On the uniqueness of the unipotent subgroups of some Moufang sets

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Dedicated to William M. Kantor on the occasion of his sixtieth birthday

Abstract. In this paper we consider all Moufang sets — or split BN-pairs of rank 1 — arising from a pair of opposite root groups in a Moufang building of rank 2, and the Moufang sets corresponding to the Suzuki groups and the Ree groups. We show that in all these cases (except for one well understood exception), the (natural) root groups are the only subgroups U of the point stabilizers G_x satisfying the following three properties: (1) U is normal in G_x; (2) U is nilpotent; (3) $UH = G_x$, for $H = G_{x,y}$, with $y \neq x$.

1. Introduction

Moufang sets are the Moufang buildings of rank 1. They are the axiomatization of the permutation groups generated by two opposite root groups (belonging to opposite roots $\mathcal{R}_0$ and $\mathcal{R}_\infty$) in a Moufang building of rank at least 2, acting on the set of roots $\mathcal{R}$ such that $\mathcal{R} \cup \mathcal{R}_0$ or $\mathcal{R} \cup \mathcal{R}_\infty$ form an apartment. Similar, but slightly different, notions are Timmesfeld's *rank one groups*, and *split* BN*-pairs of rank one*. Moufang sets were introduced some years ago by Jacques Tits as tools in the classification programme of twin buildings — a programme that has been successfully completed by Bernhard Mühlherr (and in Mühlherr's approach, Moufang sets play indeed a central role). In the present paper, we want to show a uniqueness result for all Moufang sets arising from higher rank Moufang buildings as mentioned above, and in addition also for some well known Moufang sets arising from diagram automorphisms of some rank two buildings, in casu the Suzuki groups (some of which also appear as above in Moufang octagons) and the Ree groups in characteristic 3.

The motivation for this work, and for studying Moufang sets in general, is threefold.

[*]The first author is a Postdoctoral Fellow of the Fund for Scientific Research - Flanders (Belgium)(F.W.O. - Vlaanderen)

[‡]The fourth author is partially supported by the Fund for Scientific Research – Flanders (Belgium)(F.W.O. - Vlaanderen)

- Firstly, results on Moufang sets can be applied in situations where one deals with higher rank Moufang building, as is clear from the origin of the notion. As an example, we refer to [1], where Moufang sets are used to prove the uniqueness of the splitting of any spherical split BN-pair of rank at least two. Other examples can be found in the papers [3] and [8], where Moufang sets are used to characterize Moufang quadrangles in terms of "one half" of the conditions, and [4], where an even weaker assumption is made.

- Secondly, Jacques Tits initiated in [10] (see also [11]) a geometric study of buildings of rank one, via a procedure involving the unipotent subgroups of the corresponding Moufang sets. The results of the current paper are very useful in this respect.

- Thirdly, Moufang sets also appear outside the theory of buildings in situations where permutation groups are involved. For instance, when investigating automorphism groups of (finite) rank two geometries with Moufang-like conditions, Moufang sets come naturally into play. Also, it has recently been shown [2] that every Jordan division algebra gives rise, in a very natural way, to a Moufang set with abelian root groups. Hence theorems about abelian Moufang sets immediately imply results on Jordan division algebras. The Main Result of the present paper covers all these abelian Moufang sets.

Roughly speaking, we will show in this paper that the root groups of large classes of known Moufang sets are unique as transitive nilpotent normal subgroups of the point stabilizers.

We will state this more precisely in the next section. In the course of the proof, we also complete a slight oversight in [1].

Finally we remark that all finite Moufang sets are classified. In the case where the little projective group (see below for a definition) is not sharply 2-transitive, one has either $\mathsf{PSL}_2(q)$, $q \geq 4$, $\mathsf{PSU}_3(q)$, $q \geq 3$, $\mathsf{Sz}(q) \cong {}^2\mathsf{B}_2(q)$, $q \geq 8$, and $\mathsf{Re}(q) \cong {}^2\mathsf{G}_2(q)$, for appropriate prime powers q. This has been shown by Hering, Kantor and Seitz [5] (odd characteristic) and Shult [7] (even characteristic).

2. Definitions and Statement of the Main Result

2.1. Definition of a Moufang Set.

A *Moufang Set* is a system $\mathcal{M} = (X, (U_x^+)_{x\in X})$ consisting of a set X and a family of groups of permutations (we write the action of a permutation on a point on the right, using exponential notation) of X indexed by X itself and satisfying the following conditions.

(MS1) U_x^+ fixes $x \in X$ and is sharply transitive on $X \backslash \{x\}$.

(MS2) In the full permutation group of X, each U_x^+ normalizes the set of subgroups $\{U_y^+ \mid y \in X\}$

The group U_x^+ shall be called a *root group.* The elements of U_x^+ are often called *root elations.*

If $\mathcal{M} = (X, (U_x^+)_{x \in X})$ is a Moufang set, and $Y \subseteq X$, then Y induces a sub Moufang set if for each $x \in X'$, the stabilizer $(U_y^+)_Y$ acts sharply transitively on $Y \setminus \{y\}$. In this case $(Y, ((U_y^+)_Y)_{y \in Y})$ is a Moufang set.

The group generated by the U_x^+, for all $x \in X$, is called the *little projective group* of $\mathcal{M}$. A (faithful) permutation group G of X is called a *projective group* of $\mathcal{M}$ if $U_x^+ \trianglelefteq G_x$, for all $x \in X$. A permutation of X that normalizes the set of subgroups $\{U_y^+ \mid y \in X\}$, is called an *automorphism* of the Moufang set. For a given projective group G, we shall call a subgroup V_x of G_x a *unipotent subgroup of G* if

(US1) V_x acts transitively on $X \setminus \{x\}$;

(US2) $V_x \trianglelefteq G_x$;

(US3) V_x is nilpotent.

In fact, the existence of a unipotent subgroup is equivalent with the Moufang set being a split BN-pair of rank 1. If the little projective group of a Moufang set is not sharply two-transitive, then in all known examples, the root groups are unipotent subgroups of any projective group. The question arises whether the root subgroups can be characterized in this way. We will show that it is indeed the case for all Moufang sets related to Moufang buildings of rank 2 (including the ones corresponding with the Suzuki groups) and the Ree groups in characteristic 3.

In order to provide a precise statement, we introduce these classes of Moufang sets below.

We will not prove that these well known examples are in fact Moufang sets. We will just define them. In most cases, this means that we give two root groups as permutation groups. The other root groups are obtained by conjugation.

We also provide some so-called *μ-actions.* For a given (ordered) pair of distinct root groups (U_∞^+, U_0^+), with $0, \infty \in X$, and for every element $x \in X \setminus \{0, \infty\}$, there are unique elements $u \in U_0^+$ and $u', u'' \in U_\infty^+$ such that $u(x) = \infty$ and $\mu(x) := u'uu''$ interchanges 0 with ∞. The action of $\mu(x)$ on X is called the *simple μ-action* with respect to (U_∞^+, U_0^+, x). If $x' \in X \setminus \{0, \infty\}$, then the action of $\mu(x, x') := \mu(x)^{-1}\mu(x')$ is called the *double μ-action* with respect to $(U_\infty^+, U_0^+, x, x')$. The latter fixes both 0 and ∞, and so it normalizes both U_0^+ and U_∞^+.

2.2. Projective lines over skew fields.

Let $\mathbb{K}$ be any skew field. Put X equal to the set of vector lines of the 2-dimensional (left) vector space $V(2, \mathbb{K})$ over $\mathbb{K}$. After a suitable coordinatization, let 0 denote the vector line spanned by $(1, 0)$ and ∞ the vector line spanned by $(0, 1)$. Then, with regard to the usual (right) action of matrices on vectors (and hence on vector lines), we define U_0^+ as the group of matrices $\left(\begin{smallmatrix} 1 & 0 \\ k & 1 \end{smallmatrix}\right)$, for all $k \in \mathbb{K}$. The group U_∞^+ consists of the matrices $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$, $k \in \mathbb{K}$.

The little projective group is here $PSL_3(\mathbb{K})$ in its natural action. We denote this Moufang set as $\mathcal{M}\mathsf{PL}(\mathbb{K})$ and call it *the projective line over* $\mathbb{K}$. The corresponding set is sometimes denoted by $\mathsf{PG}(1,\mathbb{K})$.

It is an elementary exercise to compute the μ-actions for the pair (U_∞^+, U_0^+). One obtains $\mu((1,a)) : \mathbb{K}(x,y) \mapsto \mathbb{K}(ya^{-1}, -xa)$. The double μ-action is now easy: $\mu((1,1),(1,a)) : \mathbb{K}(x,y) \mapsto \mathbb{K}(xa^{-1}, ya)$.

Identifying $\mathbb{K}(1,k)$ with $k \in \mathbb{K}$ and $\mathbb{K}(0,1)$ with the element ∞, we may also write the above actions as follows:

$$U_\infty^+ = \{u : x \mapsto x + a, \infty \mapsto \infty \mid a \in \mathbb{K}\},$$
$$U_0^+ = \{u : x \mapsto (x^{-1} + a^{-1})^{-1}, -a \mapsto \infty, \infty \mapsto a, 0 \mapsto 0 \mid a \in \mathbb{K}\},$$
$$\mu(a) : x \mapsto -ax^{-1}a, 0 \leftrightarrow \infty,$$
$$\mu(1,a) : x \mapsto axa.$$

We will refer to this as the *non-homogeneous representation.*

2.3. Projective lines over alternative division rings.

An *alternative division ring* is a ring $\mathbb{A}$ with identity 1 in which the following laws hold.

(ADR1) For each non-zero element a, there exists an element b such that $b \cdot ac = c$ and $ca \cdot b = c$ for all $c \in \mathbb{A}$.

(ADR2) $(ab \cdot a)c = a(b \cdot ac)$ and $b(a \cdot ca) = (ba \cdot c)a$ for all $a, b, c \in \mathbb{A}$.

(ADR3) $ab \cdot ca = a(bc \cdot a) = (a \cdot bc)a$ for all $a, b, c \in \mathbb{A}$.

An alternative division ring is associative if and only if it is a skew field. The only non-associative alternative division rings are the Cayley division rings. Axiom (ADR2) implies that every two elements of an alternative division ring are contained in a sub skew field.

As, by (ADR1), each element a of $\mathbb{A}$ has a unique inverse a^{-1}, we may define the root groups U_0^+ and U_∞^+ in the same way as we did for the skew fields in the non-homogeneous representation. We then also obtain the same results for the simple and double μ-actions (and note that expressions like axa are unambiguous by (ADR2)). The corresponding Moufang set is denoted by $\mathcal{M}\mathsf{PL}(\mathbb{A})$ and called *the projective line over* $\mathbb{A}$.

