Complete arcs on the parabolic quadric $\mathrm{Q}(4, q)$

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Abstract

Using the representation $T_2(\mathcal{O})$ of $\mathrm{Q}(4, q)$ and algebraic methods, we prove that complete $(q^2 - 1)$-arcs of $\mathrm{Q}(4, q)$ do not exist when $q = p^h$, p odd prime and $h > 1$. As a by-product we prove an embeddability theorem for the direction problem in $\mathrm{AG}(3, q)$.

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1 Introduction

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of objects called points and lines (respectively), and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

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(i) each point is incident with $1+t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1+s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{~I} M \mathrm{~I} y \mathrm{~I} L$.

The integers s and t are the parameters of the GQ and $\mathcal{S}$ is said to have order (s,t). If $s=t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s,t), then $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t+1)(st+1)$ (see e.g. [11]). The *dual* $\mathcal{S}^D$ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is the incidence structure $(\mathcal{B}, \mathcal{P}, \mathrm{I})$. It is again a GQ.

A GQ $\mathcal{S}' = (\mathcal{P}', B', \mathrm{I}')$ of order (s', t') is called a *subquadrangle* of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t) if $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{B}' \subset \mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$.

An *ovoid* of a GQ $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that every line is incident with exactly one point of the ovoid. An ovoid of a GQ of order (s,t) has necessarily size $1+st$. An *arc* or a *partial ovoid* of a GQ is a set $\mathcal{K}$ of points such that every line contains *at most* one point of $\mathcal{K}$. An arc $\mathcal{K}$ is called *complete* if and only if $\mathcal{K} \cup \{p\}$ is not an arc for any point $p \in \mathcal{P} \setminus \mathcal{K}$, in other words, if $\mathcal{K}$ cannot be extended. It is clear that any arc of a GQ of order (s,t) contains $1 + st - \rho$ points, $\rho \geq 0$, with $\rho = 0$ if and only if $\mathcal{K}$ is an ovoid.

In this paper we consider a classical finite generalized quadrangle of order q, which consists of the points and lines, together with the natural incidence, of the non-singular parabolic quadric $\mathrm{Q}(4,q)$ in $\mathrm{PG}(4,q)$. It is well-known, (see e.g. [11]) that this GQ has ovoids. A particular example of an ovoid is any elliptic quadric $\mathrm{Q}^-(3,q)$ contained in it. When q is prime, these are the only ovoids [2], when q is a prime power, other examples are known, see e.g. [15] for a list of references.

Concerning complete arcs of a GQ of order (s,t), we state the following results from [11].

Theorem 1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t). Any $(st-\rho)$-arc of $\mathcal{S}$ with $0 \leq \rho < t/s$ is contained in a uniquely defined ovoid of $\mathcal{S}$.*

Theorem 2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t). Let $\mathcal{K}$ be a maximal partial ovoid of size $st - t/s$ of $\mathcal{S}$. Let $\mathcal{B}'$ be the set of lines incident with no point of $\mathcal{K}$, and let $\mathcal{P}'$ be the set of points on at least one line of $\mathcal{B}'$ and let I' be the restriction of* I *to points of $\mathcal{P}'$ and lines of $\mathcal{B}'$. Then $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order $(s, \rho = t/s)$.*

Applying these theorems to the *GQ* $\mathrm{Q}(4,q)$ implies only that an arc of size q^2 cannot be complete. When $\mathcal{K}$ is a complete (q^2-1)-arc of $\mathrm{Q}(4,q)$, then the second theorem implies that the subquadrangle $\mathcal{S}'$ is a hyperbolic quadric $\mathrm{Q}^+(3,q)$ contained in $\mathrm{Q}(4,q)$.

When q is even, it is known that no complete arcs of size $q^2-\rho$, $0<\rho<q$ exist [5]. No analogous theorem is known for q odd. It is even not known whether or not $Q(4,q)$, q odd, has complete (q^2-1)-arcs. It is probably a bit unexpected but this problem seems to be quite hard. Using a combination of geometrical and algebraic methods, the problem can be solved when $q=p^h$, p odd prime, $h>1$. The main result is as follows.

Theorem 3 *Suppose that $q=p^h$, p an odd prime, $h>1$. Then $\mathrm{Q}(4,q)$ has no complete (q^2-1)-arcs.*

The restriction $h>1$ seems to be unexpected, but complete (q^2-1)-arcs of $\mathrm{Q}(4,q)$ are known when $q=3,5,7,11$. These examples were first found by T. Penttila. References and more information on particular examples will be given in the last section.

