Characterization results on small blocking sets of the polar spaces $Q^+(2n+1,2)$ and $Q^+(2n+1,3)$

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Abstract

In [8], De Beule and Storme characterized the smallest blocking sets of the hyperbolic quadrics $Q^+(2n+1,3)$, $n \geq 4$; they proved that these blocking sets are truncated cones over the unique ovoid of $Q^+(7,3)$. We continue this research by classifying all the minimal blocking sets of the hyperbolic quadrics $Q^+(2n+1,3)$, $n \geq 3$, of size at most $3^n + 3^{n-2}$. This means that the three smallest minimal blocking sets of $Q^+(2n+1,3)$, $n \geq 3$, are now classified. We present similar results for $q = 2$ by classifying the minimal blocking sets of $Q^+(2n+1,2)$, $n \geq 3$, of size at most $2^n + 2^{n-2}$. This means that the two smallest minimal blocking sets of $Q^+(2n+1,2)$, $n \geq 3$, are classified.

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1 Introduction

The *finite classical polar spaces* are the non-singular symplectic polar spaces $W(2n+1,q)$, the non-singular parabolic quadrics $Q(2n,q)$, $n \geq 2$, the non-singular elliptic and hyperbolic quadrics $Q^-(2n+1,q)$, $n \geq 2$, and $Q^+(2n+1,q)$, $n \geq 1$, and the non-singular hermitian varieties $H(d,q^2)$, $d \geq 3$. For q even, the parabolic polar spaces $Q(2n,q)$ are isomorphic to the symplectic polar spaces $W(2n-1,q)$.

The *generators* of a classical polar space are the subspaces of maximal dimension contained in these polar spaces. If the generators are of dimension $r-1$, then the polar space is said to be of *rank* r.

A *blocking set* of a finite classical polar space $\mathcal{P}$ is a set of points intersecting every generator in at least one point. A blocking set B of $\mathcal{P}$ is called *minimal* when no proper subset of B still is a blocking set of $\mathcal{P}$.

In recent years, much research has been done to classify blocking sets of the classical finite polar spaces. We refer to [13] for a survey of the known results. For recent results, we also refer to [4, 5, 6, 7, 14].

An *ovoid* $\mathcal{O}$ of a classical polar space $\mathcal{P}$ is a set of points of $\mathcal{P}$ such that every generator contains exactly one point of $\mathcal{O}$.

One of the main problems in the theory of ovoids is the problem of the existence of ovoids of the hyperbolic quadrics $Q^+(2n+1,q)$, $n > 3$. Only for $q=2$ and $q=3$ is it known that the quadrics $Q^+(2n+1,q)$, $n > 3$, have no ovoids [1, 11, 15]. The known fact of the non-existence of ovoids of $Q^+(2n+1,2)$, $n > 3$, and of ovoids of $Q^+(2n+1,3)$, $n > 3$, now implies the question of the characterization of the smallest blocking sets of these quadrics.

To state the results we need the notion of truncated cones. Consider in $\mathrm{PG}(d,q)$ two skew subspaces U and V, and let M be a set of points of U. If $M \neq \emptyset$, then the *cone* VM with *vertex* V and *base* M is the union of the subspaces $\langle V,P\rangle$ with $P \in M$. If $M = \emptyset$, then the cone VM is by definition equal to V. The *truncated cone* V^*M with vertex V and base M is obtained from the cone by removing the points of V.

In [8], De Beule and Storme characterized the smallest blocking sets of $Q^+(2n+1,3)$, $n > 3$. They proved that these blocking sets have size $q^n + q^{n-3}$, and that the blocking sets of size $3^n + 3^{n-3}$ are truncated cones

$\pi_{n-4}^*\mathcal{O} = \pi_{n-4}\mathcal{O}\setminus\pi_{n-4}$, with vertex π_{n-4} and base $\mathcal{O}$, where π_{n-4} is an $(n-4)$-dimensional space contained in $Q^+(2n+1,3)$, and where $\mathcal{O}$ is an ovoid of $Q^+(7,3) \subseteq \pi_{n-4}^{\perp}$, with $\perp$ the orthogonal polarity defined by $Q^+(2n+1,3)$.

We continue this research by classifying all minimal blocking sets of $Q^+(2n+1,3)$ of size at most 3^n+3^{n-2}. This amounts to a classification of three types of blocking sets. The smallest blocking sets are the truncated cones $\pi_{n-4}^*\mathcal{O}$ of size 3^n+3^{n-3} described by De Beule and Storme, and the third smallest ones are the truncated cones $\pi_{n-3}^*\mathcal{O}$, with $\mathcal{O}$ an ovoid of $Q^+(5,3)$, having size 3^n+3^{n-2}. Note that $Q^+(5,3)$ has two ovoids. The first type of ovoid is the 3-dimensional elliptic quadric, corresponding under the Klein correspondence to a regular spread and hence to the Desarguesian projective plane $PG(2,9)$. The second type of ovoid is equal to a set $(Q^-(3,3)\setminus C)\cup C^{\perp}$, where $Q^-(3,3)$ is a 3-dimensional elliptic quadric contained in $Q^+(5,3)$ and where C is a conic contained in $Q^-(3,3)$. Here $\perp$ is the polarity related to $Q^+(5,3)$. This ovoid corresponds under the Klein correspondence to a derived spread giving the Hall plane of order 9.