2.4. Polar lines.

Let $\mathbb{K}$ be a skew field and let σ be an involution of $\mathbb{K}$ (so $(ab)^\sigma = b^\sigma a^\sigma$, for all $a, b \in \mathbb{K}$). Define $\mathbb{K}_\sigma = \{a + a^\sigma \mid a \in \mathbb{K}\}$ and $\mathrm{Fix}_{\mathbb{K}}(\sigma) = \{a \in \mathbb{K} \mid a^\sigma = a\}$. Let K_0 be an additive subgroup of $\mathbb{K}$ such that

(IS1) $\mathbb{K}_\sigma \subseteq K_0 \subseteq \mathrm{Fix}_{\mathbb{K}}(\sigma)$, (IS2) $a^\sigma K_0 a \subset K_0$ for all $a \in \mathbb{K}$,

(IS3) $1 \in K_0$.

Then $(\mathbb{K}, K_0, \sigma)$ is called an *involutory set*. The restriction of $\mathcal{M}\mathrm{PL}(\mathbb{K})$ to $K_0 \cup \{\infty\}$ in the non-homogeneous representation is well defined and is a Moufang set, called a *polar line*, denoted by $\mathcal{M}\mathrm{PL}(\mathbb{K}, K_0, \sigma)$. Hence, again, the root group actions and the μ-actions can be copied from the non-homogeneous representation of projective lines over a skew field given above.

2.5. Hexagonal Moufang sets.

We recall the notion of a hexagonal system, which is essentially equivalent to the notion of a quadratic Jordan division algebra of degree three.

A *hexagonal system* is a tuple $(\mathbb{J}, \mathbb{F}, \mathsf{N}, \#, \mathsf{T}, \times, 1)$, where $\mathbb{F}$ is a commutative field, $\mathbb{J}$ is a vector space over $\mathbb{F}$, N is a function from $\mathbb{J}$ to $\mathbb{F}$ called the *norm*, $\#$ is a function from $\mathbb{J}$ to itself called the *adjoint*, T is a symmetric bilinear form on $\mathbb{J}$ called the *trace*, $\times$ is a symmetric bilinear map from $\mathbb{J} \times \mathbb{J}$ to $\mathbb{J}$ and 1 is a distinguished element of $\mathbb{J} \setminus \{0\}$ called the *identity* such that for all $t \in \mathbb{F}$ and all $a, b, c \in \mathbb{J}$, the following identities hold.

1.	$1^\# = 1$,	7.	$\mathsf{T}(a \times b, c) = \mathsf{T}(a, b \times c)$,
2.	$(ta)^\# = t^2 a^\#$,	8.	$\mathsf{N}(a+b) = \mathsf{N}(a) + \mathsf{T}(a^\#, b) + \mathsf{T}(a, b^\#) + \mathsf{N}(b)$,
3.	$\mathsf{N}(ta) = t^3 \mathsf{N}(a)$,	9.	$(a+b)^\# = a^\# + a \times b + b^\#$,
4.	$\mathsf{T}(a, a^\#) = 3\mathsf{N}(a)$,	10.	$a^\# \times (a \times b) = \mathsf{N}(a)b + \mathsf{T}(a^\#, b)a$,
5.	$a^{\#\#} = \mathsf{N}(a)a$,	11.	$a^\# \times b^\# + (a \times b)^\# = \mathsf{T}(a^\#, b)b + \mathsf{T}(a, b^\#)a$,
6.	$b = \mathsf{T}(b, 1) \cdot 1 - 1 \times b$,	12.	$\mathsf{N}(a) = 0$ if and only if $a = 0$.

If we define the *inverse* a^{-1} of an arbitrary nonzero $a \in \mathbb{J}$ as $a^{-1} = \mathsf{N}(a)^{-1} a^\#$, then we can define the Moufang set $\mathcal{M}\mathsf{H}(\mathbb{J})$ related to $\mathbb{J}$ in exactly the same way as before for the projective line over a field $\mathbb{K}$, in its non-homogeneous representation. These Moufang sets are called *hexagonal Moufang sets*.

Hexagonal systems are classified by the work of various people. We refer to [12] for more details.

2.6. Orthogonal Moufang sets.

Let $\mathbb{K}$ be a commutative field and let L_0 be vector space over $\mathbb{K}$. An *anisotropic quadratic form* q on L_0 is a function from L_0 to $\mathbb{K}$ such that

(QF1) $q(ta) = t^2 q(a)$ for all $t \in \mathbb{K}$ and all $a \in L_0$,

(QF2) the function $f : L_0 \times L_0 \to \mathbb{K}$ given by $f(a, b) = q(a+b) - q(a) - q(b)$, for all $a, b \in L_0$, is bilinear,

(QF3) $q^{-1}(0) = \{0\}$.

The map f is called the *bilinear form associated with* q. Now embed L_0 in a vector space L over $\mathbb{K}$ as a codimension 2 subspace; hence we may put $L = \mathbb{K} \times L_0 \times \mathbb{K}$, and we define X as the set of all vector lines $\mathbb{K}(x_-, v_0, x_+)$ in the vector space $\mathbb{K} \times L_0 \times \mathbb{K}$) such that $x_- x_+ = q(v_0)$. Then $U^+_{(0,0,1)}$ consists of the maps u_w, $w \in L_0$, fixing $\mathbb{K}(0,0,1)$ and mapping $\mathbb{K}(1, v, q(v))$ onto $\mathbb{K}(1, v + w, q(v + w))$. Likewise, $U^+_{(1,0,0)}$ consists of the maps u'_w, $w \in L_0$, fixing $\mathbb{K}(1,0,0)$ and mapping $\mathbb{K}(q(v), v, 1)$ onto $\mathbb{K}(q(v + w), v + w, 1)$. This defines a Moufang set, called an *orthogonal Moufang set over* K, and denoted by $\mathcal{M}\mathsf{O}(\mathbb{K}, q)$.

One calculates that u_w maps $\mathbb{K}(q(v), v, 1)$ onto the vector line $\mathbb{K}(q(z), z, 1)$, with

$$z = q(v)q(q(v)w + v)^{-1}(q(v)w + v).$$

If L_0 has dimension 1, then we may put $q(x) = x^2$ and $\mathcal{M}\mathsf{O}(\mathbb{K}, q)$ is isomorphic with the projective line $\mathcal{M}\mathsf{PL}(\mathbb{K})$. If L_0 has dimension 2, then q defines a field extension $\mathbb{F}$ of $\mathbb{K}$ and $\mathcal{M}\mathsf{O}(\mathbb{K}, q)$ is isomorphic with $\mathcal{M}\mathsf{PL}(\mathbb{F})$.

This class of Moufang sets also comprises the ones related to indifferent sets (with the terminology of [12], see [11].

2.7. Hermitian Moufang sets.

Let $(\mathbb{K}, K_0, \sigma)$ be an involutory set, let L_0 be a right vector space over $\mathbb{K}$ and let q be a function from L_0 to $\mathbb{K}$. Then q is an *anisotropic pseudo-quadratic form* on L_0 with respect to K_0 and σ if there is a skew-hermitian form (with respect to σ) f on L_0 such that

(PF1) $q(a + b) \equiv q(a) + q(b) + f(a, b) \pmod{K_0}$,

(PF2) $q(at) \equiv t^\sigma q(a)t \pmod{K_0}$ for all $a, b \in L_0$ and all $t \in \mathbb{K}$,

(PF3) $q(a) \equiv 0 \pmod{K_0}$ only for $a = 0$.

An *anisotropic pseudo-quadratic space* is a quintuple $(\mathbb{K}, K_0, \sigma, L_0, q)$ such that $(\mathbb{K}, K_0, \sigma)$ is an involutory set, L_0 is a right vector space over $\mathbb{K}$ and q is an anisotropic pseudo-quadratic form on L_0 with respect to K_0 and σ.

Let $(\mathbb{K}, K_0, \sigma, L_0, q)$ be some anisotropic pseudo-quadratic space and let f denote the corresponding skew-hermitian form. Following [12, (11.24)], let $(T, \cdot)$ denote the group $\{(a, t) \in L_0 \times K \mid q(a) - t \in K_0\}$ with $(a, t) \cdot (b, u) = (a + b, t + u + f(b, a))$ and choose $(a, t) \in T \setminus \{(0, 0)\}$ and $s \in \mathbb{K} \setminus \{0\}$. Then we may put $X = T \cup \{\infty\}$, and the group U^+_∞ is given by the right action of T on itself. The double μ-action is given by $\mu((0, 1), (a, t) =: \mu_{(a,t)} : (b, v) \mapsto ((b - at^{-1}f(a, b))t^\sigma, tvt^\sigma)$.

These Moufang sets are called *Hermitian Moufang sets.*

2.8. An exceptional Moufang set of type E_7.

There is a Moufang set corresponding with an algebraic group of absolute type E_7 and which also arises from an exceptional Moufang quadrangle of type E_8. We will not give a detailed description here, but in the course of our proof, we will refer to [12] for a precise definition of this *exceptional Moufang set of type* E_7.

2.9. Suzuki-Tits Moufang sets.

Let $\mathbb{K}$ be a field of characteristic 2, and denote by $\mathbb{K}^2$ its subfield of all squares. Suppose that $\mathbb{K}$ admits some Tits endomorphism θ, i.e., the endomorphism θ is such that it maps x^θ to x^2, for all $x \in \mathbb{K}$. Let $\mathbb{K}^\theta$ denote the image of $\mathbb{K}$ under θ. Let L be a vector space over $\mathbb{K}^\theta$ contained in $\mathbb{K}$, such that $\mathbb{K}^\theta \subseteq L$ and such that $L \setminus \{0\}$ is closed under taking multiplicative inverse. For a unique standard notation, we also assume that L generates K as a ring. The *Suzuki-Tits Moufang set* $\mathcal{M}\mathsf{Sz}(\mathbb{K}, L, \theta)$ can be defined as the action of a certain subgroup of the centralizer of a polarity of a mixed quadrangle $\mathsf{Q}(\mathbb{K}, \mathbb{K}^\theta; L, L^\theta)$ on the corresponding set of absolute points. A more precise and explicit description can be extracted from Section 7.6 of [13], as follows.