In the second section an alternative representation of $\mathrm{Q}(4,q)$ is given and a complete (q^2-1)-arc of $\mathrm{Q}(4,q)$ will be described in this representation. In the third section algebraic techniques are applied to prove the main result.

2 Tits' representation of $\mathrm{Q}(4,q)$

An *oval* of $\mathrm{PG}(2,q)$ is a set of $q+1$ points $\mathcal{C}$, such that no three points of $\mathcal{C}$ are collinear. When q is odd, it is known that all ovals of $\mathrm{PG}(2,q)$ are conics. When q is even, several other examples and infinite families are known, see e.g. [9]. The GQ $T_2(\mathcal{C})$ is defined as follows. Let $\mathcal{C}$ be an oval of $\mathrm{PG}(2,q)$, embed $\mathrm{PG}(2,q)$ as a plane in $\mathrm{PG}(3,q)$ and denote this plane by π_∞. Points are defined as follows:

(i) the points of $\mathrm{PG}(3,q)\setminus\mathrm{PG}(2,q)$;
(ii) the planes π of $\mathrm{PG}(3,q)$ for which $|\pi\cap\mathcal{C}|=1$;
(iii) one new symbol (∞).

Lines are defined as follows:

(a) the lines of $\mathrm{PG}(3,q)$ which are not contained in $\mathrm{PG}(2,q)$ and meet $\mathcal{C}$ (necessarily in a unique point);
(b) the points of $\mathcal{O}$.

Incidence between points of type (i) and (ii) and lines of type (a) and (b) is the inherited incidence of $\mathrm{PG}(3,q)$. In addition, the point (∞) is incident with no line of type (a) and with all lines of type (b). It is straightforward to show that this incidence structure is a GQ of order q. The following theorem (see e.g. [11]) allows us to use this representation.

Theorem 4 *The GQs* $T_2(\mathcal{C})$ *and* $\mathrm{Q}(4,q)$ *are isomorphic if and only if* $\mathcal{C}$ *is a conic of the plane* $\mathrm{PG}(2,q)$.

Since all ovals of $\mathrm{PG}(2,q)$, q odd, are conics, $T_2(\mathcal{C}) \cong \mathrm{Q}(4,q)$ when q is odd. From now on we suppose that q is odd.

Let $\mathcal{K}$ be a complete k-arc of $T_2(\mathcal{C})$. Since $\mathrm{Q}(4,q) \cong T_2(\mathcal{C})$ has a collineation group acting transitively on the points (see e.g. [10]), we can suppose $(\infty) \in \mathcal{K}$. This implies that $\mathcal{K}$ contains no points of type (ii). It is clear that no two points of type (i) of $\mathcal{K}$ determine a line meeting $\mathcal{C}$ in a point. Hence the existence of $\mathcal{K}$ implies the existence of a set U of $k-1$ points of type (i) such that no two points determine a line meeting π_∞ in $\mathcal{C}$. It is easy to see that the converse is also true: from a set U of $k-1$ points in $AG(3,q)$ with the property that all lines joining at least two points of U are disjoint from $\mathcal{C}$, we can find an arc $\mathcal{K}$ of $T_2(\mathcal{C})$ of size k by adding ∞ to U. The completeness of $\mathcal{K}$ is equivalent to the maximality of U.

In the next section it is proved that such a set U cannot be maximal when $|U| = q^2 - 2$, $q = p^h$, p prime, $h > 1$.

3 Directions in $\mathrm{AG}(3,q)$

This chapter is devoted to the direction problem in $\mathrm{AG}(3,q)$, where $q = p^h$ is a prime power. Embed $\mathrm{AG}(3,q)$ in $\mathrm{PG}(3,q)$ in such a way that the infinite plane is $X_3 = 0$. Consider a set $U = \{(a_i, b_i, c_i, 1) : i = 1, \ldots, k\} \subset \mathrm{AG}(3,q)$. Let $D = \{(a_i - a_j, b_i - b_j, c_i - c_j, 0) : i \neq j\}$ denote the *set of directions determined by* U. The set D is a subset of the infinite plane $X_3 = 0$ and consists of those points through which there is an affine line with at least two points of U. It is easy to see that if $|U| \geq q^2 + 1$, then it determines all directions: for any direction there are q^2 lines in the parallel class, so by the pigeon-hole principle, there is a line with at least two points. In the previous section we saw that if $|U| = q^2$ and D is disjoint from a conic $\mathcal{C}$, then by adding to U the point ∞ we find an ovoid of $T_2(\mathcal{C})$ and vice versa.