The second smallest minimal blocking sets have size $3^n+2\cdot 3^{n-3}$, and are described in the following way.

We construct a blocking set in the quadrics $Q^+(2n+1,q)$, $n\geq 3$. First we explain the construction for $n=3$.

Example 1.1 Consider the tangent hyperplane $P^{\perp}$ for a point $P\in Q^+(7,q)$. This hyperplane meets $Q^+(7,q)$ in a cone with vertex P and base a hyperbolic quadric $Q^+(5,q)$. Let S be a solid in $P^{\perp}$ meeting $Q^+(7,q)$ in a 3-dimensional elliptic quadric $Q^-(3,q)$. Then $S^{\perp}$ is a solid meeting $Q^+(7,q)$ in a 3-dimensional elliptic quadric $Q^-(3,q)^{\perp}$ containing P and P lies on q^2+1 lines meeting this $Q^-(3,q)$ in S. Let $PQ^-(3,q)$ be the cone with vertex P and base $Q^-(3,q)$.

The point set $B=((PQ^-(3,q)\setminus Q^-(3,q))\cup Q^-(3,q)^{\perp})\setminus\{P\}$ is a minimal blocking set B of $Q^+(7,q)$ of size q^3+q-1.

Now we consider the general case $n\geq 3$. Consider an $(n-4)$-dimensional subspace π_{n-4} contained in the hyperbolic quadric $Q^+(2n+1,q)$, and consider in its quotient geometry $Q^+(7,q)$ with respect to $Q^+(2n+1,q)$ the blocking set B. Then the truncated cone π_{n-4}^*B is a minimal blocking set of size $q^n+q^{n-2}-q^{n-3}$.

For $q=3$, these blocking sets are blocking sets of size $3^n+2\cdot 3^{n-3}$.

For $Q^+(2n+1,2)$, we prove similar results. We characterize all minimal blocking sets of size at most $2^n + 2^{n-2}$. We prove that they are either a blocking set of size $2^n + 2^{n-3}$ which is a truncated cone $\pi^*_{n-4}\mathcal{O}$ over an ovoid $\mathcal{O}$ of $Q^+(7,2)$, or a blocking set of size $2^n + 2^{n-2}$ which is a truncated cone $\pi^*_{n-3}Q^-(3,2)$.

We first present some general results. Then the classification for $q = 3$ is given, and the article ends with the classification results for $q = 2$. We use later on that $Q^+(7,3)$ has a unique ovoid [12]; this ovoid lives in fact in a 6-dimensional parabolic quadric $Q(6,3)$ contained in $Q^+(7,3)$.

2 General results

We state here only two easy results that hold for general q and that might be useful in other situations as well.

Lemma 2.1 *Let B be a minimal blocking set of $Q^+(2n+1,q)$, $n \geq 3$, and suppose that $|B| = q^n + \delta$ with $\delta \leq q^{n-2}$.*

(a) *If $P \in B$, then $|P^\perp \cap B| \leq \delta$.*

(b) *Let $R \in Q^+(2n+1,q) \setminus B$. If R lies on a totally singular line with exactly $t > 0$ points in B, then $|R^\perp \cap B| \leq tq^{n-1} + \delta$.*

Proof. (a) As B is minimal, there exists a generator π on P meeting B only in P. This generator has q^n hyperplanes not containing P. Each such hyperplane H lies in a second generator, which must meet B. Clearly different hyperplanes H yield different points of B, since no point of B (except for P) can be perpendicular to π.

(b) Let l be a totally singular line on R meeting B in exactly $t > 0$ points. Assume that all generators on l meet B again in a point outside l. Then the planes of $Q^+(2n+1,q)$ on l meeting B in a point outside l give a blocking set of the $Q^+(2n-3,q)$ seen in the quotient geometry on l. Thus, there are at least $q^{n-2}+1$ such planes. Hence each point of $l \cap B$ is perpendicular to at least $q^{n-2}+1$ further points of B, but this contradicts (a). Hence, there exists a generator π on l meeting B only in the t points of $l \cap B$. Then π contains $(q-t)q^{n-1}$ hyperplanes not containing P nor any of the points of $\pi \cap B = l \cap B$. As in (a) this implies that $|R^\perp \cap B| \leq |B| - (q-t)q^{n-1} = tq^{n-1} + \delta$. □

Corollary 2.2 *Let B be a minimal blocking set of $Q^+(2n+1,q)$, $n \geq 3$, and suppose that $|B| = q^n + \delta$ with $\delta \leq q^{n-2}$. Consider a point $R \in Q^+(2n+1,q)$ with $R \notin B$.*

Then the lines of $Q^+(2n+1,q)$ on R that meet B form a minimal blocking set of the quadric $Q^+(2n-1,q)$ seen in the quotient geometry of R in $R^\perp \cap Q^+(2n+1,q)$. Hence, there are at least $q^{n-1}+1$ such lines and equality holds if and only if they form an ovoid in this quotient geometry.