Let X be the following set of points of $\mathsf{PG}(3, \mathbb{K})$, given with coordinates with respect to some given basis:

$$X = \{(1,0,0,0)\} \cup \{(a^{2+\theta} + aa' + a'^\theta, 1, a', a) \mid a, a' \in L\}.$$

Let $(x, x')_\infty$ be the collineation of $\mathsf{PG}(3, \mathbb{K})$ determined by

$$(x_0\ x_1\ x_2\ x_3) \mapsto (x_0\ x_1\ x_2\ x_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^{2+\theta} + xx' + x'^\theta & 1 & x' & x \\ x & 0 & 1 & 0 \\ x^{1+\theta} + x' & 0 & x^\theta & 1 \end{pmatrix},$$

and let $(x, x')_0$ be the collineation of $\mathsf{PG}(3, \mathbb{K})$ determined by

$$(x_0\ x_1\ x_2\ x_3) \mapsto (x_0\ x_1\ x_2\ x_3) \begin{pmatrix} 1 & x^{2+\theta} + xx' + x'^\theta & x & x' \\ 0 & 1 & 0 & 0 \\ 0 & x^{1+\theta} + x' & 1 & x^\theta \\ 0 & x & 0 & 1 \end{pmatrix}.$$

The group $\mathsf{Sz}(\mathbb{K}, L, \theta)$ is generated by the subgroups

$$U_\infty^+ = \{(x, x')_\infty \mid x, x' \in L\} \text{ and } U_0^+ = \{(x, x')_0 \mid x, x' \in L\}.$$

Both subgroups U_∞^+ and U_0^+ indeed act on X, as an easy computation shows, and they act sharply transitively on $X \setminus \{(1,0,0,0)\}$ and $X \setminus \{(0,1,0,0)\}$, respectively. Moreover, it can be checked easily that $(U_0^+)^{(x,x')_\infty} = (U_\infty^+)^{(y,y')_0}$, with

$$y = \frac{x'}{x^{2+\theta} + xx' + x'^\theta} \text{ and } y' = \frac{x}{x^{2+\theta} + xx' + x'^\theta}.$$

It now follows rather easily that we indeed obtain a Moufang set. When emphasizing one particular point, namely $(\infty) := (1,0,0,0)$, we can write $(a, a') :=$

$(a^{2+\theta}+aa'+a'^{\theta}, 1, a', a)$, and the unique element of U_∞^+ that maps $(0,0)$ to (b,b') is given by $(b,b')_\infty : (a,a') \mapsto (a+b, a'+b'+ab^\theta)$. The root group U_∞^+ is given by the set $\{(a,a')_\infty \mid a,a' \in L\}$ with operation $(a,a')_\infty \oplus (b,b')_\infty = (a+b, a'+b'+ab^\theta)_\infty$.

We remark that, if $L = \mathbb{K}$, then the Moufang set can also be obtained from a Moufang octagon, unlike the case $L \neq \mathbb{K}$.

2.10. Ree-Tits.

Let $\mathbb{K}$ be a field of characteristic 3, and denote by $\mathbb{K}^3$ its subfield of all third powers. Suppose that $\mathbb{K}$ admits some Tits endomorphism θ, i.e., the endomorphism θ is such that it maps x^θ to x^3, for all $x \in \mathbb{K}$. Let $\mathbb{K}^\theta$ denote the image of $\mathbb{K}$ under θ. The *Ree-Tits Moufang set* $\mathcal{M}\mathsf{Re}(\mathbb{K},\theta)$ can be defined as the action of a certain subgroup of the centralizer of a polarity of a mixed Moufang hexagon $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$ on the corresponding set of absolute points. A more precise and explicit description can be extracted from Section 7.7 of [13], as follows.

For $a, a', a'' \in \mathbb{K}$, we put

$$\begin{aligned}
f_1(a,a',a'') &= -a^{4+2\theta} - aa''^{\theta} + a^{1+\theta}a'^{\theta} + a''^2 + a'^{1+\theta} - a'a^{3+\theta} - a^2a'^2,\\
f_2(a,a',a'') &= -a^{3+\theta} + a'^{\theta} - aa'' + a^2a',\\
f_3(a,a',a'') &= -a^{3+2\theta} - a''^{\theta} + a^{\theta}a'^{\theta} + a'a'' + aa'^2.
\end{aligned}$$

Let X be the following set of points of $\mathsf{PG}(6,\mathbb{K})$, given with coordinates with respect to some given basis:

$$\begin{aligned}
X = \{&(1,0,0,0,0,0,0)\} \cup\\
&\{(f_1(a,a',a''), -a', -a, -a'', 1, f_2(a,a',a''), f_3(a,a',a'')) \mid a,a',a'' \in \mathbb{K}\}.
\end{aligned}$$

Let $(x,x',x'')_\infty$ be the collineation of $\mathsf{PG}(6,\mathbb{K})$ determined by

$$(x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6) \mapsto (x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6) \cdot C$$

where

$$C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
p & 1 & 0 & -x & 0 & x^2 & -x''-xx'\\
q & x^\theta & 1 & x'-x^{1+\theta} & & r & s\\
x'' & 0 & 0 & 1 & 0 & x & -x'\\
f_1(x,x',x'') & -x' & -x & -x'' & 1 & f_2(x,x',x'') & f_3(x,x',x'')\\
x'-x^{1+\theta} & 0 & 0 & 0 & 0 & 1 & -x^\theta\\
x & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

with

$$\begin{aligned}
p &= x^{1+\theta} - x'^{\theta} - xx'' - x^2x', \\
q &= x''^{\theta} + x^{\theta}x'^{\theta} + x'x'' - xx'^2 - x^{2+\theta}x' - x^{1+\theta}x'' - x^{3+2\theta}, \\
r &= x'' - xx' + x^{2+\theta}, \\
s &= x'^2 - x^{1+\theta}x' - x^{\theta}x'',
\end{aligned}$$

and put $(x,x',x'')_0 := (x,x',x'')_\infty^g$, with g the collineation of $\mathsf{PG}(6,\mathbb{K})$ determined by

$$(x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6) \mapsto (x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6)\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The group $\mathsf{Re}(\mathbb{K},\theta)$ is generated by the subgroups

$$U_\infty^+ = \{(x,x',x'')_\infty \mid x,x',x'' \in \mathbb{K}\} \text{ and } U_0^+ = \{(x,x',x'')_0 \mid x,x',x'' \in \mathbb{K}\}.$$

Both subgroups U_∞^+ and U_0^+ indeed act on X, as the reader can verify with a straightforward but tedious computation, and they act regularly on the sets $X \setminus \{(1,0,0,0,0,0,0)\}$ and $X \setminus \{(0,0,0,0,1,0,0)\}$, respectively. Moreover, it can be checked that $(U_0^+)^{(x,x',x'')_\infty} = (U_\infty^+)^{(y,y',y'')_0}$, with

$$\begin{aligned}
y &= -\frac{f_3(x,x',x'')}{f_1(x,x',x'')}, \\
y' &= -\frac{f_2(x,x',x'')}{f_1(x,x',x'')}, \\
y'' &= -\frac{x''}{f_1(x,x',x'')}.
\end{aligned}$$

It now follows that we indeed obtain a Moufang set. When emphasizing one particular point, namely $(\infty) := (1,0,0,0,0,0,0)$, we can write, following 7.7.7 of [13],

$$(a,a',a'') := (f_1(a,a',a''), -a', -a, -a'', 1, f_2(a,a',a''), f_3(a,a',a'')),$$

and the unique element of U_∞^+ that maps $(0,0,0)$ to (b,b',b'') is given by

$$(b,b',b'')_\infty : (a,a',a'') \mapsto (a+b, a'+b'+ab^\theta, a''+b''+ab'-a'b-ab^{1+\theta}).$$

The root group U_∞^+ is now the set $\{(a,a',a'')_\infty \mid a,a',a'' \in \mathbb{K}\}$ with operation

$$(a,a',a'')_\infty \oplus (b,b',b'')_\infty = (a+b, a'+b'+ab^\theta, a''+b''+ab'-a'b-ab^{1+\theta})_\infty.$$

2.11. Main Result.

Our main result reads as follows.

Main Result. *Let $\mathcal{M}$ be a projective line over a skew field or over a division ring, a polar line, an orthogonal Moufang set, a Hermitian Moufang set, an exceptional Moufang set of type E_7, a hexagonal Moufang set, a Suzuki-Tits Moufang set, a Ree-Tits Moufang set, or any sub Moufang set of any projective line over a skew field and let G be any projective group of $\mathcal{M}$. Then every unipotent subgroup of G is a root group of $\mathcal{M}$, except if $\mathcal{M}$ is the Hermitian Moufang set acting on* 9 *points with little projective group* $\mathrm{PSU}_4(2)$*, and G is* $\mathrm{P\Sigma U}_4(2)$*. In the latter case G_x is not a root group, but it is a unipotent subgroup, for all $x \in X$.*

In the next section, we will prove the Main Result. Note that [1] already contains a partial proof of the Main Result, namely for the cases of a projective line, a hexagonal Moufang set and an orthogonal Moufang set. However, due to some change in the arguments for a projective line in the revised version of that paper, the proof there became in fact incomplete for the hexagonal Moufang sets, the projective lines over proper alternative division rings, and for orthogonal Moufang sets. For this reason, we reprove these cases here, using a slightly more direct and shorter argument.

We also note that Timmesfeld proved part of the Main Result in [9]. The reason why we insist on giving yet a full proof here is that our arguments are more streamlined, more general, and more elementary.

3. Proof of the Main Result

3.1. Moufang sets with commutative root groups.

The Moufang sets introduced in the previous section that have commutative root groups are those isomorphic to a sub Moufang set of a projective line over a (skew) field (which we shall refer to as *semi projective lines (over a (skew) field)*, the hexagonal and orthogonal Moufang sets, and the projective lines over proper alternative division rings. We treat all these cases simultaneously. The arguments are different from the ones in [1] in that we need a slightly more complicated computation for the case of a projective line over a skew field, but this argument can be copied for the other cases (this type of reasoning was alluded to in [1], but unfortunately, the authors of the latter reference overlooked the fact that the arguments given for the projective line over a skew field was not extendable). Note that we also include some other type of Moufang sets with commutative root groups, and contained in a projective line over a field in characteristic 2, see [6], not explicitly mentioned here.

So let $\mathcal{M} = (X, (U_x^+)_{x\in X})$ be a Moufang set with commutative root groups as in the previous paragraph. Each of these Moufang sets is defined using (or "over") a field $\mathbb{K}$. We choose arbitrarily two elements of X and call them 0 and ∞. Then

U_∞^+ is an abelian group and we denote the composition low in this group by $+$. Let G be any projective group of $\mathcal{M}$.

Suppose U_∞ is a second unipotent subgroup of G, contained in G_∞. Since the product of two normal nilpotent subgroups is nilpotent, we may assume without loss of generality that $U_\infty^+ \subsetneqq U_\infty$. Since U_∞^+ acts sharply transitively on the set $X \setminus \{\infty\}$, there exists $\varphi \in U_\infty$ fixing some element of X, and we may assume without loss of generality that φ fixes 0.

