu(e_1)}$ for all $w \in \tilde{W}$ and $x_2(v) := [e_1, x_4(v)^{-1}]_2$ and $x_0(v) := x_4(v)^{\mu(e_4)}$ for all $v \in \tilde{V}$. Then $\tilde{U}_i = x_i(\tilde{V})$ and $U_i = x_i(V)$ for $i \in \{0, 2, 4\}$, and $\tilde{U}_i = x_i(\tilde{W})$ and $U_i = x_i(W)$ for $i \in \{1, 3, 5\}$.

As in [2, Section 5], we define a map $\tau_{\tilde{V}}$ from $\tilde{V} \times \tilde{W}$ to $\tilde{V}$ and a map $\tau_{\tilde{W}}$ from $\tilde{W} \times \tilde{V}$ to $\tilde{W}$, both of which are usually denoted by $\cdot$ or by juxtaposition, by setting

$$[x_1(w), x_4(v)^{-1}]_2 = x_2(\tau_{\tilde{V}}(v, w)) = x_2(vw)\,,$$
$$[x_1(w), x_4(v)^{-1}]_3 = x_3(\tau_{\tilde{W}}(w, v)) = x_3(wv)\,,$$

for all $w \in \tilde{W}$ and all $v \in \tilde{V}$. It follows from the previous paragraph that $\tau_{\tilde{V}}(V, W) \subseteq V$ and $\tau_{\tilde{W}}(W, V) \subseteq W$. We denote these restrictions of $\tau_{\tilde{V}}$ and $\tau_{\tilde{W}}$ by τ_V and τ_W, respectively.

By the remaining part of [2, Section 5], $\tilde{\Omega} := (\tilde{V}, \tilde{W}, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \epsilon, \delta)$ and $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ are quadrangular systems, $\Gamma_1 \cong \Gamma(\Theta_Q(\Omega))$ and $\Gamma_2 \cong \Gamma(\Theta_Q(\tilde{\Omega}))$. It is clear from the previous paragraph that Ω is a subsystem of $\tilde{\Omega}$.

(ii) Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_V, \tau_W, \epsilon, \delta)$. Let $\Theta_2 = \Theta_Q(\tilde{\Omega})$ be the root group sequence parametrized by $\tilde{\Omega}$ via some isomorphisms x_i. Now let $\Theta_1 = \Theta_Q(\Omega)$ be the root group sequence parametrized by Ω via the restriction of these same isomorphisms x_1 and x_3 to W and the restriction of x_2 and x_4 to V. Then Θ_1 is a subsequence of Θ_2, and hence, by Theorem 2.9.(ii), $\Gamma(\Theta_1)$ is isomorphic to a subquadrangle of $\Gamma(\Theta_2)$. □

Definition 6.7. Let Γ_1 and Γ_2 be two Moufang quadrangles, and suppose that Γ_1 is a subquadrangle of Γ_2. Let Σ be an apartment of Γ_1, labeled in such a way that the statements of Theorem 6.4 hold for the root groups $\tilde{U}_i$ of Γ_2. Suppose that the statements of Theorem 6.4 do *not* hold for the root groups U_i of Γ_1; then by this theorem, they do hold after the relabeling $i \mapsto 5 - i$. Then we say that Γ_1 is *dually included* in Γ_2. Note that this definition is independent of the choice of Σ.

If there exists a quadrangular system $\tilde{\Omega}$ and a subsystem Ω of $\tilde{\Omega}$ such that $\Gamma_1 \cong \Gamma(\Theta_Q(\Omega))$ and $\Gamma_2 \cong \Gamma(\Theta_Q(\tilde{\Omega}))$, then we say that Γ_1 is *algebraically included* in Γ_2.

Remark 6.8. In section 8, we will give an example of an inclusion which is not algebraic but which is dual (8.1), as well as an example which is neither algebraic nor dual (8.2).

Definition 6.9. A quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is called *indifferent* if $F \equiv 0$ and $H \equiv 0$, *reduced* if $F \not\equiv 0$ and $H \equiv 0$, *co-reduced* if $F \equiv 0$

and $H \not\equiv 0$ and *wide* if $F \not\equiv 0$ and $H \not\equiv 0$. A Moufang quadrangle is called *indifferent*, *reduced* or *wide* if it is parametrized by a quadrangular system which is indifferent, (co-)reduced or wide, respectively.

Remark 6.10. If $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a co-reduced quadrangular system, then $\Omega^* := (W, V, \tau_W, \tau_V, \delta, \epsilon)$ is a reduced quadrangular system; see [2, Theorem 8.12]. Note that this can only occur if V and W are both 2-torsion groups, i.e. every element of V and W has order 1 or 2.

Lemma 6.11. (i) *If $\delta \in W$ has order 2 (in particular, if W is a 2-torsion group), then V is a 2-torsion group.*

(ii) *If V contains an element of order 2, then V is a 2-torsion group.*

Proof. From the defining axioms ($\mathbf{Q}_9$) and ($\mathbf{Q}_{12}$) in [2, Section 2], we get that $v(\delta \boxplus \delta) = v + v$ for all $v \in V$.

(i) If δ has order 2, then $v + v = v(\delta \boxplus \delta) = 0$ for all $v \in V$.

(ii) Let $c \in V^*$ be an element for which $c + c = 0$, then $c(\delta \boxplus \delta) = 0$, and hence $\delta \boxplus \delta = 0$, so by (i), V is a 2-torsion group.

□

Lemma 6.12. *If $\mathrm{Rad}(F) \neq 0$, then V and W are 2-torsion groups.*

Proof. See [2, Lemma 8.10]. □

The quadrangular systems have been classified; see [9, (17.4)] or [2]. It turns out that there are six different classes of quadrangular systems, which we will list now. We refer to [2] for more details about their definition.

Theorem 6.13. *Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be an arbitrary quadrangular system, and assume that Ω is not co-reduced. Then (at least) one of the following holds:*

(i) *Ω is indifferent, and $\Omega \cong \Omega_D(K, K_0, L_0)$ for some indifferent set (K, K_0, L_0). We say that Ω is of* indifferent type.

(ii) *Ω is reduced, and $\Omega \cong \Omega_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ). We say that Ω is of* involutory type.

(iii) *Ω is reduced, and $\Omega \cong \Omega_Q(K, V_0, q)$ for some anisotropic quadratic space (K, V_0, q). We say that Ω is of* quadratic form type.

(iv) *Ω is wide, and $\Omega \cong \Omega_P(K, K_0, \sigma, V_0, \pi)$ for some anisotropic pseudo-quadratic space $(K, K_0, \sigma, V_0, \pi)$. We say that Ω is of* pseudo-quadratic form type.

(v) *Ω is wide, and $\Omega \cong \Omega_E(K, V_0, q)$ for some quadratic space (K, V_0, q) of type E_6, E_7 or E_8. We say that Ω is of* type E_6, E_7 or E_8.

(vi) *Ω is wide, and $\Omega \cong \Omega_F(K, V_0, q)$ for some quadratic space (K, V_0, q) of type F_4. We say that Ω is of* type F_4.

Moreover, if we are in case (i) *or* (vi)*, then V and W are 2-torsion groups.*

Remark 6.14. It is obvious from the defining commutator relations in 6.1 that a (co-)reduced quadrangle can never be included in an indifferent quadrangle, and that a wide quadrangle cannot be included in a (co-)reduced quadrangle.

The conditions in Theorem 6.6.(i) look very restrictive, but in fact, they are satisfied quite often:

Lemma 6.15. *Let Γ_1 and Γ_2 be two Moufang quadrangles, and suppose that Γ_1 is a subquadrangle of Γ_2. Let Σ be an apartment of Γ_1, labeled in such a way that the statements of Theorem 6.4 hold for the root groups $\tilde{U}_i$ of Γ_2.*

(i) *If $\tilde{U}_4$ is not a 2-torsion group, then the inclusion of Γ_1 in Γ_2 is either algebraic or dual.*

(ii) *If Γ_2 is of indifferent type or of involutory type, or if $\Gamma_2 \cong \Gamma(\Omega_Q(K, V_0, q))$ for some* regular *quadratic form q, then the inclusion of Γ_1 in Γ_2 is algebraic.*

Proof. (i) Suppose that the inclusion is not dual, so that the statements of Theorem 6.4 hold for the root groups U_i of Γ_1 as well. Since $\tilde{U}_4$ is not 2-torsion, Lemma 6.11.(ii) implies that U_4 is not 2-torsion either. We are in one of the cases (ii), (iii), (iv) or (v) of Theorem 6.13, and in each case, the fact that $U_4 \cong V$ is not 2-torsion implies that the defining (skew) field K has characteristic different from 2. It is easily checked from their definition that $\mathrm{Im}(F) = \mathrm{Rad}(H) \neq 0$ in each case, and hence $V_1 = Y_1 \neq 1$. Similarly, $\tilde{V}_1 = \tilde{Y}_1 \neq 1$. Since $V_i \leq \tilde{V}_i$ for all i, it follows that $Y_1 \cap \tilde{Y}_1 \neq 1$.

By Lemma 6.12, $\mathrm{Rad}(\tilde{F}) = 0$, hence $\tilde{Y}_4 = 1$, so condition (a) of Theorem 6.6.(i) is satisfied. In particular, Γ_1 is algebraically included in Γ_2.

(ii) If Γ_2 is indifferent or reduced, then $[\tilde{U}_1, \tilde{U}_3] = 1$, and in particular, $[U_1, U_3] = 1$. However, this is equivalent to $\tilde{Y}_1 = \tilde{U}_1$ and $Y_1 = U_1$, respectively, and in particular, $Y_1 \cap \tilde{Y}_1 = U_1 \neq 1$.

If Γ_2 is indifferent, then the same argument applies to $\tilde{U}_4$, and in particular $Y_4 \neq 1$, $\tilde{Y}_4 \neq 1$ and $Y_4 \cap \tilde{Y}_4 = U_4 \neq 1$, so condition (c) of Theorem 6.6.(i) is satisfied.

If Γ_2 is of involutory type but not indifferent, then $\mathrm{Rad}(\hat{F}) = 0$. If $\Gamma_2 \cong \Gamma(\Omega_Q(K, V_0, q))$ for some quadratic form q with corresponding bilinear form f, then $\mathrm{Rad}(\hat{F}) = \mathrm{Rad}(f)$, so the condition on q to be regular implies that $\mathrm{Rad}(\hat{F}) = 0$ in this case as well. In both cases, $\tilde{Y}_4 = 1$, so condition (a) of Theorem 6.6.(i) is satisfied.

In all these cases, we can conclude by Theorem 6.6.(i) that Γ_1 is algebraically included in Γ_2.

□

7 Algebraic inclusions of Moufang quadrangles

In this section, we will always assume that Γ_1 and Γ_2 are Moufang quadrangles such that Γ_1 is algebraically included in Γ_2. Moreover, we will assume that none of the root groups is 2-torsion. The goal of this section is to classify these inclusions.

By Definition 6.7, there exists a quadrangular system $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \epsilon, \delta)$ and a subsystem $\Omega = (V, W, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \epsilon, \delta)$ of $\tilde{\Omega}$ such that $\Gamma_1 \cong \Gamma(\Theta_Q(\Omega))$ and $\Gamma_2 \cong \Gamma(\Theta_Q(\tilde{\Omega}))$. Since the root groups are not 2-torsion, neither are the groups V, W, $\tilde{V}$ and $\tilde{W}$. In particular, we are in one (or more than one) of the cases (ii), (iii), (iv) or (v) of Theorem 6.13.

Definition 7.1. A quadrangular system which is of involutory type but not of quadratic form type will be called of *proper involutory type*. An involutory set (K, K_0, σ) is *proper* if and only if $\sigma \neq 1$ and K is generated by K_0 as a ring. By [9, (21.10)], a quadrangular system $\Omega_I(K, K_0, \sigma)$ is of proper involutory type if and only if (K, K_0, σ) is proper.