Proof. Every generator on R contains a point of B, which lies in $R^\perp$. Thus the lines of $Q^+(2n+1,q)$ on R meeting B block all generators on R. As we have seen in the proof of Lemma 2.1 (b), any totally isotropic line l on R meeting B lies in a generator meeting B only in points of l. This proves the minimality. □

3 Blocking sets in $Q^+(2n+1,3)$

In what follows, we study minimal blocking sets B of $Q^+(2n+1,q)$, $q = 3$ and $n \geq 3$, of size $q^n + \delta$ with

$$\delta \leq q^{n-2}.$$

We also assume that $\delta \geq q^{n-3} + 1$ since De Beule and Storme proved that every blocking set B of $Q^+(2n+1,3)$, $n \geq 3$, contains at least $q^n + q^{n-3}$ points. In case $|B| = q^n + q^{n-3}$, B is a truncated cone $\pi^*_{n-4}\mathcal{O}$, with $\mathcal{O}$ an ovoid of $Q^+(7,3)$, see [8].

This ovoid of $Q^+(7,3)$ is an ovoid of a parabolic quadric $Q(6,3)$ contained in $Q^+(7,3)$. Regarding this ovoid of $Q(6,3)$, we will use the following properties, found by computer [10] .

Lemma 3.1 *An ovoid of $Q(6,3)$ is intersected by*

(1) *the tangent hyperplanes to $Q(6,3)$ in 1 or 10 points,*

(2) *the 5-dimensional hyperbolic quadrics of $Q(6,3)$ in 10 points,*

(3) *the 5-dimensional elliptic quadrics of $Q(6,3)$ in 7 or 16 points.*

Remark 3.2 In [3], it was proven that a partial ovoid of $Q^-(5,q)$ has size at most $(q^3+q+2)/2$. For $q=3$, this reduces to the upper bound 16 on the size of a partial ovoid on $Q^-(5,3)$. The preceding computer search shows that the ovoid of $Q(6,3)$ contains partial ovoids of the largest possible size on $Q^-(5,3)$.

There exists a unique type of partial ovoids of size 16 on $Q^-(5,3)$. This partial ovoid is dual to the unique partial spread of size 16 on the Hermitian generalized quadrangle $H(3,9)$. It was shown by Ebert and Hirschfeld that partial spreads of $H(3,9)$ have size at most 16, and that partial spreads of size 16 are projectively unique. Moreover, they can be linked to the Kummer surface [9].

Lemma 3.3 *The set B does not contain a line.*

Proof. Assume on the contrary that B contains the line l. Then l is totally isotropic since it contains at least three points of the quadric. If $P \in l$, then P is perpendicular to the $q+1=4$ points of l and to at most $\delta-4$ further points in B (Lemma 2.1 (a)). Hence $\delta \geq 4$. As every point of $B \setminus l$ is perpendicular to at least one point in l, we conclude that $|B| \leq 4+4(\delta-4)$. As $|B| = q^n+\delta$, this shows $3\delta \geq q^n+12$, that is $\delta \geq q^{n-1}+4$. This is a contradiction. □

Lemma 3.4 *If there exists a 3-secant l to B, and P is the point of $l \setminus B$, then B is a truncated cone P^*B', where B' is a minimal blocking set in the quotient geometry $Q^+(2n-1,q)$ of P in $P^\perp \cap Q^+(2n+1,q)$.*

Proof. Put $q=3$ and let $|B| = q^n+\delta$, with $\delta \leq q^{n-2}$. For each of the three points P_i, $i=1,2,3$, of B on a 3-secant l, at most δ points of B lie in its perp $P_i^\perp$ (Lemma 2.1). So,

$$|(P_1^\perp \cup P_2^\perp \cup P_3^\perp) \cap B| \leq \delta + (\delta-3) + (\delta-3) = 3\delta - 6.$$

Hence, for the point P on the 3-secant l that does not lie in B, we have for $q=3$ that

$$|P^\perp \cap B| \geq |B| - (3\delta-6) + 3 = q^n - 2\delta + 9.$$

As $\delta \leq q^{n-2}$, then Lemma 2.1 (b) implies that every totally singular line on P that meets B must meet B in three points.