Let z be a nontrivial element of the center of U_∞, and let $u \in U_\infty^+$ be such that zu fixes 0. Since zu centralizes U_∞^+, it must fix X pointwise, hence $z = u^{-1}$. In all cases, the three elements $0, 0^z$ and ∞ are contained in a semi projective line over a field $\mathbb{F}$. Without loss of generality, we may put 0^z equal to the multiplicative identity element 1 of $\mathbb{F}$. Indeed, we pass to the new multiplication $a \cdot b = a(0^z)^{-1}b$ if necessary. Moreover, in the case of hexagonal Moufang sets and projective lines over alternative division rings, we may assume that 1 is the identity element of the division algebra (this amounts to passing to a isotopic algebra; the Moufang sets do not change). Hence z maps 0 to 1, i.e., $u : x \mapsto x+1$. Now let $a \in X \setminus \{0, 1, \infty\}$ be arbitrary. Then $0, 1, \infty$ and a are contained in a semi projective line over some skew field $\mathbb{F}'$. The restriction to the points of X of the addition with respect to ∞ in $\mathbb{F}'$ coincides with the $+$ of our root group. Since we are in a skew field now, the double μ-actions are well defined; hence a^2 is well defined in particular (and one can check that it is independent of the chosen sub Moufang set by considering the intersection of all of them). Since the center is a characteristic subgroup of U_∞, it is normal in G_∞, and since G contains the little projective group, it contains all double μ-actions with respect to $(U_\infty^+, U_0^+, 1, a)$, for every $a \in X \setminus \{0, \infty\}$. It follows that $x \mapsto x + a^2$ also belongs to the center of U_∞, and hence φ fixes all squares in X.

First suppose that the characteristic of $\mathbb{K}$ is not equal to 2. Then every $a \in X$ can be written as $a = \frac{1}{4}((a+1)^2 - (a-1)^2)$. Note that $a+1, a-1 \in X$, and so $x \mapsto x + (a+1)^2 - (a-1)^2$ belongs to the center of U_∞. Applying now the double μ-action with respect to $(U_\infty^+, U_0^+, 1, 1/2)$, we see that $x \mapsto x + a$ belongs to the center of U_∞, and hence φ must fix all $a \in X$, a contradiction.

Next suppose that $\mathbb{K}$ has characteristic 2. Define $U_\infty^{[0]} := U_\infty$, $U_\infty^{[j]} := [U_\infty, U_\infty^{[j-1]}]$ for $j \geq 1$ and take i such that $U_\infty^{[i]}$ does not act freely on $X \setminus \{\infty\}$, but $U_\infty^{[i+1]}$ does (i exists by nilpotency of U_∞). We may clearly assume that $\varphi \in U_\infty^{[i]}$. We prove some properties of φ.

Observation 1. *The map φ is additive, i.e., for all $a, b \in X$, we have $(a+b)^\varphi = a^\varphi + b^\varphi$.*

Proof. Denote $U_\infty^+ \ni t_a : x \mapsto x + a$, $a \in X \setminus \{\infty\}$. We have $(t_a t_b)^\varphi = t_a^\varphi t_b^\varphi$ and $0^{t_a^\varphi} = a^\varphi$, so $t_a^\varphi = t_{a^\varphi}$. We get $(t_{a+b})^\varphi = (t_a t_b)^\varphi = t_a^\varphi t_b^\varphi = t_{a^\varphi} t_{b^\varphi}$. Taking the image of 0 , we obtain the result. $\square$

Observation 2. *For all $a, b \in X$ such that $\{0, 1, a, b, \infty\}$ is contained in a semi projective line over some skew field $\mathbb{L}$ with the property that φ fixes the multiplica-*

tive identity 1*, we have* $(aba)^\varphi = a^\varphi b^\varphi a^\varphi$ *(where juxtaposition is multiplication in the skew field* $\mathbb{L}$*). Consequently,* $(a^{-1})^\varphi = (a^\varphi)^{-1} =: a^{-\varphi}$*.*

Proof. Denote the element of U_0^+ mapping a to ∞ by t'_a, and use the notation t_a of the previous proof, too. By definition the double μ-action $\mu_a := x \mapsto axa$ is equal to the product $t_1t'_1t_1t_at'_at_a$. As before, $t_a^\varphi = t_{a^\varphi}$ and $t'^\varphi_a = t'_{a^\varphi}$. We now have, remembering that φ fixes 1:

$$\begin{aligned}(aba)^\varphi &= b^{\mu_a\varphi} = b^{t_1t'_1t_1t_at'_at_a\varphi} = (b^\varphi)^{(t_1t'_1t_1t_at'_at_a)^\varphi}\\ &= (b^\varphi)^{t_1t'_1t_1t_{a^\varphi}t'_{a^\varphi}t_{a^\varphi}} = (b^\varphi)^{\mu_{a^\varphi}} = a^\varphi b^\varphi a^\varphi.\end{aligned}$$

So the first assertion is proved. Now put $b = a^{-1}$ and the second assertion follows. □

For every $b \in X$, we have $[\varphi, t_b] = t_{b+b^\varphi}$ and by nilpotency of U_∞ and the fact that φ cannot centralize U_∞^+, there exists $b \in X$ with $b \neq b^\varphi$ such that $[\varphi, t_{b+b^\varphi}] = 1$ and $t_{b+b^\varphi} \neq 1$. So we have $(b + b^\varphi)^\varphi = b + b^\varphi$, implying $b^{\varphi^2} = b$.

Now $[\varphi, U^{[i]}]$ acts freely on $X \setminus \{\infty\}$. Denote as above, for $a \in X$, the double μ-action $x \mapsto axa$ by μ_a. Then $[\mu_a\varphi^{-1}\mu_a^{-1}, \varphi^{-1}]$ acts freely on $X \setminus \{\infty\}$, and since both φ and μ_a fix 0, we get $[\mu_a\varphi^{-1}\mu_a^{-1}, \varphi^{-1}] = \mathrm{id}$. Now we claim that in all cases except for $\mathcal{M}$ orthogonal, the set $\{0, 1, a, a^\varphi, \infty\}$ is contained in a semi projective line over some skew field $\mathbb{F}$. This is trivial if $\mathcal{M}$ is itself a semi projective line. If it is a projective line over a proper alternative division ring, then this follows from the fact that every two elements in such a division ring generate an associative division ring. If $\mathcal{M}$ is a hexagonal Moufang set, then use [12, (30.6) and (30.17)]. The claim follows. If $\mathcal{M}$ is an orthogonal Moufang set, then, as is noted in [1], $\{0, b, b^\varphi, \infty\}$ is contained in a sub Moufang set isomorphic to a projective line over a field, which we can also denote by $\mathbb{F}$ (and which is isomorphic to a quadratic extension of $\mathbb{K}$). If this sub Moufang set does not contain the element 1 chosen before, then we can re-choose it as $b + b^\varphi$. It is fixed under φ.

We now calculate, using the multiplication in $\mathbb{F}$, and taking into account $b^{\varphi^{-2}} = b$, $b^{-\varphi^{-1}} = b^{-\varphi}$, and Observation 2,

$$\begin{aligned}b^{-1} &= (b^{-1})^{[\mu_b\varphi^{-1}\mu_b^{-1}, \varphi^{-1}]} = b^{\varphi\mu_b^{-1}\varphi\mu_b\varphi^{-1}\mu_b^{-1}\varphi}\\ &= (b^{-1})^{\varphi^{-1}} b^{\varphi^{-2}} (b^{-1})^{\varphi^{-1}} b b^{-1} b (b^{-1})^{\varphi^{-1}} b^{\varphi^{-2}} (b^{-1})^{\varphi^{-1}}\\ &= cbc,\end{aligned}$$

where $c = b^{-\varphi}bb^{-\varphi}$. So we have $cbc = b^{-1}$, which implies $(cb)^2 = 1$. Since $\mathrm{char} K = 2$, we obtain $cb = 1$. Hence $1 = b^{-1}cb^2$. But $b^2 = (b^2)^\varphi = (b^\varphi)^2$ (since φ fixes all squares and then use the first assertion of Observation 2), and we obtain $1 = b^{-1}b^{-\varphi}bb^\varphi$, resulting in $bb^\varphi = b^\varphi b$.

But now $(b + b^\varphi)^2 = b^2 + (b^\varphi)^2 + bb^\varphi + b^\varphi b = b^2 + b^2 = 0$, hence $b = b^\varphi$, a contradiction. Hence φ is already the identity and $U_\infty = U_\infty^+$.

3.2. Hermitian Moufang sets.

Let $\Xi = (\mathbb{K}, K_0, \sigma, L_0, q)$ be a proper anisotropic pseudo-quadratic space as defined above (see also [12, (11.17)]), with corresponding skew-hermitian form $f : L_0 \times L_0 \to \mathbb{K}$. By [12, (21.16)], we may assume that q is non-degenerate, i.e. $\{a \in L_0 \mid f(a, L_0) = 0\} = 0$. Let $(T, \cdot)$ be as in Subsection 2.7. Then the group U_∞^+ is isomorphic to T, and acts in a natural way on T itself by right multiplication; we will write $\tau_{(a,t)}$ for the element of U_∞^+ mapping $(b, v) \in T$ to $(b, v) \cdot (a, t)$. Then $Z(U_\infty^+) = \{\tau_{(0,t)} \mid t \in K_0\}$. We will also write T^* for $T \setminus \{(0, 0)\}$. In general, we write a superscript $*$ when we delete the 0-element of a set (0-vector, additive identity,...).

As before, let $U_\infty^+ \lneqq U_\infty$. For convenience we shall write $U = U_\infty$ and $U^+ = U_\infty^+$. Also, put $B := G_\infty$. Since $U^+ \trianglelefteq B$ and $U \leq B$, we have that $Z(U^+) \trianglelefteq B$ and $Z(U^+) \trianglelefteq U$. Let $\tilde{U} := U/Z(U^+)$, $\tilde{U}^+ := U^+/Z(U^+)$, and $\tilde{B} := B/Z(U^+)$. Then $\tilde{U}^+ \lneqq \tilde{U}$, and $\tilde{U}$ is a non-trivial nilpotent group; in particular, $\tilde{Z} := Z(\tilde{U}) \neq 1$. For every $a \in L_0$, we let $\tau_a := \tau_{(a,q(a))} Z(U^+) \in \tilde{U}^+$; then the map $a \mapsto \tau_a$ is an isomorphism from $(L_0, +)$ to $\tilde{U}^+$. Note that $\tilde{U}^+ \neq 1$ by the properness of Ξ. The natural action of B on T induces an action of $\tilde{B}$ on L_0. Since $\tilde{U}^+$ acts regularly on L_0, there exists an element φ in $\tilde{U} \setminus \tilde{U}^+$ fixing $0 \in L_0$. Then φ fixes the orbit $0^{\tilde{Z}}$ elementwise.

Since $[\tilde{U}^+, \tilde{U}] \leq \tilde{U}^+$, it follows from the nilpotency of $\tilde{U}$ that there exists a non-trivial element $\tau \in \tilde{U}^+ \cap \tilde{Z}$. Moreover, $\tilde{Z} \trianglelefteq \tilde{B}$; for every $(a, t) \in T^*$, the mapping

$$\mu_{a,t} : b \mapsto \big(b - at^{-1} f(a, b)\big) t^\sigma ,$$

for all $b \in L_0$, belongs to $\tilde{B}$. (See [12, (33.13)].) Let $F := \{c \in L_0 \mid \tau_c \in \tilde{Z}\}$. Then F is a non-trivial additive subgroup of L_0 such that $\mu_{(a,t)}(F) \subseteq F$ for all $(a, t) \in T^*$. If we can now show that $F = L_0$, then it would follow that $\varphi = 1$, which is a contradiction; hence it would follow that $U = U^+$, which is we want to obtain. We will see that there is one exception for which there really exists $U \neq U^+$.

