$PSL(3,q)$ and line-transitive linear spaces

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Abstract

We present a partial classification of those finite linear spaces $\mathcal{S}$ on which an almost simple group G with socle $PSL(3,q)$ acts line-transitively.

A *linear space* $\mathcal{S}$ is an incidence structure consisting of a set of points Π and a set of lines Λ in the power set of Π such that any two points are incident with exactly one line. The linear space is called *non-trivial* if every line contains at least three points and there are at least two lines. Write $v = |\Lambda|$ and $b = |\Pi|$.

The investigation of those finite linear spaces which admit an almost simple group that is transitive upon lines is already underway [4, 7] motivated largely by the theorem of Camina and Praeger [5]. We continue this investigation by considering the situation when the socle of a line-transitive automorphism group is $PSL(3,q)$. The statement of our theorem is as follows:

Theorem A. *Suppose that $PSL(3,q) \trianglelefteq G \leq AutPSL(3,q)$ and that G acts line-transitively on a finite linear space $\mathcal{S}$. Then one of the following holds:*

- *$S = PG(2,q)$, the Desarguesian projective plane, and G acts 2-transitively on points;*
- *$PSL(3,q)$ is point-transitive but not line-transitive on $\mathcal{S}$. Furthermore, if G_α is a point-stabilizer in G then $G_\alpha \cap PSL(3,q) \cong PSL(3,q_0)$ where $q = q_0^a$ for some integer a.*

The proof of Theorem A will depend heavily upon an unpublished result of Camina, Neumann and Praeger which classifies the line-transitive actions of $PSL(2,q)$ (a weaker version of this result has appeared in the literature, see [16]):

Theorem 1. *Let $G = PSL(2,q), q \geq 4$ and suppose that G acts line-transitively on a linear space $\mathcal{S}$. Then one of the following holds:*

- *$G = PSL(2,2^a), a \geq 3$ acting transitively on $\mathcal{S}$, a Witt-Bose-Shrikhande space. Here Π is the set of dihedral subgroups of G of order $2(q+1)$ and Λ is the set of involutions $t \in G$ with the incidence relation being inclusion.*

*This paper contains results from the author's PhD thesis. I would like to thank my supervisor, Professor Jan Saxl.

- $\mathcal{S} = PG(2,2)$, $G = PSL(2,7)$ *and the action is 2-transitive.*

In the case where $\mathcal{S}$ is a projective plane Theorem A is implied by a more general result from the author's PhD thesis [10]; this result will be published separately from the current paper. Furthermore where G is flag-transitive upon $\mathcal{S}$ Theorem A is implied by [18].

Now observe that if a linear space $\mathcal{S}$ is line-transitive then every line has the same number, k, of points and every point lies on the same number,r, of lines. Such a linear space is called *regular* and those line-transitive linear spaces for which $k = 3$ or $k = 4$ have been completely classified in [8, 13, 2, 6, 15].

Hence in order to prove Theorem A we need to consider the situation when $\mathcal{S}$ is not a projective plane, is not flag-transitive and $k \geq 5$. The rest of the paper will be occupied with this proof. The first two sections outline some background lemmas concerning linear spaces. Section 3 gives background information about $PSL(3,q)$. In Section 4 we reduce the proof to the situation when $PSL(3,q)$ is transitive upon the lines of the space $\mathcal{S}$. This reduction makes use of the notion of **exceptionality** of permutation representations, the relevance of which was pointed out by Dr Peter Neumann. The remaining sections are devoted to the situation when $PSL(3,q)$ is line-transitive.

The following notation will hold, unless stated otherwise, throughout this paper. We will take G to be a group acting on a regular linear space $\mathcal{S}$ with parameters b, v, k, r. We will write α to be a point of $\mathcal{S}$ with G_α to be the stabilizer of α in the action of G. Similarly $\mathfrak{L}$ is a line of $\mathcal{S}$ and $G_{\mathfrak{L}}$ is the corresponding line-stabilizer.

1 Known Lemmas

We list here some well-known lemmas which we will use later. The first lemma is proved easily by counting.

Lemma 2. *1.* $b = \frac{v(v-1)}{k(k-1)} \geq v$ *(Fisher's inequality);*

2. $r = \frac{v-1}{k-1} \geq k$*;*

Lemma 3. *[4, Lemma 6.5] Let p be an odd prime divisor of v.*

1. *If $b = \frac{3}{2}v$ then $p = 5$ and $25 \not| v$, or $p \equiv 1, 2, 4$ or $8(15)$;*
2. *If $b = 2v$ then $p \equiv 1(4)$.*

For the remainder of this section assume that G acts line-transitively on the linear space $\mathcal{S}$.

Theorem 4. *[3, Theorem1] If $k \big| v$ then G is flag-transitive.*

Lemma 5. *[6, Lemma 4] If g is an involution of G and g fixes no points, then $k \big| v$. In particular, G is flag-transitive.*

Lemma 6. *[6, Lemma 2] Let $\mathfrak{L}$ be a line in $\mathcal{S}$ and let $T \leq G_{\mathfrak{L}}$. Assume that T satisfies the following two conditions:*

1. $|Fix_\Pi(T) \cap \mathfrak{L}| > 1$*;*

2. *if* $U \le G_{\mathfrak{L}}$ *and* $|Fix_{\Pi}(U) \cap L| > 1$ *and* U *is conjugate to* T *in* G*, then* U *is conjugate to* T *in* $G_{\mathfrak{L}}$*.*

Then either $Fix_{\Pi}(T) \subseteq \mathfrak{L}$ *or the induced linear space on* $Fix_{\Pi}(T)$ *is regular and* $N_G(T)$ *acts line-transitively on the space.*

Lemma 7. *[7, Lemma 2.2] Let* g *be an involution in* G *and assume that there exists* N*,* $N \lhd G$ *such that* $|G : N| = 2$ *with* $g \notin N$*. Then* N *acts line-transitively also.*

Note that Lemma 7 allows us to conclude that if $PGL(2, q)$ acts transitively on the lines of a linear space $\mathcal{S}$ then $PSL(2, q)$ also acts transitively on the lines of $\mathcal{S}$ and so that space is known.

Our next result provides the framework for our analysis of the line-transitive actions of $PSL(3, q)$. Since $\mathcal{S}$ is not a projective plane then, by Fisher's inequality $b > v$ and since $b = v(v-1)/(k(k-1))$, there must be some prime p that divides both $v-1$ and b. We shall refer to such a prime as a *significant* prime.

Lemma 8. *[4, Lemma 6.1] Suppose that* $\mathcal{S}$ *is not a projective plane and let* p *be a significant prime. Let* P *be a Sylow* p*-subgroup of* G_α*. Then* P *is a Sylow* p*-subgroup of* G *and* G_α *contains the normalizer* $N_G(P)$*.*

Lemma 9. *[4, Lemma 6.3] Let* H, K *be subgroups such that*

$$G_\alpha \le H < K \le G$$

and let $c = |K : H|$*. Then* r *divides* $\frac{1}{2}(c-1)k$ *and* b *divides* $\frac{1}{2}(c-1)v$*.*

Corollary 10. *[4, Corollary 6.4] Let* H, K *be as in Lemma 9.*

1. *Let*
$$c_0 = \gcd\{(c-1) \mid c = |K : H|, \text{ where } G_\alpha \le H < K \le G\}.$$
Then r *divides* $\frac{1}{2}c_0 k$ *and* b *divides* $\frac{1}{2}c_0 v$*.*

2. *There cannot be groups* H, K *such that* $G_\alpha \le H < K \le G$ *and* $|K : H| = 2$*.*

3. *If there are groups* H, K *such that* $G_\alpha \le H < K \le G$ *and* $|K : H| = 3$ *then* $\mathcal{S}$ *is a projective plane.*

2 New Lemmas

We state a series of lemmas which will be used in our analysis of the actions of $PSL(3, q)$. The first is a generalization of the Fisher inequality to non-regular linear spaces.

2.1 General linear spaces

Lemma 11. *In any linear space* $\mathcal{S}$*, not necessarily regular, Fisher's inequality holds:* $b \ge v$*.*

Proof. We need to prove the statement under the assumption that the number of points in a line is not a constant. Let c be the maximum number of points on a line of $\mathcal{S}$. Since any two points lie on a unique line we know that

$$b \geq \frac{\binom{v}{2}}{\binom{c}{2}} = \frac{v(v-1)}{c(c-1)}.$$

Thus if $c(c-1) \leq v-1$ then we are finished. Assume to the contrary from this point on. We split into two cases:

1. **Suppose that** $(c-1)^2 \geq v$**.** Then $\frac{v+c-1}{c} \leq c-1$. Let $\mathfrak{L}_c$ be a line with c points on it and choose α a point not on $\mathfrak{L}_c$. Then the average number of points in a line containing α and intersecting $\mathfrak{L}_c$ is less than or equal to $\frac{v-c-1}{c}+2 = \frac{v+c-1}{c} \leq c-1$. Call the number of lines intersecting $\mathfrak{L}_c$, b_0, and observe that

   $$b \geq b_0 \geq \frac{(v-c)c}{c-1}.$$

   Now we know that $v > c^2$ and so $vc - c^2 > vc - v$, hence $\frac{(v-c)c}{c-1} > v$. Thus this case is covered.

2. **Suppose that** $(c-1)^2 < v \leq c(c-1)$**.** Note that $v > 2$ implies that $c > 2$. Let r_α be the number of lines incident with a point α. If $r_\alpha \geq c$ for all α then, let f be the number of flags:

   $$vc \leq f \leq bc.$$

   Thus $v \leq b$ as required. Assume then that there exists a point α such that $r_\alpha \leq c-1$. Observe that every line not passing through α must have be incident with at most r_α points. Remove α and any lines which are incident with only α and one other point. Then $v > c$ and we still have a linear space, S^*. S^* has $v-1$ points, at most b lines, and the maximum number of points on a line is $c-1$. This implies that,

   $$b_S \geq b_{S^*} \geq \frac{\binom{v-1}{2}}{\binom{c-1}{2}} = \frac{(v-1)(v-2)}{(c-1)(c-2)}.$$

   Thus we are finished so long as $(c-1)(c-2) < v-2$. But $(c-1)^2 < v$ gives us this inequality since $c \geq 3$.

All cases are proved and the result stands. □

2.2 Regular linear spaces

We return to our assumption that $\mathcal{S}$ is a regular linear space.

Lemma 12. *Let $g \in G$ be an involution. Then g fixes at least $(v-1)/k$ lines.*

Proof. If g has no fixed point then g fixes $v/k \geq (v-1)/k$ lines. If g has a fixed point, α, then let m be the number of fixed lines through α. By definition, g moves the rest of the lines through α. Apart from α these lines contain $v - m(k-1) - 1$ points. None of these points is fixed hence every one of these points lies on a fixed line. Thus the number of lines fixed by g is at least

$$m + \frac{v - m(k-1) - 1}{k} = \frac{v + m - 1}{k} \geq \frac{v-1}{k}$$

lines as required. $\square$

Lemma 13. *Let g be an involution which is an automorphism of a linear space $\mathcal{S}$. Suppose that $\mathcal{S}$ has a constant number of points on a line, k, and that g fixes d_l lines and d_p points. Then, either*

- $d_l \geq d_p$*; or*
- $v = k^2$.

