Linear representations of semipartial geometries[*][†]

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Abstract

Semipartial geometries (SPG) were introduced in 1978 by Debroey and Thas [5]. As some of the examples they provided were embedded in affine space it was a natural question to ask whether it was possible to classify all SPG embedded in affine space. In $\mathrm{AG}(2,q)$ and $\mathrm{AG}(3,q)$ a complete classification was obtained ([6]). Later on it was shown that if an SPG, with $\alpha > 1$, is embedded in affine space it is either a linear representation or $\mathrm{TQ}(4,2^h)$ (see [8],[11]). In this paper we derive general restrictions on the parameters of an SPG to have a linear representation and classify the linear representations of SPG in $\mathrm{AG}(4,q)$, hence yielding the complete classification of SPG in $\mathrm{AG}(4,q)$, with $\alpha > 1$.

1 Introduction

A semipartial geometry with parameters s, t, α and μ, denoted by $\mathrm{spg}(s,t,\alpha,\mu)$, is a connected partial linear space $\mathbb{S}$ of order (s,t) satisfying the following axioms.

(i) If a point x and a line L are not incident, then there are either 0 or α ($\alpha > 0$) points which are collinear with x and incident with L.

(ii) If two points are not collinear, then there are μ ($\mu > 0$) points collinear with both.

Semipartial geometries where introduced by Debroey and Thas in [5]. Semipartial geometries have a strongly regular point graph. A semipartial geometry such that $\alpha = 1$ is called a partial quadrangle, and were introduced in [4] by Cameron. A semipartial geometry such that for each anti-flag, i.e. non-incident point-line pair (x, L), there are exactly α points on L collinear with x is called a partial geometry [1]. In that case, condition (ii) is trivially satisfied with $\mu = \alpha(t+1)$ and, conversely, every semipartial geometry with $\mu = \alpha(t+1)$ is a partial geometry $\mathrm{pg}(s,t,\alpha)$. A $\mathrm{pg}(s,t,t)$ is also known as

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a *(Bruck) net of order* $s+1$ *and degree* $t+1$. A semipartial geometry that is not a partial geometry will be called a *proper* semipartial geometry. Several examples of partial and proper semipartial geometries are known; for an overview on these geometries we refer to [7, 9]. In the rest of this section however we shall restrict ourselves to those examples and constructions that we will need in the rest of this paper.

Consider an affine space $\mathrm{AG}(n+1,q)$ and a point set $\mathcal{K}$ in its hyperplane $\Pi := \mathrm{PG}(n,q)$ at infinity. The geometry $T_n^*(\mathcal{K})$ with point set the points of $\mathrm{AG}(n+1,q)$ and as set of lines the the union of all parallel classes of lines of $\mathrm{AG}(n+1,q)$, whose points at infinity are the points of $\mathcal{K}$ is called *the linear representation of* $\mathcal{K}$ (the incidence is the one inherited from $\mathrm{AG}(n+1,q)$).

A maximal arc $\mathcal{K}$ of degree d, with $d>0$, in a projective plane Π of order q is a non-empty set of points such that each line of Π that intersects $\mathcal{K}$ in at least one point intersects it in exactly d points, i.e., it is a nonempty set of $qd-q+d$ points in Π such that every line of Π has either 0 or d points in common with $\mathcal{K}$.

A unital $\mathcal{U}$ in a projective plane $\Pi = \mathrm{PG}(2,q^2)$ is a set of q^3+1 points such that each line of Π intersects $\mathcal{U}$ in either 1 or $q+1$ points.

We can now give an overview of the known $\mathrm{spg}(s,t,\alpha,\mu)$ which have a linear representation $T_n^*(\mathcal{K})$. We always suppose that $\mathcal{K}$ is not trivial, i.e. $\mathcal{K}$ nor its complement is empty, a point or a subspace. If $\alpha=1$ then Calderbank [2], and Tzanakis and Wolfskill [18] obtained an almost complete classification.

Theorem 1.1 *If* $\mathcal{K}$ *is a non-trivial point set in* $\mathrm{PG}(n,q)$ *such that* $T_n^*(\mathcal{K})$ *is an* $\mathrm{spg}(q-1,|\mathcal{K}|,1,\mu)$*, then only the following cases can occur:*

- $\mathcal{K}$ *is a hyperoval in* $\mathrm{PG}(2,2^m)$*;*
- $\mathcal{K}$ *is an ovoid in* $\mathrm{PG}(3,q)$*;*
- $\mathcal{K}$ *is an* 11*-cap in* $\mathrm{PG}(4,3)$*;*
- $\mathcal{K}$ *is the unique* 56*-cap in* $\mathrm{PG}(5,3)$*; or a* 78*-cap in* $\mathrm{PG}(5,4)$ *such that each external point is on* 7 *secants (at least one example is known);*
- $\mathcal{K}$ *is a* 430*-cap in* $\mathrm{PG}(6,4)$*, however it is not known whether such a cap exists.*

If $\alpha>1$ the following examples are known:

- in $\mathrm{AG}(2,q)$ every linear representation is a Bruck net;
- $\mathcal{K}$ is a maximal arc in $\mathrm{PG}(2,q)$, and then $T_2^*(\mathcal{K})$ is a partial geometry, and was constructed by Thas [16];
- $\mathcal{K}$ is a unital $\mathcal{U}$ in $\mathrm{PG}(2,q^2)$, and then $T_2^*(\mathcal{U})$ is an $\mathrm{spg}(q^2-1,q^3,q,q^2(q^2-1))$;
- $\mathcal{K}$ is a Baer-subgeometry $\mathcal{B}\cong\mathrm{PG}(n,q)$ of $\mathrm{PG}(n,q^2)$, and then $T_n^*(\mathcal{B})$ is an $\mathrm{spg}(q^2-1,\frac{q^n-1}{q-1}-1,q,q(q+1))$.