If G is any generator of $Q^+(2n+1,q)$, then $G' := \langle P, P^\perp \cap G\rangle$ is a generator and thus meets B and hence $P^\perp \cap B$. It follows that there exists a point $X \in P^\perp \cap B$ such that the line PX meets G. As PX is a 3-secant, it follows that G contains a point of $P^\perp \cap B$. Hence $P^\perp \cap B$ is a blocking set of $Q^+(2n+1,q)$, so by the minimality of B we conclude that $B \subseteq P^\perp$. As every line on P that meets B is a 3-secant, it follows that B is a truncated cone with vertex P. The base of this truncated cone must be a minimal blocking set B' of size $|B'| = |B|/3$ in the quotient geometry $Q^+(2n-1,q)$ of $P^\perp \cap Q^+(2n+1,q)$. □

Notation. A point $P \in Q^+(2n+1,q) \setminus B$ with the property that every line on P that meets B, intersects B in exactly two points will be called a *special point.*

Lemma 3.5 *Suppose that there does not exist a* 3*-secant to* B*. If* l *is a* 2*-secant to* B*, then exactly one of the two points of* $l \setminus B$ *is a special point.*

Proof. Let P_1, P_2 be the two points of $l \cap B$, with $|B| = q^n + \delta$ with $q := 3$. Then $P_i^\perp$ meets B in at most δ points, so in P_1, P_2 and in at most $\delta - 2$ other points. Hence, for at least one of the two points $R \in l \setminus B$, we have

$$|R^\perp \cap B| \geq 2 + \frac{1}{2}(|B| - 2 - 2(\delta - 2)) = \frac{1}{2}|B| + 3 - \delta.$$

Therefore Lemma 2.1 (b) shows that no totally singular line on R meets B in a unique point. In other words, R is a special point. As the lines on R that meet B form a blocking set in the quotient structure $Q^+(2n-1,q)$ on R, we see that at least $q^{n-1}+1$ totally singular lines on R meet B; hence $|R^\perp \cap B| \geq 2(q^{n-1}+1)$. Taking into account that $|B| \leq q^n + q^{n-2}$, and that $|l^\perp \cap B| \leq \delta \leq q^{n-2}$, this proves that we can not have this property for both points of $l \setminus B$. □

Lemma 3.6 *If* $n = 3$*, then* B *is an ovoid of* $Q^+(7,q)$*, a cone over an ovoid of* $Q^+(5,q)$ *or the structure described in Example* 1.1*.*

Proof. We have that $|B| \leq q^3 + q = q^3 + 3 = 30$. If $|B| = q^3 + 1$, then B is an ovoid of $Q^+(7,3)$.

Suppose then that $|B| \geq q^3 + 2$. If there exists a 3-secant, then Lemma 3.4 shows that there exists a point $P \in Q^+(7,q) \setminus B$ such that B is a truncated cone with vertex P. In other words, there are $|B|/3$ totally isotropic lines on P such that B is the union of these lines except for P. Then $|B| = q^3 + 3$ and there are $q^2 + 1$ such lines. It follows that these lines form an ovoid of the quadric $Q^+(5,q)$ in the quotient geometry on P (Corollary 2.2).

For the remainder of the proof, we assume that there does not exist a 3-secant to B. We show that B is the structure described in Example 1.1. Since B is not an ovoid, B contains perpendicular points. As there do not exist 3-secants, we find a 2-secant to B, and hence a special point (Lemma 3.5). Assume that two of the 2-secants to B on P are perpendicular. Then the four points of B on these two lines are pairwise perpendicular, but this contradicts Lemma 2.1 (a). Hence, any two 2-secants to B on P are non-perpendicular, which implies that the 2-secants to B on P form an ovoid of the hyperbolic quadric $Q^+(5,3)$ seen in the quotient geometry of B. Hence, there are $q^2 + 1$ such 2-secants through P and $|P^\perp \cap B| = 2(q^2 + 1) = 20$. Consequently, $|B \setminus P^\perp| = |B| - 20 \leq 10$.

Let $l_1, \ldots, l_{10}$ be the 2-secants to B on P, and let P_i be the second point of l_i not in B. The only points of $B \cap P^\perp$ that are perpendicular to P_i are the two points of B on the line l_i (since l_i and l_j, $i \neq j$, are not perpendicular). Corollary 2.2 shows that at least $q^2 = 9$ lines on P_i meet B in a point outside $P^\perp$.

Assume that the points P_i, $i = 1, \ldots, 10$, span more than a 3-space. Then take five points P_i spanning a 4-space S. Since each point P_i is perpendicular to at least 9 of the, at most, 10 points of $B \setminus P^\perp$, it follows that at least 5 points of $B \setminus P^\perp$ lie in the polar space of the five points P_i, and thus in the plane $S^\perp$. So three of the five points of B in $S^\perp$ are collinear. But we are in the situation that there do not exist 3-secants to B, a contradiction. Hence, the points P_i span a subspace S of dimension three.

