Generalized Quadrangles with an Abelian Singer Group*

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Abstract

In this note we characterize thick finite generalized quadrangles constructed from a generalized hyperoval as those admitting an abelian Singer group, i.e., an abelian group acting regularly on the points.

1 Introduction

One of the very fruitful theories of the past five decades in Finite Geometry, Combinatorial Group Theory and Finite Field Theory, is that of *difference sets*. We refer to [1] for a general reference. It can be shown that if G is a finite group and D a (v, k, λ)-difference set in G, there can be constructed a symmetric 2-(v, k, λ) *design* [1], on which G acts regularly, where the action is considered on the points or on the lines. Conversely we have that each symmetric 2-design with an automorphism group regular on the points defines a difference set in a natural way. Such a group will be called a "Singer group" throughout this note.

One of the most popular parts of difference set theory is that of *planar difference sets* — these are the difference sets giving rise to 2-$(n^2 + n + 1, n +$

*Mathematics Subject Classification (2000): 05B10, 05B25, 05E20, 20B25, 51E12, 51E20.

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$1,1)$ designs, that is, *projective planes* of order n [6]. There, many of the questions concerning Singer groups stand among the most fundamental ones in the theory, the most notable being perhaps the classification of planes admitting an abelian Singer group. Conjecturally, those should always be Desarguesian.

Recall that projective planes are members of the family of *generalized n-gons* [9]; they are precisely the generalized 3-gons. In the past 15 years, the question has been posed several times whether there are fruitful Singer group/difference set theories for other types of (building like) geometries, especially for the other generalized n-gons. For generalized 4-gons, or also "generalized quadrangles" (see the next section for a formal definition), such a theory was initiated by D. Ghinelli in [5], where it was shown, amongst other things, that a finite generalized quadrangle of order s (> 1) cannot admit an abelian Singer group.

In the present note, we classify all finite thick generalized quadrangles admitting an abelian Singer group, by showing that they essentially arise as generalizations of linear representations in projective spaces of even characteristic. Along the way, some generalizations will turn up. As a corollary, we also handle partial quadrangles and the other generalized n-gons, $n > 4$.

A *generalized linear representation* of a geometry $\mathcal{Q} = (P, B, I)$ in the affine space $\mathrm{AG}(n, q)$ is a monomorphism θ of $\mathcal{Q}$ into the geometry of points and subspaces of $\mathrm{AG}(n, q)$, in such a way that P^θ is the set of all points of $\mathrm{AG}(n, q)$, that B^θ is a union of parallel classes of subspaces (not necessarily of the same dimension) of $\mathrm{AG}(n, q)$, and that each point of L^θ is the image of some point of L for any line L in B. One usually identifies $\mathcal{Q}$ with its image $\mathcal{Q}^\theta$. If $\mathcal{Q}$ is a generalized quadrangle with generalized linear representation in $\mathrm{AG}(n, q)$, then it can be shown that $n = 3m$ for some natural m, that the parallel classes of the lines define a set $\mathcal{O}$ of q^m+2 disjoint $(m-1)$-dimensional spaces at infinity with the property each m-dimensional space containing an element of $\mathcal{O}$ intersects exactly one other element of $\mathcal{O}$, and that q is even. This object at infinity is then a special type of *generalized hyperoval* (see [3]). Vice versa, each set $\mathcal{O}$ as above yields a generalized quadrangle in this way.

We can now formulate our main theorem.

Main Theorem 1.1 *If $\mathcal{Q}$ is thick finite generalized quadrangle of order (s, t) admitting a regular abelian automorphism group, then $\mathcal{Q}$ is isomorphic to a generalized quadrangle arising from a generalized hyperoval.*

Remark 1.2 In the paper [2] *translation partial geometries* are introduced, and thoroughly studied. These are partial geometries with parameters (s, t, α)

admitting a regular abelian automorphism group, so that each line orbit is a normal spread and $t = \alpha(s+2)$. For $\alpha = 1$, i.e. for generalized quadrangles, this point of view is a special case of the problem under consideration. It will appear that the second and third defining property for translation partial geometries with $\alpha = 1$ will be redundant.

Acknowledgement. The main ideas of the present note were developed during the very nice conference "Incidence Geometry" (La Roche, Belgium, 23–29 May 2004), where a question of J. C. Fisher motivated us to do so.

2 Proof of the Main Theorem, Corollaries

We first recall some definitions.