To end this introduction we mention some theorems which will be of use in the following sections.

Theorem 1.2 ([15]) *Let* $\mathcal{O}$ *be a set of points in* $\mathrm{PG}(n,q)$*,* $n\geq 3$*, such that each line intersects* $\mathcal{O}$ *in either* α *or* β *points. If* $\mathcal{O}$ *nor its complement is empty, a point or a hyperplane, then* q *is an odd square and if* $\alpha\leq\beta$ *then*

$$\alpha = \frac{1}{2}(q+1-\sqrt{q}(1-\epsilon)),$$

$$\beta = \frac{1}{2}(q+1+\sqrt{q}(1+\epsilon)) \textit{ and}$$

$$|\mathcal{O}| = \frac{1}{2}(1+\frac{q^{n-1}-1}{q-1}(q+\epsilon\sqrt{q})+\delta\sqrt{q^{n-1}}),$$

where $\epsilon = \pm 1$ *and* $\delta = \pm 1$.

Theorem 1.3 ([14]) *If* $\mathcal{K}$ *is a set of points of* $\mathrm{PG}(n,q)$, $n \geq 3$, *with the property that every hyperplane of* $\mathrm{PG}(n,q)$ *intersects* $\mathcal{K}$ *in either* 0 *or* $m > 0$ *points, then* $\mathcal{K}$ *is either a unique point or the point set of the complement of a hyperplane of* $\mathrm{PG}(n,q)$.

Theorem 1.4 ([19]) *If* $\mathcal{K}$ *is a point set in* $\mathrm{PG}(n,q)$, $n \geq 3$, *with the property that* $\mathcal{K}$ *spans* $\mathrm{PG}(n,q)$ *and such that each line of* $\mathrm{PG}(n,q)$ *intersects* $\mathcal{K}$ *in either* 0, 1 *or* $\alpha \geq \sqrt{q}+1$ *points, then* $\mathcal{K}$ *is either a Baer-subgeometry, an affine subspace of* $\mathrm{PG}(n,q)$, *or* $\mathcal{K}$ *equals the point set of* $\mathrm{PG}(n,q)$.

2 General results

From now on let $\mathcal{K}$ be a non-trivial set of points in $\mathrm{PG}(n,q)$, $n \geq 3$ (i.e. $\mathcal{K}$ is not a subspace nor its complement). Embed $\mathrm{PG}(n,q)$ as a hyperplane Π in $\mathrm{PG}(n+1,q)$. We assume that the linear representation $T_n^*(\mathcal{K})$ of $\mathcal{K}$ is an $\mathrm{spg}(q-1, |\mathcal{K}|-1, \alpha, \mu)$. In this section we will derive some general results for such a set $\mathcal{K}$, which will enable us in the following sections to obtain a classification when $n = 3$. We always suppose that $\alpha > 1$.

Lemma 2.1 *Every line of* Π *intersects* $\mathcal{K}$ *in either* 0, 1 *or* $\alpha+1$ *points, and the set* $\mathcal{K}$ *consists of* $1+x\alpha$, $x \in \mathbb{N}$, *points. There exists a constant* θ *such that each point not belonging to* $\mathcal{K}$ *is incident with* θ *lines intersecting* $\mathcal{K}$ *in* 1 *point. Furthermore* $\mathcal{K}$ *has two intersection numbers with respect to hyperplanes.*

Proof. It is readily checked that the α-condition for SPG implies that a line intersecting $\mathcal{K}$ in at least two points must intersect it in $\alpha+1$ points. Now consider a fixed point of $\mathcal{K}$. Then any line through this point contains either 0 or α other points of $\mathcal{K}$. Hence $|\mathcal{K}| = 1 + x\alpha$, with x the number of lines through a given point of $\mathcal{K}$ intersecting $\mathcal{K}$ in at least 2 points. The existence of the constant θ is a consequence of the μ-condition for SPG. There holds $\mu = (|\mathcal{K}| - \theta)\alpha$. Finally, since the point graph of $T_n^*(\mathcal{K})$ is strongly regular, the last assertion of the lemma follows from a result by Delsarte, see [10] (see also [3]). □

We will call a line intersecting $\mathcal{K}$ in 0, 1, respectively $\alpha+1$ points an *exterior line*, a *tangent*, respectively an $(\alpha+1)$-*secant*.

Lemma 2.2 *(i) There exist exterior lines of* $\mathcal{K}$.

(ii) Every hyperplane of Π *has at least one point in common with* $\mathcal{K}$.

Proof.

(i) Suppose by way of contradiction that every line of Π would have at least one point in common with $\mathcal{K}$, then clearly each line intersects $\mathcal{K}$ in either 1 or $\alpha+1$ points. From Theorem 1.2 there follows that $1 = \frac{1}{2}(q+1-\sqrt{q}(1-\epsilon))$, with $\epsilon = \pm 1$, clearly a contradiction.

(ii) If there exist hyperplanes exterior to $\mathcal{K}$, then Lemma 2.1 implies that a hyperplane contains either 0 or $m > 0$ points of $\mathcal{K}$. Hence Theorem 1.3 yields that $\mathcal{K}$ is either a point or the point set of the complement of a hyperplane, in contradiction with our assumptions. □

Together with Lemma 2.1 the previous lemma implies that a hyperplane contains either $1 + y\alpha$ points or $1 + z\alpha$ points, $y, z \in \mathbb{N}$, with $y < z$ ([3]). We will call a hyperplane of the former (resp. latter) type a y-hyperplane (resp. z-hyperplane). From [3] it follows that $\mathcal{K}$ yields a two-weight code with weights $w_1 = 1 + x\alpha - (1 + z\alpha)$ and $w_2 = 1 + x\alpha - (1 + y\alpha)$.