What we are looking for is a result assuring that, if $|U| = q^2 - 2$ and D is disjoint from a conic, then U can be extended to a set of q^2 points still not determining points of the conic. We will prove a bit more, since we will only use the fact that D misses at least $p + 2$ points (this is true when $q = p^h$, $h > 1$).

The theory of directions in the affine plane has proved to be very useful in many aspects. Using and developing techniques originally due to Rédei [12], in the last years the classification of all sets of size q (which is the analogue of the size q^2 case in 3-dimensions) and determining at most half of the directions was achieved by Ball, Blokhuis, Brouwer, Storme and Szőnyi, see [1], [4].

It was perhaps surprising, when Ball and Lavrauw [3] proved theorems for the 3-dimensional case with the much weaker condition that D misses at least q from the q^2+q+1 directions. They considered the $|U| = q^2$ case, while in our result we have $|U| = q^2 - 2$. Also Ball and Lavrauw were the first to use the $T_2(\mathcal{C})$ representation of $\mathrm{Q}(4, q)$ and use results about directions. Our approach is similar to theirs, but we will use more geometrical arguments (mainly due to Sziklai [13]) to help the use of the Rédei polynomial. Also some ideas of a paper by Szőnyi [14] will be used, more details will be discussed at the end of the section.

Denote by O the complement of D in the infinite plane. From now on by "plane" we will always mean a plane different from $X_3 = 0$.

Lemma 1 *Suppose the plane π has at least one point in common with O. Then $|\pi \cap U| \leq q$.*

If $|U| = q^2$, then the points of O can be characterized by the property that $|U \cap \pi| = q$ for all planes π containing the point in question.

Proof. Let $p \in O \cap \pi$. The q affine lines in π through p cover the affine points of π and on each there is at most one point of U, hence $|\pi \cap U| \leq q$.

For the second statement, suppose that π is a plane through the infinite point $p \in O$. Note that there are q affine planes through the infinite line of π partitioning the points of the affine part, hence they all meet U in exactly q points. Finally suppose $p \notin O$, that is, p is a determined direction, but all planes through p have q points in common with U. Let l be an affine line with direction corresponding to p with $r \geq 2$ points of U. Counting the number of points of U on planes through l, we have $q^2 = |U| = r + (q+1)(q-r)$, a contradiction. □

Lemma 2 *For any $x, y, z, w \in GF(q)$, $(y, z, w) \neq (0, 0, 0)$, the multiplicity of $-x$ in the multi-set $\{ya_i + zb_i + wc_i : i = 1, \ldots, k\}$ is the same as the number of common points of U and the plane $yX_0 + zX_1 + wX_2 + xX_3 = 0$.*

Proof. The point $(a_i, b_i, c_i, 1)$ is on the plane if and only if $ya_i + zb_i + wc_i + x = 0$. □

Define the Rédei polynomial of U as follows

$$R(X,Y,Z,W)=\prod_{i=1}^{k}(X+a_iY+b_iZ+c_iW)=X^k+\sum_{i=1}^{k}\sigma_i(Y,Z,W)X^{k-i}$$

Here $\sigma_i(Y,Z,W)$ is the i-th elementary symmetric polynomial of the multi-set $\{a_iY+b_iZ+c_iZ : i\}$ and is either zero or has degree i. The use of R is that it translates intersection properties of U with planes to algebraic conditions.

Lemma 3 *Consider three fixed field elements y, z and w. Substitute $Y=y$, $Z=z$ and $W=w$ in $R(X,Y,Z,W)$ and consider $R(X,y,z,w)$ as a polynomial in X. Then the multiplicity of the root x equals the number of common points of U and the plane with equation $yX_0+zX_1+wX_2+xX_3=0$.*

Proof. The proof follows immediately by Lemma 2. □

To demonstrate the use of R we prove the following theorem.