We start by making some observations about the maps $\mu_{(a,t)}$. Let $b \in L_0^*$ be fixed. If $(a, t) \in T^*$ is such that $f(a, b) = 0$, then we have

$$\mu_{(a,t)}(b) = bt^\sigma ; \tag{3.1}$$

in particular, if $t \in K_0$, then $\mu_{(0,t)}(b) = bt^\sigma = bt$, since $K_0 \leq \mathrm{Fix}_K(\sigma)$, and hence F is closed under right multiplication by K_0.

Lemma 3.1. *If F is a non-trivial K-subspace of L_0, then $F = L_0$.*

Proof. Suppose that F is a non-trivial K-subspace of L_0.

Let $b \in F^*$ be fixed, let $a \in L_0^*$ be arbitrary, and let $t = q(a)$; then $(a, t) \in T^*$. If $f(a, b) \neq 0$, then $b - \mu_{(a,t)}(b)t^{-\sigma} = at^{-1} f(a, b) \in F$, and hence $a \in F$. So assume that $f(a, b) = 0$. Since q is non-degenerate, there exists a $c \in L_0$ such that $f(c, b) \neq 0$, and hence also $f(a + c, b) = f(c, b) \neq 0$. Hence $c \in F$ and $a + c \in F$, so also in this case we have that $a = (a + c) - c \in F$. □

If K_0 generates $\mathbb{K}$ (as a ring), then it follows from the fact that F is closed under right multiplication by K_0, that F is a $\mathbb{K}$-subspace of L_0. So we may assume that K_0 does not generate $\mathbb{K}$ as a ring. By [12, (23.23)], this implies that K_0 is a commutative field, and either $\mathbb{K}/K_0$ is a separable quadratic extension and σ is the non-trivial element of $\mathrm{Gal}(\mathbb{K}/K_0)$, or $\mathbb{K}$ is a quaternion division algebra over K_0 and σ is the standard involution of $\mathbb{K}$. Let N and T denote the (reduced) norm and trace of $\mathbb{K}/K_0$, respectively.

Assume first that $\dim_{\mathbb{K}} L_0 = 1$; we will, in fact, identify L_0 and $\mathbb{K}$ in this case. Let $\rho := q(1) \in \mathbb{K} \setminus K_0$; then $q(t) + K_0 = t^\sigma \rho t + K_0 = t^\sigma(\rho + K_0)t$ for all $t \in \mathbb{K}$. Also, $f(1,1) = \gamma := \rho - \rho^\sigma$, and hence $f(s,t) = s^\sigma \gamma t$ for all $s,t \in \mathbb{K}$. One can now compute that

$$\mu_{(t,t^\sigma(\rho+c)t)}(s) = (\rho + c)^{-1}(\rho + c)^\sigma s t^\sigma (\rho + c)^\sigma t\,,$$

for all $s,t \in \mathbb{K}^*$ and all $c \in K_0$. Since $\mathsf{N}(\rho + c) = (\rho + c)(\rho + c)^\sigma \in K_0$, it follows that, for all $s \in F^*$, $(\rho^\sigma + c)^2 s t^\sigma (\rho^\sigma + c)t \in F$ as well, and hence

$$r^2 s t^\sigma r t \in F\,, \quad \text{for all } r \in \langle 1, \rho\rangle_{K_0} \text{ and all } t \in \mathbb{K}\,. \tag{3.2}$$

Suppose first that $\mathbb{K}/K_0$ is a separable quadratic extension; then $\mathbb{K}$ is commutative, and $\mathbb{K} = \langle 1, \rho\rangle_{K_0}$. Hence, by (3.2), $r^3 s \in F$ for all $r \in \mathbb{K}$. If $K_0 = \mathrm{GF}(2)$, then $\mathbb{K} = \mathrm{GF}(4)$, and then $r^3 \in K_0$ for all $r \in \mathbb{K}$ (this is the case which will lead to the exception). So assume that $|K_0| \geq 3$, and suppose that $\mathbb{K}^3 \subseteq K_0$. Since $K = K_0(\rho)$ is a quadratic extension field of K_0, we have $\rho^2 = a\rho + b$ for some $a, b \in K_0$. Then $\rho^3 = (a^2+b)\rho + ab$, hence $a^2 + b = 0$, and therefore $\rho^2 - a\rho + a^2 = 0$. If $\mathrm{char}(\mathbb{K}) = 3$, then this would imply $(\rho + a)^2 = 0$ and thus $\rho = -a \in K_0$, a contradiction. If $\mathrm{char}(\mathbb{K}) \neq 3$, then $(\rho + t)^3 - \rho^3 - 1 = 3\rho t(\rho + t) \in K_0$, and therefore $\rho(\rho + t) \in K_0$ for all $t \in K_0^*$. Choose a $t \in K_0 \setminus \{0, -1\}$; then $\rho = \rho(\rho + (t+1)) - \rho(\rho + t) \in K_0$, again a contradiction. We conclude that $\mathbb{K}^3 \not\subseteq K_0$, and hence $F = \mathbb{K}$.

Suppose now that $\mathbb{K}$ is a quaternion division algebra over K_0; in particular, K_0 is an infinite commutative field. If we consider (3.2) with $r = \rho + c$ for some $c \in K_0 \setminus \{0\} = Z(\mathbb{K})^*$, subtract the same expression with $r = \rho$ and $r = c$, and divide by c, then we get that

$$\rho(\rho + 2c)\mathsf{N}(t)s + (c + 2\rho)s t^\sigma \rho t \in F\,,$$

for all $c \in K_0^*$. If $\mathrm{char}(\mathbb{K}) = 2$, then it follows that $\rho^2 \mathsf{N}(t)s + c s t^\sigma \rho t \in F$, for all $c \in K_0^*$, and hence $s t^\sigma \rho t \in F$ for all $t \in \mathbb{K}$. If $\mathrm{char}(\mathbb{K}) \neq 2$, then we write $\rho^2 = a\rho + b$ with $a, b \in K_0$; if we take $r = \rho - a/2$ in (3.2), then we obtain that $s t^\sigma \rho t \in F$ for all $t \in \mathbb{K}$ since $r^2 \in K_0^*$ and $s t^\sigma (a/2) t \in sK_0 \subseteq F$. So we have shown that, in all characteristics, F is invariant under right multiplication by elements of the set $K_0 \cup \{t^\sigma \rho t \mid t \in \mathbb{K}\}$. It remains to show that the subring generated by $K_0 \cup \{t^\sigma \rho t \mid t \in \mathbb{K}\}$ is $\mathbb{K}$. Suppose that

$$K_1 := \langle K_0 \cup \{t^\sigma \rho t \mid t \in \mathbb{K}\}\rangle_{\mathrm{ring}} \neq \mathbb{K}\,.$$

Since every subring of $\mathbb{K}$ containing K_0 is a K_0-vector space of dimension 1, 2 or 4, and since $\rho \notin K_0$, we must have $\dim_{K_0} K_1 = 2$; hence we can find a $t \in \mathbb{K} \setminus K_1$ for which $\mathsf{T}(t) = 0$ and $\mathsf{T}(\rho t) \neq 0$. Then $t^\sigma = -t$ and $\rho t = -t^\sigma \rho^\sigma + r$ for some

$r \in K^*$; hence

$$t^\sigma \rho t = -t(-t^\sigma \rho^\sigma + r) = t^\sigma t \cdot \rho^\sigma - t \cdot r \notin K_1 \,,$$

a contradiction. So $K_1 = \mathbb{K}$, and hence $F = \mathbb{K}$ in this case as well.

Now suppose that $\dim_K L_0 \geq 2$. If $\mathbb{K}$ is a quaternion division algebra over K_0 or if $\mathbb{K}$ is a quadratic extension field over K_0 with $K_0 \neq \mathrm{GF}(2)$, then it follows from the result in dimension 1 that F is a $\mathbb{K}$-subspace of L_0, and hence $F = L_0$ by Lemma 3.1. It only remains to consider the case where $K_0 = \mathrm{GF}(2)$ and $\mathbb{K} = \mathrm{GF}(4)$.

Let $b \in F \setminus \{0\}$ be arbitrary. Since $\dim_K L_0 \geq 2$, there exists an $a \in L_0^*$ such that $f(a, b) = 0$; by (3.1), $bq(a)^\sigma \in F$. Since q is anisotropic, $q(a)^\sigma \notin K_0$, and it thus follows that $bK \in F$. This shows that F is a $\mathbb{K}$-subspace of L_0, and we can again conclude by Lemma 3.1 that $F = L_0$.

We will now describe the exception. So let $K_0 = \mathrm{GF}(2)$, let $\mathbb{K} = \mathrm{GF}(4)$, and let $\dim_{\mathbb{K}} L_0 = 1$; we will again identify L_0 and $\mathbb{K} = \mathrm{GF}(4)$. Then $\rho := q(1)$ is one of the two elements in $\mathbb{K} \setminus K_0$, and $f(1, 1) = \gamma := \rho - \rho^\sigma = 1$; hence $f(s, t) = s^\sigma t$ for all $s, t \in \mathbb{K}$. Then $U^+ \cong T$ is a group of order 8. In the case that the projective group is $\mathsf{P\Sigma U}(3, 2)$, we have $B_+ = T \cdot \mathrm{Gal}(\mathbb{K}/K_0)$, which is a group of order 16. If we take $U = B_+$, then U is of course a normal subgroup of B_+, but U is also nilpotent (since it is a 2-group) and transitive (since U^+ is already transitive), giving us the desired exception to the Main Theorem.

3.3. Exceptional Moufang sets of type E_7.

We now consider the case of the Moufang sets arising from a Moufang quadrangle of type E_6, E_7 or E_8. In fact, we have already handled E_6 and E_7, since these correspond to Hermitian Moufang sets, but our approach does not make any distinction between these three cases.