Lemma 2.3 *If $\mathcal{K}$ is a set of points in $\Pi := \mathrm{PG}(n, q)$, $n \geq 3$, $q = p^m$, p prime, with the property that $T_n^*(\mathcal{K})$ is an spg$(q-1, |\mathcal{K}|-1, \alpha, \mu)$ then $\alpha = p^i$, $0 \leq i \leq m$.*

Proof. By the previous lemma we can choose a subspace $\gamma := \mathrm{PG}(l, q) \subset \Pi$ exterior to $\mathcal{K}$, $0 < l < n-1$, such that no $(l+1)$-dimensional subspace is exterior to $\mathcal{K}$. Now consider any $\Gamma := \mathrm{PG}(l+2, q) \subset \Pi$ containing γ. Clearly $|\Gamma \cap \mathcal{K}| = 1 + c\alpha$, $c \in \mathbb{N} \setminus \{0\}$. Every $(l+1)$-dimensional subspace of Γ containing γ will contain $1 + c_j\alpha$, $c_j \in \mathbb{N}$, points of $\mathcal{K}$. We obtain

$$\sum_{j=0}^{q}(1 + c_j\alpha) = 1 + c\alpha$$

and hence $q + \alpha\sum_{j=0}^{q} c_j = c\alpha$. This proves the lemma. □

Lemma 2.4 *There holds that $z - y$ equals p^k for some $k > 0$*

Proof. From [3] it follows that $w_2 - w_1 = p^w$, $w \in \mathbb{N}$. Hence the previous lemma implies that $w_2 - w_1 = (z-y)p^i = p^w$. Thus $z - y = p^k$ for some $k \in \mathbb{N}$. We will now show that $k > 0$. Consider an exterior line L and let δ be the number of z-hyperplanes containing L. We count the pairs (u, η), where $u \in \mathcal{K}$, $u \in \eta$ and η a hyperplane containing L, in two ways:

$$\delta(1 + z\alpha) + (\frac{q^{n-1}-1}{q-1} - \delta)(1 + y\alpha) = (1 + x\alpha)\frac{q^{n-2}-1}{q-1}.$$

Now consider an $(\alpha+1)$-secant M and let δ' be the number of z-hyperplanes containing M. Here we count the pairs (v, ξ), where $v \in \mathcal{K} \setminus M$, $v \in \xi$ and ξ a hyperplane containing M,

$$\delta'(z-1)\alpha + (\frac{q^{n-1}-1}{q-1} - \delta')(y-1)\alpha = (x-1)\alpha\frac{q^{n-2}-1}{q-1}.$$

Subtracting the second equation from the first yields

$$(\delta' - \delta)p^k = \frac{\alpha+1}{\alpha}q^{n-2}.$$

Since $\alpha \neq q$ (because otherwise $\mathcal{K}$ would be the point set of a subspace) we find that $\frac{\alpha+1}{\alpha}q^{n-2} \geq q^{n-2} + 2q^{n-3}$. As $\delta, \delta' \in \mathbb{N}$, we see that if $k = 0$ it follows that $\delta' \geq q^{n-2} + 2q^{n-3}$, a contradiction since $\delta' \leq (q^{n-1}-1)/(q-1)$. □

Lemma 2.5 *If $\mathcal{K}$ is a set of points in $\Pi := \mathrm{PG}(n, q)$, $n \geq 3$, $q = p^m$, p prime, with the property that $T_n^*(\mathcal{K})$ is an spg$(q-1, |\mathcal{K}|-1, p^i, \mu)$ then the strongly regular point graph of $T_n^*(\mathcal{K})$ has parameters*

- $\mu = p^i \frac{x(1+xp^i)(p^m-p^i)}{p^{mn}+\cdots+p^m-xp^i}$;
- $\lambda = q - 2 + xp^i(p^i - 1)$ *and*
- $K = (xp^i + 1)(p^m - 1)$, *with K the valency of the graph.*

Proof. From the previous lemmas we know that $\mu = (xp^i + 1 - \theta)p^i$. So we should now determine θ. We count in two ways the pairs (u, v), where $u \notin \mathcal{K}$, $v \in \mathcal{K}$ and uv a tangent. We obtain $(1+xp^i)(\frac{q^n-1}{q-1} - x)q = (\frac{q^{n+1}-1}{q-1} - 1 - xp^i)\theta$ from which θ follows. It now easily follows that $\mu = p^i \frac{x(1+xp^i)(p^m-p^i)}{p^{mn}+\cdots+p^m-xp^i}$.

The values for λ and K follow trivially. □

Theorem 2.6 *Let $\mathcal{K}$ be a set of points in $\Pi := \mathrm{PG}(n, q)$, $n \geq 3$, $q = p^m$, p prime, with the property that $T_n^*(\mathcal{K})$ is an spg$(q-1, |\mathcal{K}|-1, p^i, \mu)$. If $i \geq m/2$, then $T_n^*(\mathcal{K}) \cong T_n^*(\mathcal{B})$.*

Proof. Theorem 1.4 immediately implies that $\mathcal{K}$ is a Baer subgeometry. □

From now on we may suppose that $i < m/2$. We will use the following theorem from [3].

Theorem 2.7 ([3]) *If $\mathcal{K}$ is a point set in $\mathrm{PG}(n, q)$ with the property that $T_n^*(\mathcal{K})$ has a strongly regular point graph with parameters $(v = q^{n+1}, K = |\mathcal{K}|\,(q-1), \lambda, \mu)$, then*

$$q(w_2 - w_1) = ((\lambda - \mu)^2 + 4(K - \mu))^{1/2},$$

where $w_1 < w_2$ are the two intersection numbers of $\mathcal{K}$ with respect to hyperplanes of $\mathrm{PG}(n, q)$.

Since the point graph of $T_n^*(\mathcal{K})$ is strongly regular Theorem 2.7 implies that

$$p^{2m}p^{2k+2i} = (\lambda - \mu)^2 + 4(K - \mu) \tag{1}$$

with λ, μ and K as in Lemma 2.5.