Proof. We know that if $\mathcal{S}$ is a projective plane then the result holds since the permutation character on points and lines is the same [9, 4.1.2]. Now suppose that $\mathcal{S}$ is not a projective plane and split into two cases:

1. **Suppose that $d_p \leq k$.** Assume that $d_l < d_p$. We know, by Lemma 12, that g fixes at least $\frac{v-1}{k}$ lines. Then

$$\begin{aligned} d_l < d_p \quad &\Longrightarrow \quad \frac{v-1}{k} < k \\ &\Longrightarrow \quad v - 1 < k^2. \end{aligned}$$

   Then, since $(k-1)\big|(v-1)$, we must have $\frac{v-1}{k-1} \leq k+1$. If $\frac{v-1}{k-1} \leq k$ then $b \leq v$ and so $b = v$ and $\mathcal{S}$ is a projective plane. If $\frac{v-1}{k-1} = k+1$ then $v = k^2$ as given.

2. **Suppose that $d_p > k$.** Then the fixed points and lines of g form a linear space. We may appeal to Lemma 11.

$\square$

Lemma 14. *Suppose that $b = \frac{c}{d}v$ where $(c,d) = 1$. Then the significant primes are exactly those which divide c.*

Proof. By definition a prime is significant if it divides b and $v-1$. Then we just use the fact that

$$\frac{c}{d}v = b = \frac{v(v-1)}{k(k-1)} = \frac{(v-1)/(k-1)}{k}v.$$

$\square$

Lemma 15. *Let $H < G_\alpha$. If $N_G(H) \not\leq G_\alpha$ then H is in $G_\mathfrak{L}$ for some line $\mathfrak{L}$.*

Proof. Simply take $g \in N_G(H)\backslash G_\alpha$. Then $H^g = H$ is contained in G_α and $G_{\alpha g}$. Hence H fixes the line joining α and αg. $\square$

2.3 Line-transitive linear spaces

Throughout this section we assume that G acts line-transitively on $\mathcal{S}$.

Lemma 16. *Let g be an involution of G and write $n_g = |g^G|$ for the size of a conjugacy class of involutions in G. Let $r_g = |g^G \cap G_{\mathfrak{L}}|$ be the number of such involutions in a line-stabilizer $G_{\mathfrak{L}}$. Then the following inequality holds:*

$$\frac{n_g(v-1)}{br_g} \leq k \leq \frac{r_g v}{n_g} + 1.$$

Proof. Count pairs of the form $(\mathfrak{L}, g)$ where $\mathfrak{L}$ is a line and g is an involution fixing $\mathfrak{L}$, in two different ways. Then

$$|\{(\mathfrak{L}, g)\}| = br_g \geq n_g c$$

where c is the minimum number of lines fixed by an involution. Now, by the previous lemma, $c \geq \frac{v-1}{k}$ thus we have

$$r_g \geq \frac{n_g c}{b} \geq \frac{n_g(v-1)}{bk} = \frac{n_g(k-1)}{v}.$$

This implies two inequalities:

$$k - 1 \leq \frac{r_g v}{n_g} \quad , \quad k \geq \frac{n_g(v-1)}{br_g}$$

and the result follows. □

Lemma 17. *Suppose that $|G_\alpha| = \frac{c}{d}|G_{\mathfrak{L}}|$ where $(c, d) = 1$. Then the significant primes are exactly those which divide c.*

Proof. Simply use the fact that $v = |G|/|G_\alpha|$, $b = |G|/|G_{\mathfrak{L}}|$ and refer to Lemma 14. □

Lemma 18. *Suppose that p^a is a prime power dividing $v - 1$ and that p does not divide into $|G|$. Then p^a divides $k(k-1)$.*

Proof. Since p does not divide $|G|$, p cannot divide into b. Since $b = \frac{v(v-1)}{k(k-1)}$ and p^a divides into $v - 1$ we must have p^a dividing into $k(k-1)$. □

We will often repeatedly use Lemma 18, with different primes, to exclude the possibility of a particular group, G, acting line-transitively on a space with a particular number of points, v. Our method for doing this usually involves showing that any line size k must be too large to satisfy Fisher's inequality (Lemma 2).

3 Background Information on $PSL(3, q)$

We will sometimes precede the structure of a subgroup of a projective group with ˆ which means that we are giving the structure of the pre-image in the corresponding linear group. We will also refer to elements of this linear group in terms of matrices under the standard modular representation.

3.1 Subgroup information

We need information about the subgroups of $PSL(3,q)$, $PSL(2,q)$ and $GL(2,q)$. Here $q = p^a$ for a prime p, positive integer a.

Theorem 19. *[14, 17, 1, 12] The maximal subgroups of $PSL(3,q)$ are among the following list. Conditions given are necessary for existence and maximality but not sufficient. The first three types are all maximal for $q \geq 5$.*

	Description	**Notes**
1	$\hat{}[q^2] : GL(2,q)$	*two $PSL(3,q)$-conjugacy classes*
2	$\hat{}(q-1)^2 : S_3$	*one $PSL(3,q)$-conjugacy class*
3	$\hat{}(q^2+q+1).3$	*one $PSL(3,q)$-conjugacy class*
4	$PSL(3,q_0).(q-1,3,b)$	*$q = q_0^b$ where b is prime*
5	$PSU(3,q_0)$	$q = q_0^2$
6	A_6	*q odd*
7	$3^2.SL(2,3)$	*q odd*
8	$3^2.Q_8$	*q odd*
9	$SO(3,q)$	*q odd*
10	$PSL(2,7)$	*q odd*

We will refer to maximal subgroups of $PSL(3,q)$ as being of type x, where x is a number between 1 and 10 corresponding to the list above.

Referring to [1, 14] we state the following lemma:

Lemma 20. *Suppose that H is a subgroup of $PSL(3,q)$ lying in a maximal subgroup of type 4 or 5 and H does not lie in any other maximal subgroup of $PSL(3,q)$. Then one of the following holds:*

- *H has a cyclic normal subgroup of index less than or equal to 3.*
- *H contains $PSL(3,q_1)$ with index less than or equal to 3. Here $q = q_1^c$, c an integer.*
- *H contains $PSU(3,q_1)$ with index less than or equal to 3. Here $q = q_1^c$, c an integer.*
- *H is isomorphic to $A_6.2$ or A_7 and $q = 5^a, a$ even.*

We state a result given by Suzuki [19, Theorem 6.25] which gives the structure of all the subgroups of $PSL(2,q)$:

Theorem 21. *Let q be a power of the prime p. Let $d = (q-1,2)$. Then a subgroup of $PSL(2,q)$ is isomorphic to one of the following groups.*

1. *The dihedral groups of order $2(q \pm 1)/d$ and their subgroups.*
2. *A parabolic group P_1 of order $q(q-1)/d$ and its subgroups. A Sylow p-subgroup P of P_1 is elementary abelian, $P \lhd P_1$ and the factor group P_1/P is a cyclic group of order $(q-1)/d$.*
3. *$PSL(2,r)$ or $PGL(2,r)$, where r is a power of p such that $r^m = q$.*

4. A_4, S_4 *or* A_5.

Note that when $p = 2$, the above list is complete without the final entry. Furthermore, referring to [14], we see that there are unique $PSL(2, q)$ conjugacy classes of the maximal dihedral subgroups of size $2(q \pm 1)/d$ as well as a unique $PSL(2, q)$ conjugacy class of parabolic subgroups P_1.

We will also need the subgroups of $GL(2, q)$ which can be easily obtained from the subgroups of $PSL(2, q)$ (for the odd characteristic case see [1, Theorem 3.4].)

Theorem 22. *H, a subgroup of $GL(2, q)$, $q = p^a$, is amongst the following up to conjugacy in $GL(2, q)$. Note that the last two cases may be omitted when $p = 2$.*

1. *H is cyclic;*

2. *$H = AD$ where*
$$A \leq \left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} : \lambda \in GF(q) \right\}$$
and $D \leq N(A)$, is a subgroup of the group of diagonal matrices;

3. *$H =< c, S >$ where $c | q^2 - 1$, S^2 is a scalar 2-element in c;*

4. *$H =< D, S >$ where D is a subgroup of the group of diagonal matrices, S is an anti-diagonal 2-element and $|H : D| = 2$;*

5. *$H =< SL(2, q_0), V >$ or contains $< SL(2, q_0), V >$ as a subgroup of index 2 and here $q = q_0^c$, V is a scalar matrix. In the second case, $q_0 > 3$;*

6. *$H/ < -I >$ is isomorphic to $S_4 \times C$, $A_4 \times C$, or (with $p \neq 5$) $A_5 \times C$, where C is a scalar subgroup of $GL(2, q)/ < -I >$;*

7. *$H/ < -I >$ contains $A_4 \times C$ as a subgroup of index 2 and A_4 as a subgroup with cyclic quotient group, C is a scalar subgroup of $GL(2, q)/ < -I >$.*

We will refer to maximal subgroups of $GL(2, q)$ as being of type x, where x is a number between 1 and 7 corresponding to the list above.

Finally observe that $PSL(3, q)$ contains a single conjugacy class of involutions. This class is of size $q^2(q^2 + q + 1)$ for q odd and of size $(q^2 - 1)(q^2 + q + 1)$ for q even. In addition note that we will write μ for $(q - 1, 3)$.

3.2 The subgroup D

We define D to be the centre of a Levi complement of a particular parabolic subgroup. Typically D is the projective image of

$$\left\{ \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{F}_q^* \right\}.$$

Suppose that $G = PSL(3,q)$ acts line-transitively on a linear space. Since D normalizes a Sylow t-subgroup of $PSL(3,q)$ for many different t, D often lies inside a point-stabilizer G_α. Furthermore, since D has a large normalizer, $\hat{\ } GL(2,q)$, by Lemma 15, D often lies inside a line-stabilizer, $G_{\mathfrak{L}}$.

We exploit this fact using Lemma 6 since if D satisfies the conditions given in the lemma and the fixed points of D are not collinear then we induce a line-transitive action of $PGL(2,q)$ on a linear space. All such actions on a non-trivial linear space are known. In the event that the fixed set is a trivial linear space (that is, $k = 2$) line-transitivity is equivalent to 2-homogeneity on points and these actions are also all well-known.

We need information about the occurrence of D in various subgroups and about how G-conjugates of D intersect. We state the relevant facts below; proofs are omitted as the results are easily derived from matrix calculations.

Lemma 23. *The $PSL(3,q)$ conjugates of D intersect trivially.*

Lemma 24. *Let $U : \hat{\ } GL(2,q)$ be a parabolic subgroup of $PSL(3,q)$, $q > 7$, U an elementary abelian p-group. We can choose $\hat{\ } GL(2,q)$ conjugate to*

$$C_G(D) = \hat{\ } \left\{ \begin{pmatrix} \frac{1}{DET} & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{pmatrix} : \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GL(2,q), DET = eh - fg \right\}.$$

Let H be a maximal subgroup of $\hat{\ } GL(2,q)$ in $PSL(3,q)$. Write a for a primitive element of $GF(q)$.

1. *If H is of type 2 in $\hat{\ } GL(2,q)$ then some $\hat{\ } GL(2,q)$ conjugate of H contains one individual conjugate, and two families of conjugates, of D, generated by the projective images of the following matrices, for $f \in GF(q)$:*

$$\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & f & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & f & \frac{1}{a^2} \end{pmatrix}.$$

2. *If H is of type 3 in $\hat{\ } GL(2,q)$ then H contains only D.*

3. *If H is of type 4 in $\hat{\ } GL(2,q)$ then some $\hat{\ } GL(2,q)$-conjugate of H contains three conjugates of D, generated by the projective images of the following matrices:*

$$\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}.$$

4. *If H is of type 5 in $\hat{\ } GL(2,q)$ then one of the following holds:*

   - *H contains only D;*
   - *$H \geq SL(2,q)$;*
   - *$H \geq SL(2,q_0)$ where $q = q_0^2$ and $q_0 = 3, 4$ or 7.*

5. *If H is of type 6 or 7 in $\hat{}GL(2,q)$ then one of the following holds:*
    - *H contains only the central copy of D;*
    - *$q = 13, 16$ or 19.*

Corollary 25. *A subgroup of $PSL(3,q)$ of type 3 contains only the 3 diagonal conjugates of D as listed above for H of type 4 in $\hat{}GL(2,q)$.*

4 Reducing to the Simple Case

Let B be a normal subgroup in a group G which acts upon a set Π. Then (G, B, Π) is called *exceptional* if the only common orbital of B and G in their action upon Π is the diagonal (see [11]).