As the lines $l_i = PP_i$ form an ovoid in the quotient geometry on P, we have $P \notin S$ and the ovoid is an elliptic quadric $Q^-(3,q)$. In other words, the $q^2 + 1$ points P_i are the points of $S \cap Q^+(7,q) = Q^-(3,q)$ for some elliptic 3-subspace. In the cone with vertex P over this $Q^-(3,q)$, we see the 10 points P_i, the point P, and the 20 points of B, and that are all points of $Q^+(7,q)$ in this cone.

Consider any point $X \in B \setminus P^\perp$. Then $X^\perp$ meets $\langle S, P \rangle$ in an elliptic 3-space.

If this would not be S, then it would meet S in at most a conic, so X would be perpendicular to at most four points P_i and thus perpendicular to at least six points of $B \cap P^\perp$ since the elliptic 3-space $X^\perp \cap \langle S, P \rangle$ has 10 points in $Q^+(7,q)$. This contradicts Lemma 2.1 (a). Hence, all points of $B \setminus P^\perp$ lie in $S^\perp$. So B is a subset of the blocking set described in Example 1.1. As B as well as Example 1.1 are minimal blocking sets, they are equal. □

Lemma 3.7 *The number of points lying only on* 2*-secants to* B *is at most* $\frac{1}{2}|B|(\delta-1)/(q^{n-1}+1)$. *For* $n=4$, *this implies that there are at most* 12 *such points.*

Proof. Since every point of B is perpendicular to at most δ points of B, the number of 2-secants is at most $\frac{1}{2}|B|(\delta-1)$. A point lying only on 2-secants lies on at least $q^{n-1}+1$ different 2-secants, because these 2-secants form a blocking set of the $Q^+(2n-1,q)$ seen in the quotient geometry on that point. □

Lemma 3.8 *Suppose that the point* $P \in Q^+(2n+1,q) \setminus B$ *only lies on* 2*-secants to* B. *If* $n=4$, *then the* 2*-secants on* P *form an ovoid in the* $Q^+(2n-1,q)$ *of the quotient geometry at* P.

Proof. The 2-secants on P form an ovoid if and only if they are pairwise not perpendicular.

Assume that there exist two 2-secants on P that are perpendicular. Then they span a plane π meeting B in at least four points. Since there does not exist a 3-secant (nor a 4-secant), π meets B in exactly four points forming a conic. Then there are 2-secants in π not containing P, and so we find a second point P' only lying on 2-secants to B (Lemma 3.5).

Thus, there exist two perpendicular special points. Now consider any two perpendicular special points X, X', with $X \neq X'$. Then the line XX' does not meet B (Lemma 3.5). In the quotient geometry of the line XX', we see a hyperbolic quadric $Q^+(5,q)$, and $XX' \cap B$ projects to a blocking set of this $Q^+(5,q)$. As such a blocking set has at least $q^2+1=10$ points, we find at least 10 totally isotropic planes on XX' meeting B. If τ is such a plane, then $|\tau \cap B| \geq 4$, as every line on X or X' meeting B intersects B in at least two points. Then τ contains a 2-secant to B not passing through P and not passing through P'. Therefore τ contains a third special point. Thus, we find in each of the, at least, 10 planes a new special point. Since there are at most 12 special points (Lemma 3.7), it follows that there are exactly 12 special points and all are perpendicular to X and X'. Also, exactly 10 planes on XX' meet B and these are the 10 planes on one of the 10 new special points.

As in the preceding argument, X and X' have been arbitrary perpendicular special points, the same argument shows that the 12 special points are pairwise perpendicular. Hence, the 12 special points generate a totally isotropic subspace. So, they lie together in a generator G. But this is impossible since the 10 special points different from X and X' project from XX' onto an ovoid of the hyperbolic quadric $Q^+(5,q)$ in the quotient geometry of XX'. We have a contradiction. $\square$

Theorem 3.9 *The three smallest minimal blocking sets B of $Q^+(2n+1,3)$, $n \geq 3$, are as follows:*

(1) *blocking sets of size $3^n + 3^{n-3}$ which are truncated cones $\pi^*_{n-4}\mathcal{O}$, where π_{n-4} is an $(n-4)$-dimensional space contained in $Q^+(2n+1,3)$, and where $\mathcal{O}$ is an ovoid contained in a 6-dimensional parabolic quadric $Q(6,3)$ contained in $\pi^{\perp}_{n-4}$;*

(2) *blocking sets of size $3^n + 2 \cdot 3^{n-3}$ which are truncated cones $\pi^*_{n-4}B$, where π_{n-4} is an $(n-4)$-dimensional space contained in $Q^+(2n+1,3)$, and where B is a minimal blocking set of size 3^3+2, as described in Example 1.1, contained in a 7-dimensional hyperbolic quadric $Q^+(7,3)$ contained in $\pi^{\perp}_{n-4}$;*

(3) *blocking sets of size $3^n + 3^{n-2}$ which are truncated cones $\pi^*_{n-3}\mathcal{O}$, where π_{n-3} is an $(n-3)$-dimensional space contained in $Q^+(2n+1,3)$, and where $\mathcal{O}$ is an ovoid contained in a 5-dimensional hyperbolic quadric $Q^+(5,3)$ contained in $\pi^{\perp}_{n-3}$.*

Proof. We use induction on n. For $n = 3$, the theorem is proven in Lemma 3.6. Suppose then that $n \geq 4$. If there exists a 3-secant to B, then Lemma 3.4 proves that B is a truncated cone P^*B', where B' is a minimal blocking set in the quotient geometry $Q^+(2n-1,q)$ of $P^{\perp} \cap Q^+(2n+1,q)$.