Finally we show that from $\mathcal{K}$ we can construct a point set in $\mathrm{PG}(n-1, q)$ having two intersection numbers with respect to hyperplanes. Let u be any point of $\mathcal{K}$ and consider a hyperplane Δ of Π not containing u. As every hyperplane through u contains either y or z $(\alpha+1)$-secants through u, we see that the projection of $\mathcal{K}$ from u on Δ yields a point set $\mathcal{L}$ of cardinality x in Δ with the property that every hyperplane of Δ contains either y or z points of $\mathcal{L}$. Notice that both intersection numbers occur.

Lemma 2.8 *There holds*

$$x^2(q^{n-2} - 1) + x(q^{n-2}(q - 1) - (y + z)(q^{n-1} - 1)) + yz(q^n - 1) = 0 \tag{2}$$

Proof. In [12] this is shown for $n = 3$ (not in the context of projections of a set $\mathcal{K}$). The proof we give for the general case is analogous. Let $\mathcal{L}$ and Δ be as above. Denote by τ_y, respectively τ_z, the number of hyperplanes of Δ containing y, respectively z, points of $\mathcal{L}$. We obtain

$$\tau_y + \tau_z = \frac{q^n - 1}{q - 1}$$

$$\tau_y y + \tau_z z = x\frac{q^{n-1} - 1}{q - 1}$$

$$\tau_y y(y - 1) + \tau_z z(z - 1) = x(x - 1)\frac{q^{n-2} - 1}{q - 1}$$

Eliminating τ_y and τ_z from these equations yields equation (2). □

3 The case $n = 3$

In this section we suppose that the setup is as in the previous section with $n = 3$, $\alpha = p^i > 1$ and $i < m/2$. Furthermore we use the same notations. We start by handling some special cases.

We need the following theorem, which is due to Thas.

Theorem 3.1 ([17]) *Suppose $\mathcal{K}$ is a point set in $\mathrm{PG}(n, q)$, $n \geq 3$, with the property that a hyperplane contains either 1 or $m > 1$ points of $\mathcal{K}$ and such that there exists at least one hyperplane containing exactly 1 point of $\mathcal{K}$. Then $\mathcal{K}$ is the point set of a line of $\mathrm{PG}(n, q)$ or $\mathcal{K}$ is an ovoid of $\mathrm{PG}(3, q)$.*

In our setup this immediately translates into the following.

Theorem 3.2 *The case $y = 0$ cannot occur.*

Next we exclude the other end of the spectrum.

Theorem 3.3 *The case $z = q + 1$ cannot occur.*

Proof. In a z-plane of Π every point of $\mathcal{K}$ is clearly contained in $q+1$ $(\alpha+1)$-secants. There follows that a z-plane contains no tangent lines and hence that $\mathcal{K}$ induces a maximal arc in every z-plane. This implies that $\alpha = p^l - 1$ in contradiction with Lemma 2.3. □

We will now start with an analysis of equation (1), but first we introduce a new notation. We will denote by $\mathcal{O}(p^f)$ any polynomial in p of degree at least p^f with coefficients in $\mathbb{N}$. The calculations in the rest of this section are tedious, and can easily be carried out in MAPLE. That is the reason why in most steps we only mention the terms we need and use shortened expressions.

If $n = 3$ equation (1) becomes, after multiplying both sides with $(p^{3m} + p^{2m} + p^m - xp^i)^2$,

$$p^{2m+4i+2k}(p^{3m-i} + p^{2m-i} + p^{m-i} - x)^2 = U \tag{3}$$

with

$$U = (x^2 - 2x^3 + x^4)p^{4i} + \mathcal{O}(p^{4i+1})$$

Considering this equation modulo p^{4i+1} we find that p divides $x^2(x - 1)^2$. There follows

Lemma 3.4 *Either p divides x, or p divides $x-1$.*

Lemma 3.5 *There holds that $x = y = z \pmod p$.*

Proof. We first show that every plane contains exterior lines. Assume that a z-plane π would contain no exterior lines. It then follows that $\mathcal{K}$ induces in π either a line, a Baer subgeometry or a unital (see Chapter 12 of [12]) yielding $\alpha \geq p^{m/2}$, a contradiction. Now let L be an exterior line to $\mathcal{K}$, and suppose that there are δ z-planes containing L. We obtain

$$\delta(1 + z\alpha) + (q + 1 - \delta)(1 + y\alpha) = 1 + x\alpha$$

which yields

$$\delta p^k + y + p^{m-i} + yp^m = x$$

Hence $x = y \pmod p$. As $z - y = p^k$, with $k > 0$ the result follows. □

The fact that $\delta \neq 0$ will be of use later on.

3.1 The case p divides x

We write $x = ap$ and substitute this in equation (3). We obtain

$$\begin{aligned} U &= p^{4m} - 2ap^{2m+2i+1} + a^2p^{4i+2} + \mathcal{O}(p^{4m+1}) + a\mathcal{O}(p^{3m+i+1}) \\ &\quad + a^2\mathcal{O}(p^{m+3i+2}) + a^3\mathcal{O}(p^{4i+3}) + a^4\mathcal{O}(4i+4). \end{aligned}$$

Lemma 3.6 *There holds that $p^{2m-2i-1}$ divides a.*

Proof. Suppose that p^j divides a with $0 \leq j < 2m-2i-1$.We write $a = a'p^j$. As $4i+2+2j < 2m+2i+1+j$ and $4i+2+2j < 4m$ if $j < 2m-2i-1$, clearly $U = a'^2p^{4i+2+2j} + \mathcal{O}(p^{4i+2j+3})$. There are two possibilities. If $j < m-i$, the left hand side of equation 3 becomes

$$p^{2m+4i+2k+2j}(p^{3m-i-j} + p^{2m-i-j} + p^{m-i-j} - a')^2.$$

If $j \geq m - i$ we obtain

$$p^{4m+2i+2k}(p^{2m} + p^m + 1 - a'p^{j-m+i})^2.$$

In either case there holds that the left hand side of equation (3) is 0 modulo $p^{4i+2j+3}$. This implies that a'^2 is 0 modulo p and hence that p^{j+1} divides a. The lemma follows. □

As an immediate consequence we can write from now on $x = bp^{2m-2i}$. We will now turn to the analysis of equation (2) which will allow us to exclude the case p divides x.