Lemma 26. *Suppose a group G acts line-transitively on a linear space $\mathcal{S}$; suppose furthermore that B is a normal subgroup in G which is not line-transitive on $\mathcal{S}$; finally suppose that $|G : B| = t$, a prime.*

Then either $\mathcal{S}$ is a projective plane or (G, B, Π) is exceptional.

Proof. The suppositions mean that, for a line $\mathfrak{L}$ of $\mathcal{S}$, $G_{\mathfrak{L}} = B_{\mathfrak{L}}$. We have two possibilities:

- **Suppose that B is point-transitive on $\mathcal{S}$.** Then let α and β be members of Π, the set of points of $\mathcal{S}$. Let $\mathfrak{L}$ be the line connecting them. Then, since $G_{\alpha,\beta} \leq G_{\mathfrak{L}}$ and $B_{\alpha,\beta} \leq B_{\mathfrak{L}}$, we know that $G_{\alpha,\beta} = B_{\alpha,\beta}$.

  We know furthermore that $|G_\alpha : B_\alpha| = t$, hence we may conclude that, for all pairs of points α and β, $|B_\alpha : B_{\alpha,\beta}| < |G_\alpha : G_{\alpha,\beta}|$. In other words (G, B, Π) is **exceptional**.

- **Suppose that B is not point-transitive on $\mathcal{S}$.** Then, by the Frattini argument, $G = N_G(P)B$ for all $P \in Syl_pB$ where p is any prime dividing into $|B|$. If $G_\alpha \geq N_G(P)$ then B is point-transitive which is a contradiction. Thus, by Lemma 15, if a Sylow p-subgroup of B stabilizes a point then it also stabilizes a line.

  Now let $b_B = |B : B_{\mathfrak{L}}|$, $v_B = |B : B_\alpha|$. Then primes dividing into b_B are a subset of the primes dividing into v_B. Furthermore $b = tb_B$ and $v = tv_B$. Thus primes dividing into b are a subset of the primes dividing into v. Thus there are no significant primes and $\mathcal{S}$ is a projective plane.

$\square$

Now suppose that $PSL(3,q) \trianglelefteq G \leq AutPSL(3,q)$ and G acts line-transitively on a space $\mathcal{S}$ which is not a projective plane. Suppose furthermore that $PSL(3,q)$ is not line-transitive on $\mathcal{S}$. Then there exist groups G_1, G_2 such that $PSL(3,q) \trianglelefteq G_1 \trianglelefteq G_2 \leq G \leq AutPSL(3,q)$ where $|G_2 : G_1|$ is a prime, G_1 is not line-transitive on $\mathcal{S}$ while G_2 is. By the above argument, (G_2, G_1, Π) is an exceptional triple and [11, Theorem 1.5] implies that, for $q > 2$, G_α only lies inside maximal subgroups of type $PSL(3, q_0)$ where $q = q_0^a, a > 3$.

Appealing to Lemma 20 we have only four possibilities for G_α. The last two on the list lie inside $PSU(3,q_0)$ with $q=q_0^2$ and so can be excluded. The first can be excluded similarly. Hence Theorem A holds in this situation for $q>2$.

When $q=2$, $PSL(3,2)\cong PSL(2,7)$. Since $AutPSL(2,7)=PGL(2,7)$, Lemma 7 implies, under the given suppositions, that $PSL(2,7)$ is line-transitive on $\mathcal{S}$. Then Theorem 1 implies that Theorem A holds in this situation.

In order to prove Theorem A it is now sufficient to proceed under the assumptions of the following hypothesis. Our aim is to show that this hypothesis leads to a contradiction. We will need to consider different possibilities for a linear space $\mathcal{S}$ having a significant prime dividing $|PSL(3,q)|=q^3(q-1)^2(q+1)(q^2+q+1)/\mu$ where $\mu=(q-1,3)$.

Hypothesis. *Suppose that $G=PSL(3,q)$ acts line-transitively but not flag-transitively on a linear space $\mathcal{S}$ which is not a projective plane. Let b,v,k,r be the parameters of the space. Let D be the subgroup of $PSL(3,q)$ as defined in the previous section. We suppose, by Lemma 5, that every involution of $PSL(3,q)$ fixes a point. Finally we assume that $q>2$.*

5 Preliminary Cases

5.1 Significant prime: $t|q^2+q+1, t\neq 3$

Suppose first that some $t|q^2+q+1, t\neq 3$ is a significant prime. Lemma 8 implies that $G_\alpha\geq\hat{}(q^2+q+1).3$ which is the normalizer of a Sylow t-subgroup of $PSL(3,q)$. Now $\hat{}(q^2+q+1).3$ is maximal in $PSL(3,q)$ for $q\neq 4$ and so, in this case, $G_\alpha=\hat{}(q^2+q+1).3$ This is a contradiction since then G_α doesn't contain any involution, contradicting our Hypothesis.

When $q=4$ the only other possibility is that $G_\alpha=PSL(2,7)$ and $v=120$. Then $17|v-1$ and by Lemma 18, $k\geq 17$ which contradicts Fisher's inequality (Lemma 2).

5.2 Significant prime: $t=p$

Suppose now that p is significant. Lemma 8 implies that $G_\alpha\geq\hat{}[q^3]:(q-1)^2$, a Borel subgroup, which is the normalizer of a Sylow p-subgroup of $PSL(3,q)$. Then G_α is either a Borel subgroup or a parabolic subgroup of $PSL(3,q)$.

In the latter case the action of G is 2-transitive on points and hence flag-transitive. Thus this case is already covered.

When G_α is a Borel subgroup $v=(q^2+q+1)(q+1)$ and, by Corollary 10, b divides into $\frac{1}{2}q(q+1)(q^2+q+1)$. This implies that $r>k>q+1$. Then $r=\frac{v-1}{k-1}<q^2+q+1$.

Consider the set of lines through the point α. These lines contain all points of $\mathcal{S}$ and so the points of $\mathcal{S}\backslash\{\alpha\}$ can be thought of as making up a rectangle with dimensions r by $k-1$. The area of this rectangle (that is, the number of points in the rectangle) is $v-1=r(k-1)=q^3+2q^2+2q$.

Now G_α has five orbits on $\mathcal{S}\backslash\{\alpha\}$ of size q,q,q^2,q^2 and q^3. Each of these orbits forms a rectangle of points in $\mathcal{S}\backslash\{\alpha\}$. Thus we have a rectangle of area q^3+2q^2+2q made out of rectangles of area q,q,q^2,q^2 and q^3 with integer dimensions. We investigate this situation.

Write $[q^a]_l$ and $[q^a]_w$ for the length and width of the rectangles of area q^a. Define the length of the rectangle of area q^a to be the dimension in the direction of the side of length r in the big rectangle. Observe first that $q \leq k-1 < [q^3]_w \leq [q^3]_l < r < q^2$.

Now if, for $a = 1, 2$, there exists a rectangle such that $[q^a]_w > [q^3]_w$ or $[q^a]_l > [q^3]_l$ then $v - 1 \geq pq^6$ which is a contradiction.

Suppose that there is a rectangle of area q^a such that $[q^a]_w < [q^3]_w$. This rectangle must combine with others to make the total width $k-1$. It either combines with a rectangle of width at least $[q^3]_w$ or it combines with several of width less than $[q^3]_w$. Given that there are only five rectangles in total the latter possibility can only occur for $p = 2, 3$. In fact a cursory examination can rule out the case where $p = 3$. A similar argument works if we consider lengths instead widths.

Thus, for $p > 2$, we are reduced to factorising $q^2 + 2q + 2$ in $\mathbb{Z}[x]$. But this polynomial is clearly irreducible by Eisenstein's criterion.

If $p = 2$ then a slight modification of the above argument reduces to the factorisation problem once again and the possibility that G_α is a Borel subgroup is excluded.

Remark. *Note that we have excluded the possibility that G_α is a parabolic or a Borel subgroup, no matter what prime is significant.*

5.3 Remaining Cases

We wish to enumerate the remaining cases that we need to examine. First of all note that when q is small applications of Lemmas 18 and 2 can be used to exclude all possibilities. We will assume from here on therefore that $q \geq 8$.

In addition one case in particular is worth mentioning now: When q is odd and when both 2 and $3|(q-1)$ are significant primes.

The only maximal subgroups which have index not divisible by 2 and 3 in this case are those of type 2 and 4. Suppose that G_α lies in a subgroup M of type 2, Without loss of generality the diagonal subgroup normalized by the group of permutation matrices isomorphic to S_3. Now D normalizes a Sylow 2-subgroup of M. In addition $Q \in Syl_3 G$ is conjugate to $H : C_3$ where H is a diagonal subgroup, C_3 a group of permutation matrices. Q does not normalize D hence G_α contains at least two conjugates of D. Since these intersect trivially, by Lemma 23, these generate a subgroup of index dividing μ in the diagonal subgroup. Our group G_α must therefore be the full subgroup of type 2.

If G_α is contained in a subgroup, M, of type 4 then in order to contain an element of order $\frac{q-1}{\mu}$, $M = PSL(3, q_0), q = (q_0)^2$. But then the index of M in G is even which is a contradiction.

Thus the cases which we need to examine are, for $q \geq 8$:

	Significant primes t	**Possible stabilizers**
I	$\exists t \big\vert (q+1), t \neq 2$	$\hat{}(q^2-1).2 \leq G_\alpha < q^2 : \hat{}GL(2,q)$
II	$\exists t \big\vert (q-1), t \neq 2, 3$ OR $2, 3 \big\vert (q-1)$ both significant	$G_\alpha = \hat{}(q-1)^2 : S_3$
III	$3 \big\vert (q-1)$ is uniquely significant	G_α is a subgroup of a maximal subgroup of type 2, 4, 5 or 8
IV	$2 \big\vert (q-1)$ is uniquely significant	G_α is a subgroup of a maximal subgroup of types 1, 2 or 4

6 Case I: $\exists t|(q+1), t \neq 2$ significant

In this case G_α contains a subgroup H of order $2(q^2-1)/\mu$ which itself has a cyclic subgroup of size $(q^2-1)/\mu$ and G_α lies inside a parabolic subgroup of G.

Now observe that H lies inside a copy of $\hat{}GL(2,q)$ and that $\hat{}GL(2,q)$ normalizes an elementary abelian subgroup, U, of $PSL(3,q)$, of order q^2. In its conjugation action on the non-identity elements of U, $\hat{}GL(2,q)$ has stabilizers of order $q(q-1)$. Thus our group H must, if it normalizes any subgroup of U, normalize a subgroup of order $1+x(q+1)$ for some integer x. Now for such a value to divide q^2, as required, x must be 0 or $q-1$.

Thus $G_\alpha = \hat{}A.B$ where A is trivial or of size q^2 and $H \leq B \leq GL(2,q)$. Now, in the characteristic 2 case, $GL(2,q) = PSL(2,q) \times (q-1)$ and $H = D_{2(q+1)} \times (q-1)$. Since $D_{2(q+1)}$ is maximal in $PSL(2,q)$ for all even $q \geq 8$, we know that $B = H$ or $B = GL(2,q)$. In the odd characteristic case, $GL(2,q) =< -I > .(PSL(2,q) \times (\frac{q-1}{2})).2$ and $H =< -I > .(H \times (\frac{q-1}{2})).2$. Now, for all odd $q > 9$, $D_{2(q+1)}$ is maximal in $GL(2,q)$ and, once again we conclude that $B = H$ or $B = GL(2,q)$.