Let $(\mathbb{K}, L_0, q)$ be a quadratic space of type E_6, E_7 or E_8 as defined in [12, (12.31)], with corresponding bilinear form $f : L_0 \times L_0 \to \mathbb{K}$ and with base point $\epsilon \in L_0^*$. Let X_0 be the vector space over $\mathbb{K}$ and $(a, v) \mapsto av$ be the map from $X_0 \times L_0 \to X_0$ as defined in [12, (13.9)]. Let h be the bilinear map from $X_0 \times X_0$ to L_0 defined in [12, (13.18) and (13.19)], let g be the bilinear map from $X_0 \times X_0$ to $\mathbb{K}$ defined in [12, (13.26)], and let π be the map from X_0 to L_0 as defined in [12, (13.28)]. Moreover, let $\pi(a, t) := \pi(a) + t\epsilon$ for all $(a, t) \in X_0$. Following [12, (16.6)], let $(S, \cdot)$ be the group with underlying set $X_0 \times \mathbb{K}$ and with group operation

$$(a, t) \cdot (b, u) = (a + b, t + u + g(a, b))$$

for all $(a, t), (b, u) \in S$. Then the group $U^+ := U_\infty^+$ is isomorphic to S, and acts in a natural way on S itself by right multiplication; we will write $\tau_{(a,t)}$ for the element of U^+ mapping $(b, v) \in S$ to $(b, v) \cdot (a, t)$. Then $Z(U^+) = \{\tau_{(0,t)} \mid t \in \mathbb{K}\}$.

Let U be a second unipotent subgroup in G_∞, and assume, as before, $U^+ \leq U$.

Exactly as in section 3.2, we let $\tilde{U} := U/Z(U^+)$, $\tilde{U}^+ := U^+/Z(U^+)$, $\tilde{B} := B/Z(U^+)$, and $\tilde{Z} := Z(\tilde{U}) \neq 1$. For every $a \in X_0$, we let $\tau_a := \tau_{(a,0)}Z(U^+) \in \tilde{U}^+$;

then the map $a \mapsto \tau_a$ is an isomorphism from $(X_0, +)$ to $\tilde{U}^+$. The natural action of U on S induces an action of $\tilde{U}$ on X_0. Since $\tilde{U}^+$ acts regularly on X_0, there exists an element φ in $\tilde{U} \setminus \tilde{U}^+$ fixing $0 \in X_0$, and hence fixing the orbit $0^{\tilde{Z}}$ elementwise. Again, there exists a non-trivial element $\tau \in \tilde{U}^+ \cap \tilde{Z}$. For every $(a,t) \in S^*$, the mapping

$$\mu_{a,t} : b \mapsto b\pi(a,t) + ah(b,a) - \frac{f(h(b,a), \pi(a,t))}{q(\pi(a,t))} a\pi(a,t)\,,$$

for all $b \in X_0$, belongs to $\tilde{B}$. (The computation of this expression requires some calculation, similar to the other cases in [12, Chapter 33]. Observe also that $q(\pi(a,t)) \neq 0$ by [12, (13.49)].) Let $F := \{c \in X_0 \mid \tau_c \in \tilde{Z}\}$. Then F is a non-trivial additive subgroup of X_0 such that $\mu_{(a,t)}(F) \subseteq F$ for all $(a,t) \in S^*$. We will again show that $F = X_0$ to obtain the required contradiction.

First of all, observe that it follows from the fact that $\mu_{(0,t)}(b) = tb$ for all $t \in \mathbb{K}$ and all $b \in X_0$ that F is a $\mathbb{K}$-subspace of X_0.

Lemma 3.2. *Let $b \in X_0^*$. If $b \in F$, then $b\pi(b) \in F$.*

Proof. Let $b \in F$. Then, for all $t \in \mathbb{K}$, also $\mu_{b,t}(b) \in F$, that is,

$$\mu_{b,t}(b) = \left(1 - \frac{f(h(b,b), \pi(b,t))}{q(\pi(b,t))}\right) b\pi(b,t) + bh(b,b) \in F\,. \tag{3.3}$$

Note that $h(b,b) = 2\pi(b)$ if $\mathrm{char}(\mathbb{K}) \neq 2$ and that $h(b,b) = f(\pi(b), \epsilon)\epsilon$ if $\mathrm{char}(\mathbb{K}) = 2$, by [12, (13.28) and (13.45)]. Also observe that we have already shown that $b \cdot s\epsilon \in F$ for all $s \in \mathbb{K}$.

Assume first that $\mathrm{char}(\mathbb{K}) \neq 2$. Then it follows from (3.3) that

$$\left(3 - \frac{f(2\pi(b), \pi(b,t))}{q(\pi(b,t))}\right) b\pi(b) \in F\,,$$

for all $t \in \mathbb{K}$, and it is easily checked that this expression is zero if and only if $q(\pi(b)) = 3t^2$. Choose any t for which $q(\pi(b)) \neq 3t^2$; then it follows that $b\pi(b) \in F$ since F is a $\mathbb{K}$-subspace of X_0.

Now assume that $\mathrm{char}(\mathbb{K}) = 2$. It now follows from (3.3) that

$$\left(1 + \frac{f(f(\pi(b), \epsilon)\epsilon, \pi(b,t))}{q(\pi(b,t))}\right) b\pi(b) \in F\,,$$

for all $t \in \mathbb{K}$, and this expression is zero if and only if

$$t^2 + f(\pi(b), \epsilon)t + q(\pi(b)) + f(\pi(b), \epsilon)^2 = 0\,.$$

This quadratic equation has at most 2 solutions; let t be any element of K which is not a solution of this equation. Then it follows that $b\pi(b) \in F$ in this case as well. □

Lemma 3.3. *Let $b \in X_0^*$. If there exist elements $s, t \in \mathbb{K}$, not both zero, such that $b(s\pi(b) + t\epsilon) \in F$, then $b \in F$.*

Proof. Let $b \in X_0^*$ and $s, t \in \mathbb{K}$ (not both zero) be such that $b(s\pi(b) + t\epsilon) \in F$. If $s = 0$, then $t \neq 0$, and then $tb \in F$, hence $b \in F$. So assume that $s \neq 0$; then $b\pi(b, s^{-1}t) \in F$. Assume without loss of generality that $s = 1$. It is shown in the proof of [12, (13.67)] that $\pi(b\pi(b,t)) = q(\pi(b,t))\pi(b)$. By [12, (13.49)], $q(\pi(b,t)) \neq 0$. If we now apply Lemma 3.2 on the element $b\pi(b,t) \in F$, then we get that $b\pi(b,t)\pi(b) \in F$, and since $\overline{\pi(b,t)} = f(\epsilon, \pi(b,t))\epsilon - \pi(b,t)$, it also follows that $b\pi(b,t)\overline{\pi(b,t)} \in F$. But $b\pi(b,t)\overline{\pi(b,t)} = q(\pi(b,t))b$ by [12, (13.7)], so $b \in F$, and we are done. □

As in [12, (13.42)], we define $P(a,b) := f(h(a,b), \epsilon)$ for all $a, b \in X_0$; then P is an alternating bilinear form, which is non-degenerate. (This form is called F in [12], but we choose P to avoid confusion with our set F.)

Lemma 3.4. *Let $a, b \in X_0^*$. If $b \in F$ and $P(b,a) \neq 0$, then $a \in F$.*

Proof. Let $a \in X_0^*$ and let $b \in F$ such that $P(b,a) \neq 0$. Then for all $s, t \in \mathbb{K}$, we have that $\mu_{a,t}(b) - \mu_{a,s}(b) \in F$. It follows that

$$\frac{f(h(b,a), \pi(a,s))}{q(\pi(a,s))} a\pi(a,s) - \frac{f(h(b,a), \pi(a,t))}{q(\pi(a,t))} a\pi(a,t) \in F\,,$$

for all $s, t \in \mathbb{K}$. Let $x := f(h(b,a), \pi(a)) \in \mathbb{K}$ and let $y := P(b,a) \in \mathbb{K}^*$; then this can be rewritten as

$$\left(\frac{x+sy}{q(\pi(a,s))} - \frac{x+ty}{q(\pi(a,t))}\right) a\pi(a) + \left(s\frac{x+sy}{q(\pi(a,s))} - t\frac{x+ty}{q(\pi(a,t))}\right) a \in F\,.$$

By [12, (13.41)], a and $a\pi(a)$ are linearly independent. On the other hand, since $y \neq 0$, there exists only one element $s \in \mathbb{K}$ for which $x + sy = 0$. If we now choose $s \neq t$ such that $x + sy \neq 0$ and $x + ty \neq 0$, then the expression above cannot be zero, and hence we have found constants $c, d \in \mathbb{K}$, not both zero, such that $a(c\pi(a) + d\epsilon) \in F$. It follows from Lemma 3.3 that $a \in F$, which is what we had to show. □

We are now in a position to show that $X_0 = F$. We already know that F is non-trivial, so choose some fixed element $b \in F^*$. Now let $c \in X_0^*$ be arbitrary. If $P(b,c) \neq 0$, then $c \in F$ by Lemma 3.4. If $P(b,c) = 0$, then choose an element $a \in X_0$ such that $P(b,a) \neq 0$ (such an element exists since P is non-degenerate). But now the elements a and $a + c$ both satisfy the hypotheses of Lemma 3.4, and hence they both belong to F. It follows that also $c = (a + c) - a$ belongs to F, and hence we have shown that $X_0 = F$.

3.4. Suzuki-Tits Moufang sets.

We start with some observations. We use the notation of Subsection 2.9.

Observation 3. *The mapping $x \mapsto x^{1+\theta}$ induces a permutation of L. Also, the Tits endomorphism $x \mapsto x^\theta$ is a bijection from L onto L^θ.*

Proof. Indeed, if $x \in L$, then $x^\theta \in L\theta \subseteq \mathbb{K}^\theta$, so $x^{1+\theta} = x^\theta x \in \mathbb{K}^\theta L = L$. Moreover, for given nonzero $u \in L$, the element $u^{\theta-1}$ is mapped onto $\frac{u^\theta}{u}.\frac{u^2}{u^\theta} = u$. Since $u^{-1} \in L$, also $u^{\theta-1} = u^\theta u^{-1} \in L$. The mapping $x \mapsto x^{1+\theta}$ is injective since $x \mapsto x^{\theta-1}$ is its inverse.

If $x^\theta = y^\theta$, then applying θ, we get $x^2 = y^2$, so $x = y$. □

Observation 4. *For each nonzero $t \in L^\theta$, the mapping h_t fixing (∞) and mapping (a, a') onto $(ta, t^{1+\theta}a')$ belongs to $\mathsf{Sz}(\mathbb{K}, L, \theta)$.*

Proof. This follows from a calculation similar to one culminating in the formulae of (33.17) of [12], using the matrices in Subsection 2.9. □

Observation 5. *For $|\mathbb{K}| = 2$, every projective group of $\mathcal{M}\mathsf{Sz}(\mathbb{K}, \mathbb{K}, \mathrm{id})$ is isomorphic to the little projective group G. Also, in this Moufang set the stabilizer G_∞ related to (∞) is isomorphic to U_∞^+ and hence this Moufang set has unique transitive nilpotent normal subgroups.*

Proof. This readily follows from the well known fact that, in this case, the Moufang set is a Frobenius group of order 20 acting on 5 elements, and that this group is a maximal subgroup of the full symmetric group on five letters. □

From now on, we may assume that $|\mathbb{K}| \geq 8$. The following observation is well known for the classical case $L = \mathbb{K}$.

