The valuations of the near octagon $\mathbb{G}_4$

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Abstract

In [4] it was shown that the dual polar space $DH(2n-1,4)$, $n \geq 2$, has a sub near-$2n$-gon $\mathbb{G}_n$ with a large automorphism group. In this paper, we classify the valuations of the near octagon $\mathbb{G}_4$. We show that each such valuation is either classical, the extension of a non-classical valuation of a $\mathbb{G}_3$-hex or is associated with a valuation of Fano-type of an $\mathbb{H}_3$-hex. In order to describe the latter type of valuation we must study the structure of $\mathbb{G}_4$ with respect to an $\mathbb{H}_3$-hex. This study also allows us to construct new hyperplanes of $\mathbb{G}_4$. We also show that each valuation of $\mathbb{G}_4$ is induced by a (classical) valuation of the dual polar space $DH(7,4)$.

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1 Introduction

1.1 Basic definitions

Let $\mathcal{S}$ be a *dense near $2n$-gon*, i.e. $\mathcal{S}$ is a partial linear space which satisfies the following properties:

(i) For every point p and every line L, there exists a unique point $\pi_L(p)$ on L nearest to p. Here, distances $\mathrm{d}(\cdot,\cdot)$ are measured in the point graph or collinearity graph of $\mathcal{S}$.

(ii) Every line of $\mathcal{S}$ is incident with at least three points.

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(iii) Every two points of $\mathcal{S}$ at distance 2 from each other have at least two common neighbours.

(iv) The maximal distance between two points of $\mathcal{S}$ is equal to n.

A dense near 0-gon is a point, a dense near 2-gon is a line and a dense near quadrangle is a generalized quadrangle (Payne and Thas [10]).

For every point y of $\mathcal{S}$ and every non-empty set X of points, we define $\mathrm{d}(y, X) := \min\{\mathrm{d}(x, y) \,|\, x \in X\}$. If X is a non-empty set of points of $\mathcal{S}$, then for every $i \in \mathbb{N}$, $\Gamma_i(X)$ denotes the set of points y of $\mathcal{S}$ at distance i from X. If X is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(X)$.

One of the following two cases occurs for two lines K and L of $\mathcal{S}$ (see e.g. [5, Theorem 1.3]): (i) there exist unique points $k^* \in K$ and $l^* \in L$ such that $\mathrm{d}(k, l) = \mathrm{d}(k, k^*) + \mathrm{d}(k^*, l^*) + \mathrm{d}(l^*, l)$ for all $k \in K$ and $l \in L$; (ii) the map $K \to L; x \mapsto \pi_L(x)$ is a bijection and its inverse is equal to the map $L \to K; y \mapsto \pi_K(y)$. If the latter possibility occurs, then K and L are called *parallel*.

By Theorem 4 of Brouwer and Wilbrink [2], every two points x and y of $\mathcal{S}$ at distance $\delta \in \{0, \ldots, n\}$ from each other are contained in a unique convex subspace $\langle x, y \rangle$ of diameter δ. These convex subspaces are called *quads*, respectively *hexes*, if $\delta = 2$, respectively $\delta = 3$. The lines and quads through a given point x of $\mathcal{S}$ define a linear space which is called the *local space at* x. If $X_1, X_2, \ldots, X_k$ are non-empty sets of points, then $\langle X_1, X_2, \ldots, X_k \rangle$ denotes the smallest convex subspace containing $X_1 \cup X_2 \cup \cdots \cup X_k$. A convex subspace F of $\mathcal{S}$ is called *classical* in $\mathcal{S}$ if for every point x of $\mathcal{S}$, there exists a necessarily unique point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. If every quad of $\mathcal{S}$ is classical in $\mathcal{S}$, then $\mathcal{S}$ is a so-called *dual polar space* (Cameron [3]). The near polygon $\mathcal{S}$ is then isomorphic to a geometry Δ whose points and lines are the maximal and next-to-maximal singular subspaces of a given polar space Π (natural incidence). A proper convex subspace F of $\mathcal{S}$ is called *big* in $\mathcal{S}$ if every point of $\mathcal{S}$ has distance at most 1 from F. If F is big in $\mathcal{S}$, then F is also classical in $\mathcal{S}$. If F is big in $\mathcal{S}$ and if every line of $\mathcal{S}$ is incident with precisely 3 points, then we can define a *reflection* $\mathcal{R}_F$ *about* F which is an automorphism of $\mathcal{S}$. If $x \in F$, then we define $\mathcal{R}_F(x) := x$. If $x \notin F$, then $\mathcal{R}_F(x)$ is the unique point on the line $x\pi_F(x)$ different from x and $\pi_F(x)$. Near polygons were introduced by Shult and Yanushka [11]. We refer to (Chapter 2 of) De Bruyn [5] for more background information on (dense) near polygons.

A function f from the point-set of $\mathcal{S}$ to $\mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties:

(V1) $f^{-1}(0) \neq \emptyset$;

(V2) every line L of $\mathcal{S}$ contains a unique point x_L such that $f(x) = f(x_L)+1$

for every point x of L different from x_L;

(V3) every point x of $\mathcal{S}$ is contained in a necessarily unique convex subspace F_x such that the following properties are satisfied for every $y \in F_x$:

(i) $f(y) \leq f(x)$;

(ii) if z is a point collinear with y such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations of dense near polygons were introduced in De Bruyn and Vandecasteele [6]. If f is a valuation of $\mathcal{S}$, then we denote by O_f the set of points with value 0. A quad Q of $\mathcal{S}$ is called *special (with respect to f)* if it contains two distinct points of O_f, or equivalently (see [6]), if it intersects O_f in an ovoid of Q. We denote by G_f the partial linear space with points the elements of O_f and with lines the special quads (natural incidence).

Proposition 1.1 (Proposition 2.12 of [6]) *Let $\mathcal{S}$ be a dense near polygon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a (not necessarily convex) subpolygon of $\mathcal{S}$ for which the following holds:* (1) *F is a dense near polygon;* (2) *F is a subspace of $\mathcal{S}$;* (3) *if x and y are two points of F, then $d_F(x, y) = d_{\mathcal{S}}(x, y)$. Let f denote a valuation of $\mathcal{S}$ and put $m := \min\{f(x) \,|\, x \in \mathcal{P}'\}$. Then the map $f_F : \mathcal{P}' \to \mathbb{N}; x \mapsto f(x) - m$ is a valuation of F.*

Definition. The valuation f_F of F defined in Proposition 1.1 is called the valuation of F *induced* by f.

Examples. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon.

(1) For every point x of $\mathcal{S}$, the map $f_x : \mathcal{P} \to \mathbb{N}; y \mapsto \mathrm{d}(x, y)$ is a valuation of $\mathcal{S}$ which we call a *classical valuation.*

(2) Suppose O is an ovoid of $\mathcal{S}$, i.e. a set of points meeting each line in a unique point. For every point x of $\mathcal{S}$, we define $f_O(x) = 0$ if $x \in O$ and $f_O(x) = 1$ otherwise. Then f_O is a valuation of $\mathcal{S}$ which we call an *ovoidal valuation.*

(3) Let x be a point of $\mathcal{S}$ and let O be a set of points at distance n from x having a unique point in common with every line at distance $n - 1$ from x. For every point y of $\mathcal{S}$, we define $f(y) = \mathrm{d}(x, y)$ if $\mathrm{d}(x, y) \leq n - 1$, $f(y) = n - 2$ if $y \in O$ and $f(y) = n - 1$ otherwise. Then f is a valuation of $\mathcal{S}$ which we call a *semi-classical valuation.*

(4) Suppose $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a convex subspace of $\mathcal{S}$ which is classical in $\mathcal{S}$. Suppose that $f' : \mathcal{P}' \to \mathbb{N}$ is a valuation of F. Then the map $f : \mathcal{P} \to \mathbb{N}; x \mapsto f(x) := \mathrm{d}(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $\mathcal{S}$. We call f the extension of f'.