We need to consider the case where $q = 9$ and $H < B < GL(2,q)$. In fact this case cannot occur since the only proper subgroup of $PSL(2,9)$ containing D_{10} is A_5, but $< -I > .(A_5 \times (\frac{q-1}{2}))$ is not normalized by any element of $GL(2,q)$ of non-square determinant.

Thus we can summarize the cases that we need to examine:

1. $G_\alpha = U.\hat{}(q^2-1).2$ where $U = [q^2]$;
2. $G_\alpha = \hat{}GL(2,q)$;
3. $G_\alpha = \hat{}((q^2-1).2)$.

Note that we exclude the case where $G_\alpha = \hat{}U : GL(2,q)$, as then G_α is maximal parabolic and this case is already excluded. We will consider the remaining cases in turn.

Remark. *These cases also arise when $2|(q+1)$ is the only significant prime (see Section 9). The arguments given below are general and apply in that situation as well.*

6.1 Case 1: $G_\alpha = U.(\hat{}(q^2-1).2)$.

Now we know that $v = \frac{1}{2}(q^2+q+1)q(q-1)$ and, since G_α lies inside a parabolic subgroup, we can appeal to Corollary 10 to observe that

$$b\Big|\frac{1}{8}(q^2+q+1)q(q-1)(q+1)(q-2) \text{ and } b\Big|\frac{1}{4}(q^2+q+1)q(q-1)(q+1)q.$$

Thus $b\big|\frac{1}{4(2,q-1)}(q^2+q+1)q(q-1)(q+1)$ and so $4(2,q-1)q^2(q-1)/\mu$ divides $|G_{\mathfrak{L}}|$. For $q > 7$ this means that $G_{\mathfrak{L}}$ lies in a parabolic subgroup. Observe that we can presume that $U : D$ lies in $G_{\mathfrak{L}}$ for some $\mathfrak{L}$ since $U : D$ lies in G_α and is normalized by the full parabolic subgroup (Lemma 15).

Suppose that U is non-normal in $G_{\mathfrak{L}} = \hat{}A.B$ where A is an elementary abelian p-group and $B \leq GL(2,q)$. Then $G_{\mathfrak{L}}$ must lie in a parabolic subgroup which is not conjugate to $N_G(U)$ and $|U \cap A| = q$. If $A \backslash U$ is non-empty then U acts by conjugation on these elements with an orbit, Ω, of size q. Then $U \cap A$ and Ω lie inside A and generate q^2 elements.

Hence we must have A of size q or q^2. The latter would make $|G_{\mathfrak{L}}| \geq 4q^3(q-1)/\mu$ which is larger than $|G_\alpha|$ which is a contradiction. Hence we conclude that $|A| = q$.

Since A is normal in $G_{\mathfrak{L}}$ we must have $G_{\mathfrak{L}}$ a subgroup of a Borel subgroup. However in this case U is normal in $G_{\mathfrak{L}}$. This is a contradiction.

Hence we have U normal in $G_{\mathfrak{L}}$. Furthermore there are no other G-conjugates of U in $G_{\mathfrak{L}}$, since $U \cap U^g$ is trivial for all g in $G \backslash N_G(U)$. Hence we may appeal to Lemma 6. Then either $U : \hat{}GL(2,q)$ acts line-transitively on the fixed set of U, which is itself a linear space, or this fixed set lies completely in one line. In the first case, such an action of $U : \hat{}GL(2,q)$ has a kernel $U : \hat{}D$ and corresponds to a line-transitive action of $PGL(2,q)$ with stabilizer a dihedral group $D_{2(q+1)}$.

Examining the results of line-transitive and 2-transitive actions of $PGL(2,q)$ we find that there is one such action to consider. We have q even and $PGL(2,q)$ acts line-transitively upon a Witt-Bose-Shrikhande space with line-stabilizer an elementary abelian group of order q. In $PSL(3,q)$ this corresponds to $G_{\mathfrak{L}}$ having order $\frac{q^3(q-1)}{\mu}$ and $b = (q-1)(q+1)(q^2+q+1)$. Then we must have,

$$\begin{aligned} & k(k-1) = \frac{v(v-1)}{b} = \frac{1}{4}q(q^3-q^2+q-2) \\ \Longrightarrow \quad & 2k(2k-2) = q^4 - q^3 + q^2 - 2q. \end{aligned}$$

Now observe that,

$$(q^2 - \frac{1}{2}q + 1)(q^2 - \frac{1}{2}q - 1) < q^4 - q^3 + q^2 - 2q < (q^2 - \frac{1}{2}q + 2)(q^2 - \frac{1}{2}q).$$

Thus this case is excluded.

We can assume therefore that the set of fixed points of U lies completely in one line. This fixed set has size $\frac{1}{2}q(q-1)$ and thus k is at least this large. Now the subgroups conjugate to U intersect trivially. Thus U lying in $G_{\mathfrak{L}}$ has orbits on the points of $\mathfrak{L}$ of size 1 ($\frac{1}{2}q(q-1)$ such) or q^2 (for q odd) or $\frac{q^2}{2}$ (for q even.)

If $k \geq q^2 + \frac{1}{2}q(q-1)$ then $k(k-1) > v$ which is a contradiction. If $k = \frac{1}{2}q(q-1)$ then $k - 1 = \frac{1}{2}(q+1)(q-2)$ divides into $v - 1 = \frac{1}{2}(q+1)(q^3 - q^2 + q - 2)$. This is possible only for $q \leq 4$ which is a contradiction. Thus we are left with the possibility that q is even and $k = \frac{1}{2}q(q-2)$. Once again $k-1$ dividing into $v-1$ implies that $q \leq 4$.

6.2 Case 2: $G_\alpha = \hat{}GL(2,q)$

Since $v = q^2(q^2+q+1)$ and G_α lies inside a parabolic subgroup, we can appeal to Corollary 10 to observe that

$$b \Big| \frac{1}{2}q^2(q^2+q+1)(q-1)(q+1) \text{ and } b \Big| \frac{1}{2}q^2(q^2+q+1)(q+1)q.$$

Thus $b \big| \frac{1}{2}q^2(q^2+q+1)(q+1)$ and so $2q(q-1)^2/\mu$ divides $|G_{\mathfrak{L}}|$.

This implies that, for $q > 7$, $G_{\mathfrak{L}}$ lies in a parabolic subgroup or $q = 16$. When $q = 16$ we find that the prime 4111 divides into v-1=69888 which, using Lemma 18, contradicts Lemma 2.

Thus $G_{\mathfrak{L}}$ lies in a parabolic subgroup and we write $G_{\mathfrak{L}} = \hat{}A.B$ as usual. If $A = \{1\}$ then we must have $G_{\mathfrak{L}} = \hat{}B \leq \hat{}GL(2,q)$. Examining the subgroups of $GL(2,q)$ given in

Theorem 22 we find that $|G_{\mathfrak{L}}|$ is divisible by $\frac{|GL(2,q)|}{2\mu}$. Now if $\mu = 3$ and 3 is significant then G_α does not lie in a parabolic subgroup. Hence we must have $|G_{\mathfrak{L}}| = \frac{1}{2}|G_\alpha|$ with 2 uniquely significant. But then Lemma 3 implies that any prime dividing into v must be equivalent to $1(4)$. Now in our current situation any significant prime divides into $\frac{q+1}{2}$ thus 2 is not a significant prime; this is a contradiction.

Now if $1 \neq g \in A$ then $|C_{PSL(3,q)}(g)| = q^3(q-1)/\mu$. Thus B must act on the non-trivial elements of A with orbits of size divisible by $q-1$. Thus $|A| = q$ or q^2.

If $|A| = q^2$ then $|G_{\mathfrak{L}}| \geq 2q^2(q-1)^2/\mu > |G_\alpha|$ which cannot happen. If $|A| = q$ then $p = 2$ (since, if p is odd, B must act on the non-trivial elements of A with orbits of size divisible by $2(q-1)$.) For $q > 4$ we must have B either maximal in $GL(2,q)$ of type 4 or a subgroup of the Borel subgroup of $GL(2,q)$. In the first case $\hat{}B$ has orbits of size at least $2(q-1)$ on the non-identity elements of A, thus this case can be excluded.

If B lies inside a Borel subgroup of $GL(2,q)$ then $B = B_1.B_2$ where $2 < B_1$ and $B_2 = (q-1)^2$. In fact we must have $|B| = \frac{q(q-1)^2}{\mu}$ since B_2 acts by conjugation on the non-identity elements of B_1 with orbits of size $q-1$. Hence $|G_{\mathfrak{L}}| = \frac{q^2(q-1)^2}{\mu}$ and $b = q(q+1)(q^2+q+1)$. Hence we must have

$$k(k-1) = q^4 + q^2 - q.$$

Now observe that,

$$q^2(q^2-1) < q^4 + q^2 - q < (q^2+1)q^2.$$

Thus this case is excluded.

6.3 Case 3: $G_\alpha = \hat{}(q^2-1).2$

Since $v = \frac{1}{2}q^3(q^2+q+1)(q-1)$ and G_α lies inside a parabolic subgroup, we can appeal to Corollary 10 to observe that b divides into both

$$\frac{1}{4}q^3(q^2+q+1)(q-1)(q+1)q \text{ and } \frac{1}{8}q^3(q^2+q+1)(q-1)(q+1)(q^3-2q^2+2q-2).$$

Thus $b|\frac{1}{4(2,q-1)}q^3(q^2+q+1)(q-1)(q+1)$ and so $4(2,q-1)(q-1)/\mu$ divides $|G_{\mathfrak{L}}|$.

To begin with note that all cases where $11 < q \leq 16$ and $q = 9, 19, 25, 31, 37, 64$ can be ruled out using Lemma 18. When $q = 11$, Lemma 18 leaves one possibility, namely that $k = 444$. But then b is not an integer and so this situation can be excluded. When $q = 8$, Lemma 18 leaves one possibility, namely that $k = 171$. But then $k-1$ does not divide into $v-1$ and so this situation too can be excluded.

Using these facts, and recalling that $4(2,q-1)(q-1)/\mu$ divides $|G_{\mathfrak{L}}| < |G_\alpha|$, we can exclude the possibility that $G_{\mathfrak{L}}$ lies in a subgroup of $PSL(3,q)$ of type 3-10. Hence we assume that $q \geq 17$ and $G_{\mathfrak{L}}$ lies inside a subgroup of type 1 or 2 for the rest of this section.

Now $D < G_\alpha$ and, by Lemma 15, D lies in $G_{\mathfrak{L}}$ for some line $\mathfrak{L}$. We refer to Lemma 6 to split our investigation into three cases:

- **Case 3.A:** All G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and the fixed set of D is a linear-space acted on line-transitively by $\hat{}GL(2,q)$, the normalizer of D.

- **Case 3.B:** All G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and the fixed points of D, of which there are $\frac{1}{2}q(q-1)$, lie on one line;
- **Case 3.C:** $G_{\mathfrak{L}}$ contains at least two $G_{\mathfrak{L}}$-conjugacy classes of G-conjugates of D.