Observation 6. *The center Z_∞^+ of U_∞^+ coincides precisely with the set of elements of U_∞^+ of order less than or equal to 2. The orbit of $(0, 0)$ under Z_∞^+ is equal to $\{(0, a') \mid a' \in L\}$, while the orbit of (∞) under the center Z_0^+ of U_0^+ is equal to $\{(a, 0) \mid a \in L^*\} \cup \{(\infty)\}$.*

Proof. An easy and straightforward computation shows that $Z_\infty^+ = \{(0, a')_\infty \mid a' \in L\}$, and also that $(a, a')_\infty$ has order two if and only if $a = 0$ and $a' \neq 0$. Using the matrices of Subsection 2.9, one now sees that $Z_0^+ = \{(0, x')_0 \mid x \in L\}$, but the element $(0, a^{-1-\theta})_0$ maps $(1, 0, 0, 0)$ to $(1, (a^{-1-\theta})^\theta, 0, a^{-1-\theta})$, which coincides with $(a^{2+\theta}, 1, 0, a) = (a, 0)$. □

Our Main Result will strongly depend on the following lemma.

Lemma 3.5. *Let φ be an automorphism of the Moufang set $\mathcal{M}\mathsf{Sz}(\mathbb{K}, L, \theta)$ fixing (∞) and all elements $(0, a')$ with $a' \in L^\theta$. Then φ is necessarily the identity.*

Proof. By the definition of automorphism, the permutation φ normalizes U_∞^+, and hence also Z_∞^+. Likewise, it normalizes Z_0^+. Using Observation 6, this immediately implies that φ stabilizes the sets $\{(0, a') \mid a \in L\}$ and $(a, 0) \mid a \in L$. Hence we may write $(a, 0)^\varphi = (a^{\varphi_1}, 0)$, with φ_1 a permutation of L fixing 0, and $(0, a')^\varphi = (0, a'^{\varphi_2})$, with φ_2 a permutation of L fixing L^θ pointwise. Since φ fixes $(0, 0)$, we may interpret the foregoing formulae as conjugation of elements of U_∞^+ with φ. Hence, we obtain

$$(a,a')_\infty^\varphi = (a,0)_\infty^\varphi \oplus (0,a')_\infty^\varphi = (a^{\varphi_1}, a'^{\varphi_2})_\infty.$$

We now use the fact that φ induces an automorphism of $U^+\infty$ by conjugation. The equality $(a,0)_\infty^\varphi \oplus (b,0)_\infty^\varphi = (a+b, ab^\theta)_\infty^\varphi$ translates implies

$$a^{\varphi_1}(b^{\varphi_1})^\theta = (ab^\theta)^{\varphi_2} \tag{3.4}$$

Putting $a=1$, and taking into account that $b^\theta \in L^\theta$ is fixed by φ_2, we see that $1^{\varphi_1}(b^{\varphi_1})^\theta = b^\theta$. Putting $b=1$, this implies $1^{\varphi_1}(1^{\varphi_1})^\theta = 1$, hence $1^{\varphi_1} = 1$ by Observation 3. The previous equality now gives us $(b^{\varphi_1})^\theta = b^\theta$. Again using Observation 3 we conclude $\varphi_1 = \mathrm{id}$.

Now putting $b=1$ in Equation (3.4), we deduce $a^{\varphi_1} = a^{\varphi_2}$. The assertion now follows. □

Theorem 3.6. *Let G be an arbitrary projective group of $\mathcal{M}\mathsf{Sz}(\mathbb{K}, L, \theta)$, and let U_∞ be a unipotent subgroup of G. Then $U_\infty \equiv U_\infty^+$.*

Proof. We may assume $U_\infty^+ \le U_\infty$. Let $u \in Z(U_\infty)$. Then u acts fixed point freely on $X \setminus \{(\infty)\}$, and it commutes with every element of U_∞^+. Identifying the element (a,a') with the group element $(a,a')_\infty$, and noting that the action of U_∞^+ can hence be identified with the right action on itself, the action of u can be described as left action on U_∞^+. So, if u maps $(0,0)$ onto (c,c'), then we may write $u : (a,a')_\infty \mapsto (c,c')_\infty \oplus (a,a')_\infty$. Hence, if $c \neq 0$, then the map $\varphi : (a,a')_\infty \mapsto (c,c')_\infty \oplus (a,a')_\infty \oplus (c, c'+c^{1+\theta})_\infty$ is nontrivial, belongs to U_∞ and fixes all elements of the form $(0,a')$, with $a' \in L$. This contradicts Lemma 3.5.

So $c=0$. Considering the isomorphic Moufang set $\mathcal{M}\mathsf{Sz}(\mathbb{K}, Lc'^{-1}, \theta)$, we may assume that $c'=1$. Since the center of U_∞ is invariant under each mapping h_t, $t \in L^\theta$. Observation 4 implies that $(0,t^\theta)_\infty \in Z(U_\infty)$. If $U_\infty \neq U_\infty^+$, then there exists a nontrivial element $\varphi \in U_\infty$ fixing $(0,0)$. Since φ commutes with $(0,t^\theta)$, $t \in L$, it fixes all elements $(0,t^\theta)$, with $t \in L$. Lemma 3.5 shows that φ is the identity, a contradiction. Hence U_∞ must coincide with U_∞^+.

The theorem is proved. □

3.5. Ree-Tits Moufang sets.

We start again with some observations, using the notation of Subsection 2.10.

Observation 7. *The mapping $x \mapsto x^{2+\theta}$ is a permutation of $\mathbb{K}$, inducing a permutation of $\mathbb{K}^2$. Also, the Tits endomorphism $x \mapsto x^\theta$ is a bijection from $\mathbb{K}$ onto $\mathbb{K}^\theta$. Finally, the set $\{t^{1+\theta} \mid t \in \mathbb{K}\}$ contains $\mathbb{K}^2$.*

Proof. The inverse of $x \mapsto x^{2+\theta}$ is given by $x \mapsto x^{2-\theta}$, for $x \neq 0$, and $0 \mapsto 0$.

Also, if $x^\theta = y^\theta$, then applying θ, we get $x^3 = y^3$, so $x = y$.

Finally, for any $x \in \mathbb{K}$, the element $(x^{-1+\theta})^{1+\theta}$ is the arbitrary but prescribed square $x^2 \in \mathbb{K}^2$, which proves the last assertion. □

Observation 8. *For each nonzero $t \in \mathbb{K}$, the mapping h_t fixing (∞) and mapping (a, a', a'') onto $(t^{\theta-1}a, t^2a', t^{1+\theta}a'')$ belongs to $\mathsf{Re}(\mathbb{K}, \theta)$.*

Proof. The subgroups $\{(0, x', 0)_\infty \mid x' \in \mathbb{K}\} \leq U_\infty^+$ and $\{(0, x', 0)_0 \mid x' \in \mathbb{K}\} \leq U_0^+$ preserve the set $\{(0, a', 0) \mid a' \in \mathbb{K}\} \cup \{(\infty)\}$, inducing a Moufang set $\mathcal{M}'$ isomorphic to a projective line over $\mathbb{K}$. Using the matrices above related to the mapping $(0, x', 0)_\infty$ and $(0, x', 0)_0$, one now calculates that the mapping $(0, a', 0) \mapsto (0, t^2a', 0)$, for any $t \in \mathbb{K}^*$, which belongs to $\mathcal{M}'$, acts on X as h_t. $\square$

Observation 9. *The center Z_∞^+ of U_∞^+ consists precisely of the elements $(0, 0, a'')_\infty$, with $a'' \in \mathbb{K}$. Also, the elements of U_∞^+ of order less than or equal to 3 form a subgroup $V_\infty^+ = \{(0, a', a'') \mid a', a'' \in \mathbb{K}\}$ which coincides precisely with the commutator subgroup $[U_\infty^+, U_\infty^+]$, and also with the set of elements $u \in U_\infty^+$ satisfying $[u, U_\infty^+] \leq Z_\infty^+$. The orbit of $(0, 0, 0)$ under Z_∞^+ is equal to $\{(0, 0, a'') \mid a'' \in \mathbb{K}\}$, while the orbit of (∞) under the center Z_0^+ of U_0^+ is equal to $\{(a, 0, -a^{2+\theta}) \mid a \in \mathbb{K}^*\} \cup \{(\infty)\}$.*

Proof. The first assertion follows from an easy and straightforward computation using the operation $\oplus$ introduced above.

The second assertion follows from the identities

$$(a, a', a'')_\infty \oplus (a, a', a'')_\infty \oplus (a, a', a'')_\infty = (0, 0, -a^{2+\theta})_\infty$$

and

$$[(a, a', a'')_\infty, (b, b', b'')_\infty] = (0, ab^\theta - a^\theta b, ab^{1+\theta} - a^{1+\theta}b + a^\theta b^2 - a^2b^\theta + a'b - ab')_\infty,$$

and from the following two claims: (1) for arbitrary $a \in \mathbb{K}$, the identity $ab^\theta - a^\theta b = 0$, for all $b \in \mathbb{K}$, implies $a = 0$, and (2) the additive subgroup A of $\mathbb{K}$ generated by the elements $ab^\theta - a^\theta b$, for $a, b \in \mathbb{K}$, coincides with $\mathbb{K}$ itself.

We prove Claim (1). Putting $b = 1$, Observation 7 implies $a = 1$, a contradiction since $b^\theta - b = 0$ is not an identity in $\mathbb{K}$. We now prove Claim (2). Putting $a \neq b$, we see that A is nontrivial. Let $x \in A$, $x \neq 0$, with $x = ab^\theta - a^\theta b$, for some $a, b \in \mathbb{K}$. Substituting ta and tb for a and b, respectively, with $t \in \mathbb{K}^*$ arbitrary, we see that $t^{1+\theta}x \in A$. Observation 7 implies that, for all $k \in \mathbb{K}$, the element xk^2 belongs to A. For arbitrary $y \in \mathbb{K}$, we now have

$$y = x(x^{-1} - y)^2 - x(x^{-1})^2 - xy^2 \in A.$$

The claim is proved.

The explicit form (using matrices as in Subsection 2.10) of $(0, 0, a'')_0 = (0, 0, a'')_\infty^g$ shows that

$$\begin{aligned}
&(\infty)^{(0,0,a'')_0} \\
&= (-f_3(0, 0, a'')f_1(0, 0, a'')^{-1}, -f_2(0, 0, a'')f_1(0, 0, a'')^{-1}, -a''f_1(0, 0, a'')^{-1}), \\
&= ({a''}^{\theta-2}, 0, -{a''}^{-1}),
\end{aligned}$$

and the last assertion follows by putting $a'' = a^{-2-\theta}$. $\square$

We need one more observation before we can prove the analogue of Lemma 3.5 for Ree-Tits Moufang sets.