In the literature, valuations have been used for the following important applications: (i) classification of dense near polygons; (ii) constructions of new hyperplanes of dense near polygons, in particular of dual polar spaces; (iii) classification of certain hyperplanes of dense near polygons, in particular of dual polar spaces; (iv) study of isometric full embeddings between dense near polygons.

We will now define two classes of dense near polygons which will be important throughout this paper.

(I) Let X be a set of size $2n+2$, $n \geq 2$, and let $\mathbb{H}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following incidence structure:

(i) $\mathcal{P}$ is the set of all partitions of X in $n+1$ subsets of size 2;

(ii) $\mathcal{L}$ is the set of all partitions of X in $n-1$ subsets of size 2 and one subset of size 4;

(iii) a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition determined by the point p is a refinement of the partition determined by L.

By Brouwer, Cohen, Hall and Wilbrink [1], see also De Bruyn [5, Section 6.2], $\mathbb{H}_n$ is a dense near $2n$-gon.

(II) Let $H(2n-1, 4)$, $n \geq 2$, denote the hermitian variety $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$ of $\mathrm{PG}(2n-1, 4)$ (with respect to a given reference system). The number of nonzero coordinates (with respect to the same reference system) of a point p of $\mathrm{PG}(2n-1, 4)$ is called the *weight* of p. The maximal and next-to-maximal subspaces of $H(2n-1, 4)$ define a dual polar space $DH(2n-1, 4)$. Let $\mathbb{G}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following substructure of $DH(2n-1, 4)$:

(i) $\mathcal{P}$ is the set of all generators of $H(2n-1, 4)$ containing n points with weight 2;

(ii) $\mathcal{L}$ is the set of all $(n-2)$-dimensional subspaces of $H(2n-1, 4)$ containing at least $n-2$ points of weight 2;

(iii) incidence is reverse containment.

By De Bruyn [4], see also De Bruyn [5, 6.3], $\mathbb{G}_n$ is a dense near $2n$-gon and its above-defined embedding in $DH(2n-1, 4)$ is isometric, i.e. preserves distances.

1.2 The main result

The near octagon $\mathbb{G}_4$ has hexes isomorphic to $\mathbb{G}_3$ and $\mathbb{H}_3$. Every $\mathbb{G}_3$-hex F is big in $\mathbb{G}_4$ and hence every valuation f of F will give rise to a valuation of $\mathbb{G}_4$, namely the extension of f. No $\mathbb{H}_3$-hex is big in F. We will later show (Proposition 6.11) that if f is a valuation of an $\mathbb{H}_3$-hex F such that G_f is a

Fano-plane, then there exists a unique valuation $\overline{f}$ of $\mathbb{G}_4$ such that $O_{\overline{f}} = O_f$. We will call $\overline{f}$ a valuation of *Fano-type* of $\mathbb{G}_4$. In this paper, we classify all valuations of $\mathbb{G}_4$. We will show the following.

Theorem 1.2 (Section 6) *If f is a valuation of $\mathbb{G}_4$, then f is one of the following:*
(1) *f is a classical valuation of $\mathbb{G}_4$;*
(2) *f is the extension of a non-classical valuation in a $\mathbb{G}_3$-hex of $\mathbb{G}_4$;*
(3) *f is a valuation of Fano-type of $\mathbb{G}_4$.*
Each of these valuations is induced by a unique (classical) valuation of $DH(7,4)$.

Notice that all valuations of $DH(7,4)$ are classical by Theorem 6.8 of De Bruyn [5]. In order to describe the valuations of Fano-type of $\mathbb{G}_4$ (see Section 5), we must study the structure of $\mathbb{G}_4$ with respect to an $\mathbb{H}_3$-hex (Section 4). This study allows us to construct a class of hyperplanes of $\mathbb{G}_4$ (Corollary 4.13).

2 The valuations of the near hexagons $\mathbb{G}_3$, $\mathbb{H}_3$, $Q(5,2) \times \mathbb{L}_3$ and $W(2) \times \mathbb{L}_3$

The valuations of the near hexagons $\mathbb{G}_3$, $\mathbb{H}_3$, $Q(5,2) \times \mathbb{L}_3$ and $W(2) \times \mathbb{L}_3$ were determined in De Bruyn and Vandecasteele [7].

There are two types of valuations in $\mathbb{G}_3$: the classical valuations and the non-classical valuations. If f is a non-classical valuation of $\mathbb{G}_3$, then $G_f \cong \overline{W(2)}$, the linear space obtained from the generalized quadrangle $W(2)$ by adding its ovoids as extra lines. Moreover, every point with value 1 is contained in a unique special quad and every $Q(5,2)$-quad of $\mathbb{G}_3$ contains a unique point with value 0.

The near hexagon $\mathbb{H}_3$ has $W(2)$-quads and grid-quads. Every $W(2)$-quad is big in $\mathbb{H}_3$. Every point is incident with precisely 6 lines and every local space is isomorphic to the Fano-plane in which a point has been removed. There are four types of valuations in the near hexagon $\mathbb{H}_3$: the classical valuations, the extensions of the ovoidal valuations in $W(2)$-quads (*valuations of extended type*), the valuations f for which G_f is a line of size 3 (*valuations of grid-type*) and the valuations f for which G_f is a Fano-plane (*valuations of Fano-type*). In the following lemma, we collect some known facts about valuations of grid-type and Fano-type.

Lemma 2.1 ([7]) *(i) Let f be a valuation of grid-type of $\mathbb{H}_3$. Then O_f is an ovoid in a grid-quad Q of $\mathbb{H}_3$. If $d(x, O_f) \leq 2$, then $f(x) = d(x, O_f)$. If $d(x, O_f) = 3$, then $f(x) = 1$.*

(ii) Let f be a valuation of Fano-type of $\mathbb{H}_3$. Then every $W(2)$-quad contains a unique point of O_f and every grid-quad intersects O_f in either the empty set or an ovoid of the grid-quad. If a grid-quad Q is disjoint from O_f, then Q intersects the set of points with value 1 in an ovoid of Q. The 3 special grid-quads through a point $x \in O_f$ partition the set of lines through x.

Lemma 2.2 *Let f be a valuation of Fano-type of $\mathbb{H}_3$. Let Q be a $W(2)$-quad of $\mathbb{H}_3$ and let G_2 and G_3 be two grid-quads of $\mathbb{H}_3$ such that (i) Q, G_2 and G_3 are mutually disjoint, and (ii) $\mathcal{R}_Q(G_2) = G_3$. Put $G_1 := \pi_Q(G_2) = \pi_Q(G_3)$. Then one of the following cases occurs:*

(1) *There exists precisely one $i \in \{2, 3\}$ such that $|G_i \cap O_f| = 3$ and $|G_{5-i} \cap O_f| = 0$. Moreover, the unique point in $O_f \cap Q$ is not contained in G_1.*

(2) *$|G_2 \cap O_f| = |G_3 \cap O_f| = 0$ and the unique point in $O_f \cap Q$ is contained in G_1.*

Proof. We distinguish two cases.

(1) Suppose that the unique point x^* in $O_f \cap Q$ is not contained in G_1. Put $x^{*\perp} \cap G_1 = \{x_1, x_2, x_3\}$ and let L_i, $i \in \{1, 2, 3\}$, denote the unique line through x_i meeting G_2 and G_3. Since $x^* \notin G_1$, $\mathrm{d}(x^*, G_2) = \mathrm{d}(x^*, G_3) = 2$. Hence, every quad through x^* meeting G_2 and G_3 is a grid. Hence, $\langle x^* x_1, L_1 \rangle$, $\langle x^* x_2, L_2 \rangle$ and $\langle x^* x_3, L_3 \rangle$ are the only grid-quads through x^* meeting G_2 (G_3) in a point. These three grid-quads are special with respect to the valuation f. Hence, $|L_1 \cap O_f| = 1$. Choose $i \in \{2, 3\}$ such that $G_i \cap L_1 \cap O_f \neq \emptyset$. Then $|G_i \cap O_f| = 3$. Since every point of $G_1 \setminus \{x_1, x_2, x_3\}$ has value 2, $G_i \cap O_f = (G_i \cap L_1) \cup (G_i \cap L_2) \cup (G_i \cap L_3)$. Hence, none of the points $G_{5-i} \cap L_1$, $G_{5-i} \cap L_2$, $G_{5-i} \cap L_3$ belongs to O_f. Since every point of $G_1 \setminus \{x_1, x_2, x_3\}$ has value 2, no point of G_{5-i} has value 0. So, we have case (1) of the lemma.