6.3.1 Case 3.A

This situation corresponds to a line-transitive action of $PGL(2,q)$ with stabilizer $D_{2(q+1)}$. Then Theorem 1 implies that $p=2$ and the fixed set of D is a Witt-Bose-Shrikhande space. The corresponding line-stabilizer in $PGL(2,q)$ has size q and so $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)}{\mu}$ in $PSL(3,q)$. Suppose that $|G_{\mathfrak{L}}| = \frac{q(q-1)}{\mu}$ and so

$$\begin{aligned} k(k-1) &= \frac{v(v-1)}{b} = \frac{1}{4}(q^6 - q^5 + q^4 - 2q^3 + 2q^2 - 2q) \\ \implies \quad (2k)(2k-2) &= q^6 - q^5 + q^4 - 2q^3 + 2q^2 - 2q. \end{aligned}$$

But now observe that

$$(q^3 - \frac{1}{2}q^2 + \frac{3}{8}q + 2)(q^3 - \frac{1}{2}q^2 + \frac{3}{8}q) < 2k(2k-2) < (q^3 - \frac{1}{2}q^2 + \frac{3}{8}q)(q^3 - \frac{1}{2}q^2 + \frac{3}{8}q - 2).$$

For $q > 16$ this gives a contradiction.

The only other possibility is that $|G_{\mathfrak{L}}| = \frac{2q(q-1)}{\mu}$ and $[q] \times \frac{q-1}{\mu} = G_{\mathfrak{L}} \cap C_G(D)$. This implies that $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $PSL(3,q)$.

Now $[q] \times \frac{q-1}{\mu}$ is normal in $G_{\mathfrak{L}}$ and so $[q]$ is normal in $G_{\mathfrak{L}}$ and $G_{\mathfrak{L}}$ lies inside a Borel subgroup of $PSL(3,q)$. Then D acts on the normal subgroup of $G_{\mathfrak{L}}$ of order $2q$. Furthermore D centralizes at most q of these elements and has orbits on the rest of size at least $\frac{q-1}{\mu}$. These orbits intersect cosets of $[q] \trianglelefteq C_G(D) \cap G_{\mathfrak{L}}$ with a size of at most 1. This gives a contradiction.

6.3.2 Case 3.B

Observe that all $PSL(3,q)$-conjugates of D intersect trivially. Observe too that all elements of G_α are of form TS where $T \in \hat{}(q^2-1)$ and S^2 lies in D. Then $(TS)^2$ lies in D and hence if E is some other conjugate of D then $E \cap G_\alpha$ is of size at most $(2, q-1)$. Thus the orbits of D on $\mathfrak{L}$, a line which it fixes, are either of size $\frac{q-1}{(2,q-1)\mu}$ or of size 1 and there are $\frac{1}{2}q(q-1)$ of these. We conclude that k is a multiple of $\frac{q-1}{(2,q-1)\mu}$.

Now we find that $(v-1, |G|) = \frac{q+1}{(2,q-1)}$. Since $\frac{q-1}{(2,q-1)\mu} \big| k$ and $b = \frac{v(v-1)}{k(k-1)}$ divides into $|G|$ then $b \big| \frac{\mu}{2}(q^2+q+1)q^3(q+1)$.

Thus, for $q \not\equiv 1(3)$, $|G_{\mathfrak{L}}| = 2(q-1)^2 \geq 512$. If $q \equiv 1(3)$ then $|G_{\mathfrak{L}}| = \frac{2}{9}(q-1)^2.a$ where $a = 1, 2$ or 3.

Suppose first that p is odd. Consider the possibility that $G_{\mathfrak{L}}$ lies inside a subgroup of type 2 and not in a parabolic subgroup. So $G_{\mathfrak{L}}$ is a subgroup of $\hat{}(q-1)^2 : S_3$ and must have either 3 or S_3 on top. The former case is impossible as then b does not divide into $\frac{\mu}{2}(q^2+q+1)q^3(q+1)$. Now $G_{\mathfrak{L}} = \hat{}(A \times A) : S_3$ or $(\frac{A}{\mu} \times \frac{A}{\mu}) : S_3$. Then, since $G_{\mathfrak{L}}$ must contain a subgroup conjugate to D, we find that $G_{\mathfrak{L}} = (\frac{q-1}{\mu} \times \frac{q-1}{\mu}) : S_3$, $\mu = 3$ or $G_{\mathfrak{L}} = \hat{}(q-1)^2 : S_3$. The latter case violates Fisher's inequality and can be excluded.

In the former case $G_{\mathfrak{L}}$ contains at most $q+2$ involutions. Appealing to Lemma 16, we observe that

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{1}{2}q(q+2)(q-1) + 1.$$

This means that $b = \frac{v(v-1)}{k(k-1)} > q^5(q-3)$ which is a contradiction.

Thus $G_{\mathfrak{L}}$ lies inside a parabolic subgroup; in fact $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$. In order for Fisher's inequality to hold, we must have one of the following cases:

- $b = \frac{1}{2}q^3(q^2+q+1)(q+1)$ and $|G_{\mathfrak{L}}| = \frac{2(q-1)^2}{\mu}$. Thus $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 4 (in which case $G_{\mathfrak{L}}$ contains more than one $G_{\mathfrak{L}}$-conjugacy class of G-conjugates of D which is a contradiction) or $G_{\mathfrak{L}}$ is isomorphic to a subgroup of type 6 or 7. This latter case requires that $2(q-1)$ divides into 24 or 60. These possibilities have already been excluded.

- $b = \frac{3}{4}q^3(q^2+q+1)(q+1)$. Hence $|G_{\mathfrak{L}}| = \frac{4}{9}(q-1)^2$ and $q \equiv 7(12)$. Thus $G_{\mathfrak{L}}$ is isomorphic to a subgroup of type 6 or 7 in $\hat{}GL(2,q)$ and $\frac{4(q-1)}{3}$ must divide 24 or 60. This is impossible.

- $b = \frac{3}{2}q^3(q^2+q+1)(q+1)$. Then $|G_{\mathfrak{L}}| = \frac{2}{9}(q-1)^2$ and $q \equiv 1(3)$. Thus $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 4, 6 or 7.

  If $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 4 then $r_g \leq \frac{q+8}{3}$. Using Lemma 16 we see that
  $$k \geq \frac{n_g(v-1)}{br_g} > q^2(q-9).$$
  Since $(k-1)^2 < v$ this implies that
  $$q^4(q-9)^2 < \frac{1}{2}q^3(q^2+q+1)(q-1)$$
  which means that $q < 31$. Then $q = 25$, but this possibility has already been excluded using Lemma 18.

  If $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 6 or 7 then we require that $\frac{2(q-1)}{3}$ divides into 24 or 60. Hence $q = 31$ or 37. These possibilities have already been excluded.

If $p = 2$ then, in order for Fisher's inequality to hold and so that $4(q-1)/\mu$ divides into $|G_{\mathfrak{L}}|$, we have $|G_{\mathfrak{L}}| = \frac{4}{9}(q-1)^2$ and $q \equiv 1(3)$. Thus $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $PSL(3,q)$ and $G_{\mathfrak{L}} = \hat{}A.B$ as usual.

If A is trivial then $G_{\mathfrak{L}}$ is a subgroup of type 2 in $\hat{}GL(2,q)$. Then $G_{\mathfrak{L}}$ has a normal 2-group and, by Schur-Zassenhaus, $G_{\mathfrak{L}}$ also contains a subgroup of size $\frac{(q-1)^2}{9}$. This subgroup has orbits in its conjugation action on 2-elements of $G_{\mathfrak{L}}$ of size at least $\frac{q-1}{3}$. This implies that $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)^2}{9}$ which is a contradiction.

If A is non-trivial then $G_{\mathfrak{L}}$ must have orbits in its conjugation action on non-identity elements of A of size at least $\frac{q-1}{3}$. Once again this implies that $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)^2}{9}$ which is a contradiction.

6.3.3 Case 3.C

Now consider the possibility that $G_{\mathfrak{L}}$ contains at least two $G_{\mathfrak{L}}$-conjugacy classes of G-conjugates of D.

Suppose first that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}(q-1)^2 : S_3$ and does not lie in a parabolic subgroup. We know that q is odd since $4(2, q-1)(q-1)/\mu$ divides into $|G_{\mathfrak{L}}|$. Since $G_{\mathfrak{L}}$ is not in a parabolic subgroup we must have a non-trivial part of S_3 on top, of order 3 or 6. Thus all G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate which is a contradiction.

Thus we may conclude that $G_{\mathfrak{L}}$ is in a parabolic subgroup. Write $G_{\mathfrak{L}} = \hat{}A.B$ as usual. If A is trivial then, referring to Lemma 24, we conclude that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2, q)$ of type 2,4 or 5. If $G_{\mathfrak{L}}$ is of type 5 then $q = 49$ and this can be ruled out using Lemma 18.

If $G_{\mathfrak{L}}$ is of type 2 and not of type 4 then it must contain non-trivial p-elements. Some conjugate of D in $G_{\mathfrak{L}}$ must have orbits in its conjugation action on these elements of size $\frac{q-1}{\mu}$. Thus $A_1 : \frac{q-1}{\mu} \leq |G_{\mathfrak{L}}|$ where A_1 is a p-group of size divisible by q. We will consider this possibility together with the case when A is non-trivial.

So suppose that A is non-trivial. Now either all G-conjugates of D in $G_{\mathfrak{L}}$ lie in $C_G(A)$ or else $|A| \geq q$. Consider the first possibility. In this case $A : D$ and $A : E$ lie inside $C_G(A)$ where E is a G-conjugate of D. Now $C_G(A) \leq C_G(g)$ for g an element or order p. Since $C_G(g) \cong [q^3] : \frac{q-1}{\mu}$, we know that D and E are conjugate in $C_G(A) \cap G_{\mathfrak{L}}$ by Schur-Zassenhaus. This is a contradiction and so we assume that $|A| \geq q$; thus, in both cases that we have considered so far, $Q : D \leq G_{\mathfrak{L}}$ where Q is a p-group of order divisible by q.

Now let E be a G-conjugate of D in $G_{\mathfrak{L}}$ which is not $G_{\mathfrak{L}}$-conjugate to D. Suppose $E \cap (Q : D)$ is non-trivial and $1 \neq h \in E \cap (Q : D)$. Then h lies inside a $Q : D$-conjugate of D by applying Sylow theorems to $Q : D$. But this is impossible since Lemma 23 implies that either $E = D$ or $E \cap D$ is trivial. Hence $|G_{\mathfrak{L}}| \geq \frac{q(q-1)^2}{\mu^2} > |G_\alpha|$ which is also impossible.

Finally we must consider the possibility that $G_{\mathfrak{L}}$ is of type 2 in $\hat{}GL(2, q)$; that is, $G_{\mathfrak{L}}$ is a subgroup of $\hat{}(q-1)^2 : 2$. We must have q odd since $4(2, q-1)(q-1)/\mu$ divides into $|G_{\mathfrak{L}}|$. Furthermore the G-conjugates of D in $\hat{}(q-1)^2 : 2$ normalize each other and so $\frac{(q-1)^2}{\mu^2}$ divides into $|G_{\mathfrak{L}}|$. There are three possibilities to consider:

- $G_{\mathfrak{L}} \leq \hat{}(q-1)^2$. In this case $G_{\mathfrak{L}}$ contains at most 3 involutions. Appealing to Lemma 16, we observe that
$$k \leq \frac{r_g v}{n_g} + 1 = \frac{3}{2}q(q-1) + 1.$$
This is too small to satisfy $b = \frac{v(v-1)}{k(k-1)}$ hence we have a contradiction.

- $G_{\mathfrak{L}} = (\frac{q-1}{\mu} \times \frac{q-1}{\mu}) : 2$. Then $G_{\mathfrak{L}}$ contains $\frac{q+8}{3}$ involutions. Once again using Lemma 16, we observe that
$$k \leq \frac{r_g v}{n_g} + 1 = \frac{1}{6}q(q+8)(q-1) + 1.$$
But this is too small to satisfy $b = \frac{v(v-1)}{k(k-1)}$ hence we have a contradiction.