Theorem 3.7 *The case p divides x cannot occur.*

Proof. Write y and z in p-ary representation: $y = y_fp^f + \cdots$ and $z = z_lp^l + \cdots$ with $y_f \neq 0$ (because of Theorem 3.2) and $z_l \neq 0$. After division by $q-1$ equation (2) becomes

$$x^2 - x(q(y + z - 1) + y + z) + yz(q^2 + q + 1) = 0 \tag{4}$$

with $q = p^m$. Because of the previous lemma p^{4m-4i} divides x^2, the terms of lowest degree in $x(q(y+z-1)+y+z)$ are

$$\overline{by_f}p^{2m-2i+f} \quad \text{and} \quad \overline{bz_l}p^{2m-2i+l}$$

while the term of lowest degree in $yz(q^2+q+1)$ is

$$\overline{y_f z_l}p^{f+l}$$

where $\overline{uv}$ denotes multiplication of u and v modulo p. Clearly $f+l < 4m-4i$, as f and m are at most m (y and z are a number of lines through a point in a plane) and $i < m/2$. Furthermore $2m-2i+f \leq f+l$ would imply $l \geq m+1$, a contradiction. In an analogous way $2m-2i+l \leq f+l$ cannot occur. Hence if we consider equation (4) modulo p^{f+l+1} we obtain that $\overline{y_f z_l} = 0$, the final contradiction. □

3.2 The case p divides $x-1$

The basic ideas for handling this case are the same as in the previous subsection, but it will turn out that there are more subcases to deal with. We will write $x = ap+1$.

Lemma 3.8 *There holds that p^{m-i-1} divides a.*

Proof. Substituting $x = ap+1$ in U we obtain

$$\begin{aligned} U &= a^2p^{4i+2} - 2ap^{m+3i+1} + p^{2m+2i} + \mathcal{O}(p^{2m+3i}) + a\mathcal{O}(p^{m+4i+1}) \\ &\quad + a^2\mathcal{O}(p^{m+3i+2}) + a^3\mathcal{O}(p^{4i+3}) + a^4\mathcal{O}(p^{4+4i}). \end{aligned}$$

Suppose that p^j divides a with $j < m-i-1$. We write $a = a'p^j$. As $4i+2+2j < m+3i+1+j$ and $4i+2+2j < 2m+2i$ whenever $j < m-i-1$, clearly we obtain $U = a'^2p^{4i+2+2j} + \mathcal{O}(p^{4i+3+2j})$. The left hand side of equation (3) becomes

$$p^{2m+4i+2k}(p^{3m-i} + p^{2m-i} + p^{m-i} - 1 - a'p^j)^2$$

and is clearly 0 (mod $p^{4i+3+2j}$). We find that p must divide a'^2, that is, p^{j+1} divides a. The lemma follows. □

From now on we write $x = bp^{m-i}+1$.

Lemma 3.9 *There holds that $b = cp+1$ with $c \in \mathbb{N} \setminus \{0\}$.*

Proof. Substituting $x = 1 + bp^{m-i}$ in U we obtain

$$U = p^{2m+2i} - 2bp^{2m+2i} + b^2p^{2m+2i} + \mathcal{O}(p^{2m+2i+1}).$$

As the left hand side of equation (3) is 0 (mod $p^{2m+2i+1}$) we see that p must divide $(b-1)^2$. Hence $b = cp+1$ with $c \in \mathbb{N}$. If $c = 0$ we find $x < 1 + p^m$, a contradiction as any non-trivial two-weight set in Δ must contain at least $2 + p^m$ points, i.e. $|\mathcal{L}| = x \geq 2 + p^m$. □

We obtain $x = 1 + p^{m-i} + cp^{m-i+1}$.

Lemma 3.10 *There holds that p^{i-1} divides c.*

Proof. The lemma becomes of course trivial if $i = 1$, so we suppose that $i > 1$. Again in the same spirit we obtain

$$\begin{aligned} U &= 4p^{2m+4i} - 4cp^{2m+3i+1} + c^2p^{2m+2i+2} + \mathcal{O}(p^{3m+2i}) + \\ &\quad c\mathcal{O}(p^{3m+i+1}) + c^2\mathcal{O}(p^{3m+i+2}) + c^3\mathcal{O}(p^{3m+3+i}) + \\ &\quad c^4\mathcal{O}(p^{4m+4}) \end{aligned}$$

We suppose that $c = c'p^j$ with $j < i-1$. There holds that $2m+2i+2+2j < 2m+3i+1+j$ and $2m+2i+2+2j < 2m+4i$ if $j < i-1$. Furthermore the left hand side of equation (3) is $0 \pmod{p^{2m+2i+3+2j}}$, implying that p divides c' and hence that p^{j+1} divides c. The lemma follows. □

There follows that $x = 1 + p^{m-i} + dp^m$ with $d \in \mathbb{N} \setminus \{0\}$.