Observation 10. *Let φ be an automorphism of the Moufang set $\mathcal{M}\mathsf{Re}(\mathbb{K},\theta)$ fixing (∞) and $(0,0,0)$. Then φ stabilizes the set $\{(0,a',0) \mid a' \in \mathbb{K}\}$.*

Proof. Let $a' \in \mathbb{K}^*$ be arbitrary and let (b,b',b'') be the image of $(0,a',0)$ under φ. Then $(0,a',0)_\infty^\varphi = (b,b',b'')_\infty$. Since $(0,a',0)_\infty \in [U_\infty^+, U_\infty^+]$, also $(b,b',b'')_\infty$ belongs to the commutator subgroup. It follows that $b = 0$. This argument means in fact that (b,b',b'') must belong to the orbit of $(0,0,0)$ under $[U_\infty^+, U_\infty^+]$. Now we remark that $(0,-b'^{-1},0)_0$ maps (∞) onto $(0,b',0)$. Hence, similarly as above, $(0,b',b'')$ must belong to the orbit of (∞) under $[U_0^+, U_0^+]$. Using the same technique as in the proof of the previous observation, one shows that this orbit consists of, besides (∞), the elements

$$(-f_3(0,x',x'')f_1(0,x',x'')^{-1}, -f_2(0,x',x'')f_1(0,x',x'')^{-1}, -x''f_1(0,x',x'')^{-1}),$$

for $x',x'' \in \mathbb{K}$. Such an element also belongs to the orbit of $(0,0,0)$ under $[U_\infty^+, U_\infty^+]$ if and only if $f_3(0,x',x'') = 0$, hence if and only if $x''^\theta = x'x''$. If $x'' = 0$, then the assertion follows. If $x'' \neq 0$, then $x' = x''^{\theta-1}$ and we have $f_1(0,x',x'') = x''^2 + x''^{(\theta-1)(\theta+1)} = -x''^2$, hence $(0,b',b'') = (0,x''^{1-\theta},x''^{-1})$, for some $x'' \in \mathbb{K}^*$. In this case, the image of $(0,-a',0)$ must be equal to, in view of $(0,-a',0)_\infty = (0,a',0)_\infty^{-1}$, the element $(0,-x''^{1-\theta},-x''^{-1})$. But then

$$-x''^{1-\theta} = (-x'')^{1-\theta},$$

a contradiction. $\square$

Our Main Result will strongly depend on the following lemma.

Lemma 3.7. *Let φ be an automorphism of the Moufang set $\mathcal{M}\mathsf{Re}(\mathbb{K},\theta)$ fixing (∞) and all elements $(0,0,a'')$ with $a'' \in \mathbb{K}$. Then φ is necessarily the identity.*

Proof. By assumption, we have $(0,0,a'')^\varphi = (0,0,a'')$, for all $a'' \in \mathbb{K}$. By Observation 10, there is a permutation φ_1 of $\mathbb{K}$ such that $(0,a',0)^\varphi = (0,a'^{\varphi_1},0)$, for all $a' \in \mathbb{K}$. Now, by definition of automorphism of a Moufang set, φ normalizes U_0^+, and hence also its center Z_0^+. Using Observation 9, this implies that there is a permutation φ_2 of $\mathbb{K}$ such that $(a,0,-a^{2+\theta})^\varphi = (a^{\varphi_2},0,-(a^{\varphi_2})^{2+\theta})$.

This implies

$$\begin{aligned}(a,a',a'')_\infty^\varphi &= (a,0,-a^{2+\theta})_\infty^\varphi \oplus (0,a',0)_\infty^\varphi \oplus (0,0,a''+a^{2+\theta}-aa')_\infty^\varphi,\\ &= (a^{\varphi_2}, a'^{\varphi_1}, a''-(a^{\varphi_2})^{2+\theta}+a^{\varphi_2}a'^{\varphi_1}+a^{2+\theta}-aa')_\infty. \qquad (3.5)\end{aligned}$$

Let $a,b \in \mathbb{K}$ be arbitrary. Equating the second positions of $(a,0,0)_\infty^\varphi \oplus (b,0,0)_\infty^\varphi$ and $(a+b,ab^\theta,-ab^{1+\theta})_\infty^\varphi$, we obtain, using the general formulae (3.5),

$$a^{\varphi_2}(b^{\varphi_2})^\theta = (ab^\theta)^{\varphi_1}, \qquad (3.6)$$

for all $a,b \in \mathbb{K}$.

Similarly, equating the third positions of $(0,c,0)^{\varphi}_{\infty} \oplus (d,0,0)^{\varphi}_{\infty}$ and $(d,c,-cd)^{\varphi}_{\infty}$, we obtain, again using the general formulae (3.5),

$$-(d^{\varphi_2})^{2+\theta} + d^{2+\theta} - d^{\varphi_2}c^{\varphi_1} = cd - (d^{\varphi_2})^{2+\theta} + d^{\varphi_2}c^{\varphi_1} + d^{2+\theta},$$

for all $c,d \in \mathbb{K}$, which implies

$$cd = c^{\varphi_1}d^{\varphi_2}, \tag{3.7}$$

for all $c,d \in \mathbb{K}$. Putting $a=b=1$ in Equation (3.6), we see that $1^{\varphi_2}(1^{\varphi_2})^{\theta} = 1^{\varphi_1}$, which implies, in view of Equation (3.7) with $c=d=1$, that $(1^{\varphi_2})^{2+\theta}=1$. Consequently, Observation 7 shows $1^{\varphi_2}=1$. Putting $d=1$ in Equation (3.7), we now see $c=c^{\varphi_1}$, for all $c \in \mathbb{K}$, so φ_1 is the identity. The same Equation (3.7), now again with general $d \in \mathbb{K}$, now also shows that φ_2 is the identity. Formula (3.5) now implies that φ is trivial. □

Theorem 3.8. *Let G be an arbitrary projective group of $\mathcal{M}\mathsf{Re}(\mathbb{K},\theta)$, and let $U_\infty \leq G_\infty$ be a unipotent subgroup of G. Then $U_\infty \equiv U^+_\infty$.*

Proof. We may assume $U^+_\infty \leq U_\infty$. Let $u \in Z(U_\infty)$. If u maps $(0,0,0)$ onto (c,c',c''), then, similarly as in the beginning of the proof of Theorem 3.6, u can be presented as $u:(a,a',a'')_\infty \mapsto (c,c',c'')_\infty \oplus (a,a',a'')_\infty$. Hence, if $(c,c') \neq (0,0)$, then the map $\varphi:(a,a',a'')_\infty \mapsto (c,c',c'')_\infty \oplus (a,a',a'')_\infty \oplus (-c,-c'+c^{1+\theta},-c''+cc'-c^{2+\theta})_\infty$ belongs to U_∞ and fixes all elements of the form $(0,0,a'')$, with $a' \in L$. This contradicts Lemma 3.5.

So we may assume that $(c,c')=(0,0)$. Then $u=(0,0,c'')_\infty$, for some $c'' \in \mathbb{K}$. Since the center of U_∞ is invariant under each mapping h_t, $t \in \mathbb{K}$, Observation 8 implies that $(0,0,t^{1+\theta}c'')_\infty \in Z(U_\infty)$. Hence by Observation 7 $(0,0,k^2c'')_\infty \in Z(U_\infty)$, for all $k \in \mathbb{K}$. For arbitrary $x \in \mathbb{K}$, we see that

$$(0,0,x)_\infty = (0,0,(x-c''^{-1})c'')_\infty \oplus (0,0,x^2c'')^{-1}_\infty \oplus (0,0,(c''^{-1})^2c'')^{-1}_\infty,$$

which implies $Z(U_\infty)=Z^+_\infty$. Standard group theory now implies that φ fixes all elements $(0,0,x)$, with $x \in \mathbb{K}$. Lemma 3.7 shows that φ is the identity, a contradiction. Hence U_∞ must coincide with U^+_∞.

The theorem, and also our Main result, are proved. □

4. Final remarks

The present paper treats almost all known Moufang sets that do not arise from sharply 2-transitive groups. Only the cases of an algebraic group of relative rank 1 and exceptional absolute type, or absolute type D_4, were left out, as are the new Moufang sets discovered in [6] and which arise from a polarity of an exceptional Moufang quadrangle of type F_4. However, all Moufang sets that appear in higher rank 2-spherical Moufang buildings as permutation groups generated by opposite root groups are covered by our Main Result, and by [1], this provides a new proof of the main result of *loc.cit.*, namely the fact that *in every split spherical BN-pair of irreducible rank* ≥ 2 *the unipotent subgroups are unique as transitive nilpotent normal subgroups of the Borel subgroups.*

We end by noting that our Main Result implies that for the Moufang sets under consideration, and for every projective group G, the root groups U_x are characteristic subgroups of the point stabilizers G_x. It is this fact that we expect to be very useful in geometric approach to the rank 1 buildings defined by Moufang sets with nonabelian root groups, as proposed by Jacques Tits [10].

References

[1] T. De Medts, F. Haot, K. Tent and H. Van Maldeghem, Split Bn-pairs of rank at least 2 and the uniqueness of the splitting, *J. Group Theory* **8** (2005), no. 1, 1 – 10.

[2] T. De Medts and R. Weiss, Moufang sets and Jordan division algebras, to appear in *Math. Ann.*

[3] F. Haot and H. Van Maldeghem, Some characterizations of Moufang generalized quadrangles, *Glasgow J. Math.* **46** (2004), 335 – 343.

[4] F. Haot and H. Van Maldeghem, A half 3-Moufang quadrangle is Moufang, to appear in *Bull. Belg. Math. Soc. Simon Stevin.*

[5] C. Hering, W. M. Kantor and G. M. Seitz, Finite groups with a split BN-pair of rank 1, I, *J. Algebra* **20** (1972), 435 – 475.

[6] B. Mühlherr and H. Van Maldeghem, Moufang sets from groups of mixed type, to appear in *J. Algebra*.

[7] E. Shult, On a class of doubly transitive groups, *Illinois J. Math.* **16** (1972), 434 – 455.

[8] K. Tent, Half moufang implies Moufang for generalized quadrangles, *J. Reine Angew. Math.* **566** (2004), 231 – 236.

[9] F. G. Timmesfeld, A note on groups with a BN-pair of spherical type, *Arch. Math. (Basel)* **82** (2004), no. 6, 481 – 487.

[10] J. Tits, *Résumé de cours (Annuaire du Collège de France),* 97^e année, 1996-1997, 89 – 102.

[11] J. Tits, *Résumé de cours (Annuaire du Collège de France),* 100^e année, 1999-2000, 93 – 109.

[12] J. Tits and R. Weiss, *Moufang Polygons*, Springer-Verlag, *Springer Monographs in Mathematics*, Berlin, Heidelberg, New York, 2002.

[13] H. Van Maldeghem, *Generalized Polygons*, Birkhauser Verlag, Basel, Boston, Berlin, *Monographs in Mathematics*, **93**, 1998.

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