A *finite generalized quadrangle* (GQ) of *order* (s,t) is a finite geometry $\mathcal{Q} = (P, B, I)$ such that two distinct lines intersect in at most one point, such that every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines, and most importantly, with the property that given any non-incident point-line pair (p, L), there is a unique line incident with p and concurrent with L.
More generally, a *finite* $(0,1)$*-geometry* of *order* (s,t) is a finite geometry such that two distinct lines intersect in at most one point, such that every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines, and most importantly, with the property that given any non-incident point-line pair (p, L), there is at most one line incident with p and concurrent with L.
If a $(0,1)$-geometry of order (s,t) has a strongly regular point graph it is called a *partial quadrangle*, denoted by $\mathrm{PQ}(s,t,\mu)$ (here μ is the number of points collinear with any pair of non-collinear points).
If $s, t \geq 2$ the considered geometries are called *thick*.

In the rest of this paper, we use standard notation on GQ's (and $(0,1)$-geometries) — see the monograph [7].

Lemma 2.1 *Let $\mathcal{Q}$ be a finite $(0,1)$-geometry with the property that every line contains at least three points. Furthermore suppose that $G \leq \mathrm{Aut}(\mathcal{Q})$ is an abelian group of automorphisms acting regularly on the point set of $\mathcal{Q}$. Then the stabilizer in G of any line L of $\mathcal{Q}$, $\mathrm{Stab}_G(L)$, is a group of order $|L|$.*

Proof. Consider a line L of $\mathcal{Q}$ and choose any point x on L. Suppose that $|L| = s+1$. Let S denote the unique set $S = \{g_0 = id, g_1, \ldots, g_s\} \subset G$ with the property that $x^{g_i} \in L$, $i = 0, 1, \ldots, s$. It is our aim to show that S is a subgroup of G stabilizing L. Consider any two distinct elements, g_i and g_j, of S, with $i \neq 0 \neq j$. From $x \sim x^{g_i}$ it follows that $x^{g_j} \sim x^{g_i g_j}$. Analogously we obtain from $x \sim x^{g_j}$ that $x^{g_i} \sim x^{g_j g_i} = x^{g_i g_j}$. Hence the point $x^{g_i g_j}$ is collinear with two distinct points, x^{g_i} and x^{g_j}, on L and consequently belongs to L. From this it follows that $L^{g_i} = \langle x, x^{g_j} \rangle^{g_i} = \langle x^{g_i}, x^{g_i g_j} \rangle = L$, so that $S \subset \mathrm{Stab}_G(L)$. Since $|\mathrm{Stab}_G(L)| \leq s+1$ by the regularity of G, it follows that $S = \mathrm{Stab}_G(L)$. $\square$

The concept of *spread of symmetry* of a $(0,1)$-geometry $\mathcal{Q}$ will be essential for the rest of this paper. This is a partition T of the point set of $\mathcal{Q}$ into lines for which there is an automorphism group of $\mathcal{Q}$ (called the "associated group") fixing T linewise and acting regularly on the points of any of its lines.

Suppose that $\mathcal{Q}$ is a finite connected $(0,1)$-geometry of order (s,t), with $s, t \geq 2$. Further let G be a regular abelian automorphism group of $\mathcal{Q}$. Choose any fixed point x of $\mathcal{Q}$ and denote the $t+1$ lines through x by $L_0, L_1, \ldots, L_t$. By the previous lemma there exist $t+1$ subgroups of order $s+1$ of G, denoted by $S_0, S_1, \ldots, S_t$, with the property that S_i stabilizes the line L_i. It is easily seen that the orbit of L_i under G, $i = 0, 1, \ldots, t$, is a spread of symmetry of $\mathcal{Q}$.

Remark 2.2 Note that it follows already that no finite generalized n-gons with $n > 4$ admit abelian Singer groups since the existence of a spread of symmetry implies that the geometry has (ordinary) quadrangles.[1]

Every line of $\mathcal{Q}$ belongs to exactly one such spread of symmetry. This allows us to reconstruct the geometry $\mathcal{Q}$ in terms of the group G. For, define the geometry $S(G)$ to be the geometry with point set the elements of G, with line set the cosets $S_i g$, $g \in G$, and with containment as incidence relation. It is not difficult to see that the map

$$\phi : S(G) \to \mathcal{Q}, g \mapsto x^g$$

is an isomorphism. The partition of the line set of $\mathcal{Q}$ into spreads (of symmetry) introduces in a natural way a parallelism on the lines of $\mathcal{Q}$: two lines of

[1]The finiteness assumption is not essential here; Lemma 2.1 clearly has an infinite analogue with similar proof.