Lemma 3.11 *If $p \neq 2$, then $d = 2 \pmod{p}$. If $p = 2$, then d is even; furthermore in this case there holds that if 2^f, with $f > 1$ divides d, and $k > 1$, then $i = (m-1)/2$.*

Proof. There holds that

$$U = (d^2 - 4d + 4)p^{2m+4i} + (-4 + 2d)p^{3m+2i} + p^{4m} + \mathcal{O}(p^{3m+3i}).$$

As the left hand side of equation (3) is $0 \pmod{p^{2m+4i+1}}$, we see that $(d-2)^2$ is divisible by p. The first two assertions follow. Suppose that $p = 2$, 2^f, with $f > 1$ divides d, and $k > 1$. Then

$$U = 2^{2m+4i+2} + 2^{4m} + \mathcal{O}(2^{2m+4i+3})$$

(notice that $2m + 4i + 3 \leq \min(3m + 2i + 2, 3m + 3i)$, unless $i = 1$ and $m = 3$, in which case we have the desired property $i = (m-1)/2$). Since $k > 1$ the left hand side of equation 3 is $0 \pmod{2^{2m+4i+3}}$. This implies that $2m + 4i + 2 = 4m$, i.e. $i = (m-1)/2$. □

Lemma 3.12 *The case $y = 1$ cannot occur.*

Proof. Suppose that $y = 1$. Then the set $\mathcal{L}$ in Δ must either be a line, a Baer-subplane or a unital [12]. As $z = p^m + 1$ is impossible by Theorem 3.3 $\mathcal{L}$ cannot be a line. If $\mathcal{L}$ is a Baer-subplane, respectively a unital, there follows that $x = 1 + p^{m/2} + p^m$, respectively $1 + p^{3m/2}$, both in contradiction with the derived form of x. □

Lemma 3.13 *There holds that p^k divides $y - 1$.*

Proof. First notice that, since $y, z \leq p^m$, there holds that $k < m$. Using the equation obtained in the proof of Lemma 3.5 and the form of x we find

$$\delta p^k + (y-1) + p^m + (y-1)p^m = dp^m$$

with δ the number of z-planes through an exterior line of $\mathcal{K}$. The lemma follows immediately since $\delta \neq 0$. □

There are three possibilities for y and z:

(I) $y = 1 + up^l$ and $z = 1 + p^k + up^l$, with $u \in \mathbb{N}$, u not divisible by p and $l > k$ (notice that $l < m$);

(II) $y = 1 + y_k p^k + up^{k+1}$ and $z = 1 + (1 + y_k)p^k + up^{k+1}$, with $0 < y_k < p - 1$ and $u \in \mathbb{N}$ (notice that $p \neq 2$ in this case);

(III) $y = 1 - p^k + up^l$ and $z = 1 + up^l$, with $u \in \mathbb{N}$, u not divisible by p and $l > k$.

Lemma 3.14 *There holds that $k \leq m - i$.*

Proof. Consider a tangent line L to $\mathcal{K}$ and let β be the number of y-planes through L. We find

$$\beta y + (p^m + 1 - \beta)z = x$$

which yields

$$-\beta p^k + zp^m + (z - 1) = p^{m-i} + dp^m.$$

Since p^k divides $z - 1$, we see that $k \leq m - i$. □

Lemma 3.15 *The following cases cannot occur: $l < m - i$ in* (**I**)*, $k < m - i$ in* (**II**) *and $l < m - i$ in* (**III**).

Proof. Assume by way of contradiction that $l < m-i$, respectively $k < m-i$. Substituting the obtained values for x, y and z in equation (4) we find

$$up^{l+k} + \mathcal{O}(p^{l+k+1}) = 0 \text{ in case } (\mathbf{I});$$

$$y_k(y_k + 1)p^{2k} + \mathcal{O}(p^{2k+1}) = 0 \text{ in case } (\mathbf{II}) \text{ and}$$

$$up^{k+l} + \mathcal{O}(p^{k+l+1}) = 0 \text{ in case } (\mathbf{III}).$$

In case (**I**) and (**III**) we obtain a contradiction modulo p^{k+l+1}, since p does not divide u. In case (**II**) we see that $y_k \in \{0, p - 1\}$, also a contradiction. □

Lemma 3.16 *In case* (**I**) *$l > m - i$ cannot occur.*

Proof. Assume by way of contradiction that $l > m - i$. First suppose that $k < m - i$. Then equation (4) becomes

$$p^{m-i+k} + \mathcal{O}(p^{m-i+k+1}) = 0$$

yielding a contradiction modulo $p^{m-i+k+1}$.

If $k = m - i$ we obtain

$$up^{m-i+l} + \mathcal{O}(p^{m-i+l+1}) = 0$$

yielding a contradiction modulo $p^{m-i+l+1}$ since p does not divide u. □

Lemma 3.17 *Case* (**I**) *cannot occur.*

Proof. We are left with showing that also $l = m - i$ is impossible in this case, so assume the contrary. Substituting the appropriate values for x, y and z in equation (4) we obtain

$$up^{m-i+k} - p^{m-i+k} + \mathcal{O}(p^{m-i+k+1}) = 0$$

implying $u = 1 + vp$ with $v \in \mathbb{N}$. If $v \neq 0$ equation (4) becomes

$$-dp^{m+k} + vp^{m-i+k+1} + \mathcal{O}(p^{2m-i}) + v\mathcal{O}(p^{2m-i+1}) + v^2\mathcal{O}(p^{2m-2i+2}) = 0$$

from which easily follows that p^{i-1} divides v. We find $y \geq 1 + p^{m-i}(1 + p^i) = 1 + p^{m-i} + p^m$, a contradiction. Hence $v = 0$. Equation (4) becomes, modulo p^{2m}:

$$-dp^{m+k} - p^{2m-i} = 0$$

implying that $p^{m-i-k} > 1$ divides d, which in view of Lemma 3.11 yields that $p = 2$. Now first suppose that $k = 1$. It follows that the number of $(\alpha + 1)$-secants (with respect to $\mathcal{K}$) in a y-plane equals