(2) Suppose that the unique point x^* in $O_f \cap Q$ is contained in G_1. Suppose y^* is a point of $O_f \cap G_2$. Then since $\mathrm{d}(x^*, y^*) = 2$, y^* is collinear with the unique point z^* of G_2 collinear with x^*. It follows that $\langle x^*, y^* \rangle$ and G_2 are two special grid-quads meeting in a line, a contradiction. Hence, $G_2 \cap O_f = \emptyset$. In a similar way, one shows that $G_3 \cap O_f = \emptyset$. ∎

The near hexagon $Q(5, 2) \times \mathbb{L}_3$ is obtained by taking three isomorphic copies of the generalized quadrangle $Q(5, 2)$ and joining the corresponding points to lines of size 3. There are two types of valuations in $Q(5,2) \times \mathbb{L}_3$: the classical valuations and the extensions of the ovoidal valuations in grid-quads.

The near hexagon $W(2) \times \mathbb{L}_3$ is obtained by taking three isomorphic copies of the generalized quadrangle $W(2)$ and joining the corresponding points to lines of size 3. There are four types of valuations in $W(2) \times \mathbb{L}_3$: the classical valuations, the extensions of the ovoidal valuations in grid-quads, the extensions of the ovoidal valuations in $W(2)$-quads and the semi-classical valuations.

3 Properties of the near octagon $\mathbb{G}_4$

We start with some properties of the near $2n$-gon $\mathbb{G}_n$, $n \geq 3$. Let U denote the set of points of weight 1 and 2 of $\mathrm{PG}(n-1,4)$ (with respect to a certain reference system) and let $\mathcal{L}_U$ denote the linear space induced on the set U by the lines of $\mathrm{PG}(n-1,4)$. Then every local space of $\mathbb{G}_n$ is isomorphic to $\mathcal{L}_U$. Every quad of $\mathbb{G}_n$, $n \geq 3$, is isomorphic to either the (3×3)-grid, $W(2)$ or $Q(5,2)$. The near polygon $\mathbb{G}_n$, $n \geq 3$, has two types of lines:

(i) SPECIAL LINES: these are lines which are not contained in a $W(2)$-quad.

(ii) ORDINARY LINES: these are lines which are contained in at least one $W(2)$-quad.

There are two possible grid-quads in $\mathbb{G}_n$, $n \geq 3$.

(i) GRID-QUADS OF TYPE I: these grid-quads contain three ordinary and three special lines; the lines of each type partition the point set of the grid.

(ii) GRID-QUADS OF TYPE II: these grid-quads contain six ordinary lines. If $n = 3$, then every grid-quad is of type I. If $n \geq 4$, then both types of grid-quads occur.

The automorphism group of $\mathbb{G}_n$, $n \geq 3$, acts transitively on the set of special lines, the set of ordinary lines, the set of $Q(5,2)$-quads, the set of $W(2)$-quads, the set of grid-quads of type I and the set of grid-quads of type II.

The above facts readily follow from De Bruyn [5, Section 6.3]. In the following lemma we collect some properties of the near octagon $\mathbb{G}_4$. The proof is straightforward and we leave it as an exercise to the reader.

Lemma 3.1 (1) *The near octagon $\mathbb{G}_4$ has 8505 points, each line of $\mathbb{G}_4$ contains 3 points and each point of $\mathbb{G}_4$ is contained in 22 lines.*

(2) *Every quad of $\mathbb{G}_4$ is isomorphic to either the (3×3)-grid, $W(2)$ or $Q(5,2)$. Every $Q(5,2)$-quad is classical in $\mathbb{G}_4$.*

(3) *Every hex of* $\mathbb{G}_4$ *is isomorphic to either* $\mathbb{G}_3$, $\mathbb{H}_3$, $W(2)\times\mathbb{L}_3$ *or* $Q(5,2)\times\mathbb{L}_3$. *Every* $\mathbb{G}_3$*-hex is big in* $\mathbb{G}_4$.

(4) *Every point is contained in* 4 *special lines,* 18 *ordinary lines,* 36 *grid-quads of type I,* 27 *grid-quads of type II,* 36 $W(2)$*-quads,* 6 $Q(5,2)$*-quads,* 4 $\mathbb{G}_3$*-hexes,* 18 $Q(5,2)\times\mathbb{L}_3$*-hexes,* 36 $W(2)\times\mathbb{L}_3$*-hexes and* 27 $\mathbb{H}_3$*-hexes.*

(5) *Every special line is contained in* 9 *grid-quads of type I,* 0 *grid-quads of type II,* 0 $W(2)$*-quads,* 3 $Q(5,2)$*-quads,* 0 $\mathbb{H}_3$*-hexes,* 3 $\mathbb{G}_3$*-hexes,* 9 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 9 $W(2)\times\mathbb{L}_3$*-hexes.*

(6) *Every ordinary line is contained in* 2 *grid-quads of type I,* 3 *grid-quads of type II,* 6 $W(2)$*-quads,* 1 $Q(5,2)$*-quad,* 9 $\mathbb{H}_3$*-hexes,* 2 $\mathbb{G}_3$*-hexes,* 4 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 6 $W(2)\times\mathbb{L}_3$*-hexes.*

(7) *Every* $W(2)$*-quad is contained in precisely* 1 $\mathbb{G}_3$*-hex,* 1 $W(2)\times\mathbb{L}_3$*-hex,* 0 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 3 $\mathbb{H}_3$*-hexes.*

(8) *Every* $Q(5,2)$*-quad is contained in precisely* 2 $\mathbb{G}_3$*-hexes,* 3 $Q(5,2)\times\mathbb{L}_3$*-hexes,* 0 $W(2)\times\mathbb{L}_3$*-hexes and* 0 $\mathbb{H}_3$*-hexes.*

(9) *Every grid-quad of type I is contained in* 1 $\mathbb{G}_3$*-hex,* 0 $\mathbb{H}_3$*-hexes,* 1 $Q(5,2)\times\mathbb{L}_3$*-hex and* 3 $W(2)\times\mathbb{L}_3$*-hexes.*

(10) *Every grid-quad of type II is contained in* 0 $\mathbb{G}_3$*-hexes,* 3 $\mathbb{H}_3$*-hexes,* 2 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 0 $W(2)\times\mathbb{L}_3$*-hexes.*

4 Structure of $\mathbb{G}_4$ with respect to an $\mathbb{H}_3$-hex

In this section, H denotes a given $\mathbb{H}_3$-hex of $\mathbb{G}_4$.

Lemma 4.1 *It holds that* $|H| = 105$, $|\Gamma_1(H)| = 3360$, $|\Gamma_2(H)| = 5040$ *and* $|\Gamma_i(H)| = 0$ *for every* $i \geq 3$. *If* $x \in \Gamma_2(H)$, *then there are two possibilities:*

(*a*) $\Gamma_2(x) \cap H$ *is an ovoid in a* $W(2)$*-quad of* H*;*

(*b*) $\Gamma_2(x) \cap H$ *is an ovoid in a grid-quad of* H*.*

Proof. Obviously, $|H| = 105$, $|\Gamma_1(H)| = |H| \cdot 2 \cdot (22-6) = 3360$ and $|\Gamma_i(H)| = 0$ for every $i \geq 4$. Since H does not have ovoids (see e.g. [7, Lemma 5.5]), $\Gamma_3(H) = \emptyset$. Hence, $|\Gamma_2(H)| = 8505 - |H| - |\Gamma_1(H)| = 5040$. If $x \in \Gamma_2(H)$, then the map $g : H \to \mathbb{N}; y \mapsto \mathrm{d}(x,y) - 2$ is a non-classical valuation of H. (Apply Proposition 1.1 to the classical valuation f of $\mathbb{G}_4$ with $O_f = \{x\}$.) By Section 2, there are three possibilities:

(a) $O_g = \Gamma_2(x) \cap H$ is an ovoid in a $W(2)$-quad of H;

(b) $O_g = \Gamma_2(x) \cap H$ is an ovoid in a grid-quad of H;

(c) $O_g = \Gamma_2(x) \cap H$ is a set of 7 points and G_g is a Fano-plane.