$\mathcal{Q}$ are *parallel* if they belong to the same spread of symmetry. The existence of such a parallelism gives us a hint as to whether it might be possible to embed $\mathcal{Q}$ as a (generalized) linear representation in an affine space. Under a small restriction this is indeed possible. The basic ideas for the proof of this observation are taken from De Winter [4], see also De Clerck, Gevaert and Thas [2].

Before proceeding, let us note that from Lemma 2.1, it is possible to derive a nice divisibility condition.

Theorem 2.3 *Suppose $\mathcal{Q}$ is as in Lemma 2.1, and suppose that some point of $\mathcal{Q}$ is incident with at least three lines. If every line has $s+1$ points for some constant s, then $(s+1)^3$ divides $|G|$. If in particular $\mathcal{Q}$ is a thick GQ of order (s,t), then $t = s+2$.*

Proof. Let L be a line of $\mathcal{Q}$, and suppose $M \sim L \neq M$. Denote by T_L (T_M) the spread of symmetry containing L (M) as described prior to the lemma, and by S_L (S_M) its associated group. Then there is an $(s+1)\times(s+1)$-grid Γ containing L and M which is fixed by $S_LS_M = \langle S_L, S_M\rangle$, so that S_LS_M acts transitively on each regulus of Γ. Now consider a line N which meets Γ in a unique point. Then $S_N \cap S_LS_M = \{id\}$, so the subgroup $S_LS_MS_N$ of G has size $(s+1)^3$. If now $\mathcal{Q}$ is a GQ of order (s,t), then $(s+1)^3$ divides $|G| = (s+1)(st+1)$, so that $s+1$ divides $t-1$, and whence

$$s+1 \quad | \quad \frac{t-1}{s+1} - 1.$$

So either $\frac{t-1}{s+1} - 1 = 0$, or $s^2+3s+3 \leq t$, contradicting Higman's inequality $t \leq s^2$ (cf. 1.2.3 of [7]). $\square$

Now define K to be the set of all endomorphisms β of G with the property that $S_i^\beta \subset S_i$, for all $i \in \{0,1,\ldots,t\}$. Then, since G is abelian, $K,+,.$, that is, K endowed with the usual addition and multiplication of endomorphisms, is a ring.

Theorem 2.4 *Suppose that the setting and the notations are as in the above paragraph. If for any two points y and z of $\mathcal{Q}$ that are at distance two from each other in the collinearity graph of $\mathcal{Q}$ it holds that $\left|\{y,z\}^\perp\right| > 2$, then $K,+,.$ is a field.*

Proof. It is sufficient to show that every element of $K\backslash\{0\}$ is a bijection from G to G, and hence by the finiteness of G we only need to show injectivity.

Suppose that $\beta \in K$ is such that $s_0^\beta = id$ for some $s_0 \in S_0 \setminus \{id\}$; then we must show that $\beta = 0$. (The choice of S_0 is arbitrary. Further, if β has a fixed point not in $\bigcup_i S_i$, then it has a fixed point in each $S_i \setminus \{id\}$ as well.) Assume the contrary. Choose any element $s_i \in S_i \setminus \{id\}$, with $i \neq 0$. Then the point $s_0 s_i$ is at distance two from id in the collinearity graph of $S(G)$. Because of our assumption there exist elements $s_l \in S_l \setminus \{id\}$ and $s_k \in S_k \setminus \{id\}$, $l, k \notin \{0, i\}$ and $l \neq k$, such that $s_0 s_i = s_l s_k$. Note that $|\{y, z\}^\perp|$, $y \not\sim z$, must be even since G is abelian (if $y \sim y^\theta \sim z$, with $y^{\theta\gamma} = z$, then $y \sim y^\gamma \sim z$, with $\theta, \gamma \in G$). Letting β act yields $s_i^\beta = s_l^\beta s_k^\beta$. First suppose that $s_l^\beta = id$; then $s_i^\beta = s_k^\beta$. Since $S_i \cap S_k = \{id\}$ we obtain that $s_i^\beta = id$. Analogously $s_k^\beta = id$ implies that $s_i^\beta = id$. Next suppose that neither s_l^β nor s_k^β equals id. In this case the line $S_l s_k^\beta$ (of $S(G)$) intersects the line S_k in $s_k^\beta \neq id$ and intersects the line S_i in $s_i^\beta \neq id$. Hence we have found a triangle in the $(0,1)$-geometry $S(G)$, a contradiction. We conclude that $s_i^\beta = id$, and henceforth that $S_j^\beta = id$, for all $j \in \{0, 1, \ldots, t\}$. By the connectedness of the geometry $\mathcal{Q} \cong S(G)$ we know that $G = \langle S_0, S_1, \ldots, S_t \rangle$, and hence it follows that $G^\beta = id$, that is, $\beta = 0$. $\square$