- $G_{\mathfrak{L}} = \hat{}((q-1)\times(q-1)) : 2$. Then $G_{\mathfrak{L}}$ contains $q+2$ involutions and we have that,

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{1}{2}q(q+2)(q-1)+1.$$

  Once again this is too small to satisfy $b = \frac{v(v-1)}{k(k-1)}$.

Hence we may conclude that no line-transitive actions exist with primes dividing $q+1$ significant.

7 $G_\alpha = \hat{}(q-1)^2 : S_3$

In this case $v = \frac{1}{6}q^3(q+1)(q^2+q+1)$ and any significant prime t must divide into $q-1$.

Note first that, by using Lemma 18, we can assume that $q > 25$ and that $q \neq 31$, 37, 43, 49, 64, 109 or 271. Furthermore a conjugate of D lies in G_α and D is normalized by $\hat{}GL(2,q)$. Thus, by Lemma 15, a conjugate of D lies inside $G_{\mathfrak{L}}$. We split into three cases:

- **Case A**: A G-conjugate of D is normal in $G_{\mathfrak{L}}$ and $G_{\mathfrak{L}}$ contains no other G-conjugates of D;
- **Case B**: A G-conjugate of D is normal in $G_{\mathfrak{L}}$ and $G_{\mathfrak{L}}$ contains other G-conjugates of D. Thus $|G_{\mathfrak{L}}|$ is divisible by $(\frac{q-1}{\mu})^2$ and so b divides into $6\mu v$;
- **Case C**: All G-conjugates of D in $G_{\mathfrak{L}}$ are non-normal in $G_{\mathfrak{L}}$.

We examine these possibilities in turn.

7.1 Case A

In this case we know, by Lemma 6, that either $\hat{}GL(2,q)$ acts line-transitively on the linear-space which is the fixed set of D or all fixed points of D lie on a single line. The first possibility cannot occur however as this would correspond to $PGL(2,q)$ acting line-transitively on a linear-space (possibly having $k=2$ and so being a 2-homogeneous action) with line-stabilizer a dihedral group of size $2(q-1)$ which is impossible. Hence we may assume that all fixed points of D lie on a single line. There are $\frac{1}{2}q(q+1)$ of these.

If E is some other conjugate of D then $E \cap G_\alpha$ is of size at most 2. We conclude that $k = \frac{1}{2}q(q+1) + n\frac{q-1}{2\mu}$ for some integer n. This implies that $k-1$ is divisible by $\frac{q-1}{2\mu}$. Now, since $v - 1 = \frac{q-1}{2}\frac{q^5+3q^4+5q^3+6q^2+6q+6}{3}$, we observe that $b\big|(q^5+3q^4+5q^3+6q^2+6q+6)v$. Now, for p odd, $(|G|, q^5+3q^4+5q^3+6q^2+6q+6)$ is a power of 3, hence 3 is the only significant prime and $3|q-1$. For $p=2$, $(|G|, q^5+3q^4+5q^3+6q^2+6q+6)$ is divisible, at most, by the primes 2 and 3. However we know that 2 is not a significant prime here thus, again, 3 is the only significant prime. Note that $q^5+3q^4+5q^3+6q^2+6q+6$ is divisible by 27 if and only if $q \equiv 28(81)$. Thus, if 3^a is the highest power of 3 in $q-1$ then $a \neq 3$ implies that $b|27v$. If $a=3$ then we know already that $b|81v$.

This case will be completed below.

7.2 Case A and B

Now we examine the remaining possibilities of Case A along with Case B. Thus $G_{\mathfrak{L}} < \hat{}GL(2, q)$ and one of the following holds:

- $q \equiv 28(81)$, $\frac{2(q-1)^2}{81}$ divides into $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ contains precisely one G-conjugate of D;
- $q \equiv 1(3)$, $\frac{2(q-1)^2}{27}$ divides into $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ contains precisely one G-conjugate of D;
- $\frac{(q-1)^2}{\mu^2}$ divides into $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ contains more than one G-conjugate of D.

Observe also that $k(k-1) = \frac{v(v-1)}{b}$ is even and that

$$|v(v-1)|_2 = \frac{(q,2)}{4}|q^3(q+1)(q-1)|_2.$$

Thus if p is odd then we need $|G_{\mathfrak{L}}|$ divisible by $8(q-1)/\mu$.

Suppose that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2, q)$ of type 6 or 7. Since $q > 25$, Lemma 24 implies that $G_{\mathfrak{L}}$ contains at most one conjugate of D. Thus $\frac{2(q-1)}{9}$ must divide 24 or 60 or $\frac{2(q-1)}{27}$ divides 24 or 60 and $q \equiv 28(81)$. The prime powers we need to check are, therefore, 13, 19, 31, 37, 109 and 271. These cases are already all excluded.

If $G_{\mathfrak{L}}$ lies inside a group of type 3 then $G_{\mathfrak{L}}$ contains at most one conjugate of D and either $q \cong 28(81)$ and $\frac{2(q-1)}{27}$ divides into 4 or $\frac{2(q-1)}{9}$ divides into 4. Both yield values for q which are less than 25 and so can be excluded.

Suppose that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2, q)$ of type 5, $G_{\mathfrak{L}} \cong \hat{} < SL(2, q_0), V >$. Then $\frac{(q-1)^2}{81}$ divides into $2q_0(q_0^2 - 1)\frac{q_0-1}{3}$ and so $q - 1$ divides into $54(q_0^2 - 1)$. For $q \geq q_0^3$ we find that this is impossible for $q_0 > 2$. If $q_0 = 2$ then $q < 32$ and so all cases have been excluded. For $q = q_0$, $|G_{\mathfrak{L}}| < |G_\alpha|$ implies a contradiction. For $q = q_0^2$, $|G_{\mathfrak{L}}| < |G_\alpha|$ implies that $\sqrt{q} \leq 5$ and all possibilities have been excluded.

Suppose that $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $\hat{}GL(2, q)$ and not of type 4. Then $|G_{\mathfrak{L}}|$ is divisible by p for $q = p^a$, integer a. If $|G_{\mathfrak{L}}|$ is divisible by $\frac{(q-1)^2}{\mu^2}$ then $G_{\mathfrak{L}}$ has orbits on the non-identity elements of its normal p-Sylow subgroup divisible by $\frac{q-1}{\mu}$. Thus $G_{\mathfrak{L}}$ contains the entire Sylow p-subgroup of $\hat{}GL(2, q)$ and $|G_{\mathfrak{L}}| \geq q\frac{(q-1)^2}{\mu^2}$; this implies that $q < 6\mu$ which is impossible. So assume that $3\big|(q-1)$ is the only significant prime. If $\frac{2(q-1)^2}{81}$ divides into $|G_{\mathfrak{L}}|$ we must have $p = 2$ and $G_{\mathfrak{L}} = \hat{}A : B$ where A is a non-trivial 2-group. Then $q \geq 2^a$ and $q - 1$ has a primitive prime divisor s greater than 3 and $\frac{s(q-1)}{3}$ divides into $|B|$. Then B acts on the non-identity elements of A by conjugation with orbits of size divisible by s and so $|A| = q$. Thus $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)s}{3}$ which means s must be 5 and so $q = 16$. This is already excluded.

We are left with the possibility that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2, q)$ of type 4. If 2 is significant then p is odd and $G_{\mathfrak{L}}$ contains at most 3 involutions since $G_{\mathfrak{L}} \leq \hat{}(q-1)^2$. By Lemma 16 we know that $k \leq \frac{3v}{n} + 1 = \frac{1}{2}q(q+1) + 1$. This is inconsistent with our value for b. If 2 is not significant then $|G_{\mathfrak{L}}| = 2|D|e$ where e is a constant dividing $q - 1$. Then the number of involutions in $G_{\mathfrak{L}}$ is at most $e + 3$. We appeal to Lemma 16 to conclude that,

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{(e+3)(q+1)q}{6} + 1.$$

Thus,

$$\frac{3(q-1)}{e} = \frac{b}{v} = \frac{v-1}{k(k-1)} \geq \frac{6(q^6+2q^5+2q^4+q^3-6)}{(e+3)^2q^2(q^2+3q+2)} > \frac{6q^2}{(e+3)^2}.$$

This implies that $\frac{(e+3)^2}{e} > 2q$ and so $e+15 > 2q$. Since $e < q$ this must mean that $q < 15$ which is a contradiction.

7.3 Case C

Finally we consider the possibility that no conjugate of D is normal in $G_{\mathfrak{L}}$. We must have at least two conjugates of D in $G_{\mathfrak{L}}$ and so $|G_{\mathfrak{L}}| > \frac{(q-1)^2}{\mu^2}$.

Suppose first that $G_{\mathfrak{L}}$ lies in a parabolic subgroup. Then $G_{\mathfrak{L}} = \hat{}A.B$ where A is an elementary abelian p-group, $B \leq GL(2,q)$.

Suppose that A is trivial and refer to Lemma 24. Then $G_{\mathfrak{L}}$ lies in a subgroup of $\hat{}GL(2,q)$ of types 2, 4 or 5. If $G_{\mathfrak{L}}$ lies in a subgroup of type 5 then $G_{\mathfrak{L}} \geq SL(2,q)$ in which case $|G_{\mathfrak{L}}| > |G_\alpha|$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies in a subgroup of $\hat{}GL(2,q)$ of type 4 then conjugates of D in $G_{\mathfrak{L}}$ normalize each other and so $\frac{(q-1)^2}{\mu^2}$ divides into $|G_{\mathfrak{L}}|$. In this case some conjugate of D must be normal in $G_{\mathfrak{L}}$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies in a subgroup of $\hat{}GL(2,q)$ of type 2 then we must have p dividing $|G_{\mathfrak{L}}|$ otherwise all conjugates of D are normal in $G_{\mathfrak{L}}$. But then some conjugate of D acts by conjugation on the non-trivial elements of the normal p-subgroup with orbits of size $\frac{q-1}{\mu}$. Thus q divides $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ has a normal subgroup Q of size q. We will deal with this situation at the end of the section.

Thus A is non-trivial. Suppose that all conjugates of D in $G_{\mathfrak{L}}$ centralize all elements of A. Then these conjugates lie in a subgroup of order $q^3(q-1)/\mu$. Now if $G_{\mathfrak{L}} \cap C_G(A)$ only contains p-elements centralized by D then $G_{\mathfrak{L}} \cap C_G(A)$ contains only one conjugate of D. By our supposition this means that $G_{\mathfrak{L}}$ contains only one conjugate of D which is a contradiction. Thus $G_{\mathfrak{L}} \cap C_G(A)$ contains p-elements not centralized by D. Then the normal p-subgroup of $G_{\mathfrak{L}} \cap C_G(A)$ has size $|A| + n\frac{q-1}{\mu}$ for some n. Thus $G_{\mathfrak{L}} \geq Q : D$ for a p-group Q of size at least q.

If a conjugate of D in $G_{\mathfrak{L}}$ does not act trivially in its action on elements of A then A must be of order divisible by q. Once again $G_{\mathfrak{L}} \geq Q : D$ where $|Q| \geq q$. We deal with this situation at the end of the section.

Now suppose that $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 2. In order for there to be two conjugates, D and E, of D in $G_{\mathfrak{L}}$ we must have D, E in $\hat{}(q-1)^2$. Hence $\frac{(q-1)^2}{\mu^2}||G_{\mathfrak{L}}|$. For D, E to be non-normal, we must have $G_{\mathfrak{L}} \geq (\frac{q-1}{\mu} \times \frac{q-1}{\mu}) : 3$. If 2 is significant then p is odd and $G_{\mathfrak{L}} \leq \hat{}(q-1)^2 : 3$ and $G_{\mathfrak{L}}$ contains at most 3 involutions. By Lemma 16, we know that $k \leq \frac{3v}{n} + 1 = \frac{1}{2}q(q+1)+1$. This is inconsistent with our value for b. If 2 is not significant then $G_{\mathfrak{L}} = (\frac{q-1}{3} \times \frac{q-1}{3}) : S_3$ and $b = 3v$.