We will now show that possibility (c) cannot occur. Suppose the contrary. Let u denote an arbitrary point of O_g and let Q_1, Q_2 and Q_3 denote the three grid-quads of H through u which are special with respect to g. Since Q_i, $i \in \{1,2,3\}$, is contained in an $\mathbb{H}_3$-hex, it is a grid-quad of type II by Lemma 3.1 (9). Now, $\langle x, Q_i \rangle$ is a hex containing a grid-quad Q_i which is not big. It follows that $\langle x, Q_i \rangle$ is isomorphic to either $\mathbb{H}_3$ or $\mathbb{G}_3$. But the latter possibility cannot occur by Lemma 3.1 (10), since Q_i is a grid-quad of type II. Hence, $\langle x, Q_i \rangle \cong \mathbb{H}_3$ and $\langle x, u \rangle$ is a (3×3)-grid. Since $\langle x, u \rangle$ is contained in an $\mathbb{H}_3$-hex, $\langle x, u \rangle$ is a grid-quad of type II. Let L_1 and L_2 denote the two lines of $\langle x, u \rangle$ through u. Since $\langle x, u \rangle$ is a grid-quad of type II, L_1 and L_2 are ordinary lines. Let Q denote the unique $Q(5,2)$-quad through L_1. Then Q intersects H in a line. Take $i \in \{1,2,3\}$ such that $Q \cap H \subseteq Q_i$. Then the hex $\langle x, Q_i \rangle$ contains a $Q(5,2)$-quad, contradicting $\langle x, Q_i \rangle \cong \mathbb{H}_3$. Hence, either possibility (a) or (b) occurs. ∎

Lemma 4.2 *(a) Let $x \in \Gamma_2(H)$ such that $\Gamma_2(x) \cap H$ is an ovoid in a $W(2)$-quad Q of H. Then $\langle x, Q \rangle \cong \mathbb{G}_3$.*

(b) Let $x \in \Gamma_2(H)$ such that $\Gamma_2(x) \cap H$ is an ovoid in a grid-quad Q of H. Then $\langle x, Q \rangle \cong \mathbb{H}_3$.

Proof. (a) The hex $\langle x, Q \rangle$ has a $W(2)$-quad Q which is not big. It follows that $\langle x, Q \rangle \cong \mathbb{G}_3$.

(b) Since the grid-quad Q is contained in an $\mathbb{H}_3$-hex, it is a grid-quad of type II. So, the hex $\langle x, Q \rangle$ has a grid-quad of type II which is not big. As in the proof of Lemma 4.1, it follows that $\langle x, Q \rangle \cong \mathbb{H}_3$. ∎

Definition. A point x of $\Gamma_2(H)$ is said to be of type (a), respectively (b), if case (a), respectively case (b), of Lemma 4.1 (Lemma 4.2) occurs.

Lemma 4.3 *In $\Gamma_2(H)$, there are 3360 points of type (a) and 1680 points of type (b).*

Proof. In a given $\mathbb{G}_3$-hex, there are 120 points at distance 2 from a $W(2)$-quad. There are 28 $W(2)$-quads in $\mathbb{H}_3$ and each such quad is contained in a unique $\mathbb{G}_3$-hex. Hence, the number of points of type (a) in $\Gamma_2(H)$ is equal to $28 \cdot 1 \cdot 120 = 3360$.

In a given $\mathbb{H}_3$-hex, there are 24 points at distance 2 from a given grid-quad. Now, there are 35 grid-quads (of type II) in H and each of these grid-quads is contained in precisely 2 $\mathbb{H}_3$-hexes different from H. Hence, the number of points of type (b) in $\Gamma_2(H)$ is equal to $35 \cdot 2 \cdot 24 = 1680$.

(Check: The total number of points of $\Gamma_2(H)$ is indeed equal to $3360 + 1680 = 5040$ as shown in Lemma 4.1). ∎

Lemma 4.4 (Chapter 7 of [5]) *Suppose one of the following cases occurs: (i) Q is a grid-quad of $\mathcal{S} \cong \mathbb{H}_3$; (ii) Q is a $W(2)$-quad of $\mathcal{S} \cong \mathbb{G}_3$. Let x be a point of $\mathcal{S}$ at distance 2 from Q. Then every line of $\mathcal{S}$ through x has a unique point in common with $\Gamma_1(Q)$.*

Let S denote the set of lines of $\mathbb{G}_4$ contained in $\Gamma_2(H)$.

Lemma 4.5 *(a) Through every point x of type (a) of $\Gamma_2(H)$, there are precisely 10 lines contained in S.*

(b) Through every point x of type (b) of $\Gamma_2(H)$, there are precisely 16 lines contained in S.

Proof. Let x be a point of $\Gamma_2(H)$ and let Q be the quad $\langle \Gamma_2(x) \cap H \rangle$. If x is a point of type (a), then $Q \cong W(2)$ and $\langle x, Q \rangle \cong \mathbb{G}_3$. If x is a point of type (b), then Q is a grid-quad and $\langle x, Q \rangle \cong \mathbb{H}_3$. By Lemma 4.4, every line through x contained in $\langle x, Q \rangle$ contains a unique point of $\Gamma_1(Q)$. Clearly, all remaining lines through x cannot contain points of $\Gamma_1(H)$ and are contained in $\Gamma_2(H)$. So, if x is a point of type (a), then x is contained in 22-12=10 lines of S. If x is a point of type (b), then x is contained in 22-6=16 lines of S. ■

From Lemmas 4.3 and 4.5, we readily obtain:

Corollary 4.6 $|S| = \frac{1}{3}[3360 \cdot 10 + 1680 \cdot 16] = 20160.$

Lemma 4.7 *Let $L = \{x_1, x_2, x_3\}$ be a line of S. For every $i \in \{1, 2, 3\}$, put $Q_i := \langle \Gamma_2(x_i) \cap H \rangle$ and $H_i := \langle x_i, Q_i \rangle$. Then H_1, H_2 and H_3 are mutually disjoint hexes.*

Proof. By symmetry, it suffices to show that $H_1 \cap H_2 = \emptyset$. Suppose that u is a point of $H_1 \cap H_2$. Every point on a shortest path between $u \in H_1 \cap H_2$ and x_1 belongs to H_1. If $x_1 \notin H_2$, then x_2 lies on such a shortest path. Hence, $x_1 \in H_2$ or $x_2 \in H_1$. So, the line x_1x_2 is contained in H_1 or H_2. By Lemma 4.4, L contains a point of $\Gamma_1(H)$, contradicting the fact that $L \in S$. ■

Lemma 4.8 *Let $L = \{x_1, x_2, x_3\}$ be a line of S, put $Q_i = \langle \Gamma_2(x_i) \cap H \rangle$ and $H_i = \langle x_i, Q_i \rangle$. If x_1 is of type (a), then x_2 and x_3 have the same type and $\mathcal{R}_{H_1}(H_2) = H_3$.*

Proof. By Lemma 4.2, $Q_1 \cong W(2)$ and $H_1 \cong \mathbb{G}_3$. So, H_1 is big in $\mathbb{G}_4$. By Lemma 4.7, H_1 and H_2 are mutually disjoint. Let H_3' be the reflection of H_2 about H_1 (in the near octagon $\mathbb{G}_4$) and let Q_3' denote the reflection of Q_2 about Q_1 (in the near hexagon H). Then $Q_3' \cong Q_2$, $H_3' \cong H_2$ and $Q_3' \subset H_3$.

Since $x_3 \in H_3'$, we have that $Q_3 = Q_3'$ and $H_3 = H_3'$. Hence, x_3 is of the same type as x_2. ■

Lemma 4.9 *Every point x of type (a) of $\Gamma_2(H)$ is contained in precisely 6 lines of S which only contains points of type (a).*

Proof. Put $Q := \langle \Gamma_2(x) \cap H \rangle$.