$$\frac{(1 + p^i + p^m)(1 + p^{m-i})}{1 + p^i}$$

from which we deduce that $1 + p^i$ divides $1 + p^{m-i}$. If we now count the number of $(\alpha + 1)$-secants in a z-plane we find

$$\frac{(1 + p^i + p^{i+1} + p^m)(1 + p + p^{m-i})}{1 + p^i}$$

which implies (using the fact that $1+p^i$ divides $1+p^{m-i}$ and rewriting $p+p^{m-i}$ as $p - 1 + 1 + p^{m-i}$) that $1 + p^i$ divides $p - 1$, clearly a contradiction. Now suppose that $k > 1$ with $m - i - k \neq 1$. From Lemma 3.11 it follows that $i = (m - 1)/2$, and hence the number of intersecting lines (with respect to $\mathcal{K}$) in a y-plane can never be an integer, a contradiction. Finally suppose that $k > 1$ with $m - i - k = 1$. Here as well, $1 + p^i$ divides $1 + p^{m-i}$ and since the number of intersecting lines in a z-plane must be an integer we find that $1+p^i$ divides $p^{i+k} + p^m = p^{i+k}(1+p)$ and hence $1+p^i$ divides $1+p$, i.e. $i = 1$ and hence $k = m - 2$. Equation (4) becomes $2^{2m}d^2 - 5.2^{2m-2}(2^m + 1)d + 2^{2m-1}(3.2^{2m-2}+2^{m+1}+1) = 0$. As this equation must have at least one integer solution in d it follows that the square root of its discriminant D must be an integer. We obtain $D = 2^{4m-4}(2^{2m} - 7.2^{m+1} - 7)$ and hence $2^{2m} - 7.2^{m+1} - 7$ has an integer square root. Rewriting this we see that $(2^m - 7)^2 - 56$ is a^2 for some $a \in \mathbb{N}$ with $a + \beta = 2^m - 7$. There follows that $a = \frac{56-\beta^2}{2\beta}$. We find that β must divide 56 and must be even, so $\beta \in \{2, 4, 6, 8, -2, -4, -6, -8\}$. We now easily see that $\beta = 4$, $a = 5$ and $m = 4$ is the only solution. The unique solution for equation (4) is then given by $x = 169$, $y = 9$, $z = 13$ and $q = 16$. As we supposed that $\mathcal{K}$ yields a semipartial geometry these values should also satisfy equation (1). Plugging in these values in this equation we obtain the final contradiction (an other way to obtain a contradiction here is to check that $\mu \notin \mathbb{N}$). □

Lemma 3.18 *Case* (**II**) *cannot occur.*

Proof. We are left with showing that also $k = m - i$ is impossible in this case, so assume the contrary. Equation (4) becomes

$$y_k(y_k - 1)p^{2m-2i} + \mathcal{O}(p^{2m-2i+1}) = 0$$

from which we see that $y_k = 1$. Assume that $u \neq 0$. We find

$$up^{2m-2i+1} - (d + 1)p^{2m-i} + \mathcal{O}(p^{2m}) + u\mathcal{O}(p^{2m-i+1}) + u^2\mathcal{O}(p^{2m-2i+2}) = 0$$

which enables us to deduce that p^{i-1} divides u. Hence $y \geq 1 + p^{m-i}(1 + p^i) > 1 + p^m$, a contradiction.

Now suppose that $u = 0$. We count the number of intersecting lines (with respect to $\mathcal{K}$) in a y-plane:

$$\frac{(1 + p^i + p^m)(1 + p^{m-i})}{1 + p^i}$$

which implies that $1 + p^i$ divides $1 + p^{m-i}$. Now we count the number of intersecting lines in a z-plane:

$$\frac{(1 + p^i + 2p^m)(1 + 2p^{m-i})}{1 + p^i}$$

from which we deduce (using the fact that $1 + p^i$ divides $1 + p^{m-i}$) that $1 + p^i$ divides $2p^{2m-i}$ and hence that $2/(1 + p^i) \in \mathbb{N}$, a contradiction. □

Lemma 3.19 *In case* **(III)** $l > m - i$ *cannot occur.*

Proof. Suppose $l > m - i$. Since equation (4) becomes

$$p^{m-i+k} + p^{2m-2i} + \mathcal{O}(p^{2m-2i+1}) = 0$$

we see that $k = m - i$ and $p = 2$. The equation now is of the form

$$2^{2m-2i+1} + (3d - 1)2^{2m-i} - 3u2^{m-i+l} + u^2 2^{2l} + \mathcal{O}(2^{2m-i+1}) = 0.$$

Recall that $u \neq 0$. If $2m - 2i + 1 = 2m - i$, then $i = 1$ and $z \geq 1 + p^m$, a contradiction. Hence $l = m - i + 1$ and $u = 1 + 2v$ (since $l < m - i + 1$ or $l > m - i + 1$ would yield a contradiction modulo $p^{m-i+l+1}$, respectively modulo $p^{2m-2i+2}$). If $v \neq 0$, we obtain

$$-v2^{2m-2i+2} - v2^{2m-2i+3} + v(v+1)2^{2m-2i+4} + (3d-1)2^{2m-i} + \mathcal{O}(2^{2m-i+1}) = 0$$

and hence 2^{i-2} must divide v, yielding $z \geq 1 + 2^{m-i+1} + 2^m$, a contradiction. Consequently $v = 0$, but then, as d is even, we find a contradiction modulo p^{2m-i+1}. □