When p is odd, $G_{\mathfrak{L}}$ contains at most $q+2$ involutions and, by Lemma 16, this implies that $k \leq \frac{(q+2)v}{q^2(q^2+q+1)} + 1$. We therefore conclude that

$$k(k-1) \leq \frac{q(q+1)(q+2)(q+3)(q^2+2)}{36}.$$

However this implies that $\frac{b}{v} = \frac{v-1}{k(k-1)} > 4$ which is a contradiction.

When $p = 2$, $G_{\mathfrak{L}}$ contains at most $q-1$ involutions and we find that $k(k-1) \leq \frac{1}{36}q^3(q^3+6)$. Once again $\frac{b}{v} = \frac{v-1}{k(k-1)} > 4$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 4 or 5 then we have two possibilities. If $G_{\mathfrak{L}} = A_6.2$ or A_7 then, in order to satisfy $|G_{\mathfrak{L}}| > \frac{(q-1)^2}{\mu^2}$, we must have $q = 25$. This has already been excluded. If $G_{\mathfrak{L}}$ contains a subgroup of index less than or equal to 3 isomorphic to $PSU(3,q_0)$ or $PSL(3,q_0)$ where $q = q_0^a$ then we require that $q_0^3(q_0^2-1)(q_0^3-1) < 6(q-1)^2$. Thus we need $q \geq q_0^4$. This implies that either $\frac{q-1}{\mu}$ does not divide into $|G_{\mathfrak{L}}|$ or that $q = 64$. Both cases give contradictions.

If $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 6,7,8 or 10 then $\frac{(q-1)^2}{\mu^2} < 360$. This implies that $q \leq 19$ or $q \equiv 1(3)$ and $q \leq 49$. All of these cases have been excluded already.

If $G_{\mathfrak{L}}$ is in a group of type 9 then $|G_{\mathfrak{L}}| < |G_\alpha|$ implies that $G_{\mathfrak{L}}$ is a proper subgroup. Since $|G_{\mathfrak{L}}| > \frac{(q-1)^2}{\mu^2}$ we must have $G_{\mathfrak{L}} \leq [q] : (q-1)$. Thus $G_{\mathfrak{L}} = A : B$ where $A \leq [q]$, $B \leq (q-1)$. All conjugates of B in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and B contains a conjugate of D. Thus $\frac{q-1}{\mu}$ divides into $|B|$. Since B acts semi-regularly on the non-trivial elements of A this means that $|A| = q$. Once more we conclude that $G_{\mathfrak{L}}$ has a normal subgroup of order q.

We have reduced all cases to the situation where $G_{\mathfrak{L}} \geq Q : D$ where Q is a p-group of order divisible by q. Observe that all conjugates of D in $Q : D$ are $G_{\mathfrak{L}}$ conjugate. If $G_{\mathfrak{L}}$ contains E, another G-conjugate of D which is not $G_{\mathfrak{L}}$-conjugate, then $E \cap (Q : D)$ is trivial; hence $|G_{\mathfrak{L}}| \geq \frac{q(q-1)^2}{\mu^2}$ which is too large. Thus all G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and we can apply Lemma 6 as in Case A. As in Case A this implies that 3 is uniquely significant and either $2\frac{(q-1)^2}{81} \big| |G_{\mathfrak{L}}|, q \equiv 28(81)$ or $2\frac{(q-1)^2}{27} \big| |G_{\mathfrak{L}}|, q \equiv 1(3)$. If p is odd then this means that either $q < 81$ and $q \equiv 28(81)$ or $q < 27$ and $q \equiv 1(3)$. If $p = 2$ then this means that either $q < 162$ and $q \equiv 28(81)$ or $q < 54$ and $q \equiv 1(3)$. All such possibilities have already been excluded.

Hence we may conclude that no new line-transitive action of $PSL(3,q)$ exists where $G_\alpha = \hat{}(q-1)^2 : S_3$.

Remark. *The argument in this section deals with Case II in our analysis of significant primes.*

8 Case III: $3|q-1$ is uniquely significant

In this case G_α lies inside a subgroup of $PSL(3,q)$ of type 2, 4, 5 or 8.

8.1 Case 1: G_α is a proper subgroup of a group of type 2

Then $G_\alpha = A.B$ where $B = C_3$ or S_3 and $A = \hat{}(u \times u)$ (this structure for A follows since it is normalized by C_3.) We can conclude, using Corollary 10, that $B = S_3$. Now observe that $A.2$ lies inside a copy of $\hat{}GL(2,q)$, hence is centralized by $Z(\hat{}GL(2,q))$. Thus, by Lemma 15, $A.2$ lies in $G_{\mathfrak{L}}$. Thus $|G_{\mathfrak{L}}| = 2|A|$ or $|G_{\mathfrak{L}}| = 4|A|$ while $b\big|3v$. When $p = 2$ we know that $v-1$ is odd. Since $k(k-1)$ is even and $\frac{b}{v} = \frac{v-1}{k(k-1)}$, this means that $|G_{\mathfrak{L}}| = 4|A|$ and $b = \frac{3}{2}v$.

Consider first the case where $b = \frac{3}{2}v$. Then $\frac{b}{v} = \frac{3}{2} = \frac{v-1}{k(k-1)}$ and so

$$k(k-1) = \frac{2}{3}(v-1) = \frac{1}{9u^2}[q^8 - q^6 - q^5 + q^3 - 6u^2].$$

Now observe that, for $q > 8$,

$$[\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)+\frac{1}{2}][\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)-\frac{1}{2}] > \frac{2}{3}(v-1);$$

$$[\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)\frac{1}{3}][\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)-\frac{2}{3}] < \frac{2}{3}(v-1).$$

Since $\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q) = \frac{1}{6}a$ for some integer a, this is a contradiction. Thus p is odd and $b = 3v$.

Now suppose that 4 does not divide into u. Then $|G_\alpha|_2 \leq 8$ while $|G|_2 \geq 16$, hence $v-1$ is odd. This implies that $|b|_2 < |v|_2$ which is a contradiction. Hence $12|u$.

Now $G_{\mathfrak{L}} = \hat{}(u \times u).2 < \hat{}(q-1)^2 : 2 < \hat{}GL(2,q)$ and so contains at most $u+3$ involutions. We appeal to Lemma 16 to observe that,

$$k \leq \frac{(u+3)q(q+1)(q-1)^2}{6u^2} + 1.$$

We can conclude therefore that, for $u \geq 12$,

$$k(k-1) \leq \frac{q^2(q+1)^2(q-1)^4(u+3)(u+4)}{36u^4}.$$

This is strictly smaller than $\frac{v-1}{3}$ which is a contradiction.

8.2 Case 2: G_α is a subgroup of type 4 or 5

We refer to Lemma 20. Consider first the possibility that G_α is isomorphic to $A_6.2$ or A_7 and $p = 5$. We exclude $q = 25$ using Lemma 18.

Observe that, since 3 divides $q-1$, there is a group of order 3 normal in a group isomorphic to $\hat{}(q-1)^2$. Hence a line-stabilizer contains a subgroup of order 3 or else contains the group $\hat{}(q-1)^2$ (by Lemma 15). The latter possibility is not possible, hence we may assume that $3\big||G_{\mathfrak{L}}|$. We may therefore conclude that $b = 3v$ or $b = \frac{3}{2}v$.

Now suppose that m is an integer dividing v and $b = \frac{3}{x}v$ where x is 1 or 2. We have that

$$\begin{aligned} & \frac{v-1}{k(k-1)} = \frac{3}{x} \\ \implies \quad & 3k(k-1) + x \equiv 0 \ (mod\ m) \\ \implies \quad & 36k^2 - 36k + 12x \equiv 0 \ (mod\ m) \\ \implies \quad & 9(2k-1)^2 \equiv 9 - 12x \ (mod\ m) \end{aligned}$$

Thus $9 - 12x$ is a square modulo m and m is not divisible by 3. If $G_\alpha = A_6.2$ then we know that 25 divides v. For both values of x we find that $9-12x$ is not a square modulo 25.

Thus we assume that either $G_\alpha = PSL(3, q_0), q = q_0^a, 3\big|q_0 - 1, a \not\equiv 0(mod\ 3)$; or $G_\alpha = PSU(3, q_0), q = q_0^a, 3\big|q_0 + 1, a \not\equiv 0(mod\ 6)$.

Then in the first instance we have a subgroup of G_α, $\hat{}(q_0 - 1)^2$; in the second instance we have a subgroup of G_α, $\hat{}(q_0 + 1)^2$. Such subgroups are normal in the subgroup of $PSL(3, q)$, $\hat{}(q - 1)^2$. Thus these subgroups of G_α lie in $G_\mathfrak{L}$ and we may conclude that $b\big|3v$. Once again when $p = 2$ we know that $v - 1$ is odd and so $b = \frac{3}{2}v$.

We know that $q_0^3\big||G_\mathfrak{L}|$, hence $G_\mathfrak{L}$ is not a subgroup of a group of type 2,3,6,7,8 or 10. If $G_\mathfrak{L}$ is a subgroup of a group of type 9 then $\frac{(q_0^3 \pm 1)}{3}\big|(q^2 - 1)$. Since $q = q_0^a, a \not\equiv 0(3)$ we must have $q_0 = 2$ and $G_\alpha = PSU(3, 2)$. But then $|G_\alpha| = 72$ which is the same size as in Case 1 with $u = 6$. The arguments given there exclude both $b = 3v$ and $b = \frac{3}{2}v$.

If $G_\mathfrak{L}$ is only a subgroup of a group of type 4 or 5 then either $G_\mathfrak{L} = A_6.2$ or A_7 (and 25 divides into v which is a contradiction), or $G_\mathfrak{L}$ is one of $PSL(3, q_1)$ or $PSU(3, q_1)$. Since $b|3v$ we must have $q_0 = q_1$ and $\frac{q_0^3+1}{q_0^3-1}$ equal to 3 or $\frac{3}{2}$. This is impossible.

Thus $G_\mathfrak{L}$ is a subgroup of a parabolic subgroup. Then we require that $(q_0^3 \pm 1)\big|(q^2 - 1)(q - 1)$. This implies that $q_0 = 2$ which can be excluded as in Case 1 setting u to be 6.

8.3 Case 3: G_α is a maximal subgroup of type 8

Note that p is odd here and, using Lemma 18, $q \geq 43$. Here $G_\alpha \cong 3^2.Q_8$ and $|q-1|_3 = 3$. Observe that, since 3 divides $q-1$, there is a group of order 3 normal in a group isomorphic to $\hat{}(q - 1)^2$ and so, by Lemma 15, $3 \leq G_\mathfrak{L}$. Thus $b\big|3v$. Now G_α has the same size as G_α in Case 1 with $u = 6$. The arguments given there exclude both $b = 3v$ and $b = \frac{3}{2}v$ and we are done.

Thus we have ruled out all possible actions of line-transitive actions of $PSL(3, q)$ where 3 is the unique significant prime.

9 Case IV: $2|q - 1$ is uniquely significant

In this case G_α either lies in a parabolic subgroup or in a subgroup of $PSL(3, q)$ of type 2 or 4. Since D normalizes a Sylow 2-subgroup of $PSL(3, q)$, we know that G_α contains D for some α. Furthermore, by Lemma 15, either $G_\alpha \geq \hat{}GL(2, q)$ or $D < G_\mathfrak{L}$.