Definition 7.2. A quadrangular system of pseudo-quadratic form type is not necessarily wide. If it is, then it is called of *proper pseudo-quadratic form type.*

Remark 7.3. If a quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is of involutory type $\Omega = \Omega_I(K, K_0, \sigma)$, and V and W are not 2-torsion, then $\mathrm{char}(K) \neq 2$; in particular, $K_0 = \mathrm{Fix}_K(\sigma)$, and Ω is completely determined by K and σ. Hence we will denote the involutory set by (K, σ) in this case, and we will write $\Omega = \Omega_I(K, \sigma)$.

By [9, (21.14)], (K, σ) is always proper, unless $\mathrm{Fix}_K(\sigma)$ is a commutative field F, and either $K = F$ and $\sigma = 1$, or K is a separable quadratic extension over F (and then $\sigma \in \mathrm{Gal}(K/F)^*$), or K is a quaternion division algebra over F (and then σ is the standard involution of K/F).

Remark 7.4. If a quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is of pseudo-quadratic form type $\Omega = \Omega_P(K, K_0, \sigma, V_0, \pi)$, and V and W are not 2-torsion, then $\mathrm{char}(K) \neq 2$; in particular, $K_0 = \mathrm{Fix}_K(\sigma)$, and Ω is completely determined by K, σ, V_0 and π. Hence we will write $\Omega = \Omega_P(K, \sigma, V_0, \pi)$.

By [9, (21.16)], $\Omega_P(K, \sigma, V_0, \pi)$ with $\mathrm{char}(K) \neq 2$ is of proper pseudo-quadratic form type if and only if $V_0 \neq 0$ and $\sigma \neq 1$.

Using Theorem 6.13, we can now conclude that Ω and $\tilde{\Omega}$ are of exactly one of the following types: proper involutory, quadratic form, proper pseudo-quadratic form, E_6, E_7 or E_8. We have summarized the different combinations in Table 1, and we will consider each of the cases separately.

$\leq$	$\tilde{\Omega}_I^{\text{proper}}$	$\tilde{\Omega}_Q$	$\tilde{\Omega}_P^{\text{proper}}$	$\tilde{\Omega}_E$
Ω_I^{proper}	7.5	7.10	7.8	7.11
Ω_Q	7.18	7.6	7.20	7.21
Ω_P^{proper}	6.14	6.14	7.9	7.23
Ω_E	6.14	6.14	7.22	7.23

Table 1: Algebraic inclusions of Moufang quadrangles

From now on, assume that $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \tilde{\epsilon}, \tilde{\delta})$ are quadrangular systems with V, W, $\tilde{V}$ and $\tilde{W}$ not 2-torsion. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if there exist group monomorphisms

$\phi : V \hookrightarrow \tilde{V}$ and $\psi : W \hookrightarrow \tilde{W}$ such that

$$\phi(\epsilon) = \tilde{\epsilon}\,, \tag{6}$$

$$\psi(\delta) = \tilde{\delta}\,, \tag{7}$$

$$\phi(v)\psi(w) = \phi(vw)\,, \tag{8}$$

$$\psi(w)\phi(v) = \psi(wv)\,, \tag{9}$$

for all $v \in V$ and all $w \in W$. In each case, the groups V, W, $\tilde{V}$ and $\tilde{W}$ will be parametrized by a certain algebraic structure, and we will denote the isomorphisms by square brackets, as in [2].

Theorem 7.5. *Let $\Omega \cong \Omega_I(K,\sigma)$ and $\tilde{\Omega} \cong \Omega_I(\tilde{K},\tilde{\sigma})$ for some involutory sets (K,σ) and $(\tilde{K},\tilde{\sigma})$.*

(i) *Suppose that (K,σ) is proper. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if there exists a field monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$.*

(ii) *Suppose that (K,σ) is not proper. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if there exists a field monomorphism or a field anti-monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$.*

Proof. Let $F := \mathrm{Fix}_K(\sigma)$ and let $\tilde{F} := \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$. By the definition of the operator Ω_I, we have that $V = [K]$, $W = [F]$, $\tilde{V} = [\tilde{K}]$ and $\tilde{W} = [\tilde{F}]$. If (K,σ) is not proper, then we are in one of the three cases of Remark 7.3, so in particular $F \leq Z(K)$ and $t^\sigma t = tt^\sigma$ for all $t \in K$; it follows from these two observations that $tst^\sigma = t^\sigma st$ for all $t \in K$ and all $s \in F$.

First assume that there exists a field monomorphism α from K into $\tilde{K}$ or — but only if (K,σ) is not proper — a field anti-monomorphism α from K into $\tilde{K}$, such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$. We define a map ϕ from $V = [K]$ to $\tilde{V} = [\tilde{K}]$ and a map ψ from $W = [F]$ to $\tilde{W} = [\tilde{F}]$ by setting $\phi[t] := [\alpha(t)]$ and $\psi[s] := [\alpha(s)]$ for all $t \in K$ and all $s \in F$. Since α is an (additive) monomorphism, so are ϕ and ψ. Moreover,

$$\phi(\epsilon) = \phi[1] = [\tilde{1}] = \tilde{\epsilon}\,,$$
$$\psi(\delta) = \psi[1] = [\tilde{1}] = \tilde{\delta}\,,$$

and if α is a field monomorphism, then

$$\phi[t]\psi[s] = [\alpha(t)][\alpha(s)] = [\alpha(s)\alpha(t)] = [\alpha(st)] = \phi[st] = \phi([t][s])\,,$$
$$\psi[s]\phi[t] = [\alpha(s)][\alpha(t)] = [\alpha(t)^{\tilde{\sigma}}\alpha(s)\alpha(t)] = [\alpha(t^\sigma st)] = \psi[t^\sigma st] = \psi([s][t])\,,$$

for all $t \in K$ and all $s \in F$, whereas if α is a field anti-monomorphism and (K,σ) is not proper, then $ts = st$ and $tst^\sigma = t^\sigma st$ for all $t \in K$ and all $s \in F$, and therefore

$$\phi[t]\psi[s] = [\alpha(t)][\alpha(s)] = [\alpha(s)\alpha(t)] = [\alpha(ts)] = [\alpha(st)] = \phi[st] = \phi([t][s])\,,$$
$$\begin{aligned}\psi[s]\phi[t] &= [\alpha(s)][\alpha(t)] = [\alpha(t)^{\tilde{\sigma}}\alpha(s)\alpha(t)] \\ &= [\alpha(tst^\sigma)] = [\alpha(t^\sigma st)] = \psi[t^\sigma st] = \psi([s][t])\,,\end{aligned}$$

for all $t \in K$ and all $s \in F$. In both cases, we can conclude that Ω is isomorphic to a subsystem of $\tilde{\Omega}$.

Now assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map α from K to $\tilde{K}$ and a map β from F to $\tilde{F}$ by setting $\phi[t] := [\alpha(t)]$ and $\psi[s] := [\beta(s)]$ for all $t \in K$ and all $s \in F$. Since ϕ and ψ are group monomorphisms, α and β are additive monomorphisms as well. The conditions (6)–(9) translate into the following:

$$\alpha(1) = \tilde{1}\,, \tag{10}$$

$$\beta(1) = \tilde{1}\,, \tag{11}$$

$$\beta(s)\alpha(t) = \alpha(st)\,, \tag{12}$$

$$\alpha(t)^{\tilde{\sigma}}\beta(s)\alpha(t) = \beta(t^{\sigma}st)\,, \tag{13}$$

for all $t \in K$ and all $s \in F$. If we set $t = 1$ in (12), then we get that β is the restriction of α to F. By (12) again, $\alpha(s)\alpha(t) = \alpha(st)$ for all $t \in K$ and all $s \in F$. If we set $s = 1$ in (13), then we get that $\alpha(t)^{\tilde{\sigma}}\alpha(t) = \alpha(t^{\sigma}t)$ for all $t \in K$. If we replace t by $t + 1$ and subtract the original identity, then it follows that $\alpha(t)^{\tilde{\sigma}} + \alpha(t) = \alpha(t^{\sigma} + t)$, and since α is additive, we get that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$.

If (K, σ) is proper, then K is generated (as a ring) by F, and it follows by induction on the identity $\alpha(s)\alpha(t) = \alpha(st)$ for all $t \in K$ and all $s \in F$ that α is multiplicative on K, hence it is a field monomorphism.

If (K, σ) is not proper, we have to proceed in a different way. It follows from (12) and (13) that $\alpha(t^{\sigma}s)\alpha(t) = \alpha(t^{\sigma}st)$ for all $t \in K$ and all $s \in F$. If we set $s = (t^{\sigma})^{-1}t^{-1} \in F$, then it follows that $\alpha(t^{-1})\alpha(t) = \alpha(1) = \tilde{1}$, and hence α preserves inverses. It then follows from Hua's identity

$$aba = a - (a^{-1} + (b^{-1} - a)^{-1})^{-1}$$

that $\alpha(aba) = \alpha(a)\alpha(b)\alpha(a)$ for all $a, b \in K$, and hence α is a Jordan homomorphism. It follows from a result by Jacobson and Rickart [4] that α is a homomorphism or an anti-homomorphism. Since we already now that α is injective, the proof is finished. □

Theorem 7.6. *Let $\Omega \cong \Omega_Q(K, V_0, q)$ and $\tilde{\Omega} \cong \Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q})$ for some anisotropic quadratic spaces (K, V_0, q) and $(\tilde{K}, \tilde{V}_0, \tilde{q})$ with base points e and $\tilde{e}$, respectively. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if there exists a vector space monomorphism (β, α) from (K, V_0) into $(\tilde{K}, \tilde{V}_0)$ such that $\alpha(e) = \tilde{e}$ and $\tilde{q}(\alpha(v)) = \beta(q(v))$ for all $v \in V_0$.*

Proof. By the definition of the operator Ω_Q, we have that $V = [V_0]$, $W = [K]$, $\tilde{V} = [\tilde{V}_0]$ and $\tilde{W} = [\tilde{K}]$.

First assume that there exists a vector space monomorphism (β, α) from (K, V_0) into $(\tilde{K}, \tilde{V}_0)$ such that $\alpha(e) = \tilde{e}$ and $\tilde{q}(\alpha(v)) = \beta(q(v))$ for all $v \in V_0$. We define a map ϕ from $V = [V_0]$ to $\tilde{V} = [\tilde{V}_0]$ and a map ψ from $W = [K]$ to $\tilde{W} = [\tilde{K}]$ by setting $\phi[v] := [\alpha(v)]$ and $\psi[t] := [\beta(t)]$ for all $v \in V_0$ and all $t \in K$.

Since α and β are additive monomorphisms, so are ϕ and ψ. Moreover,

$$\phi(\epsilon) = \phi[e] = [\tilde{e}] = \tilde{\epsilon}\,,$$
$$\psi(\delta) = \psi[1] = [\tilde{1}] = \tilde{\delta}\,,$$
$$\phi[v]\psi[t] = [\alpha(v)][\beta(t)] = [\beta(t)\alpha(v)] = [\alpha(tv)] = \phi[tv] = \phi([v][t])\,,$$
$$\psi[t]\phi[v] = [\beta(t)][\alpha(v)] = [\beta(t)\tilde{q}(\alpha(v))] = [\beta(t)\beta(q(v))] = \psi[tq(v)] = \psi([t][v])\,,$$

for all $v \in V_0$ and all $t \in K$, and we can conclude that Ω is isomorphic to a subsystem of $\tilde{\Omega}$.