Since $|B'| = |B|/3 \leq q^{n-1} + q^{n-3}$, we can assume, by induction on n, that B' is as described in this theorem. So B' is either a truncated cone $\pi^*_{n-5}\mathcal{O}_1$, $\pi^*_{n-5}B''$, or $\pi^*_{n-4}\mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are ovoids in respectively $Q^+(7,q)$ and $Q^+(5,q)$, and where B'' is the minimal blocking set of $Q^+(7,q)$, described in Example 1.1. The description of B as presented in the statement of the theorem now follows immediately.

It suffices therefore to prove the existence of a 3-secant. We do this indirectly. Assume that there does not exist a 3-secant. By Lemma 3.3, no line is contained in B. Also, as $n \geq 4$, the quadric $Q^+(2n+1,q)$ does not have ovoids [15]. It follows that there exists a 2-secant to B. Then we find a special point. To derive a contradiction, we distinguish between the cases $n = 4$ and $n \geq 5$.

<u>The case $n = 4$</u>. Let P be a special point. By the previous lemma, the 2-secants on P form an ovoid in the hyperbolic quadric $Q^+(7,q)$ seen in the quotient geometry of P. As mentioned in the introduction, $Q^+(7,3)$ has a unique ovoid, which is in fact an ovoid lying in a parabolic quadric $Q(6,3)$ contained in $Q^+(7,3)$.

This ovoid has $q^3+1 = 28$ points, so there are 28 different 2-secants on P, which shows that $|P^\perp \cap B| = 56$. Then $|B \setminus P^\perp| \leq 34$, so $|B \setminus P^\perp| = 34 - k$ for some $k \geq 0$.

Consider the 2-secants $l_1, \ldots, l_{28}$ on P. Each such line l_i contains P and a second point P_i not in B. At least $q^3+1 = 28$ lines of $Q^+(9,3)$ on P_i meet B. Only the line $P_iP = l_i$ of these lines lies in $P^\perp$, as the lines l_i are pairwise non-perpendicular. Thus, at least 27 points of $P_i^\perp \cap B$ lie outside $P^\perp$. In other words, from the $34-k$ points of B outside $P^\perp$, at most $7-k$ are perpendicular to one of the two points of $l_i \cap B$. Thus, the number of incident pairs (X,Y), with $X \in P^\perp \cap B$ and $Y \in B \setminus P^\perp$, is at most $28 \cdot (7-k)$.

Consider the points $Y_1, \ldots, Y_{34-k}$ of $B \setminus P^\perp$, and let y_i be the number of points in $P^\perp \cap B$ perpendicular to Y_i. The hyperplane $Y_i^\perp$ meets each of the lines $l_1, \ldots, l_{28}$ in a unique point; exactly y_i of these points lie in B, so $28 - y_i$ of them lie in $M := \{P_1, \ldots, P_{28}\}$. For two different points Y_i and Y_j, we have

$$28 = |M| \geq |Y_i^\perp \cap M| + |Y_j^\perp \cap M| - |Y_i^\perp \cap Y_j^\perp \cap M|,$$

which implies that $|Y_i^\perp \cap Y_j^\perp \cap M| \geq 28 - y_i - y_j$. If $y_i + y_j \leq 11$, then the two subspaces generated by $Y_i^\perp \cap M$ and $Y_j^\perp \cap M$ have at least 17 points in common. So, $Y_i^\perp \cap M$ and $Y_j^\perp \cap M$ are two 6-dimensional spaces containing at least 17 points of an ovoid of $Q(6,q)$, $q=3$. From Lemma 3.1 and Remark 3.2, $Y_i^\perp \cap M$ and $Y_j^\perp \cap M$ are equal, so Y_i and Y_j lie in the perp of this 6-space $Q(6,q)$, which is a conic $C = Q(2,q)$.

From counting incidences, we have

$$\sum y_i \leq 28(7-k).$$

We may assume that $y_1 \leq y_2 \leq \ldots \leq y_{34-k}$. Then $y_1 \leq 28(7-k)/(34-k)$, so $y_1 < 6$, that is, $y_1 \leq 5$. Assume that $y_4 \geq 12 - y_1$. Then

$$3y_1 + (31-k)(12-y_1) \leq 28(7-k),$$

a contradiction to $y_1 \leq 5$. Hence $y_1 + y_2 \leq y_1 + y_3 \leq y_1 + y_4 \leq 11$. But then $Y_1, \ldots, Y_4$ lie in the perpendicular conic C of the 6-space, which is a conic also containing P. Since a conic has $q+1=4$ points, we have a contradiction.