Let $\{x, x_1, x_2\}$ be a line of S through x which only contains points of type (a) and let $Q_i = \langle \Gamma_2(x_i) \cap H \rangle$, $i \in \{1, 2\}$. Then by Lemmas 4.7 and 4.8, the $W(2)$-quads Q, Q_1 and Q_2 are mutually disjoint and Q_2 is the reflection of Q_1 about Q (in the near hexagon H).

Now, there are 12 $W(2)$-quads in H disjoint with Q. Let Q' be such a $W(2)$-quad and let H' denote the unique $\mathbb{G}_3$-hex through Q'. Then the $\mathbb{G}_3$-hexes $\langle x, Q \rangle$ and H' are disjoint. Hence, the line $x\pi_{H'}(x)$ belongs to S. Since x and $\pi_{H'}(x)$ are points of type (a), also the third point of $x\pi_{H'}(x)$ has type (a) by Lemma 4.8. If we denote by $Q'' \cong W(2)$ the reflection of Q' about Q (in H) and by H'' the unique $\mathbb{G}_3$-hex through Q'', then $H'' = \mathcal{R}_{H'}(\langle x, Q \rangle)$ and $x\pi_{H'}(x) = x\pi_{H''}(x)$. It follows that there are $\frac{12}{2}$ lines of S through x containing only points of type (a). ■

From Lemmas 4.3 and 4.9, we readily obtain:

Corollary 4.10 *There are $\frac{3360 \cdot 6}{3} = 6720$ lines of S containing precisely three points of type (a).*

Lemma 4.11 *There are 13440 lines of S containing one point of type (a) and two points of type (b).*

Proof. Let x be one of the 3360 points of type (a). By Lemmas 4.5, 4.8 and 4.9, x is contained in 4 lines of S which contain a unique point of type (a). Hence, the required number is equal to $3360 \cdot 4 = 13440$. ■

By Corollary 4.6, Corollary 4.10 and Lemma 4.11, we obtain:

Corollary 4.12 *There are two types of lines in S:*

(1) *Lines of S only containing points of type (a).*

(2) *Lines of S containing a unique point of type (a) and two points of type (b).*

Corollary 4.13 *Let X denote the set of points of $\mathbb{G}_4$ consisting of the points of H, the points of $\Gamma_1(H)$ and the points of type (a) of $\Gamma_2(H)$. Then X is a hyperplane of $\mathbb{G}_4$.*

5 A new class of valuations of $\mathbb{G}_4$

Let H denote a hex of $\mathbb{G}_4$ isomorphic to $\mathbb{H}_3$ and let f denote a valuation of Fano-type of H. For every point $x \in \Gamma_1(H)$, let $\pi_H(x)$ denote the unique point of H collinear with x. We define the following function $\overline{f}$ from the point-set of $\mathbb{G}_4$ to $\mathbb{N}$:

(i) If $x \in H$, then we define $\overline{f}(x) := f(x)$.

(ii) If $x \in \Gamma_1(H)$, then we define $\overline{f}(x) := 1 + f(\pi_H(x))$.

(iii) If x is a point of type (a) of $\Gamma_2(H)$, then $\overline{f}(x) := \mathrm{d}(x, x^*)$, where x^* is the unique point of O_f contained in the $W(2)$-quad $\langle \Gamma_2(x) \cap H \rangle$.

(iv) Let x be a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 3$, where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. Then $\overline{f}(x) := 2$ if $\Gamma_2(x) \cap (O_f \cap Q) \neq \emptyset$ and $\overline{f}(x) := 1$ otherwise.

(v) Let x be a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 0$ where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. Let X denote the ovoid of Q consisting of all points with f-value 1. We define $\overline{f}(x) := 3$ if $\Gamma_2(x) \cap X \neq \emptyset$ and $\overline{f}(x) := 2$ otherwise.

In Section 6 a proof will be given for the fact that $\overline{f}$ is a valuation of $\mathbb{G}_4$. This proof is very indirect. It relies for instance on the classification of all valuations of $\mathbb{G}_4$. It is possible to give a direct proof, but this forces us to consider many cases. We will therefore only verify that property (V2) is satisfied and leave the verification of property (V3) as a (long) exercise to the interested reader. (Notice that property (V1) is trivially satisfied.)

Lemma 5.1 *The map $\overline{f}$ satisfies property (V2).*

Proof. Let L be a line of $\mathbb{G}_4$. There are different possibilities:

(1) L is contained in H. Then L satisfies property (V2) with respect to $\overline{f}$ since L satisfies property (V2) with respect to f.

(2) L intersects H in a unique point x_L. Then $\overline{f}(x) = f(x_L) + 1 = \overline{f}(x_L) + 1$ for every point x of $L \setminus \{x_L\}$. So, L satisfies property (V2).

(3) $L \subseteq \Gamma_1(H)$. Then $\pi_H(L) := \{\pi_H(x) \,|\, x \in L\}$ is a line of H parallel with L. For every point x of L, $\overline{f}(x) = f(\pi_H(x)) + 1$. Since $\pi_H(L)$ satisfies property (V2) with respect to f, L satisfies property (V2) with respect to $\overline{f}$.

(4) $|L \cap \Gamma_1(H)| = 1$ and $L \setminus \Gamma_1(H) \subseteq \Gamma_2(H)$. Let x denote an arbitrary point of $L \cap \Gamma_2(H)$ and let Q denote the unique quad of H containing $\Gamma_2(x) \cap H$. Then $\langle x, Q \rangle$ is a hex. By the definition of $\overline{f}$, there exists a constant $\epsilon \in \{-1, 0\}$ such that the map $u \mapsto \overline{f}(u) + \epsilon$ defines a valuation of $\langle x, Q \rangle$. It follows that L satisfies property (V2) with respect to $\overline{f}$.

(5) $L \subseteq \Gamma_2(H)$ and every point of L is of type (a). Put $L = \{x_1, x_2, x_3\}$ and let Q_i, $i \in \{1,2,3\}$, denote the unique $W(2)$-quad of H containing $O_i = \Gamma_2(x_i) \cap H$. The set O_i is an ovoid of Q_i. Put $H_i := \langle x_i, Q_i \rangle$, $i \in \{1,2,3\}$. Then H_1, H_2 and H_3 are three $\mathbb{G}_3$-hexes and $\mathcal{R}_{H_1}(H_2) = H_3$. From this it follows that $\pi_{Q_2}(O_1) = O_2$ and $\pi_{Q_3}(O_1) = O_3$. Let u_i^*, $i \in \{1,2,3\}$, denote the unique point of Q_i with f-value 0. Since $\mathrm{d}(u_1^*, u_2^*) = \mathrm{d}(u_1^*, u_3^*) = \mathrm{d}(u_2^*, u_3^*) = 2$, u_1^*, u_2^* and u_3^* are contained in a special grid-quad which intersects Q_1, Q_2 and Q_3 in lines. It now follows that $O_1 \cup O_2 \cup O_3$ has a unique point u^* in common with $\{u_1^*, u_2^*, u_3^*\}$. If $i \in \{1,2,3\}$ such that $u^* = u_i^*$, then $\overline{f}(x_i) = 2$ and $\overline{f}(x_j) = 3$ for all $j \in \{1,2,3\} \setminus \{i\}$. This proves that L satisfies property (V2).