Now assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map α from V_0 to $\tilde{V}_0$ and a map β from K to $\tilde{K}$ by setting $\phi[v] := [\alpha(v)]$ and $\psi[t] := [\beta(t)]$ for all $t \in K$ and all $s \in F$. Since ϕ and ψ are group monomorphisms, α and β are additive monomorphisms as well. The conditions (6)–(9) translate into the following:

$$\alpha(e) = \tilde{e}\,, \tag{14}$$
$$\beta(1) = \tilde{1}\,, \tag{15}$$
$$\beta(t)\alpha(v) = \alpha(tv)\,, \tag{16}$$
$$\beta(t)\tilde{q}(\alpha(v)) = \beta(tq(v))\,, \tag{17}$$

for all $v \in V_0$ and all $t \in K$. It only remains to show that β is multiplicative; it will then follow from (16) that (β, α) is a vector space morphism, and the condition on the quadratic forms follows from (17) with $t = 1$. But by repeated use of (16), we get that $\beta(s)\beta(t)\alpha(e) = \beta(s)\alpha(te) = \alpha(ste) = \beta(st)\alpha(e)$ for all $s, t \in K$, and hence β is multiplicative. □

Remark 7.7. Since the *Moufang quadrangles* arising from anisotropic quadratic spaces (K, V_0, q) and $(K, V_0, \gamma q)$ are isomorphic, for every value of $\gamma \in K^*$, the base points e and $\tilde{e}$ in Theorem 7.6 can be chosen arbitrarily (there is no restriction, since $\mathrm{char}(K) \neq 2$, and hence $\mathrm{Rad}(f) = 0$). Therefore the condition $\alpha(e) = \tilde{e}$ is not really a restriction.

Theorem 7.8. *Let $\Omega \cong \Omega_I(K, \sigma)$ and $\tilde{\Omega} \cong \Omega_P(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$ for some involutory set (K, σ) and some anisotropic pseudo-quadratic space $(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if Ω is already isomorphic to a subsystem of $\Omega_I(\tilde{K}, \tilde{\sigma})$.*

Proof. Let $F := \mathrm{Fix}_K(\sigma)$ and let $\tilde{F} := \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$. Denote the skew-hermitian form corresponding to $\tilde{\Omega}$ by $\tilde{h}$. By the definition of the operators Ω_I and Ω_P, we have that $V = [K]$, $W = [F]$, $\tilde{V} = [\tilde{K}]$ and $\tilde{W} = [\tilde{T}]$, where $(\tilde{T}, \boxplus)$ is the group with underlying set $\{(a, t) \in \tilde{V}_0 \times \tilde{K} \mid \tilde{\pi}(a) - t \in \tilde{F}\}$, and with group action $(a, t) \boxplus (b, s) = (a + b, t + s + \tilde{h}(b, a))$ for all $(a, t), (b, s) \in \tilde{T}$.

First assume that Ω is isomorphic to a subsystem of $\Omega_I(\tilde{K}, \tilde{\sigma})$. Since $\Omega_I(\tilde{K}, \tilde{\sigma})$ is obviously a subsystem of $\tilde{\Omega}$, it then follows that Ω is isomorphic to a subsystem of $\tilde{\Omega}$.

So assume now that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map α from K to $\tilde{K}$, a map β from F to $\tilde{V}_0$ and a map γ from F to $\tilde{K}$ by setting $\phi[t] := [\alpha(t)]$ and $\psi[s] := [\beta(s), \gamma(s)] \in [\tilde{T}]$ for all $t \in K$ and all $s \in F$. Since ϕ and ψ are group morphisms, α and β are additive morphisms as well (but note that it does not follow that γ is additive). The conditions (7) and (9) imply the following:

$$\beta(1) = 0\,, \tag{18}$$

$$\beta(s)\alpha(t) = \beta(t^\sigma st)\,, \tag{19}$$

for all $t \in K$ and all $s \in F$. We only need to show that $\beta = 0$, since it will then follow that $\psi(F) \leq (0, \tilde{F})$, and hence Ω is in fact isomorphic to a subsystem of $\Omega_I(\tilde{K}, \tilde{\sigma})$. If we set $s = 1$ in (19), then we get that $\beta(t^\sigma t) = 0$ for all $t \in K$. Replacing t by $t+1$ and subtracting the original equation, we get that $\beta(t + t^\sigma) = 0$ for all $t \in K$. But since $\mathrm{char}(K) \neq 2$, this implies that $\beta(s) = \beta((s/2) + (s/2)^\sigma) = 0$ for all $s \in F$, and we are done. $\square$

Theorem 7.9. *Let $\Omega \cong \Omega_P(K, \sigma, V_0, \pi)$ and $\tilde{\Omega} \cong \Omega_P(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$ for some proper anisotropic pseudo-quadratic spaces (K, σ, V_0, π) and $(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$. Denote the skew-hermitian forms corresponding to Ω and $\tilde{\Omega}$ by h and $\tilde{h}$, respectively. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if there exists a vector space monomorphism (β, α) from (K, V_0) into $(\tilde{K}, \tilde{V}_0)$ such that $\beta \circ \sigma = \tilde{\sigma} \circ \beta$ and $\beta(h(a,b)) = \tilde{h}(\alpha(a), \alpha(b))$ for all $a, b \in V_0$.*

Proof. Let $F := \mathrm{Fix}_K(\sigma)$ and let $\tilde{F} := \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$. By the definition of the operator Ω_P, we have that $V = [K]$, $W = [T]$, $\tilde{V} = [\tilde{K}]$ and $\tilde{W} = [\tilde{T}]$, where the groups $(T, \boxplus)$ and $(\tilde{T}, \boxplus)$ are defined as in the previous theorem.

First assume that there exists a vector space monomorphism (β, α) from (K, V_0) into $(\tilde{K}, \tilde{V}_0)$ such that $\beta \circ \sigma = \tilde{\sigma} \circ \beta$ and $\beta(h(a,b)) = \tilde{h}(\alpha(a), \alpha(b))$ for all $a, b \in V_0$. We define a map ϕ from $V = [K]$ to $\tilde{V} = [\tilde{K}]$ and a map ψ from $W = [T]$ to $\tilde{W} = [\tilde{T}]$ by setting $\phi[s] := [\beta(s)]$ and $\psi[a,t] := [\alpha(a), \beta(t)]$ for all $(a,t) \in T$ and all $s \in K$. Since α and β are additive monomorphisms, so are ϕ and ψ, because of the condition that $\beta(h(a,b)) = \tilde{h}(\alpha(a), \alpha(b))$ for all $a, b \in V_0$. Moreover,

$$\begin{aligned}
\phi(\epsilon) &= \phi[1] = [\tilde{1}] = \tilde{\epsilon}\,, \\
\psi(\delta) &= \psi[0,1] = [0, \tilde{1}] = \tilde{\delta}\,, \\
\phi[s]\psi[a,t] &= [\beta(s)][\alpha(a), \beta(t)] = [\beta(t)\beta(s)] = [\beta(ts)] = \phi[ts] = \phi([s][a,t])\,, \\
\psi[a,t]\phi[s] &= [\alpha(a), \beta(t)][\beta(s)] = [\alpha(a)\beta(s), \beta(s)^{\tilde{\sigma}}\beta(t)\beta(s)] \\
&= [\alpha(as), \beta(s^\sigma ts)] = \psi[as, s^\sigma ts] = \psi([a,t][s])\,,
\end{aligned}$$

for all $s \in K$ and all $(a,t) \in T$, and we can conclude that Ω is isomorphic to a subsystem of $\tilde{\Omega}$.

Now assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map β from K to $\tilde{K}$, a map α from T to $\tilde{V}_0$ and a map γ from T to $\tilde{K}$ by setting $\phi[t] := [\beta(t)]$ and $\psi[a,t] := [\alpha(a,t), \gamma(a,t)] \in [\tilde{T}]$ for all $t \in K$ and all $(a,t) \in T$. If we restrict ψ to $[0, F]$, then we are back in the situation of Theorem 7.8, and therefore β is a field monomorphism from K into $\tilde{K}$ such that $\beta \circ \sigma = \tilde{\sigma} \circ \beta$; moreover, $\alpha(0,t) = 0$ and $\gamma(0,t) = \beta(t)$ for all $t \in F$.

Define $\alpha(a) := \alpha(a, \pi(a))$ for all $a \in V_0$. Since ψ is a group morphism, so is

$\alpha : T \to \tilde{V}_0$, and hence

$$\begin{aligned}\alpha(a,t) &= \alpha((a,\pi(a)) \boxplus (0, t-\pi(a))) \\ &= \alpha(a,\pi(a)) + \alpha(0, t-\pi(a)) \\ &= \alpha(a,\pi(a)) = \alpha(a)\end{aligned}$$

for all $(a,t) \in T$. Again using the fact that ψ is a group morphism, we have that

$$\gamma(a+b, t+s+h(b,a)) = \gamma(a,t) + \gamma(b,s) + \tilde{h}(\alpha(b), \alpha(a)) \tag{20}$$

for all $(a,t),(b,s) \in T$. By condition (8), we have that $\gamma(a,t)\beta(s) = \beta(ts)$ for all $s \in K$ and all $(a,t) \in T$. Since β is multiplicative, we get that $\gamma(a,t) = \beta(t)$ for all $(a,t) \in T$. If we apply this on equation (20), then we get, using the fact that β is additive, that $\beta(h(b,a)) = \tilde{h}(\alpha(b),\alpha(a))$ for all $a,b \in V_0$. It follows from (9) that $\alpha(a)\beta(t) = \alpha(at)$ for all $a \in V_0$ and all $t \in K$, so (β,α) is a vector space morphism.

It only remains to show that $\alpha : V_0 \to \tilde{V}_0$ is injective. So suppose that $\alpha(a) = \alpha(b)$ for some $a,b \in V_0$. Note that $(a,\pi(a))$ and $(b,\pi(b))$ are contained in T. Since $\psi[T] \leq [\tilde{T}]$, we have that $[\alpha(a), \beta(\pi(a))] = \psi[a,\pi(a)] \in [\tilde{T}]$, and hence $\tilde{\pi}(\alpha(a)) - \beta(\pi(a)) \in \tilde{F}$, and similarly $\tilde{\pi}(\alpha(b)) - \beta(\pi(b)) \in \tilde{F}$. Since $\alpha(a) = \alpha(b)$, this implies that $\beta(\pi(b)-\pi(a)) \in \tilde{F}$. It follows that $\beta(\pi(b)-\pi(a))$ is fixed under $\tilde{\sigma}$, and hence $\pi(b)-\pi(a)$ is fixed under σ since $\beta \circ \sigma = \tilde{\sigma} \circ \beta$ and since β is injective. So $\pi(b)-\pi(a) \in F$, and hence $(a,\pi(b)) \in T$. But then

$$\psi[a,\pi(b)] = [\alpha(a), \beta(\pi(b))] = [\alpha(b), \beta(\pi(b))] = \psi[b,\pi(b)] \ ,$$

and since ψ is injective, it follows that $a = b$. We conclude that α is injective, and we are done. □

Theorem 7.10. *Let $\Omega \cong \Omega_I(K,\sigma)$ and $\tilde{\Omega} \cong \Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q})$ for some proper involutory set (K,σ) and some anisotropic quadratic space $(\tilde{K}, \tilde{V}_0, \tilde{q})$ with base point $\tilde{e}$. Then Ω cannot be isomorphic to a subsystem of $\tilde{\Omega}$.*

Proof. Let $F := \mathrm{Fix}_K(\sigma)$. By the definition of the operators Ω_I and Ω_Q, we have that $V = [K]$, $W = [F]$, $\tilde{V} = [\tilde{V}_0]$ and $\tilde{W} = [\tilde{K}]$.

Assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map α from K to $\tilde{V}_0$ and a map β from F to $\tilde{K}$ by setting $\phi[t] := [\alpha(t)]$ and $\psi[s] := [\beta(s)]$ for all $t \in K$ and all $s \in F$. Since ϕ and ψ are group monomorphisms, α and β are additive monomorphisms as well. The conditions (6)–(9) translate into the following:

$$\alpha(1) = \tilde{e}\,, \tag{21}$$
$$\beta(1) = \tilde{1}\,, \tag{22}$$
$$\beta(s)\alpha(t) = \alpha(st)\,, \tag{23}$$
$$\beta(s)\tilde{q}(\alpha(t)) = \beta(t^\sigma st)\,, \tag{24}$$

for all $t \in K$ and all $s \in F$. If we set $t = 1$ in (23), then we get that $\alpha(s) = \beta(s)\tilde{e}$ for all $s \in F$. By induction using (23) again, $\alpha(s_1 \cdots s_n) = \beta(s_1)\cdots\beta(s_n)\tilde{e}$ for all $s_1,\ldots,s_n \in F$. Since (K,σ) is proper, K is generated (as a ring) by F, and it follows that β can be extended to a field morphism $\hat{\beta} : K \to \tilde{K}$

such that $\alpha(t) = \hat{\beta}(t)\tilde{e}$. Since α is injective, so is $\hat{\beta}$. It then follows from (24) that $\hat{\beta}(s)\hat{\beta}(t)^2 = \hat{\beta}(t^\sigma)\hat{\beta}(s)\hat{\beta}(t)$ for all $t \in K$ and all $s \in F$, and hence $\hat{\beta}(t^\sigma) = \hat{\beta}(t)$ for all $t \in K$. Since $\hat{\beta}$ is injective, it follows that $t^\sigma = t$ for all $t \in K$, which contradicts the properness of (K, σ). Hence Ω cannot be isomorphic to a subsystem of $\tilde{\Omega}$. $\square$

Theorem 7.11. *Let $\Omega \cong \Omega_I(K, \sigma)$ and $\tilde{\Omega} \cong \Omega_E(\tilde{K}, \tilde{V}_0, \tilde{q})$ for some proper involutory set (K, σ) and some quadratic space $(\tilde{K}, \tilde{V}_0, \tilde{q})$ of type E_6, E_7 or E_8. Then Ω cannot be isomorphic to a subsystem of $\tilde{\Omega}$.*

Proof. Let $F := \mathrm{Fix}_K(\sigma)$. By the definition of the operators Ω_I and Ω_E, we have that $V = [K]$, $W = [F]$, $\tilde{V} = [\tilde{V}_0]$ and $\tilde{W} = [\tilde{S}]$, where $(\tilde{S}, \boxplus)$ is the (non-abelian) group with underlying set $X_0 \times K$ as defined in [9, (16.6)]. (See [9] for more details; we only mention that X_0 is a certain vector space over K and that the group operation $\boxplus$ is additive on the X_0-component.)

Assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map β from K to $\tilde{V}_0$, a map α from F to $\tilde{X}_0$ and a map γ from F to $\tilde{K}$ by setting $\phi[t] := [\beta(t)]$ and $\psi[s] := [\alpha(s), \gamma(s)] \in [\tilde{S}]$ for all $t \in K$ and all $s \in F$. Since ϕ and ψ are group morphisms, α and β are additive morphisms as well, By (7), $\psi[1] = [0, 1]$, so in particular $\alpha(1) = 0$. Condition (9) implies that $\alpha(t^\sigma st) = \alpha(s)\beta(t)$ for all $t \in K$ and all $s \in F$. If we set $s = 1$, then it follows that $\alpha(t^\sigma t) = 0$ for all $t \in K$. Replacing t by $t + 1$ and subtracting the original equation, we obtain $\alpha(t + t^\sigma) = 0$ for all $t \in K$, and hence $\alpha(F) = 0$ since $\mathrm{char}(K) \neq 2$. It follows that $\psi(W) \leq [0, K]$, so in fact Ω is isomorphic to a subsystem of $\Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q})$. But since Ω is of proper involutory type, this contradicts Theorem 7.10, so Ω cannot be isomorphic to a subsystem of $\tilde{\Omega}$. $\square$

The next case we will deal with is by far the most interesting. We will first recall some definitions and facts from the theory of quadratic forms.

Definition 7.12. Let (K, V_0, q) be an anisotropic regular quadratic space (where K has arbitrary characteristic). Then we define the *Clifford algebra* $C(V_0, q) := T(V_0)/I(V_0, q)$, where $T(V_0)$ is the tensor algebra of the vector space V_0, i.e. $T(V_0) := K \oplus V_0 \oplus (V_0 \otimes V_0) \oplus (V_0 \otimes V_0 \otimes V_0) \oplus \cdots$, and $I(V_0, q)$ is the ideal $\langle u \otimes u - q(u) \cdot 1 \mid u \in V_0 \rangle$ of $T(V_0)$. The multiplication in $C(V_0, q)$ is usually denoted by juxtaposition, so in particular $uu = q(u) \in K$ for all $u \in V_0$. The *even Clifford algebra* $C_0(V_0, q)$ is the subalgebra of $C(V_0, q)$ generated by the set $\{uv \mid u, v \in V_0\}$. The Clifford algebra and the even Clifford algebra admit a *canonical involution* $\tau : v_1 v_2 \cdots v_k \mapsto v_k v_{k-1} \ldots v_1$ for all $v_1, \ldots, v_k \in V_0$. If $\dim_K V_0 = n$, then $\dim_K C(V_0, q) = 2^n$ and $\dim_K C_0(V_0, q) = 2^{n-1}$. Both the Clifford algebra and the even Clifford algebra are either simple, or the direct sum of two isomorphic simple algebras.

Definition 7.13. Let (K, V_0, q) be an anisotropic regular quadratic space (where K has arbitrary characteristic) with base point $e \in V_0^*$; denote the bilinear form corresponding to q by f. Let $\overline{v} := f(e, v)e - v$ for all $v \in V_0$. Then we define the *Clifford algebra with base point* $C(V_0, q, e) := T(V_0)/I(V_0, q, e)$, where $I(V_0, q, e)$ is the ideal $\langle e - 1, u \otimes \overline{u} - q(u) \cdot 1 \mid u \in V_0 \rangle$ of $T(V_0)$. The multiplication in $C(V_0, q, e)$ will also be denoted by juxtaposition, so in particular $e = 1$ and $u\overline{u} = q(u) \in K$ for all $u \in V_0$. The Clifford algebra with base point admits a *canonical involution* $\tau_e : v_1 v_2 \cdots v_k \mapsto \overline{v_k}\,\overline{v_{k-1}} \ldots \overline{v_1}$ for all $v_1, \ldots, v_k \in V_0$. If $\dim_K V_0 = n$, then $\dim_K C(V_0, q, e) = 2^{n-1}$.

Lemma 7.14. *Let (K, V_0, q) be an anisotropic regular quadratic space with base point $e \in V_0^*$. Then $C_0(V_0, q) \cong C(V_0, q, e)$. More precisely, there is a K-linear isomorphism $\chi : C(V_0, q, e) \to C_0(V_0, q)$ such that $\chi(u) = eu$ for all $u \in V_0$. Moreover, $\chi \circ \tau_e = \tau \circ \chi$.*

Proof. See, for example, [9, (12.51)], except for the last statement, which can be checked by a straightforward calculation. □

The following definition will become clear in Theorem 7.18:

Definition 7.15. Let (K, V_0, q) be an anisotropic regular quadratic space with $\mathrm{char}(K) \neq 2$. Then we will say that (K, V_0, q) is *involutoric* if one of the following two conditions is satisfied:

(i) $C_0(V_0, q) \cong D$ for some division algebra D;

(ii) $\dim_K V_0 \equiv 0 \pmod 4$ and $C_0(V_0, q) \cong D \oplus D$ for some division algebra D.

In case (i), let τ be the canonical involution of $C_0(V_0, q)$; in case (ii), it follows from [6, (8.4)] that the canonical involution of $C_0(V_0, q)$ maps each of the two components two itself, and hence induces an involution τ on D. In both cases, we call $\Omega_I^{\mathrm{env}}(K, V_0, q) := \Omega_I(D, \tau)$ the *enveloping quadrangular system of involutory type* of $\Omega_Q(K, V_0, q)$.

Remark 7.16. Let (K, V_0, q) be an involutoric quadratic space. Then $\Omega_I^{\mathrm{env}}(K, V_0, q)$ is of proper involutory type, except if $\dim_K V_0 \leq 3$, or if $\dim_K V_0 = 4$ and q is the norm form of a quaternion division algebra.

Lemma 7.17. *Let (K, V_0, q) be an involutoric quadratic space with base point e, and let $C := C(V, q, e)$. If $C \cong D$ for some division algebra D, then let π_1 be the identity map from C to D. If $C \cong D \oplus D$ for some division algebra D, then let π_1 be the projection from C onto the first D-component. In both cases, the restriction of π_1 to V_0 is injective.*

Proof. This is obvious if $C \cong D$, so assume that $C \cong D \oplus D$. Since π_1 is additive, it suffices to show that $\pi_1(v) = 0$ for some $v \in V_0$ implies that $v = 0$.

So assume that $v \in V_0$ is such that $\pi_1(v) = 0$. Then $\pi_1(q(v)e) = \pi_1(v\overline{v}) = \pi_1(v)\pi_1(\overline{v}) = 0$, but since $\pi_1(e) = 1$, this implies that $q(v) = 0$ and hence $v = 0$ since q is anisotropic. □

Theorem 7.18. *Let $\Omega \cong \Omega_Q(K, V_0, q)$ and $\tilde{\Omega} \cong \Omega_I(\tilde{K}, \tilde{\sigma})$ for some anisotropic quadratic space (K, V_0, q) with base point e and some involutory set $(\tilde{K}, \tilde{\sigma})$. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if (K, V_0, q) is involutoric and $\Omega_I^{\mathrm{env}}(K, V_0, q)$ is isomorphic to a subsystem of $\tilde{\Omega}$. In particular, Ω is isomorphic to a subsystem of $\Omega_I^{\mathrm{env}}(K, V_0, q)$ itself.*

Proof. Let $\tilde{F} := \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$. By the definition of the operators Ω_Q and Ω_I, we have that $V = [V_0]$, $W = [K]$, $\tilde{V} = [\tilde{K}]$ and $\tilde{W} = [\tilde{F}]$.

First assume that (K, V_0, q) is involutoric, and that $\Omega_I^{\mathrm{env}}(K, V_0, q)$ is isomorphic to a subsystem of $\tilde{\Omega}$. Without loss of generality, we may assume that

$\tilde{\Omega} = \Omega_I^{\mathrm{env}}(K, V_0, q)$. By Lemma 7.14, we may consider $\tilde{K} = D$ and $\tilde{F} = \mathrm{Fix}_K(\tau_e)$ as subalgebras of $C := C(V_0, q, e)$. Let $\pi_1 : C \to D$ be the morphism defined in Lemma 7.17; then $\pi_1(V_0) \leq \tilde{K}$ and $\pi_1(K) = \pi_1(Ke) \leq \tilde{F}$ since $\tau_e(e) = e$. We define a map ϕ from $V = [V_0]$ to $\tilde{V} = [\tilde{K}]$ and a map ψ from $W = [K]$ to $\tilde{W} = [\tilde{F}]$ by setting $\phi[v] := [\pi_1(v)]$ and $\psi[t] := [\pi_1(t)]$ for all $v \in V_0$ and all $t \in K$. Then it follows from Lemma 7.17 that ϕ and ψ are group monomorphisms. Moreover,

$$\begin{aligned}
\phi(\epsilon) &= \phi[e] = [\pi_1(e)] = [1] = \tilde{\epsilon}\,,\\
\psi(\delta) &= \psi[1] = [\pi_1(1)] = [1] = \tilde{\delta}\,,\\
\phi[v]\psi[t] &= [\pi_1(v)][\pi_1(t)] = [\pi_1(t)\pi_1(v)] = [\pi_1(tv)] = \phi[tv] = \phi([v][t])\,,\\
\psi[t]\phi[v] &= [\pi_1(t)][\pi_1(v)] = [\pi_1(v)^{\tau_e}\pi_1(t)\pi_1(v)]\\
&= [\pi_1(\overline{v}tv)] = [\pi_1(tq(v))] = \psi[tq(v)] = \psi([t][v])\,,
\end{aligned}$$

for all $v \in V_0$ and all $t \in K$, and we can conclude that Ω is isomorphic to a subsystem of $\tilde{\Omega}$.