The case $n \geq 5$. First consider the case that there exists a special point lying on at most $q^{n-1} + q^{n-3}$ different 2-secants. Then the 2-secants to B on this point form a blocking set in the quotient geometry of this point; this blocking set is known by the induction hypothesis. From this description in Theorem 3.9, it is deduced that it contains a 3-secant, so we have a 3-secant in the original space, a contradiction.

Now we consider the case that every special point lies on at least $q^{n-1}+q^{n-3}+1$ different 2-secants. This implies that every special point is perpendicular to at least $2q^{n-1}+2q^{n-3}+2$ points of B. This implies that two special points R and R' are necessarily perpendicular, since otherwise $|(R^\perp \cup R'^\perp) \cap B| \geq 3(q^{n-1} + q^{n-3} + 1) > q^n + q^{n-2} \geq |B|$.

Consider a special point R. The 2-secants to B on this point form a blocking set in the quotient geometry $Q^+(2n-1,q)$ of this point; as $n \geq 5$, then this $Q^+(2n-1,q)$ does not possess an ovoid, so we find two perpendicular 2-secants on R. Then we find a totally singular plane π on R that contains at least four points in B. As B has no 3-secants and does not contain lines, it follows that π has precisely four points in B and these form a conic. This conic has three exterior points, which are R and two more points R_1 and R_2. These exterior points form a triangle.

It follows from Lemma 2.1 (b) that there exists a point $P \in B$ with $P \notin R^\perp$. Then P can not be perpendicular to a point $Q \in B \cap R^\perp$, since otherwise the 2-secant PQ would contain a special point not perpendicular to R. Then the line $P^\perp \cap \pi$ does not contain the four points of the conic $\pi \cap B$ and hence $P^\perp \cap \pi$ is the line R_1R_2. As R_1 and R_2 are special points, it follows that the plane $\pi' = \langle R_1, R_2, P \rangle$ meets B in a conic. As before, R_1 and R_2 are exterior points of this conic. Also as before, this plane π' contains a third special point. This special point is not perpendicular to R, contradiction. □

4 Blocking sets in $Q^+(2n+1, 2)$

We now characterize the two smallest blocking sets of $Q^+(2n+1, 2)$. For $Q^+(5, 2)$, so $n = 2$, this amounts to classifying the unique ovoid and the unique minimal blocking set of size $6 = 2^n + 2^{n-1}$. For $Q^+(2n+1, 2)$, $n \geq 3$, this amounts to classifying a minimal blocking set of size $2^n + 2^{n-3}$ and a minimal blocking set of size $2^n + 2^{n-2}$. We characterize them both as respectively truncated cones over the unique ovoid of $Q^+(7, 2)$ [12] and over the unique ovoid of $Q^+(5, 2)$.

We first characterize the two smallest blocking sets of $Q^+(5, 2)$.

Theorem 4.1 *The two smallest blocking sets of* $Q^+(5, 2)$ *are the* 3*-dimensional elliptic quadric* $Q^-(3, 2)$*, and a truncated cone* $P^*Q(2, 2)$*, with* $Q(2, 2)$ *a conic contained in* $P^\perp$*.*

Proof. A blocking set of $Q^+(5, 2)$ corresponds via the Klein correspondence to a cover of lines of $PG(3, 2)$.

The ovoids of $Q^+(5, 2)$ correspond to the spreads of $PG(3, 2)$. It is known that $PG(3, 2)$ only has the regular spread, so the elliptic quadrics $Q^-(3, 2)$ are the only ovoids of $Q^+(5, 2)$.

A minimal blocking set of size 6 of $Q^-(5, 2)$ corresponds via the Klein correspondence to a minimal cover $\mathcal{C}$ of $PG(3, 2)$ of size 6.

Counting the incidences of the points of $PG(3, 2)$ with the lines of $\mathcal{C}$, we find that there is an excess of three; in other words, either there are three distinct points covered twice by the lines of $\mathcal{C}$, or there is one point that is covered three times and a second point covered twice by the lines of $\mathcal{C}$, or there is one point covered four times by the lines of $\mathcal{C}$.

But every plane contains at least one point covered at least twice by the lines of $\mathcal{C}$. Hence, by the Bose-Burton result [2], there must be a unique line ℓ whose points are covered twice by the lines of $\mathcal{C}$. If ℓ corresponds via the Klein correspondence to the point P of $Q^+(5,2)$, then $\mathcal{C}$ translates into a truncated cone $P^*Q(2,2)$, with $Q(2,2)$ a conic in a plane of $P^\perp$. $\square$

We now classify the two smallest blocking sets of $Q^+(7,2)$. It is known that $Q^+(7,2)$ has a unique ovoid of size 2^3+1 [12]. We now assume that B is a minimal blocking set of $Q^+(7,2)$ of size 2^3+2.