(6) $L \subseteq \Gamma_2(H)$, L contains a unique point x_1 of type (a) and two points x_2 and x_3 of type (b). Let Q_1 denote the unique $W(2)$-quad of H containing all points of $\Gamma_2(x_1) \cap H$ and put $H_1 := \langle x, Q_1 \rangle$. Let G_i, $i \in \{2,3\}$, denote the grid-quad of H containing all points of $\Gamma_2(x_i) \cap H$ and put $H_i := \langle x, G_i \rangle$. Then $H_1 \cong \mathbb{G}_3$ and $H_2 \cong H_3 \cong \mathbb{H}_3$. Moreover, H_1, H_2 and H_3 are mutually disjoint and $\mathcal{R}_{H_1}(H_2) = H_3$. Put $G_1 := \pi_{Q_1}(G_2) = \pi_{Q_1}(G_3)$. For every $i \in \{2,3\}$, the map $H_i \to H_1; x \mapsto \pi_{H_1}(x)$ preserves distances. Hence, $\pi_{Q_1}(\Gamma_2(x_2) \cap G_2) = \pi_{Q_1}(\Gamma_2(x_3) \cap G_3) = \Gamma_2(x_1) \cap G_1$. We distinguish four possibilities (cf. Lemma 2.2):

(i) $|G_2 \cap O_f| = |G_3 \cap O_f| = 0$, the unique point x^* in $O_f \cap Q_1$ is contained in G_1 and $\mathrm{d}(x^*, x_1) = 2$. Then the unique line through x^* meeting G_2 and G_3 intersects G_2 and G_3 in points with f-value 1 belonging respectively to $\Gamma_2(x_2)$ and $\Gamma_2(x_3)$. It follows that $\overline{f}(x_1) = 2$ and $\overline{f}(x_2) = \overline{f}(x_3) = 3$. So, L satisfies property (V2).

(ii) $|G_2 \cap O_f| = |G_3 \cap O_f| = 0$, the unique point x^* in $O_f \cap Q_1$ is contained in G_1 and $\mathrm{d}(x^*, x_1) = 3$. The ovoid $\Gamma_2(x_1) \cap G_1$ of G_1 contains two points with f-value 1 and one point with f-value 2 and there exists an $i \in \{2,3\}$ such that (a) the ovoid $\Gamma_2(x_i) \cap G_i$ contains two points with f-value 2 and 1 point with f-value 1, and (b) the ovoid $\Gamma_2(x_{5-i}) \cap G_{5-i}$ contains three points with f-value 2. It follows that $\overline{f}(x_1) = 3$, $\overline{f}(x_i) = 3$ and $\overline{f}(x_{5-i}) = 2$. So, L satisfies property (V2).

(iii) There exists an $i \in \{2,3\}$ such that $|G_i \cap O_f| = 3$ and $|G_{5-i} \cap O_f| = 0$. Moreover, we assume that $\mathrm{d}(x_1, x^*) = 2$, where x^* is the unique point in $O_f \cap Q_1$. (Recall $x^* \notin G_1$.) Then $\Gamma_2(x_1) \cap G_1$ only contains points with f-value 2 since none of these points is collinear with x^*. Hence, $\Gamma_2(x_i) \cap G_i$ only contains points with f-value 1 and $\Gamma_2(x_{5-i}) \cap G_{5-i}$ only contains points with f-value 2. It follows that $\overline{f}(x_1) = 2$, $\overline{f}(x_i) = 1$ and $\overline{f}(x_{5-i}) = 2$. This proves that L satisfies property (V2) with respect to $\overline{f}$.

(iv) There exists an $i \in \{2,3\}$ such that $|G_i \cap O_f| = 3$ and $|G_{5-i} \cap O_f| = 0$.

Moreover, we assume that $d(x_1, x^*) = 3$ where x^* is the unique point in $O_f \cap Q_1$. (Recall $x^* \notin G_1$.) Then $\Gamma_2(x_1) \cap G_1$ contains at least one point with f-value 1 (collinear with x^*). Hence, $\Gamma_2(x_i) \cap G_i$ contains at least one point with f-value 0 and $\Gamma_2(x_{5-i}) \cap G_{5-i}$ contains at least one point with f-value 1. It follows that $\overline{f}(x_1) = 3$, $\overline{f}(x_i) = 2$ and $\overline{f}(x_{5-i}) = 3$. This proves that L satisfies property (V2). ■

6 The classification of the valuations of $\mathbb{G}_4$

6.1 Some lemmas

During our classification of the valuations of $\mathbb{G}_4$, we will need three properties which hold for valuations of general near polygons:

Lemma 6.1 ([6]) *Let f be a valuation of a dense near $2n$-gon $\mathcal{S}$.*
(*i*) *If there exists a point with value n, then f is a classical valuation.*
(*ii*) *If $d(x, O_f) \leq 2$, then $f(x) = d(x, O_f)$.*
(*iii*) *No two distinct special quads intersect in a line.*

Now, suppose that f is a valuation of $\mathbb{G}_4$.

Lemma 6.2 *If $x, y \in O_f$, then $d(x, y)$ is even.*

Proof. By Property (V2), $d(x, y) \neq 1$. Suppose $d(x, y) = 3$. Let H denote the unique hex through x and y. If f' denotes the valuation of H induced by f, then $O_{f'}$ contains two points at distance 3 from each other. This is impossible since none of the near hexagons $\mathbb{G}_3$, $W(2) \times \mathbb{L}_3$, $Q(5, 2) \times \mathbb{L}_3$, $\mathbb{H}_3$ has such valuations. ■

Lemma 6.3 *If there exists a $\mathbb{G}_3$-hex H such that $|H \cap O_f| = 15$, then $O_f = H \cap O_f$.*

Proof. Suppose $x \in O_f \setminus H$. Then $\pi_H(x)$ has value 1 and hence is contained in a unique quad Q of H which is special with respect to the non-classical valuation of H induced by f. If y is a point of $Q \cap O_f$ at distance 2 from $\pi_H(x)$, then $d(x, y) = 3$, contradicting Lemma 6.2. ■

Lemma 6.4 *If x and y are two different points of O_f, then $d(x, y) = 2$.*

Proof. Suppose the contrary. Then $d(x, y) = 4$ by Lemma 6.2. Let H denote an arbitrary $\mathbb{G}_3$-hex through x. Since $y \in O_f \setminus H$, the valuation induced in H is classical by Lemma 6.3. Hence, $f(\pi_H(y)) = d(x, \pi_H(y)) = 3$. On the other hand, since $d(\pi_H(y), y) = 1$ and $f(y) = 0$, it holds that $f(\pi_H(y)) = 1$, a contradiction. ■

Lemma 6.5 *One of the following cases occurs:*

(A) $|O_f| = 1$;

(B) *There exists a unique* $\mathbb{G}_3$*-hex* H *such that* $O_f \subseteq H$ *and* $|H \cap O_f| = 15$;

(C) $|O_f| \geq 2$ *and every special quad is a grid of type II.*

Proof. Suppose $|O_f| \geq 2$ and let x_1 and x_2 denote two distinct points of O_f. Then $d(x_1, x_2) = 2$ by Lemma 6.4. Let Q denote the unique special quad through x_1 and x_2. Then Q is not isomorphic to $Q(5, 2)$ since this generalized quadrangle has no ovoids (Payne and Thas [10]). If Q is a $W(2)$-quad or a grid-quad of type I, then Q is contained in a unique $\mathbb{G}_3$-hex H, see Lemma 3.1. Since $Q \cap O_f \subseteq H \cap O_f$, $|H \cap O_f| = 15$ and the valuation of H induced by f is non-classical. By Lemma 6.3, it then follows that $O_f = H \cap O_f$. The lemma is now clear. ■

6.2 Treatment of case (A) of Lemma 6.5

Proposition 6.6 *If* f *is a valuation of* $\mathbb{G}_4$ *such that* $|O_f| = 1$, *then* f *is a classical valuation.*

Proof. Put $O_f = \{x\}$. Let H denote an arbitrary $\mathbb{G}_3$-hex through x, let L denote the special line through x not contained in H, let x' denote an arbitrary point of $L \setminus \{x\}$ and let H' denote the unique $\mathbb{G}_3$-hex through x' not containing L. We will show that the valuation f' of H' induced by f is classical. Suppose the contrary. Let Q denote a grid-quad of H' through $x' \in O_{f'}$ which is special with respect to f'. Then $\langle L, Q \rangle \cong Q(5, 2) \times \mathbb{L}_3$. Now, $\langle L, Q \rangle$ contains a unique point with f-value 0 and a point with f-value 1 at distance 3 from it. But $Q(5, 2) \times \mathbb{L}_3$ does not have valuations of this type. So, we have a contradiction. It follows that the valuation induced in H' is classical. This implies that every point of H' at distance 3 from x' has value 4. By Lemma 6.1 (i), it then follows that f is classical. ■

6.3 Treatment of case (B) of Lemma 6.5

Proposition 6.7 *If* f *is a valuation of* $\mathbb{G}_4$ *such that* O_f *is a set of* 15 *points in a* $\mathbb{G}_3$*-hex* H *of* $\mathbb{G}_4$, *then* f *is the extension of a non-classical valuation of* $\mathbb{G}_3$.