Now assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map α from V_0 to $\tilde{K}$ and a map β from K to $\tilde{F}$ by setting $\phi[v] := [\alpha(v)]$ and $\psi[t] := [\beta(t)]$ for all $v \in V_0$ and all $t \in K$. Since ϕ and ψ are group monomorphisms, α and β are additive monomorphisms as well. The conditions (6)–(9) translate into the following:

$$\alpha(e) = \tilde{1}\,, \tag{25}$$
$$\beta(1) = \tilde{1}\,, \tag{26}$$
$$\beta(t)\alpha(v) = \alpha(tv)\,, \tag{27}$$
$$\alpha(v)^{\tilde{\sigma}}\beta(t)\alpha(v) = \beta(tq(v))\,, \tag{28}$$

for all $v \in V_0$ and all $t \in K$. By repeated use of (27), we get that $\beta(s)\beta(t)\alpha(e) = \beta(s)\alpha(te) = \alpha(ste) = \beta(st)\alpha(e)$ for all $s, t \in K$, hence β is multiplicative; it follows that $\beta(K)$ is a commutative field which is contained in $\tilde{F}$. By (28) with $t = 1$, we get that

$$\alpha(v)^{\tilde{\sigma}}\alpha(v) = \beta(q(v)) \tag{29}$$

for all $v \in V_0$. Replacing v by $v + e$ and subtracting the original equation yields $\alpha(v) + \alpha(v)^{\tilde{\sigma}} = \beta(f(e, v))$ and hence, by (27),

$$\alpha(\overline{v}) = \alpha(f(e, v)e - v) = \beta(f(e, v))\alpha(e) - \alpha(v) = \alpha(v)^{\tilde{\sigma}} \tag{30}$$

for all $v \in V_0$. It follows from (29) and (30) that $\alpha(\overline{v})\alpha(v) = \beta(q(v))$, and since $\alpha(e) = \beta(1)$ by (25) and (26), it follows that there exists an algebra morphism $\hat{\alpha} : C := C(V_0, q, e) \to \tilde{K}$ such that β is the restriction of $\hat{\alpha}$ to K and α is the restriction of $\hat{\alpha}$ to V_0.

Suppose first that C is simple. Since α and β are injective, the kernel of $\hat{\alpha}$ cannot be equal to C, hence it has to be trivial, and therefore $\hat{\alpha}$ is an algebra monomorphism. It follows that C is a division algebra. We will write $C = D$ and $\gamma = \hat{\alpha}$ in this case, and we let τ be the standard involution τ_e of C.

Now suppose that C is the direct sum of two isomorphic simple algebras, say $C = D \oplus D$. By the structure theory of Clifford algebras (see, for example, [6, (8.2)]), this can only occur if $\dim_K V_0$ is even. Again, the kernel of $\hat{\alpha}$ cannot be equal to C. It cannot be trivial, either, since C has zero divisors but $\tilde{K}$ does not. So it has to be equal to one of the two direct summands, say $0 \oplus D$. But then the restriction of $\hat{\alpha}$ to the first direct summand is injective, and hence the induced map from D to $\tilde{K}$ — which we will denote by γ — is an algebra monomorphism, so D is a division algebra in this case as well. By (30), $\overline{v} \in \ker \hat{\alpha}$ if and only if $v \in \ker \hat{\alpha}$. But if we would have $\dim_K V_0 \equiv 2 \pmod 4$, then it would follow from [6, (8.4)] that the standard involution of the even Clifford algebra switches the two direct components, and using Lemma 7.14, we would obtain a contradiction. Hence $\dim_K V_0 \equiv 0 \pmod 4$ in this case. Now we let τ be the restriction of the standard involution τ_e of C to its first component D.

So we have shown that (K, V_0, q) is involutoric. It remains to show that $\Omega_I^{\mathrm{env}}(K, V_0, q) = \Omega_I(D, \tau)$ is isomorphic to a subsystem of $\tilde{\Omega} = \Omega_I(\tilde{K}, \tilde{\sigma})$. But γ is an injective map from D into $\tilde{K}$, and it follows from (30) that $\gamma \circ \tau = \tilde{\sigma} \circ \gamma$. By Theorem 7.5, this finishes the proof of this theorem. ☐

We have now reduced the case $\Omega_Q \leq \Omega_I$ to the case $\Omega_I \leq \Omega_I$ which we have already considered in Theorem 7.5.

Remark 7.19. The condition on a quadratic space to be involutoric looks very restrictive, and in fact, it is. Nevertheless, involutoric quadratic spaces exist in any dimension. The following example was communicated by J.-P. Tignol, and is a slight modification of the appendix of [8].

Let K be a field with $\mathrm{char}(K) \neq 2$. Suppose that $Q_1, \dots, Q_n$ are quaternion algebras over K such that $A := Q_1 \otimes \cdots \otimes Q_n$ is a division algebra, and denote by $i_1, j_1, \dots, i_n, j_n$ the usual generators of the quaternion algebras $Q_1, \dots, Q_n$; moreover let $k_\ell := i_\ell j_\ell$ for every $\ell \in \{1, \dots, n\}$. Consider the elements $u_\ell := k_1 \cdots k_{\ell-1} i_\ell$ and $v_\ell := k_1 \cdots k_{\ell-1} j_\ell$ for every $\ell \in \{1, \dots, n-1\}$, and let $w := k_1 \cdots k_n$. These elements pairwise anticommute and are square-central; denote their squares by $a_1, b_1, \dots, a_n, b_n, c$.

We can map the Clifford algebra of the $2n$-dimensional quadratic form $q = \langle a_1, b_1, \dots, a_n, b_n \rangle$ to A by carrying the basis elements of the quadratic space to the elements $u_1, v_1, \dots, u_n, v_n$. This gives us an algebra homomorphism $C(q) \to A$, which has to be injective since $C(q)$ is simple, and has to be surjective by dimension count, and hence $C(q) \cong A$. Since A is division, q is anisotropic.

If the quaternion algebras $Q_1, \dots, Q_n$ are chosen in such a way that $\mathrm{disc}(q) \notin K^2$, then it follows that $C_0(q)$ is also a division algebra, which is central over the discriminant extension field of q.

Similarly, we can consider the $(2n+1)$-dimensional anisotropic quadratic form $q' = \langle a_1, b_1, \dots, a_n, b_n, c \rangle$, and one can check that $C_0(q') \cong A$.

Theorem 7.20. *Let $\Omega \cong \Omega_Q(K, V_0, q)$ and $\tilde{\Omega} \cong \Omega_P(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$ for some anisotropic quadratic space (K, V_0, q) and some anisotropic pseudo-quadratic space $(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if Ω is already isomorphic to a subsystem of $\Omega_I(\tilde{K}, \tilde{\sigma})$.*

Proof. Let $\tilde{F} := \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$. Denote the bilinear form corresponding to Ω by f. By the definition of the operators Ω_Q and Ω_P, we have that $V = [V_0]$, $W = [K]$, $\tilde{V} = [\tilde{K}]$ and $\tilde{W} = [\tilde{T}]$, where the group $(T, \boxplus)$ is defined as in Theorem 7.8.

Assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map α from V_0 to $\tilde{K}$, a map β from K to $\tilde{V}_0$ and a map γ from K to $\tilde{K}$ by setting $\phi[v] := [\alpha(v)]$ and

$\psi[t] := [\beta(t), \gamma(t)] \in [\tilde{T}]$ for all $v \in V_0$ and all $t \in K$. Since ϕ and ψ are group morphisms, α and β are additive morphisms as well. The conditions (7) and (9) imply the following:

$$\beta(1) = 0\,, \tag{31}$$

$$\beta(t)\alpha(v) = \beta(tq(v))\,, \tag{32}$$

for all $v \in V_0$ and all $t \in K$. If we set $t = 1$ in (32), then we get that $\beta(q(v)) = 0$ for all $v \in V_0$. Linearizing this identity gives us that $\beta(f(u,v)) = 0$ for all $u, v \in V_0$. Since $\mathrm{char}(K) \neq 2$, $f \neq 0$, and hence f is surjective, so $\beta(K) = 0$. It follows that $\psi[K] \leq [0, \tilde{F}]$, and hence Ω is in fact isomorphic to a subsystem of $\Omega_I(\tilde{K}, \tilde{\sigma})$. □

Theorem 7.21. *Let $\Omega \cong \Omega_Q(K, V_0, q)$ and $\tilde{\Omega} \cong \Omega_E(\tilde{K}, \tilde{V}_0, \tilde{q})$ for some anisotropic quadratic space (K, V_0, q) and some anisotropic quadratic space $(\tilde{K}, \tilde{V}_0, \tilde{q})$ of type E_6, E_7 or E_8. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if Ω is already isomorphic to a subsystem of $\Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q})$.*

Proof. The proof of this theorem is very similar to the proof of Theorem 7.20. Denote the bilinear form corresponding to Ω by f. By the definition of the operators Ω_Q and Ω_E, we have that $V = [V_0]$, $W = [K]$, $\tilde{V} = [\tilde{V}_0]$ and $\tilde{W} = [\tilde{S}]$, where $(\tilde{S}, \boxplus)$ is the (non-abelian) group with underlying set $X_0 \times K$ as defined in [9, (16.6)], as in Theorem 7.11.

Assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map α from V_0 to $\tilde{V}_0$, a map β from K to $\tilde{X}_0$ and a map γ from K to $\tilde{K}$ by setting $\phi[v] := [\alpha(v)]$ and $\psi[t] := [\beta(t), \gamma(t)] \in [\tilde{S}]$ for all $v \in V_0$ and all $t \in K$. Since ϕ and ψ are group morphisms, α and β are additive morphisms as well. The conditions (7) and (9) imply the following:

$$\beta(1) = 0\,, \tag{33}$$

$$\beta(t)\alpha(v) = \beta(tq(v))\,, \tag{34}$$

for all $v \in V_0$ and all $t \in K$. If we set $t = 1$ in (34), then we get that $\beta(q(v)) = 0$ for all $v \in V_0$. Again, it follows that $\beta = 0$, and hence $\psi[K] \leq [0, \tilde{K}]$. Therefore Ω is in fact isomorphic to a subsystem of $\Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q})$. □

Theorem 7.22. *Let $\Omega \cong \Omega_E(K, V_0, q)$ and $\tilde{\Omega} \cong \Omega_P(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$ for some anisotropic quadratic space (K, V_0, q) of type E_6, E_7 or E_8, and some proper anisotropic pseudo-quadratic space $(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$. Then Ω cannot be isomorphic to a subsystem of $\tilde{\Omega}$.*

Proof. Assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$. Then in particular, $\Omega_Q(K, V_0, q)$ is isomorphic to a subsystem of $\tilde{\Omega}$. By Theorem 7.20, this implies that $\Omega_Q(K, V_0, q)$ is isomorphic to a subsystem of $\Omega_I(\tilde{K}, \tilde{\sigma})$. It thus follows from Theorem 7.18 that (K, V_0, q) has to be involutoric. But by [9, (12.43)], $C_0(q) \cong \mathrm{Mat}_4(E)$ if q is of type E_6, $C_0(q) \cong \mathrm{Mat}_4(D) \oplus \mathrm{Mat}_4(D)$ if q is of type E_7, and $C_0(q) \cong \mathrm{Mat}_{32}(K) \oplus \mathrm{Mat}_{32}(K)$ if q is of type E_8. In all three cases, we obtain a contradiction, and hence Ω cannot be isomorphic to a subsystem of $\tilde{\Omega}$. □

The remaining two cases are very similar to each other, and we consider them together. The group S, the map $(a, v) \mapsto av$ from $X_0 \times V_0 \to X_0$, and the maps h, θ and g (and similar objects for $\tilde{\Omega}$) are as in [9, (16.6)].