Theorem 2.5 *Let $\mathcal{Q}$ be a finite connected $(0,1)$-geometry of order (s,t), $s, t \geq 2$, in which any two points at distance two in the collinearity graph of $\mathcal{Q}$ have at least three common neighbours. Then $\mathcal{Q}$ has a generalized linear representation if and only if $\mathcal{Q}$ admits a regular abelian group of automorphisms.*

Proof. It is clear that if $\mathcal{Q}$ has a generalized linear representation, say in $\mathrm{AG}(n,q)$, then the group G of all translations of $\mathrm{AG}(n,q)$ is (elementary) abelian and acts as a regular group of automorphisms on $\mathcal{Q}$.

Conversely, if every two points at distance two in the collinearity graph of $\mathcal{Q}$ have at least three common neighbours, then the previous theorem implies that $K, +, .$ is a field. Consequently G can be seen as a K-vector space of dimension $\log_{|K|}(|G|)$, and the S_i's as vector subspaces of dimension $\log_{|K|}(s+1)$. It is clear that the line set of $\mathcal{Q}$ is exactly the set of all translates of these vector subspaces, and hence $\mathcal{Q}$ has a generalized linear representation in the affine space constructed from the K-vector space G. This generalized linear representation is a linear representation if and only if $\log_{|K|}(s+1) = 1$. $\square$

We are ready to obtain the Main Theorem. For the sake of convenience we repeat its statement.

Main Theorem 2.6 *A thick finite GQ of order (s,t) having a regular abelian automorphism group, arises from a generalized hyperoval.*

Proof. Since $t = s + 2$ by Theorem 2.3 and every two points at distance two have at least three common neighbours in a thick GQ this result is an immediate consequence of Theorem 2.5. $\square$

Note that since $\mathcal{Q}$ arises from a generalized hyperoval, q necessarily is even [3].

Corollary 2.7 *If $\mathcal{Q}$ is a thick partial quadrangle $PQ(s, t, \mu)$ admitting a regular abelian group G of automorphisms, then $\mathcal{Q}$ has a generalized linear representation.*

Proof. If $\mu > 2$, then Theorem 2.5 immediately implies the result.

So suppose that $\mu = 2$. (Notice that since G is abelian $\mu = 1$ cannot occur.) With the same notations as before, we need to show that $K, +, .$ is a field. First notice that for every point g of $S(G)$ there exist i and j such that g can be written as $s_i s_j$ for some $s_i \in S_i$ and $s_j \in S_j$. Since $\mu = 2$, it is easily seen that the grid $S_i S_j$ (in $S(G) \cong \mathcal{Q}$) determined by S_i and S_j, and the grid $S_l S_k$ determined by S_l and S_k have trivial intersection unless $\{i, j\} \cap \{l, k\} \neq \emptyset$. Suppose that $\beta \in K$ is such that $s_0^\beta = id$, with $s_0 \in S_0 \setminus \{id\}$. Consider any $s_i \in S_i \setminus \{id\}$, with $i \neq 0$. Further choose any $s_j \in S_j \setminus \{j\}$, with $j \notin \{0, i\}$. Then the point $s_0 s_i s_j$ is at distance two from id and is only collinear with the point $s_0 s_i$ in the grid $S_0 S_i$. Hence there must exist k and l, such that $\{k, l\} \cap \{0, i, j\} = \emptyset$ and such that $s_0 s_i s_j = s_l s_k$, for some $s_l \in S_l \setminus \{id\}$ and some $s_k \in S_k \setminus \{id\}$, if not there would arise triangles or the assumption $\mu = 2$ would be violated — recall that G is abelian! Letting β act yields $s_i^\beta s_j^\beta = s_l^\beta s_k^\beta$. Since $S_i S_j \cap S_l S_k = \{id\}$, we obtain that $s_i^\beta = id$. It now easily follows that $\beta = 0$ and hence $K, +, .$ is a field. The proof can be finished as before. $\square$

Final Remark 2.8 It should be noticed that the classification of (GQ's arising from) generalized hyperovals is equivalent to the classification of the so-called "translation generalized quadrangles" [7] of order s, s even, cf. Thas and Thas [8] for a recent reference.

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