Lemma 3.20 *Case* **(III)** *cannot occur.*

We may suppose that $l = m - i$. Just as in the previous lemmas we find that $u = 1 + vp$, and see that $v = 0$ since otherwise $z > 1 + p^m$. Equation (4) becomes

$$dp^{m+k} - p^{2m-i} + \mathcal{O}(p^{2m-i+1}) = 0$$

from which we deduce that $p^{m-i-k} > 1$ divides d. In view of Lemma 3.11 this implies that $p = 2$. This case now is handled very analogously as in Lemma 3.17. If $k > 1$ and $m - i - k > 1$ then $i = (m - 1)/2$ and the number of $(\alpha + 1)$-secants in a z-plane can never be an integer. If $k > 1$ and $m - i - k = 1$ we find that $1 + p^i$ divides $p - 1$, a contradiction. Finally, if $k = 1$, we see that $1 + p^i$ divides $1 + p$ and hence $i = 1$. Like in Lemma 3.17 we obtain a quadratic equation in d which should have at least one integer solution. But as its discriminant equals $-2^{5m} + 2^{4m+1} + 2^{4m} + 2^{3m+2} + 2^{2m+2}$, which is always negative (unless $m = 1$ which cannot occur), we obtain the final contradiction. □

4 Summary

Theorem 4.1 *If $\mathcal{K}$ is a non-trivial set of points in* $\mathrm{PG}(3,q)$ *such that $T_n^*(\mathcal{K})$ is an spg$(q-1,|\mathcal{K}|-1,\alpha,\mu)$, then either $\alpha=1$ and $\mathcal{K}$ is an ovoid or q is a square, $\alpha=\sqrt{q}$ and $\mathcal{K}$ is the point set of a Baer-subgeometry.*

Proof. Let $q=p^m$. If $\alpha=1$ or $\alpha\geq p^{m/2}$ Theorems 1.1 and 2.6 imply the result, so suppose that $1<\alpha<p^{m/2}$. In Lemma 3.4 it was shown that if x is the number of intersecting lines through a point of $\mathcal{K}$, then either p divides x or $x-1$. If p would divide x, then Theorem 3.7 implies that such $\mathcal{K}$ cannot exist. If p would divide $x-1$ then Lemmas 3.12, 3.17, 3.18 and 3.20 yield that such $\mathcal{K}$ cannot exist. Hence there follows that necessarily $\alpha=1$ or $\alpha\geq p^{m/2}$. The theorem is proved. □

For constructions and the embedding of the semipartial geometry $\mathrm{TQ}(4,q)$ we refer the reader to [9, 13]. This semipartial geometry is due to R. Metz (private communication).

Theorem 4.2 *If S is a semipartial geometry with $\alpha>1$, embedded in* $\mathrm{AG}(4,q)$*, then either $S\cong\mathrm{TQ}(4,q)$, with $q=2^h$, or $S\cong T_3^*(\mathcal{B})$.*

Proof. By Corrolary 3.7 of [8] and Corrolary 3.3 of [11] we know that such a semipartial geometry is either $\mathrm{TQ}(4,2^h)$ or a linear representation. The result now follows immediately from the previous theorem. □

Remark. If S is a partial quadrangle embedded in $\mathrm{AG}(4,q)$, and is of type $T_3^*(\mathcal{K})$, then $S\cong T_3^*(\mathcal{O})$, with $\mathcal{O}$ an ovoid in the hyperplane Π_∞ at infinity (see Theorem 1.1).

5 Some remarks on the case $n>3$

The objective of this final section is to prove that the conclusions of Lemmas 3.4, 3.6, 3.8, 3.9, 3.10 and 3.11 remain valid in the higher dimensional case. We use the same notations as before and we suppose that $n\geq 4$.

Theorem 5.1 *There holds that either p^{2m-2i} divides x or that $x=1+p^{m-i}+dp^m$ with $d\in\mathbb{N}\setminus\{0\}$. In the latter case $d=2 \pmod p$ if $p\neq 2$ and d is even if $p=2$; furthermore if 4 divides d and $k>1$ then $i=(m-1)/2$.*

Proof. Denote by U the right hand side of equation 3, set $D=p^{3m}+p^{2m}+p^m-xp^i$, $D(n)=p^{4m}+\ldots+p^{nm}$ and $N=p^ix(1+xp^i)(p^m-p^i)$; furthermore let λ and μ be as in Lemma 2.5. We calculate the right hand side of equation (1):

$$\begin{aligned}
&(\lambda-\tfrac{N}{D+D(n)})^2+4(K-\tfrac{N}{D+D(n)})\\
=\;&\tfrac{1}{(D+D(n))^2}\left[\lambda^2(D+D(n))^2-2\lambda(D+D(n))N\right.\\
&\left.+N^2+4K(D+D(n))-4N(D+D(n))\right]\\
=\;&\tfrac{1}{(D+D(n))^2}\left[U+D(n)\left(2D\lambda^2+D(n)\lambda^2-2\lambda N+4K-4N\right)\right]
\end{aligned}$$

Hence, after multiplication of both sides of equation (1) with $(D+D(n))^2$, we obtain

$$p^{2m+2k+2i}(p^{nm}+p^{(n-1)m}+\ldots+p^m-xp^i)^2=U_n$$

with

$$U_n = U + D(n)\left(2D\lambda^2 + D(n)\lambda^2 - 2\lambda N + 4K - 4N\right)$$

Since $U_n - U = 0 \pmod{p^{4m}}$ we immediately see that the conclusions and proofs of Lemmas 3.4, 3.6, 3.8, 3.9, 3.10 remain valid in the higher dimensional case. In order to see that Lemma 3.11 remains valid it suffices to notice that if $p = 2$ there holds that $U_n - U = 0 \pmod{p^{4m+1}}$, and so also that proof can be copied. □

Conjecture. If S is a semipartial geometry with $\alpha > 1$, with the property that S is the linear representation of a non-trivial point set in $\mathrm{PG}(n,q)$, $n \geq 4$, then $S \cong T_n^*(\mathcal{B})$.

Although the techniques applied to proof this conjecture for $n = 3$ seem suitable to attack the general case, the main problems when trying to do so arise from the fact that k can be larger than m if $n \geq 4$.

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