Theorem 7.23. (i) *Let $\Omega \cong \Omega_P(K, \sigma, X_0, \pi)$ and $\tilde{\Omega} \cong \Omega_E(\tilde{K}, \tilde{V}_0, \tilde{q})$ for some proper anisotropic pseudo-quadratic space (K, σ, V_0, π) and some quadratic space $(\tilde{K}, \tilde{V}_0, \tilde{q})$ of type E_6, E_7 or E_8 with base point $\tilde{e}$. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if (K, σ) is* not *proper, with $\Omega \cong \Omega_Q(F, K, q)$, and there exists a field monomorphism $\gamma : F \hookrightarrow \tilde{K}$ and γ-vector space monomorphisms $\beta : K \hookrightarrow \tilde{V}_0$ and $\alpha : X_0 \hookrightarrow \tilde{X}_0$ such that*

$$\begin{aligned} \beta(1) &= \tilde{e}\,, \\ \alpha(av) &= \alpha(a)\beta(v)\,, \\ \gamma(q(v)) &= \tilde{q}(\beta(v))\,, \\ \tilde{h}(\alpha(a), \alpha(b)) &= \beta(h(a, b))\,, \end{aligned}$$

for all $a, b \in X_0$ and all $v \in V_0$.

(ii) *Let $\Omega \cong \Omega_E(K, V_0, q)$ and $\tilde{\Omega} \cong \Omega_E(\tilde{K}, \tilde{V}_0, \tilde{q})$ for some quadratic spaces (K, V_0, q) and $(\tilde{K}, \tilde{V}_0, \tilde{q})$ of type E_6, E_7 or E_8, with base points e and $\tilde{e}$, respectively. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if there exists a field monomorphism $\gamma : K \hookrightarrow \tilde{K}$ and γ-vector space monomorphisms $\beta : V_0 \hookrightarrow \tilde{V}_0$ and $\alpha : X_0 \hookrightarrow \tilde{X}_0$ such that*

$$\begin{aligned} \beta(e) &= \tilde{e}\,, \\ \alpha(av) &= \alpha(a)\beta(v)\,, \\ \gamma(q(v)) &= \tilde{q}(\beta(v))\,, \\ \tilde{h}(\alpha(a), \alpha(b)) &= \beta(h(a, b))\,, \end{aligned}$$

for all $a, b \in X_0$ and all $v \in V_0$.

Proof. We start by showing that if $\Omega \cong \Omega_P(K, \sigma, X_0, \pi)$ is isomorphic to a subsystem of $\tilde{\Omega} \cong \Omega_E(\tilde{K}, \tilde{V}_0, \tilde{q})$, then the choice of the involutory set (K, σ) is very limited. So suppose that $\Omega \leq \tilde{\Omega}$, then also $\Omega_I(K, \sigma)$ is isomorphic to a subsystem of $\tilde{\Omega}$. Then Theorem 7.11 implies that (K, σ) is not proper. Also, since the pseudo-quadratic space (K, σ, X_0, π) is proper, we have $\sigma \neq 1$. It follows from Remark 7.3 that $\Omega \cong \Omega_Q(F, K, q)$, where F is a commutative field, and K is either a separable quadratic extension field over F with norm q, or a quaternion division algebra over F with norm q. In these two cases, we can reparametrize the quadrangular system Ω in exactly the same way as we do for the exceptional quadrangular systems of type E_6, E_7 and E_8 (by defining $V_0 := K$ and S being the group with underlying set $X_0 \times F$ defined as usual), using the isomorphism $T \to S : (a, t) \mapsto (a, t - \pi(a))$ for all $(a, t) \in T$; see also [9, (26.44)]. Note that the map $(a, v) \mapsto av$ from $X_0 \times V_0 = K \to X_0$ and the map $h : X_0 \times X_0 \to V_0 = K$ do not change under this isomorphism.

Only to avoid confusion in the notation, we will assume from now on that we are in case (ii), even though the proof of case (i) is now completely identical.

First assume that there exists a field monomorphism $\gamma : K \hookrightarrow \tilde{K}$ and γ-

vector space monomorphisms $\beta : V_0 \hookrightarrow \tilde{V}_0$ and $\alpha : X_0 \hookrightarrow \tilde{X}_0$ such that

$$\beta(e) = \tilde{e}\,, \tag{35}$$

$$\alpha(av) = \alpha(a)\beta(v)\,, \tag{36}$$

$$\gamma(q(v)) = \tilde{q}(\beta(v))\,, \tag{37}$$

$$\tilde{h}(\alpha(a), \alpha(b)) = \beta(h(a,b))\,, \tag{38}$$

for all $a, b \in X_0$ and all $v \in V_0$. Since $g(a,b) = f(h(b,a), e)/2$ for all $a, b \in X_0$ by [9, (13.26)], and similarly for $\tilde{g}$, it follows from (35), (37) and (38) that $g(\alpha(a), \alpha(b)) = \gamma(g(a,b))$ for all $a, b \in X_0$. We define a map ϕ from $V = [V_0]$ to $\tilde{V} = [\tilde{V}_0]$ and a map ψ from $W = [S]$ to $\tilde{W} = [\tilde{S}]$ by setting $\phi[v] := [\beta(v)]$ and $\psi[a,t] := [\alpha(a), \gamma(t)]$ for all $v \in V_0$ and all $(a,t) \in S$. Since α and β are additive monomorphisms, so are ϕ and ψ, because of the condition that $\gamma(g(a,b)) = \tilde{g}(\alpha(a), \alpha(b))$ for all $a, b \in X_0$. Since $\theta(a,v) = h(a, av)/2$ for all $a \in X_0$ and all $v \in V_0$ by [9, (13.28)], it follows from (36) and (38) that $\tilde{\theta}(\alpha(a), \beta(v)) = \beta(\theta(a,v))$ for all $a \in X_0$ and all $v \in V_0$, and since β is a γ-semilinear vector space isomorphism, we also have $\beta(tv) = \gamma(t)\beta(v)$ for all $t \in K$ and all $v \in V_0$. Using (35), (36) and (37), we thus get that

$$\begin{aligned}
\phi(\epsilon) &= \phi[e] = [\beta(e)] = [\tilde{e}] = \tilde{\epsilon}\,,\\
\psi(\delta) &= \psi[0,1] = [\alpha(0), \gamma(1)] = [0, \tilde{1}] = \tilde{\delta}\,,\\
\phi[v]\psi[a,t] &= [\beta(v)][\alpha(a), \gamma(t)] = [\tilde{\theta}(\alpha(a), \beta(v)) + \gamma(t)\beta(v)]\\
&= [\beta(\theta(a,v) + tv)] = \phi[\theta(a,v) + tv] = \phi([v][a,t])\,,\\
\psi[a,t]\phi[v] &= [\alpha(a), \gamma(t)][\beta(v)] = [\alpha(a)\beta(v), \gamma(t)\tilde{q}(\beta(v))]\\
&= [\alpha(av), \gamma(tq(v))] = \psi[av, tq(v)] = \psi([a,t][v])\,,
\end{aligned}$$

for all $v \in V_0$ and all $(a,t) \in S$, and we can conclude that Ω is isomorphic to a subsystem of $\tilde{\Omega}$.

Now assume that Ω is isomorphic to a subsystem of $\tilde{\Omega}$, with corresponding group monomorphisms ϕ and ψ satisfying (6)–(9). We define a map β from V_0 to $\tilde{V}_0$, a map α from S to $\tilde{X}_0$ and a map γ from S to $\tilde{K}$ by setting $\phi[t] := [\beta(t)]$ and $\psi[a,t] := [\alpha(a,t), \gamma(a,t)] \in [\tilde{S}]$ for all $t \in K$ and all $(a,t) \in S$. Moreover, let $\alpha(a) := \alpha(a,0)$ and $\gamma(t) := \gamma(0,t)$ for all $a \in X_0$ and all $t \in K$. By (6), $\beta(e) = \tilde{e}$. If we restrict ψ to $[0,K]$, then we are back in the situation of Theorem 7.21, and therefore $\gamma : K \to \tilde{K}$ is a field monomorphism and $\beta : V_0 \to \tilde{V}_0$ is a $\gamma-$semilinear vector space monomorphism, such that $\tilde{q}(\beta(v)) = \gamma(q(v))$ for all $v \in V_0$; moreover, $\alpha(0,t) = 0$ for all $t \in F$. Since ψ is additive,

$$\alpha(a+b, t+s+g(a,b)) = \alpha(a,t) + \alpha(b,s)\,, \tag{39}$$

$$\gamma(a+b, t+s+g(a,b)) = \gamma(a,t) + \gamma(b,s) + \tilde{g}(\alpha(a,t), \alpha(b,s))\,, \tag{40}$$

for all $(a,t), (b,s) \in X_0$. If we set $t = 0$ and $b = 0$ in (39), then we get that $\alpha(a,s) = \alpha(a,0) + \alpha(0,s)$ for all $(a,s) \in X_0$, and hence $\alpha(a,s) = \alpha(a)$ for all $(a,s) \in X_0$. Similarly, $\gamma(a,s) = \gamma(a,0) + \gamma(s)$ for all $(a,s) \in X_0$. If we substitute this last identity in (40), then we get that

$$\gamma(a+b, 0) + \gamma(g(a,b)) = \gamma(a,0) + \gamma(b,0) + \tilde{g}(\alpha(a), \alpha(b)) \tag{41}$$

for all $a, b \in X_0$. Remember that $\mathrm{char}(K) \neq 2$. If we interchange a and b in (41), and add the result to (41), then we get, using the fact that g and $\tilde{g}$ are anti-symmetric by [9, (13.47.i)], that $2\gamma(a+b,0) = 2(\gamma(a,0)+\gamma(b,0))$ for all $a,b \in X_0$, and hence $\gamma : S \to K$ is additive in the X_0-component. The identities (8) and (9) translate into the following:

$$\tilde\theta(\alpha(a),\beta(v)) + \gamma(a,t)\beta(v) = \beta(\theta(a,v)+tv)\,, \tag{42}$$

$$\alpha(a)\beta(v) = \alpha(av)\,, \tag{43}$$

$$\gamma(a,t)\tilde q(\beta(v)) = \gamma(av, tq(v))\,, \tag{44}$$

for all $v \in V_0$ and all $(a,t) \in S$. If we substitute $2v$ for v and 0 for t in (44), then we get that $\gamma(a,0)\tilde q(2\beta(v)) = \gamma(2av,0)$ for all $a \in X_0$ and all $v \in V_0$. Using the fact that $\gamma : S \to K$ is additive in the X_0-component, it follows that $4\gamma(a,0)\tilde q(\beta(v)) = 2\gamma(av,0) = 2\gamma(a,0)\tilde q(\beta(v))$, and if we choose $v \neq 0$, then $\tilde q(\beta(v)) \neq 0$ since β is injective and $\tilde q$ is anisotropic; hence $\gamma(a,0) = 0$ for all $a \in X_0$, and therefore $\gamma(a,t) = \gamma(t)$ for all $(a,t) \in S$.

Since β is a γ-vector space morphism, $\beta(te) = \gamma(t)\tilde e$ for all $t \in K$, and it thus follows from (43) that $\alpha(ta) = \alpha(a \cdot te) = \alpha(a)\beta(te) = \gamma(t)\alpha(a)$ for all $t \in K$ and all $a \in X_0$, so α is a γ-vector space morphism as well. Also, since ψ is injective, $\alpha : X_0 \to \tilde X_0$ and $\gamma : K \to \tilde K$ are injective as well.