Theorem 4.2 *Every minimal blocking set B of size 10 of $Q^+(7,2)$ is a truncated cone $P^*Q^-(3,2)$, where $Q^-(3,2)$ is a 3-dimensional elliptic quadric of $Q^+(5,2)$ contained in $P^\perp$.*

Proof. We have $|B| = q^3 + \delta$ with $\delta = 2$, so B is not an ovoid. Also, Lemma 2.1 (a) implies that B does not contain lines. Since B is not an ovoid, there exists a line ℓ of $Q^+(5,2)$ containing the two points R_1 and R_2 of B.

Let Π be a generator through ℓ, then $\Pi \cap B = \{R_1, R_2\}$. Let S be a point of Π, $S \notin \ell$. Lemma 2.1 (b) shows that $|S^\perp \cap B| \leq 6$. The lines on S meeting B form a minimal blocking set in the quotient geometry on S (Corollary 2.2). This blocking set is not an ovoid, since the lines SR_1 and SR_2 are perpendicular. Hence, there are at least six such lines. It follows that $|S \cap B| = 6$ and no line on S meets B in more than one point. Also, S projects $S^\perp \cap B$ onto a minimal blocking set of size 6 of its quotient geometry $Q^+(5,2)$. So, this projection is a truncated cone with a point vertex over a quadric $Q(2,2)$. Thus, the six points of $S^\perp \cap B$ lie in pairs in three planes through a line h on S; also $h \cap B = \emptyset$. These three totally singular planes can be written as $\langle h, l_i \rangle$, where l_1, l_2, l_3 are lines meeting B in two points.

Each of the lines l_i meets h in one of its two points other than S. Then, we find a point V on h lying in two of the lines l_i. As V lies in at least five totally singular lines meeting B (Corollary 2.2), it follows that $|V^\perp \cap B| \geq 7$. Then Lemma 2.1 (b) implies that every totally singular line on V that meets B must meet B in two points. Then V lies on exactly five lines that meet B and the ten points of B occur in pairs on these lines. In the quotient geometry at V we see thus a blocking set of a $Q^+(5,2)$ with five points, which must be an elliptic quadric $Q^-(3,2)$. Hence B is a truncated cone $V^*Q^-(3,2)$. $\square$

We now present the general characterization result for $Q^+(2n+1,2)$, for arbitrary $n \geq 3$.

Theorem 4.3 *Let B be a minimal blocking set of size at most $2^n + 2^{n-2}$ of $Q^+(2n+1,2)$, $n \geq 3$.*

Then B is either a truncated cone $\Pi^*_{n-4}\mathcal{O}$, with $\mathcal{O}$ an ovoid in the quotient geometry $Q^+(7,2)$ of Π_{n-4}, or a truncated cone $\Pi^*_{n-3}Q^-(3,2)$, with $Q^-(3,2)$ a 3-dimensional elliptic quadric in the quotient geometry $Q^+(5,2)$ of Π_{n-3}.

Proof. We use induction on n, the case $n = 3$ handled previously in this section. Suppose now that $n \geq 4$.

Put $|B| = 2^n + \delta$. It is known that $Q^+(9,2)$, and, by consequence, also $Q^+(2n+1,2)$, $n > 4$, has no ovoid [1]. Hence we find a totally singular line ℓ containing at least two points R_1, R_2 of B. Let V be the remaining point of ℓ. From Lemma 2.1 we know that $|R_i^\perp \cap B| \leq \delta$. Hence, at most $2(\delta - 2)$ points of B outside ℓ are perpendicular to R_1 or R_2. Then $|V^\perp \cap B| \geq |B| - 2(\delta - 2) > 2^{n-1} + 2^{n-2}$. Hence $V \notin B$ and every generator through V contains at least two points of B (Lemma 2.1). Furthermore, every secant to B on V meets B in exactly two points (Lemma 2.1 (b)).

Then V projects $V^\perp \cap B$ onto a minimal blocking set B' of $Q^+(2n-1,2)$ of size $|B'| = |B|/2 \leq 2^{n-1} + 2^{n-3}$. By induction, these blocking sets B' are characterized as truncated cones $\Pi^*_{n-5}\mathcal{O}$, with $\mathcal{O}$ an ovoid in the quotient geometry $Q^+(7,2)$ of Π_{n-5} with relation to $Q^+(2n-1,2)$, or a truncated cone $\Pi^*_{n-4}Q^-(3,2)$, with $Q^-(3,2)$ a 3-dimensional elliptic quadric in the quotient geometry $Q^+(5,2)$ of Π_{n-4} with relation to $Q^+(2n-1,2)$. As B is a truncated cone with vertex V over B', then B is either the truncated cone $\langle V, \Pi_{n-5}\rangle^*\mathcal{O}$ or the truncated cone $\langle V, \Pi_{n-4}\rangle^* Q^-(3,2)$. □

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