Proof. Let f' denote the valuation of H induced by f. Then f' is a non-classical valuation of H with $O_{f'} = O_f$. Hence, $f(x) = f'(x)$ for every point $x \in H$. Now, let x be an arbitrary point of $\mathbb{G}_4$ not contained in H. Let Q denote an arbitrary $Q(5,2)$-quad of H through $\pi_H(x)$. Then the hex $\langle x, Q\rangle$ is isomorphic to $\mathbb{G}_3$ or $Q(5,2) \times \mathbb{L}_3$ and contains a unique point of O_f, namely the unique point of O_f in Q. It follows that the valuation induced in $\langle x, Q\rangle$ is classical. Hence, $f(x) = \mathrm{d}(x, O_f \cap Q) = 1 + \mathrm{d}(\pi_H(x), O_f \cap Q) = 1 + f'(\pi_H(x))$. (The latter equation follows from the fact that the valuation induced in $Q \cong Q(5,2)$ is classical.) This proves that f is the extension of f'. ■

6.4 Treatment of case (C) of Lemma 6.5

In this subsection, we suppose that f is a valuation of $\mathbb{G}_4$ such that $|O_f| \geq 2$ and such that every special quad is a grid of type II. By Lemma 6.4, every two distinct points of O_f are contained in a unique special quad.

Lemma 6.8 *It holds that* $|O_f| > 3$.

Proof. Suppose to the contrary that $|O_f| = 3$. Let Q denote the unique special grid-quad of type II and put $\{x_1, x_2, x_3\} = Q \cap O_f$. Let L denote an arbitrary ordinary line through x_1 such that $\langle L, Q\rangle \cong Q(5,2) \times \mathbb{L}_3$, let $y \in L \setminus \{x_1\}$, let H' denote a $\mathbb{G}_3$-hex through y not containing the line L and let f' denote the valuation of H' induced by f. Since $\pi_{H'}(\{x_1, x_2, x_3\}) \subseteq O_{f'}$, f' is not classical. Let Q' denote a $W(2)$-quad of H' through y which is special with respect to f'. Then $\langle L, Q'\rangle$ is isomorphic to $\mathbb{H}_3$ and does not contain Q since $\langle L, Q\rangle \cong Q(5,2) \times \mathbb{L}_3$. It follows that $\langle L, Q'\rangle \cong \mathbb{H}_3$ contains a unique point with f-value 0 and a point with f-value 1 at distance 3 from it. This is impossible, since $\mathbb{H}_3$ does not have such valuations. ■

Lemma 6.9 O_f *is a set of* 7 *points in an* $\mathbb{H}_3$*-hex of* $\mathbb{G}_4$.

Proof. Let x denote an arbitrary point of O_f. By Lemmas 6.4 and 6.8, there are two distinct grid-quads G_1 and G_2 (of type II) through x. By Lemma 6.1 (iii), $G_1 \cap G_2 = \{x\}$. Since every point of $(O_f \cap G_1) \setminus \{x\}$ has distance 2 from every point of $(O_f \cap G_2) \setminus \{x\}$, G_1 and G_2 are contained in a hex H. This hex is necessarily isomorphic to $\mathbb{H}_3$ (see Lemma 3.1 (10)) and the valuation f_H of H induced by f must be of Fano-type. Hence, $|O_f \cap H| = 7$.

We show that $\Gamma_1(H) \cap O_f = \emptyset$. Suppose $y \in \Gamma_1(H) \cap O_f$ and let $\pi_H(y)$ denote the unique point of H collinear with y. Then $f(\pi_H(y)) = 1$ and hence $\pi_H(y)$ is contained in a unique quad Q of H which is special with respect to f_H. Any point of $Q \cap O_f$ at distance 2 from $\pi_H(y)$ lies at distance 3 from y, contradicting Lemma 6.4. Hence, $\Gamma_1(H) \cap O_f = \emptyset$.

We show that $f(y) \geq 2$ for every point y of type (a) of $\Gamma_2(H)$. Let Q denote the $W(2)$-quad of H containing all points of $\Gamma_2(y) \cap H$ and let H' be the $\mathbb{G}_3$-hex $\langle y, Q \rangle$. Let u denote the unique point of $O_f \cap Q$ and let L be a line of Q through u. If the valuation $f_{H'}$ of H' induced by f is not classical, then there exists a quad of H' through L which is special with respect to $f_{H'}$. This implies that there is a point of $O_{f_{H'}} \subseteq O_f$ contained in $\Gamma_1(H)$, a contradiction. Hence, $f_{H'}$ is a classical valuation of H'. It follows that $f(y) = f_{H'}(y) = \mathrm{d}(y, u) \geq 2$.

We show that $f(y) \geq 1$ for every point y of type (b) of $\Gamma_2(H)$. Let L be a line of S through y. The unique point of type (a) on this line has value at least 2. It follows that $f(y) \geq 1$. ∎

Let H denote the unique $\mathbb{H}_3$-hex of $\mathbb{G}_4$ containing all points of O_f and let f' be the valuation of H induced by f. By Lemma 6.9, f' is a valuation of Fano-type of H.

Proposition 6.10 *The valuation f is obtained from f' in the way as described in Section 5.*

Proof. Let x denote an arbitrary point of $\mathbb{G}_4$.

If $x \in H$, then $\mathrm{d}(x, O_f) \leq 2$ and hence $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, O_{f'}) = f'(x)$ by Lemma 6.1 (i).

If $x \in \Gamma_1(H)$ such that $\mathrm{d}(\pi_H(x), O_f) \leq 1$, then $\mathrm{d}(x, O_f) \leq 2$ and hence $f(x) = \mathrm{d}(x, O_f) = 1 + \mathrm{d}(\pi_H(x), O_f) = 1 + f'(\pi_H(x))$ by Lemma 6.1 (i).

Let $x \in \Gamma_1(H)$ such that $\mathrm{d}(\pi_H(x), O_f) = 2$, or equivalently, such that $f'(\pi_H(x)) = 2$. Let H' denote an arbitrary $\mathbb{G}_3$-hex through the line $x\pi_H(x)$. Then $H' \cap H$ is a $W(2)$-quad Q. The hex H' contains a unique point with f-value 0, namely the unique point of O_f in Q. Hence, the valuation induced in H' is classical. It follows that $f(x) = 3 = 1 + f'(\pi_H(x))$.

Let x denote a point of type (a) of $\Gamma_2(H)$. Let Q denote the $W(2)$-quad of H containing all points of $\Gamma_2(x) \cap H$ and let x^* denote the unique point of O_f in Q. The hex $\langle x, Q \rangle$ is isomorphic to $\mathbb{G}_3$ and contains a unique point of O_f, namely x^*. Hence, the valuation induced in $\langle x, Q \rangle$ is classical. It follows that $f(x) = \mathrm{d}(x, x^*)$.

Let x denote a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 3$, where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. The hex $\langle x, Q \rangle$ is isomorphic to $\mathbb{H}_3$ and the valuation of $\langle x, Q \rangle$ induced by f is of grid-type. It follows that $f(x) = 2$ if $\Gamma_2(x) \cap O_f \cap Q \neq \emptyset$ and $f(x) = 1$ otherwise.