It only remains to show that $\tilde h(\alpha(a),\alpha(b)) = \beta(h(a,b))$ for all $a,b \in X_0$. By [9, (26.19.i)],

$$\theta(a+b,e) - \theta(a,e) - \theta(b,e) = h(b,a) - g(a,b)e \tag{45}$$

for all $a,b \in X_0$, and a similar identity holds in $\tilde\Omega$. On the other hand, it follows from (42) with $t = 0$ that $\tilde\theta(\alpha(a),\tilde e) = \beta(\theta(a,e))$ for all $a \in X_0$. If we evaluate this identity in $a+b$ with $a,b \in X_0$, then it follows from (45) that

$$\tilde h(\alpha(b),\alpha(a)) - \tilde g(\alpha(a),\alpha(b))\tilde e = \beta(h(b,a)) - \beta(g(a,b)e)$$

for all $a,b \in X_0$. Since $\beta(g(a,b)e) = \gamma(g(a,b))\tilde e = \tilde g(\alpha(a),\alpha(b))\tilde e$ by (41), we conclude that $\tilde h(\alpha(b),\alpha(a)) = \beta(h(b,a))$ for all $a,b \in X_0$, and we are done. □

8 Some examples of non-algebraic inclusions of Moufang quadrangles

In this section, we give two examples of inclusions of Moufang quadrangles which are not algebraic.

First assume that Γ_1 and Γ_2 are two Moufang quadrangles such that Γ_1 is a subquadrangle of Γ_2, and that none of the root groups is 2-torsion. By Lemma 6.15.(i), the inclusion is either algebraic or dual. We now show that these dual inclusions do really exist.

Theorem 8.1. *Let $\tilde\Omega \cong \Omega_E(K,V_0,q)$ for some quadratic space (K,V_0,q) of type E_6, E_7 or E_8 with base point $e \in V_0^*$, and assume that* $\mathrm{char}(K) \neq 2$. *We will write $\pi(a) := \theta(a,e)$ for all $a \in X_0$. Let $a \in X_0$ be arbitrary, let V_a be the one-dimensional subspace of V_0 generated by a, and let $q_a : V_a \to K$ be the quadratic form defined by $q_a(ta) = t^2$ for all $t \in K$. Let $\Omega \cong \Omega_Q(K,V_a,q_a)$. Then $\Gamma(\Omega)$ is isomorphic to a dually included subquadrangle of $\Gamma(\tilde\Omega)$.*

Proof. Let $\phi_1, \dots, \phi_4$ be the maps defined by

$$\begin{aligned}
\phi_1 &: U_1 \to \tilde{U}_4 : x_1(t) \mapsto \tilde{x}_4(te)\,,\\
\phi_2 &: U_2 \to \tilde{U}_3 : x_2(ta) \mapsto \tilde{x}_3(-ta, 0)\,,\\
\phi_3 &: U_3 \to \tilde{U}_2 : x_3(t) \mapsto \tilde{x}_2(t\pi(a))\,,\\
\phi_4 &: U_4 \to \tilde{U}_1 : x_4(ta) \mapsto \tilde{x}_1(ta, 0)\,,
\end{aligned}$$

for all $t \in K$. Note that $\tilde{x}_3(-ta, 0) = \tilde{x}_3(ta, 0)^{-1}$ since $\mathrm{char}(K) \neq 2$. We have to show that these maps preserve the commutator relations, that is, we have to check whether

$$\begin{aligned}
\phi_2([x_1(t), x_3(s)]) &= [\phi_1(x_1(t)), \phi_3(x_3(s))]\,,\\
\phi_3([x_2(ta), x_4(sa)]) &= [\phi_2(x_2(ta)), \phi_4(x_4(sa))]\,,\\
\phi_2([x_1(t), x_4(sa)]_2) \cdot \phi_3([x_1(t), x_4(sa)]_3) &= [\phi_1(x_1(t)), \phi_4(x_4(sa))]\,,
\end{aligned}$$

for all $s, t \in K$.

$$\begin{aligned}
\phi_2([x_1(t), x_3(s)]) &= \phi_2(x_2(0)) = \tilde{x}_3(0, 0)\,,\\
[\phi_1(x_1(t)), \phi_3(x_3(s))] &= [\tilde{x}_4(te), \tilde{x}_2(s\pi(a))] = [\tilde{x}_2(s\pi(a)), \tilde{x}_4(-te)^{-1}]^{-1}\\
&= \tilde{x}_3(0, f(s\pi(a), -te))^{-1} = \tilde{x}_3(0, 0)^{-1} = \tilde{x}_3(0, 0)\,,
\end{aligned}$$

for all $s, t \in K$, where we have used the fact that $f(e, \pi(a)) = 0$ by [9, (13.41)].

$$\begin{aligned}
\phi_3([x_2(ta), x_4(sa)]) &= \phi_3([x_2(ta), x_4(-sa)^{-1}]) = \phi_3(x_3(f_a(ta, -sa)))\\
&= \phi_3(x_3(-2st)) = \tilde{x}_2(-2st\pi(a))\,,\\
[\phi_2(x_2(ta)), \phi_4(x_4(sa))] &= [\tilde{x}_3(-ta, 0), \tilde{x}_1(sa, 0)] = [\tilde{x}_1(sa, 0), \tilde{x}_3(ta, 0)^{-1}]^{-1}\\
&= \tilde{x}_2(h(sa, ta))^{-1} = \tilde{x}_2(-2st\pi(a))\,,
\end{aligned}$$

for all $s, t \in K$, where we have used the fact that $h(a, a) = 2\pi(a)$ by [9, (13.28)].

$$\begin{aligned}
&\phi_2([x_1(t), x_4(sa)]_2) \cdot \phi_3([x_1(t), x_4(sa)]_3)\\
&\qquad= \phi_2([x_1(t), x_4(-sa)^{-1}]_2) \cdot \phi_3([x_1(t), x_4(-sa)^{-1}]_3)\\
&\qquad= \phi_2(x_2(-tsa)) \cdot \phi_3(x_3(tq_a(-sa)))\\
&\qquad= \tilde{x}_3(tsa, 0) \cdot \tilde{x}_2(ts^2\pi(a))\,,\\
&[\phi_1(x_1(t)), \phi_4(x_4(sa))] = [\tilde{x}_4(te), \tilde{x}_1(sa, 0)]\\
&\qquad= [\tilde{x}_1(sa, 0), \tilde{x}_4(-te)^{-1}]^{-1}\\
&\qquad= \big(\tilde{x}_2(\theta(sa, -te)) \cdot \tilde{x}_3(sa \cdot (-te), 0)\big)^{-1}\\
&\qquad= \tilde{x}_3(tsa, 0) \cdot \tilde{x}_2(ts^2\pi(a))\,,
\end{aligned}$$

for all $s, t \in K$, where we have used the fact that $\theta(sa, tv) = s^2t\theta(a, v)$ for all $a \in X_0$, all $v \in V_0$ and all $s, t \in K$, by [9, (13.35)]. Hence all the commutator relations are preserved. It is obvious that the maps $\phi_1, \dots, \phi_4$ are monomorphisms, so we are done. $\square$

This is just one example of a dual inclusion; there are definitely more examples, but we do not want to give a classification of all dual inclusions here.

We now give an easy example of a non-algebraic non-dual inclusion; this can only exist if some of the root groups are 2-torsion, so we have to consider algebraic structures where the characteristic of the corresponding (skew) field has characteristic equal to 2.

Theorem 8.2. *Let $F = \mathbf{GF}(2)$ be the field with 2 elements, and let $K := F(\alpha)$ for some α which is algebraically independent over F. Let V_0 be a 3-dimensional vector space over K, and let $q : V_0 \to K : (y_1, y_2, y_3) \mapsto \alpha(y_1^2 + y_1 y_2 + y_2^2) + y_3^2$; then q is an anisotropic quadratic form from V_0 to K with base point $e = (0, 0, 1) \in V_0^*$. Let V_1 be a 2-dimensional vector space over K, and let $p : V_1 \to K : (z_1, z_2) = (z_1^2 + z_1 z_2 + z_2^2)$; then q is an anisotropic quadratic form from V_1 to K with base point $d = (1, 0) \in V_1^*$. Let $\tilde{\Omega} \cong \Omega_Q(K, V_0, q)$ and $\Omega \cong \Omega_Q(K, V_1, p)$. Then $\Gamma(\Omega)$ is isomorphic to a non-algebraically and non-dually included subquadrangle of $\Gamma(\tilde{\Omega})$.*

Proof. Observe that

$$\begin{aligned} f_0((y_1, y_2, y_3), (z_1, z_2, z_3)) &= \alpha(y_1 z_2 + y_2 z_1) \quad \text{and} \\ f_1((y_1, y_2), (z_1, z_2)) &= y_1 z_2 + y_2 z_1 \end{aligned}$$

for all $y_1, y_2, y_3, z_1, z_2, z_3 \in K$. Let $\phi_1, \ldots, \phi_4$ be the maps defined by

$$\begin{aligned} &\phi_1 : U_1 \to \tilde{U}_1 : x_1(t) \mapsto \tilde{x}_1(t)\,, \\ &\phi_2 : U_2 \to \tilde{U}_2 : x_2(z_1, z_2) \mapsto \tilde{x}_2(z_1, z_2, 0)\,, \\ &\phi_3 : U_3 \to \tilde{U}_3 : x_3(t) \mapsto \tilde{x}_3(\alpha t)\,, \\ &\phi_4 : U_4 \to \tilde{U}_4 : x_4(z_1, z_2) \mapsto \tilde{x}_4(z_1, z_2, 0)\,, \end{aligned}$$

for all $t, z_1, z_2 \in K$. Note that all elements of the root groups are equal to their own inverse, since all the root groups are 2-torsion. Again, we start by showing that these maps preserve the commutator relations. The relation $[U_1, U_3] = 1$ is obviously preserved. Moreover,

$$\begin{aligned} \phi_3([x_2(y_1, y_2), x_4(z_1, z_2)]) &= \phi_3(x_3(y_1 z_2 + y_2 z_1)) \\ &= \tilde{x}_3(\alpha(y_1 z_2 + y_2 z_1))\,, \\ [\phi_2(x_2(y_1, y_2)), \phi_4(x_4(z_1, z_2))] &= [\tilde{x}_2(y_1, y_2, 0), \tilde{x}_4(z_1, z_2, 0)] \\ &= \tilde{x}_3(\alpha(y_1 z_2 + y_2 z_1))\,, \end{aligned}$$

for all $y_1, y_2, z_1, z_2 \in K$, and

$$\begin{aligned} &\phi_2([x_1(t), x_4(y_1, y_2)]_2) \cdot \phi_3([x_1(t), x_4(y_1, y_2)]_3) \\ &\qquad = \phi_2(x_2(t y_1, t y_2)) \cdot \phi_3(x_3(t(y_1^2 + y_1 y_2 + y_2^2))) \\ &\qquad = \tilde{x}_2(t y_1, t y_2, 0) \cdot \tilde{x}_3(\alpha t(y_1^2 + y_1 y_2 + y_2^2))\,, \\ &[\phi_1(x_1(t)), \phi_4(x_4(y_1, y_2))] = [\tilde{x}_1(t), \tilde{x}_4(y_1, y_2, 0)] \\ &\qquad = \tilde{x}_2(t y_1, t y_2, 0) \cdot \tilde{x}_3(t\alpha(y_1^2 + y_1 y_2 + y_2^2))\,, \end{aligned}$$

for all $t, y_1, y_2 \in K$, and hence all commutator relations are preserved.