9.1 Case 1: G_α is a subgroup of a group of type 4 only

In this case $G_\alpha = PSL(3, q_0)$ or $PSL(3, q_0).3$ for some q_0 where $q = q_0^a, a$ odd. Then $D < G_\mathfrak{L}$ and so $\frac{q-1}{\mu}$ divides into $3|PSL(3, q_0)|$. We must have $q = q_0^3$. But then $PSL(3, q_0)$ does not contain an element of order $\frac{q_0^3-1}{\mu}$ and so $D \not< PSL(3, q_0)$ and this case is also excluded.

9.2 Case 2: G_α lies inside a group of type 2

Here G_α is non-maximal, $q \equiv 1(4)$ and G_α contains a cyclic subgroup of order $q - 1/\mu$. We have two possibilities:

1. $G_\alpha = A : 2$ where $A \leq \hat{}(q-1)^2$ and $|A| = a\frac{q-1}{\mu}$. Then A is proper normal in $\hat{}(q-1)^2$ for $a < q-1$ and proper normal in $\hat{}(q-1)^2 : S_3$ for $a = q-1$. Thus we may conclude, by Lemma 15, that $G_{\mathfrak{L}} = A$. We can conclude that $G_{\mathfrak{L}}$ contains at most 3 involutions.

2. We suppose that $3|(q-1)$ and $G_\alpha = (\frac{q-1}{3} \times \frac{q-1}{3}) : S_3$. In this case, $(\frac{q-1}{3} \times \frac{q-1}{3})$ is normal in $\hat{}(q-1)^2$ and hence lies in $G_{\mathfrak{L}}$. We can conclude that $|G_{\mathfrak{L}}| = 3(\frac{q-1}{3})^2$ and $G_{\mathfrak{L}}$ contains at most 9 involutions.

Consider the first case. Since $G_{\mathfrak{L}}$ contains at most 3 involutions, we may appeal to Lemma 16 to give,

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{3q(q+1)(q-1)}{2a} + 1.$$

This implies that,

$$k(k-1) < \frac{9}{4a^2} q^3 (q+1)^2 (q-1).$$

Now we know that $k(k-1) = \frac{v-1}{2}$. Thus

$$\frac{v-1}{2} = \frac{q^3(q^2+q+1)(q+1)(q-1) - 2a}{4a} < \frac{9}{4a^2} q^3 (q+1)^2 (q-1).$$

Hence $q < \frac{9}{a}$ which is impossible.

We move on to the next possibility: $H = (\frac{q-1}{3} \times \frac{q-1}{3})$ lies inside $G_{\mathfrak{L}}$ with index 3. Now H contains 3 involutions, hence $G_{\mathfrak{L}}$ must contain at most 9 involutions. Once again we appeal to Lemma 16 to give,

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{9q(q+1)}{2} + 1.$$

This gives,

$$\frac{v-1}{2} = k(k-1) < \frac{41q^2(q+1)^2}{2}.$$

Given our value for v we may conclude that,

$$q^3(q^2+q+1)(q+1) - 2 < 41q^2(q+1)^2.$$

This is only true for $q \leq 7$ which is impossible.

9.3 Case 3: G_α lies in a parabolic subgroup

Now, for P a parabolic subgroup, $|G : P| = q^2 + q + 1$. By Lemma 9 this means that any significant prime must divide $\frac{1}{2}q(q+1)$. Since 2 is uniquely significant, we may conclude that $q \equiv 3(4)$ and $b\big|\frac{1}{2}(q+1)v$. We write $G_\alpha = A.B$ where A is an elementary abelian p-group and $B \leq \hat{}GL(2,q)$.

Suppose $q \equiv 3(8)$. Then, by Lemma 9, $b = 2v$. Then, by Lemma 3, any prime m dividing into v must be equivalent to $1(4)$. Since $p \equiv 3(4)$ we have q^3 dividing into $|G_\alpha|$. Thus $A = [q^2]$ and $B \geq \hat{}SL(2,q)$. However this means that $A.B$ is normal in the full parabolic subgroup. Hence, by Lemma 15, either $G_{\mathfrak{L}} \geq G_\alpha$ (which is impossible) or G_α is the full parabolic subgroup. This case has already been excluded.

Thus $q \equiv 7(8)$ and B is a subgroup of $\hat{\ }GL(2,q)$ of type 3 or 5. Consider the case where B is a subgroup of $\hat{\ }GL(2,q)$ of type 3. We examine the possible situations here:

1. Suppose that B is maximal in $\hat{\ }GL(2,q)$, i.e. $|B| = 2(q^2-1)/\mu$. Then B acts by conjugation on the non-trivial elements of A with orbits divisible by $q+1$. Thus $|A| = q^2$ or 1. Since 2 is uniquely significant, $A < G_{\mathfrak{L}}$. This is the same situation as in Subsections 6.1 and 6.3; precisely the same arguments as in those sections allow us to exclude the situation here.

2. Suppose that B is non-maximal in $\hat{\ }GL(2,q)$. Then B contains a cyclic group C which is normal in $\hat{\ }(q^2-1)$, hence lies in $G_{\mathfrak{L}}$. Furthermore $|A|\big||G_{\mathfrak{L}}|$ since 2 is uniquely significant. Thus $|G_{\mathfrak{L}}| = |A|.|C|$ and $G_\alpha = 2|A|.|C|$ and so $b = 2v$. However in this case, by Lemma 3, any prime m dividing into v must be equivalent to $1(4)$. Here though $p \equiv 3(4)$ and p divides into v. This is a contradiction.

Now consider the possibility that B is of type 5. Since $q \equiv 7(8)$, we must have $q = p^a$ where a is odd and so $B = \hat{\ } < SL(2,q_0), V >$.

Suppose first that $q = q_0$ and so $B \geq \hat{\ }SL(2,q)$ and either A is trivial or $A = [q^2]$.

If A is trivial then either $B \lhd \hat{\ }GL(2,q)$ or $B = \hat{\ }GL(2,q)$. The first option implies that $G_{\mathfrak{L}} \geq G_\alpha$ (which is impossible). The latter option is the same as in Subsection 6.2; precisely the same arguments as in that section allow us to exclude the situation here.

If on the other hand A is non-trivial then $A = [q^2]$ and so G_α is either the full parabolic subgroup (this possibility is already excluded) or G_α is normal in the full parabolic subgroup and $G_{\mathfrak{L}} \geq G_\alpha$ (which is impossible). Thus both possibilities are excluded when $q = q_0$. We assume that $q = q_0^a$, a is odd, $a \geq 3, p \equiv 7(8)$ and $D < G_{\mathfrak{L}}$.

Now observe that $A. < V >$ is a split extension by Schur-Zassenhaus. So we can take V to be in G_α. Furthermore G_α must contain a conjugate of D. Then, since $q \geq q_0^3$, $< V > \cong \frac{q-1}{\mu}$ is G-conjugate to D. The G-conjugates of D split into two conjugacy classes inside the parabolic subgroup with centralizers isomorphic to $\hat{\ }[q] : (q-1)^2$ and $\hat{\ }GL(2,q)$. If we factor out the unipotent subgroup of the maximal parabolic then we see that, in G_α/A, $< V > A$ is centralized by $SL(2,q_0)$ and so $< V >$ must be centralized in the maximal parabolic by $\hat{\ }GL(2,q)$. This means that $< V >$ acts by conjugation on the non-identity elements of A with orbits of size $\frac{q-1}{\mu}$. In fact B has orbits of length a multiple of $\frac{(q_0+1)(q-1)}{\mu}$ on the non-trivial elements of A. Thus $|A| = q^2$ or $|A| = 1$.

Now note that, since $b\big|\frac{1}{2}v(q+1)q$, we know that $\frac{2q_0(q_0-1)(q-1)}{\mu}\big||G_{\mathfrak{L}}|$. Thus $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 1 or 4.

If $G_{\mathfrak{L}}$ lies in a subgroup of $PSL(3,q)$ of type 9 then $G_{\mathfrak{L}} = SO(3,q)$. If A is trivial then $|G_{\mathfrak{L}}| > |G_\alpha|$ which is a contradiction. If A is non-trivial then q^2 divides into $|G_{\mathfrak{L}}|$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies in a subgroup of $PSL(3,q)$ of type 4 then $G_{\mathfrak{L}} = PSL(3,q_1)$ or $PSL(3,q_1).3$. Since $D < G_{\mathfrak{L}}$ we must have $q \leq q_1^2$. But $q = p^a$ where a is odd which is a contradiction.

Thus $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $PSL(3,q)$. So $G_{\mathfrak{L}} = A_1.B_1$ where A_1 is elementary abelian and $B_1 \leq \hat{\ }GL(2,q)$. Then $\frac{2(q_0-1)(q-1)}{\mu}$ divides into $|B_1|$ and B_1 is of type 4, 5, 6 or 7.

If B_1 is of type 5 then we must have $B_1 \geq SL(2, q_0)$. Since $D < A_1.B_1$ we require that B_1 contains a cycle of length $\frac{q-1}{2\mu}$ and so $B_1 \geq < SL(2, q_0), \frac{q-1}{2\mu} >$. If A is trivial then $|B_1| \geq \frac{1}{2}|G_\alpha|$ which is a contradiction. If $A = [q^2]$ then A_1 must be non-trivial and B_1 has orbits on the non-trivial elements of A_1 of size a multiple of $\frac{(q_0+1)(q-1)}{\mu}$. Thus $|A_1| = q^2$ and $|G_\mathfrak{L}| \geq \frac{1}{2}|G_\alpha|$. By Lemma 3, $p \equiv 1(4)$ which is a contradiction.

If B_1 is of type 4, 6 or 7 then q_0 divides into $|A_1|$ and $G_\mathfrak{L} = A_1.B_1$ is a split extension. Furthermore A is trivial since $q^2 q_0$ cannot divide into $|G_\mathfrak{L}|$.

In the case of types 6 and 7, B_1 must centralize EA_1 in $G_\mathfrak{L}/A_1$ where E is a conjugate of D. Thus E has an orbit on the non-trivial elements of A_1 of size a multiple of $\frac{(q-1)}{\mu}$. Thus $|A_1| \geq q$. But $|G_\alpha| < q_0^3 \frac{q-1}{\mu}$ and $|G_\mathfrak{L}| > q\frac{q-1}{\mu}$ which is impossible.

We are left with the possibility that B_1 is of type 4 and take D to be in $G_\mathfrak{L}$. Suppose first that DA_1 is central in $B_1 = G_\mathfrak{L}/A_1$. Since $q+1$ does not divide into b, $|B_1|_2 \geq 2|(q-1)^2|_2$. This implies that D is centralized in the full parabolic by $\hat{\ }GL(2, q)$ and D has orbits on A_1 of size a multiple of $\frac{(q-1)}{\mu}$. If, on the other hand, DA_1 is not central in $B_1 = G_\mathfrak{L}/A_1$ then it is not normal either and $|B_1|$ is divisible by $2(\frac{q-1}{\mu})^2$ Then $G_\mathfrak{L}$ has orbits on the non-trivial elements of A_1 of size a multiple of $\frac{(q-1)}{\mu}$. Thus in either case $|A_1| \geq q$. But $|G_\alpha| < q_0^3 \frac{q-1}{\mu}$ and $|G_\mathfrak{L}| > q\frac{q-1}{\mu}$ which is impossible.

This deals with all the cases where 2 is a uniquely significant prime. We conclude that $PSL(3, q)$ has no line-transitive actions in this case.

We have now dealt with all possibilities for line-transitive actions of $PSL(3, q)$ on finite linear spaces. Our proof of Theorem A is complete.

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