Let x denote a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 0$, where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. The hex $\langle x, Q \rangle$ is isomorphic to $\mathbb{H}_3$ and the valuation f' of $\langle x, Q\rangle$ induced by f is either of grid-type or of Fano-type. (Notice that Q is special with respect to f'.) We will show that the latter possibility cannot occur. Suppose that f' is a valuation of Fano-type. Let u denote a point of Q with f-value 1, let v denote a point of O_f collinear with u and let $G \neq Q$ denote a grid-quad of $\langle x, Q\rangle$ through u which is special with respect to f'. The hex $\langle v, G\rangle$ intersects H in the line uv and hence contains a unique point of O_f. [Suppose that $\langle v, G\rangle$ intersects H in a quad Q' and let w be a point of $G \cap \Gamma_2(u)$. Then $\Gamma_2(w) \cap Q'$ is an ovoid of Q' which necessarily coincides with $\Gamma_2(w) \cap H$. Similarly, since $\langle w, Q\rangle$ is a hex, $\Gamma_2(w) \cap Q = \Gamma_2(w) \cap H$ is an ovoid of Q. It follows that $Q = Q'$, a contradiction.] Since the valuation induced in $\langle v, G\rangle$ contains a unique point with value 0 and a point with value 1 at distance 3 from it, the hex $\langle v, G\rangle$ is isomorphic to $W(2) \times \mathbb{L}_3$ and the valuation induced in $\langle v, G\rangle$ is semi-classical. But in a $W(2) \times \mathbb{L}_3$-hex, every grid-quad is of type I, while the grid-quad G has type II since it is contained in the $\mathbb{H}_3$-hex $\langle x, Q\rangle$ (see Lemma 3.1). So, we have a contradiction and the valuation f' must be of grid-type. Hence, $f(x) = 3$ if $\Gamma_2(x) \cap Q$ has a point with f'-value 1 and $f(x) = 2$ otherwise.

This proves the proposition. ■

6.5 The existence of valuations of Fano-type of $\mathbb{G}_4$

The existence of valuations of Fano-type of $\mathbb{G}_4$ will be shown in the following proposition.

Proposition 6.11 *Let F be a hex of $\mathbb{G}_4$ and let f be a valuation of F. Suppose that one of the following cases occurs: (i) $F \cong \mathbb{H}_3$ and f is a valuation of Fano-type of F; (ii) $F \cong \mathbb{G}_3$ and f is a non-classical valuation of F. Suppose also that $\mathbb{G}_4$ is isometrically embedded into the dual polar space $DH(7, 4)$. Then*

(1) *there exists a unique point $x \in DH(7, 4) \setminus \mathbb{G}_4$ such that $O_f \subseteq \Gamma_1(x)$;*

(2) *there exists a unique valuation $\overline{f}$ of $\mathbb{G}_4$ such that $O_f = O_{\overline{f}}$.*

If $F \cong \mathbb{G}_3$, then $\overline{f}$ is the extension of f. If $F \cong \mathbb{H}_3$, then $\overline{f}$ is obtained from f in the way as described in Section 5.

Proof. Let $\overline{F} \cong DH(5, 4)$ denote the unique hex of $DH(7, 4)$ containing F. Then $\overline{F} \cap \mathbb{G}_4 = F$. Obviously, if $x \in DH(7, 4) \setminus \mathbb{G}_4$ such that $O_f \subseteq \Gamma_1(x)$, then $x \in \overline{F} \setminus F$.

Now, let Q be a quad of F which is special with respect to the valuation f and let $y \in O_f \setminus Q$. The set O_f is a set of points at mutually distance 2 which is completely determined by its subset $\{y\} \cup (Q \cap O_f)$. Since $\Gamma_2(y) \cap Q = Q \cap O_f$, $\mathrm{d}(y, Q) = 2$. Let $\overline{Q} \cong Q(5,2)$ denote the unique quad of $DH(7,4)$ containing Q. Then $\overline{Q} \cap \mathbb{G}_4 = Q$ and $\overline{Q} \subseteq \overline{F}$. If x is a point of $DH(7,4) \setminus \mathbb{G}_4$ such that $O_f \subseteq \Gamma_1(x)$, then $x \in \overline{Q}$ and hence x coincides with the unique point y^* of $\overline{Q}$ collinear with y. Since $Q \cap O_f \subseteq \Gamma_2(y)$, $(Q \cap O_f) \cup \{y\} \subseteq \Gamma_1(y^*)$. Now, let f_1 denote the classical valuation of $DH(7,4)$ for which $O_{f_1} = \{y^*\}$ and let $\overline{f}$, respectively f_2, denote the valuation of $\mathbb{G}_4$, respectively H, induced by f_1. Then $(Q \cap O_f) \cup \{y^*\} \subseteq O_{f_2}$. Hence, $O_f = O_{f_2}$ and $f = f_2$. It follows that $O_f \subseteq \Gamma_1(x)$. Obviously, $O_f \subseteq O_{\overline{f}}$. By the above classification of the valuations of $\mathbb{G}_4$, we have $O_f = O_{\overline{f}}$. The remaining claims of the proposition follow from Propositions 6.7 and 6.10. ∎

6.6 The valuations of $\mathbb{G}_4$ are induced by valuations of $DH(7,4)$

Let the near octagon $\mathbb{G}_4$ be isometrically embedded in $DH(7,4)$. For every point x of $DH(7,4)$, the classical valuation g_x of $DH(7,4)$ with $O_{g_x} = \{x\}$ induces a valuation f_x of $\mathbb{G}_4$. It holds that $\max\{f_x(u) \mid u \in \mathbb{G}_4\} = 4 - \mathrm{d}(x, \mathbb{G}_4)$ in view of the following result which holds for general dense near polygons.

Lemma 6.12 (Proposition 2.2 of [8]) *Let $\mathcal{S}$ be a dense near $2n$-gon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a dense near $2n$-gon which is fully and isometrically embedded in $\mathcal{S}$. Let x be a point of $\mathcal{S}$ and let f_x denote the valuation of F induced by the classical valuation g_x of $\mathcal{S}$ with $O_{g_x} = \{x\}$, then $d(x, F) = n - M$, where M is the maximal value attained by f_x.*

If $x \in \mathbb{G}_4$, then f_x is a classical valuation of $\mathbb{G}_4$ and $O_{f_x} = \{x\}$. If $x \notin \mathbb{G}_4$, then f_x is not classical and hence is either the extension of a non-classical valuation of a $\mathbb{G}_3$-hex or is a valuation of Fano-type.

Proposition 6.13 *Let f be a valuation of $\mathbb{G}_4$. Then there exists a unique point x of $DH(7,4)$ such that $f = f_x$.*

Proof. Obviously, the proposition holds if f is classical. The required point x is then the unique point contained in O_f. Suppose now that f is non-classical. Let H be the hex $\langle O_f \rangle$ of $\mathbb{G}_4$ and let $\overline{H} \cong DH(5,4)$ denote the unique hex of $DH(7,4)$ containing H. For each of the two possibilities for the non-classical valuation f, the maximal value attained by f is equal to 3. Hence, if x is a point of $DH(7,4)$ such that $f_x = f$, then $\mathrm{d}(x, \mathbb{G}_4) = 1$ and $O_f = \Gamma_1(x) \cap \mathbb{G}_4$. Now, by Proposition 6.11, there exists a unique point x in

$DH(7,4) \setminus \mathbb{G}_4$ such that $O_f \subseteq \Gamma_1(x)$. Then $O_f \subseteq O_{f_x}$. Hence $O_f = O_{f_x}$ and $f = f_x$ by the above classification of the valuations of $\mathbb{G}_4$. ■

By Proposition 6.13, the number of valuations of $\mathbb{G}_4$ is equal to the number of points of $DH(7,4)$. The number of classical valuations of $\mathbb{G}_4$ is equal to the number of points of $\mathbb{G}_4$, i.e., equal to 8505. The number of valuations of $\mathbb{G}_4$ which are extensions of non-classical valuations in $\mathbb{G}_3$-hexes is equal to (# $\mathbb{G}_3$-hexes) $\times$ (# non-classical valuations in a $\mathbb{G}_3$-hex) $= 84 \cdot 486 = 40824$. The number of valuations of Fano-type of $\mathbb{G}_4$ is equal to (# $\mathbb{H}_3$-hexes) $\times$ (# valuations of Fano-type in an $\mathbb{H}_3$-hex) $= 2178 \cdot 30 = 65610$. The number 8505+40824+65610=114939 is indeed equal to the total number of points of $DH(7,4)$.

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