It remains to show that the inclusion is not algebraic. So suppose that there exists a quadrangular system $\tilde{\Omega}' = (\tilde{V}, \tilde{W}, \tau_V, \tau_W, \epsilon, \delta)$ with corresponding maps $\tilde{F}$ and $\tilde{H}$, and a subsystem $\Omega' = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ of $\tilde{\Omega}'$ with corresponding maps F and H, such that $\Gamma(\Omega) \cong \Gamma(\Omega')$ and $\Gamma(\tilde{\Omega}) \cong \Gamma(\tilde{\Omega}')$. Since $C_{U_4}(U_2) = 1$, $\mathrm{Rad}(F) = 0$, and in particular $\epsilon \notin \mathrm{Rad}(F)$. On the other hand, $C_{\tilde{U}_4}(\tilde{U}_2) \neq 1$, since it contains the element $x_4(e)$, so $\mathrm{Rad}(\tilde{F}) \neq 0$, and by axiom $(\mathbf{Q}_{10})$ in [2, Section 2], $\epsilon \in \mathrm{Rad}(\tilde{F}) \leq \mathrm{Rad}(F)$. This contradiction finishes the proof. □

The same remark as with the previous example also holds here: There are lots of other examples of non-algebraic non-dual inclusions, but we do not classify them in this paper. In particular, one can construct some more peculiar examples in the exceptional quadrangles of type E_6, E_7 and E_8, as well as in those of type F_4, but it is out of the scope to go into detail on those examples here.

9 Full and ideal inclusions of Moufang quadrangles

In this final section, we will describe the full and ideal inclusions of two Moufang quadrangles.

Definition 9.1. Let Γ_2 be an arbitrary generalized polygon. Then a subquadrangle Γ_1 of Γ_2 is called a *full* subpolygon if every point row of Γ_1 coincides with the corresponding point row of Γ_2; it is called an *ideal* subpolygon if every line pencil of Γ_1 coincides with the corresponding line pencil of Γ_2.

Lemma 9.2. *Let Γ_1 and Γ_2 be Moufang quadrangles for which none of the root groups is 2-torsion, and suppose that Γ_1 is a full or ideal subquadrangle of Γ_2. Then Γ_1 is algebraically included in Γ_2.*

Proof. Let $\Gamma_1 \cong \Gamma(\Omega)$ and $\Gamma_2 \cong \Gamma(\tilde{\Omega})$ for some quadrangular systems $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \tilde{\epsilon}, \tilde{\delta})$. By Lemma 6.15.(i), the inclusion of Γ_1 in Γ_2 is either algebraic or dual.

So suppose that it is dual, and let $\phi_i : U_i \hookrightarrow \tilde{U}_{5-i}$ for $i \in \{1, \ldots, 4\}$ be group monomorphisms preserving the commutator relations. In particular,

$$[\phi_1(U_1), \phi_3(U_3)] = 1 \quad \text{and} \tag{46}$$

$$[\phi_2(U_2), \phi_4(U_4)] \neq 1\,. \tag{47}$$

It follows from (47) that $[\tilde{U}_1, \tilde{U}_3] \neq 1$, hence $\tilde{\Omega}$ is wide, so it is either of pseudo-quadratic form type, or of type E_6, E_7 or E_8; in particular, $\tilde{W}$ is not abelian. Since V is abelian, we cannot have that $V \cong \tilde{W}$, and in particular, $\phi_2(U_2) \neq \tilde{U}_3$ and $\phi_4(U_4) \neq \tilde{U}_1$. But since the inclusion is full or ideal, we must then have $\phi_1(U_1) = \tilde{U}_4$ and $\phi_3(U_3) = \tilde{U}_2$. It then follows from (46) that $[\tilde{U}_2, \tilde{U}_4] = 1$, which contradicts the fact that none of the root groups is 2-torsion. Hence the inclusion cannot be dual, so it must be algebraic. □

Definition 9.3. Let $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system, and let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a subsystem of $\tilde{\Omega}$ such that the corresponding inclusion of Moufang quadrangles is full or ideal. Then we say that Ω is a *full* subsystem of $\tilde{\Omega}$. If $V = \tilde{V}$, then we call Ω a *V-full* subsystem of $\tilde{\Omega}$, and if $W = \tilde{W}$, then we call it a *W-full* subsystem of $\tilde{\Omega}$.

We now consider an arbitrary quadrangular system of a certain type, and we examine which quadrangular systems can occur as full subsystems.

Theorem 9.4. (i) *Let $\tilde{\Omega} \cong \Omega_I(K,\sigma)$ for some proper involutory set (K,σ). Then $\tilde{\Omega}$ does not have full subsystems other than $\tilde{\Omega}$ itself.*

(ii) *Let $\tilde{\Omega} \cong \Omega_Q(K,V_0,q)$ for some anisotropic quadratic space (K,V_0,q). Then every W-full subsystem of $\tilde{\Omega}$ is of the form $\Omega \cong \Omega_Q(K,V_1,q|_{V_1})$ for some subspace V_1 of V_0; $\tilde{\Omega}$ does not have V-full subsystems other than $\tilde{\Omega}$ itself.*

(iii) *Let $\tilde{\Omega} \cong \Omega_P(K,\sigma,V_0,\pi)$ for some proper pseudo-quadratic space (K,σ,V_0,π). Then every V-full subsystem of $\tilde{\Omega}$ is of the form $\Omega \cong \Omega_P(K,\sigma,V_1,\pi|_{V_1})$ for some subspace V_1 of V_0 (where Ω might or might not be proper); $\tilde{\Omega}$ does not have W-full subsystems other than $\tilde{\Omega}$ itself.*

(iv) *Let $\tilde{\Omega} \cong \Omega_E(K,V_0,q)$ for some quadratic space (K,V_0,q) of type E_6, E_7 or E_8. Then there is only one proper V-full subsystem of $\tilde{\Omega}$, namely $\Omega \cong \Omega_Q(K,V_0,q)$; $\tilde{\Omega}$ does not have W-full subsystems other than $\tilde{\Omega}$ itself.*

Proof. Let $\tilde{\Omega} = (\tilde{V},\tilde{W},\tau_V,\tau_W,\epsilon,\delta)$ be a quadrangular system, and let $\Omega = (V,W,\tau_V,\tau_W,\epsilon,\delta)$ be a subsystem of $\tilde{\Omega}$.

(i) Suppose that $\tilde{\Omega} \cong \Omega_I(K,\sigma)$. Then $\tilde{F}(\tilde{V},\tilde{V}) = \tilde{W}$ since $\mathrm{char}(K) \neq 2$. In particular, if $V = \tilde{V}$, then it follows from $F(V,V) \leq W$ that $W = \tilde{W}$ as well, hence $\Omega = \tilde{\Omega}$.

On the other hand, suppose that $W = \tilde{W}$. Since (K,σ) is proper, K is generated by $\mathrm{Fix}_K(\sigma)$, and hence every element $v \in \tilde{V} = [K]$ can be written as $v = \epsilon w_1 \cdots w_n$ for some n and some elements $w_1,\dots,w_n \in \tilde{W} = [\mathrm{Fix}_K(\sigma)]$. But this implies that $v \in \epsilon W \cdots W \leq V$, so $V = \tilde{V}$ as well, and again, we conclude that $\Omega = \tilde{\Omega}$.

(ii) Suppose that $\tilde{\Omega} \cong \Omega_Q(K,V_0,q)$ for some anisotropic quadratic space (K,V_0,q). By Theorem 7.10, every subsystem of $\tilde{\Omega}$ has to be of quadratic form type as well. Since $\mathrm{char}(K) \neq 2$, the bilinear form f corresponding to q is not identically zero, and hence surjective, so $\tilde{F}(\tilde{V},\tilde{V}) = \tilde{W}$. Again, it follows that every V-full subsystem is also W-full and hence equal to $\tilde{\Omega}$ itself.

So suppose that $W = \tilde{W}$. It then follows that $\Omega \cong \Omega_Q(K,V_1,p)$ for some subspace $V_1 \leq V_0$, and since $\tau_W([t],[v]) = [tq(v)]$ for all $t \in K$ and all $v \in V_0$, and similarly $\tau_W([t],[v]) = [tp(v)]$ for all $t \in K$ and all $v \in V_1$, it follows that p is the restriction of q to V_1, which is what we had to show.

Note that, in principle, we have to require V_1 to contain the base point of (K,V_0,q), but since the base point can be chosen arbitrarily (see Remark 7.7), this restriction is obsolete.

(iii) Suppose that $\tilde{\Omega} \cong \Omega_P(K,\sigma,V_0,\pi)$ for some proper pseudo-quadratic space (K,σ,V_0,π). Suppose that $W = \tilde{W}$. Since (K,σ,V_0,π) is proper, the corresponding skew-hermitian map $h : V_0 \times V_0 \to K$ is not identically zero, and hence onto. Therefore $\tilde{H}(\tilde{W},\tilde{W}) = \tilde{V}$, and it follows that $V = \tilde{V}$ as well, so $\Omega = \tilde{\Omega}$.

So assume that $V = \tilde{V}$. Suppose first that Ω is of involutory type. Since $V = \tilde{V} = [K]$, we must have $\Omega \cong \Omega_I(K,\sigma')$ for some involution σ' of K.

But if we evaluate $\epsilon F(\epsilon, [t]) - [t]$ in Ω and in $\tilde{\Omega}$, then we get that $\sigma' = \sigma$, and hence $\Omega \cong \Omega_I(K, \sigma) \cong \Omega_P(K, \sigma, 0, 0)$.

Next, suppose that Ω is of quadratic form type. By Theorem 7.20, Ω is a subsystem of $\Omega_I(K, \sigma)$, and by Theorem 7.18, the involutory set (K, σ) is not proper, and hence $\Omega_I(K, \sigma)$ is of quadratic form type. Since Ω is a V-full subsystem of $\Omega_I(K, \sigma)$, it follows from (ii) of this theorem that we must have $\Omega \cong \Omega_I(K, \sigma) \cong \Omega_P(K, \sigma, 0, 0)$.

Now suppose that Ω is of proper pseudo-quadratic form type. Since $V = \tilde{V} = [K]$, we must have $\Omega \cong \Omega_P(K, \sigma', V_1, \pi')$ for some involution σ' of K, some vector space V_1 over K and some anisotropic pseudo-quadratic form π' from V_1 to K. Since $\Omega_I(K, \sigma')$ is then also a V-full subsystem of $\tilde{\Omega}$, it follows as before that $\sigma' = \sigma$. It follows from the relation $[a, 0][t] = [at, 0]$ for all $a \in V_0$ and all $v \in K$ that the additive subgroup $V_1 \leq V_0$ is in fact a K-subspace of V_0. Since $\pi'(v) = h(v, v)/2 = \pi(v)$ for all $v \in V_1$, π' is the restriction of π to V_1, and hence we have shown that Ω is as required.

Finally, Ω cannot be of type E_6, E_7 or E_8 by Theorem 7.22.

(iv) Suppose that $\tilde{\Omega} \cong \Omega_E(K, V_0, q)$ for some quadratic space of type E_6, E_7 or E_8. Suppose that $W = \tilde{W}$. By [9, (13.25)], the corresponding map $h : X_0 \times X_0 \to V_0$ is surjective. Therefore $\tilde{H}(\tilde{W}, \tilde{W}) = \tilde{V}$, and it follows that $V = \tilde{V}$ as well, so $\Omega = \tilde{\Omega}$.

So assume that $V = \tilde{V}$. We have that $\tilde{F}(\tilde{V}, \tilde{V}) = [0, K] < [S] = \tilde{W}$, and hence $F(V, V) = [0, K]$ as well; in particular, $[0, K] \leq W$. If $W = [0, K]$, then $\Omega \cong \Omega_Q(K, V_0, q)$. So we may now assume that there exists an element $(\xi, t) \in S$ such that $[\xi, t] \in W$. Since $[0, t] \in W$ as well, we also have that $[\xi, 0] \in W$. It then follows from the relation $[a, 0][v] = [av, 0]$ for all $a \in X_0$ and all $v \in V_0$ that $[\xi v, 0] \in W$. Continuing in this way, we obtain that $[\xi V_0 \cdots V_0, 0] \leq W$. By [9, (27.7)], however, $\xi C_0(V_0, q) = X_0$, and hence $[X_0, 0] \leq W$. Again using the fact that $[0, K] \leq W$, we conclude that $W = \tilde{W}$, and hence $\Omega = \tilde{\Omega}$.

□

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