Theorem 5 *If $|U|=q^2-1$, then it can be extended to a set of q^2 points determining the same set of directions.*

Proof. We have $|U|=q^2-1$, hence for any infinite line l with equation $yX_0+zX_1+wX_2=X_3=0$, if l meets O, then $q-1$ affine planes through l meet U in q points and one in $q-1$ points (see Lemma 1). For the Rédei polynomial this means that $R(X,y,z,w)$ has $q-1$ q-fold roots and one $(q-1)$-fold root. Hence there is a constant S depending on y,z,w for which

$$(X-S)R(X,y,z,w)=(X^q-X)^q$$

holds. It is easy to see that

$$S=\sigma_1(y,z,w)=(\sum_i a_i)y+(\sum_i b_i)z+(\sum_i c_i)w.$$

This implies, that if we add to U the point $(-(\sum_i a_i), -(\sum_i b_i), -(\sum_i c_i), 1)$, then for the Rédei polynomial of the new set, $R^*(X,y,z,w)=(X^q-X)^q$ holds for any line $yX_0+zX_1+wX_2=X_3=0$ meeting O. By Lemma 1 and 2, this means that the new set determines the same directions. □

Note that by the end of the previous section, this result implies that a partial ovoid of size q^2 in $T_2(\mathcal{C})$ can always be extended to an ovoid.

From now on suppose that q is odd and $|U|=q^2-2$. After translation (not affecting O or D) we can suppose $\sum_i a_i=\sum_i b_i=\sum_i c_i=0$. Note that this implies $\sigma_1\equiv 0$.

Lemma 4 *All planes meeting O have q, $q-1$ or $q-2$ points in common with U. If l is a line of the infinite plane with at least one point in common with O, then either we have $q-1$ planes thorugh l with q points of U and one with $q-2$ points, or $q-2$ planes with q points of U and two with $q-1$ points.*

Proof. This is an easy consequence of Lemma 1. □

Lemma 5 *If the infinite line with equation $yX_0 + zX_1 + wX_2 = X_3 = 0$ has at least one common point with O, then*

$$R(X, y, z, w)(X^2 - \sigma_2(y, z, w)) = (X^q - X)^q. \qquad (1)$$

Proof. From the previous lemma and Lemma 3 we know that

$$R(X, y, z, w)(X - S)(X - S') = (X^q - X)^q,$$

where S and S' are not necessarily different and depend on y, z, w. Considering the first three terms on both sides and taking into account that $\sigma_1 = 0$, we have $(X - S)(X - S') = X^2 - \sigma_2(y, z, w)$. □

Note that this implies that for these y, z, w, we can explicitly determine R in terms of σ_2:

$$R(X, y, z, w) = X^q(X^{q^2-q-2} + \sigma_2 X^{q^2-q-4} + \sigma_2^2 X^{q^2-q-6} + \cdots),$$

hence $\sigma_{2l+1}(y, z, w) = 0$, $\sigma_{2l}(y, z, w) = \sigma_2(y, z, w)^l$ for $l = 0, 1, ..., \frac{q^2-q-2}{2}$.

Let $S_k(Y, Z, W) = \sum_i (a_i Y + b_i Z + c_i W)^k$, the k-th power sum of the multi-set $\{a_i Y + b_i Z + c_i W : i\}$.

Lemma 6 *If the line with equation $yX_0 + zX_1 + wX_2 = X_3 = 0$ has at least one common point with O, then $S_k(y, z, w) = 0$ for odd k and $S_k(y, z, w) = -2\sigma_2^{k/2}(y, z, w)$ for even k.*

Proof. We prove the statement by induction on k. For $k = 1$ we have $S_1 = \sigma_1 = 0$. For the induction step suppose we have proved the statement for $1, ..., k-1$ and recall that by the Newton formulas, we have

$$k\sigma_k = S_1\sigma_{k-1} - S_2\sigma_{k-2} + - \ldots - (-1)^k S_k\sigma_0,$$

where σ_0 is defined to be 1. If k is odd, then except for S_k, we know that all summands are zero, so $S_k = 0$. If k is even, then replacing the terms $S_i\sigma_{k-i}$, i odd with zero and replacing the terms $S_i\sigma_{k-i}$, i even with $-2\sigma_2^{k/2}$, finally, using that the left hand side is $k\sigma_2^{k/2}$, we are done. □

Theorem 6 *If $|U| = q^2 - 2$, $q = p^h$ and $|O| \geq p + 2$, then U can be extended* *by two points to a set of q^2 points determining the same directions.*

Proof. Consider the previously found equation with $k = p + 1$. We have

$$S_{p+1}(y, z, w) + 2\sigma_2(y, z, w)^{\frac{p+1}{2}} = 0$$

for all lines $yX_o + zX_1 + wX_2 = X_3 = 0$ meeting O. This is a dual curve of degree at most $p+1$ vanishing on at least $(p+2)$ pencils, hence it is identically zero. This means that the equation $S_{p+1} = -2\sigma_2^{\frac{p+1}{2}}$ is an identity of polynomials. On the other hand, it is easy to see that S_{p+1} has only terms of the form $Y^{p+1}, Z^{p+1}, W^{p+1}$ and $Y^pW, YW^p, Z^pW, ZW^p, Y^pZ, YZ^p$. By Lemma 7 (after this proof), this implies that σ_2 is reducible. By Equation (1) (see Lemma 5) it has to be of the form $(AY + BZ + CW)^2$.

Add to U the points $(A, B, C, 1)$ and $(-A, -B, -C, 1)$. It is easy to see that for the Rédei polynomial of the new set, we have $R^*(y, z, u) = (X^q - X)^q$ whenever the (infinite) line with equation $yX_0 + zX_1 + uX_2 = X_3 = 0$ has at least one point in common with O. By Lemma 1, we are done. □

Lemma 7 *Let I consist of all linear combinations of the following monomials: Y^{p+1}, Z^{p+1}, W^{p+1}, Y^pW, YW^p, Z^pW,ZW^p,Y^pZ, YZ^p. Suppose σ is a homogeneous polynomial of degree 2 for which $\sigma^{\frac{p+1}{2}}$ is in I. Then σ is not irreducible.*

Proof. Suppose σ is irreducible. Then by replacing Y, Z and W by Y', Z' and W' which are linear combinations of the original variables, we can achieve $\sigma = Z'^2 - W'Y'$. Note that such a transformation permutes elements of I. But $(Z'^2 - W'Y')^{\frac{p+1}{2}}$ is obviously not in I, a contradiction. □

We end this section with some remarks about the connection of our technique to those used in the above mentionned papers [14] and [3].

In [14] Szőnyi proves that if a set of $q - c\sqrt{q}$ points in AG$(2, q)$ determines at most $\frac{q-1}{2}$ directions, then it can be extended to a set of q points determining the same directions. The proof is analogous to ours, but there it can be shown that the curve defined by the missing factors of R cannot have too many points, unless it has a linear component implying that there is a point that can be added to the set without changing the set of determined directions. In our situation instead of a curve we have the surface $X^2 - \sigma_2(Y, Z, W) = 0$. This might be the equation of a hyperbolic quadric having as many points as we can deduce from the conditions, so we cannot obtain a contradiction. This is why we needed some extra ideas and this is why it seems difficult to use Szőnyi's method to extend the result for sets of size $q^2 - c\sqrt{q}$. Applying our method to the planar case we can prove the following.

Theorem 7 *Suppose that U is a set of $q-2$ points in* $\mathrm{AG}(2,q)$*, q odd, missing at least $p+2$ directions. Then U can be extended to a set of q points determining the same directions.*

Finally, some words about the connection to the paper [3]. There the authors use the very same Rédei polynomial, but recognize lines with a clever evaluation in R, while we only recognize planes. The advantage in this is that the Rédei polynomial (after plugging in y, z, w) becomes a bit nicer. Our method also applies to every problem attacked in [3], but of course only together with the extra ideas the authors develop there.

4 Related results

As mentioned in the introduction, the only examples of complete (q^2-1)-arcs of $\mathrm{Q}(4,q)$ are known for $q = 3, 5, 7$ and 11. It is not clear if examples exist for all odd primes. The examples for $q = 5, 7$ and 11 were found using a computer and were recently described in detail in [7]. The example for $q = 3$ can easily be constructed from an elliptic quadric contained in $\mathrm{Q}(4,q)$.

We mention that the non-existence of complete (q^2-1)-arcs of $\mathrm{Q}(4,q)$, $q = 9$ was shown using an exhaustive computer search [6].

For general q and complete arcs of different size, the following theorem is part of a slightly more general theorem in [8].

Theorem 8 *Suppose that $\mathcal{K}$ is a complete $(q^2+1-\delta)$-arc of* $\mathrm{Q}(4,q)$*, $\delta \leq \sqrt{q}$. Then δ is even.*

Putting this together with Theorem 3, a maximal partial ovoid which is not an ovoid of $\mathrm{Q}(4,q)$, $q = p^h$, p odd, $h > 1$, has size at most $q^2 - 3$.

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