Partial ovoids and partial spreads in symplectic and orthogonal polar spaces

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Abstract

We present improved lower bounds on the sizes of small maximal partial ovoids and small maximal partial spreads in the classical symplectic and orthogonal polar spaces, and improved upper bounds on the sizes of large maximal partial ovoids and large maximal partial spreads in the classical symplectic and orthogonal polar spaces. An overview of the status regarding these results is given in tables. The similar results for the hermitian classical polar spaces are presented in [10].

1 Introduction

The *classical finite polar spaces* are the non-singular symplectic polar spaces $W(2n+1,q)$, the non-singular parabolic quadrics $Q(2n,q)$, $n \geq 2$, the non-singular elliptic and hyperbolic quadrics $Q^-(2n+1,q)$, $n \geq 2$, and $Q^+(2n+1,q)$, $n \geq 1$, and the non-singular hermitian varieties $H(d,q^2)$, $d \geq 3$. For q even, the parabolic polar spaces $Q(2n,q)$ are isomorphic to the symplectic polar spaces $W(2n-1,q)$.

The *generators* of a classical polar space are the subspaces of maximal dimension contained in these polar spaces. If the generators are of dimension $r-1$, then the polar space is said to be of *rank* r.

The polar spaces of rank $r=2$ coincide with the generalized quadrangles.

A *generalized quadrangle* $\mathcal{Q}$ of order (s,t), also denoted by GQ(s,t), is an incidence structure $\mathcal{Q}=(P,B,I)$, consisting of a set P of *points*, a set B of *lines*, and a symmetric incidence relation $I \subset (P \times B) \cup (B \times P)$ satisfying the following four axioms:

- Every point is incident with exactly $t+1$ lines and two points are both incident with at most one line.
- Every line is incident with exactly $s+1$ points, and two distinct lines are both incident with at most one point.
- If a point R is not incident with a line ℓ, then there is a unique point-line pair (T, m), such that $RImITI\ell$.
- There exists a non-incident point-line pair.

Interchanging the roles of points and lines in a GQ(s,t) gives the *dual* generalized quadrangle of order (t,s).

The finite classical generalized quadrangles are the non-singular parabolic quadric $Q(4,q)$ of order (q,q), the non-singular elliptic quadric $Q^-(5,q)$ of order (q,q^2), the non-singular hyperbolic quadrics $Q^+(3,q)$ of order $(q,1)$, the non-singular hermitian varieties $H(3,q^2)$ and $H(4,q^2)$ of respective orders (q^2,q) and (q^2,q^3), and the symplectic generalized quadrangle $W(3,q)$ in $PG(3,q)$ of order (q,q). The generalized quadrangles $Q(4,q)$ and $W(3,q)$ are dual to each other. The generalized quadrangles $Q(4,q)$ and $W(3,q)$ are self-dual if and only if q is even. Finally, $H(3,q^2)$ and $Q^-(5,q)$ also are dual to each other.

An *ovoid* of a classical polar space $\mathcal{P}$ is a set $\mathcal{O}$ of points of $\mathcal{P}$ such that every generator contains exactly one point of $\mathcal{O}$. A *partial ovoid* of a classical polar space $\mathcal{P}$ is a set $\mathcal{O}$ of points of $\mathcal{P}$ such that every generator contains at most one point of $\mathcal{O}$. A *spread* of a classical polar space $\mathcal{P}$ is a set $\mathcal{S}$ of generators of $\mathcal{P}$ partitioning the point set of $\mathcal{P}$. A *partial spread* of a classical polar space $\mathcal{P}$ is a set $\mathcal{S}$ of pairwise disjoint generators of $\mathcal{P}$. A partial ovoid or spread is called *maximal* when it is not contained in a larger partial ovoid or spread of the same polar space.

Let $X := |\mathcal{P}|/|\Pi|$, where Π is a generator of $\mathcal{P}$. Then X is the size of an ovoid or spread in $\mathcal{P}$, in case $\mathcal{P}$ effectively contains an ovoid or spread. Assume that $\mathcal{O}$ is a partial spread or partial ovoid of $\mathcal{P}$, then $X - |\mathcal{O}|$ is called the *deficiency* of $\mathcal{O}$.

The first natural problem regarding ovoids and spreads in finite classical polar spaces is that of the existence of these ovoids and spreads. In [18, 32, 33], the known results on the existence or non-existence of ovoids and spreads in finite classical polar spaces are given.

Then research was focussed on the size of the largest partial ovoids and spreads of finite classical polar spaces that do not have ovoids or spreads, and to the problem of the extendability of partial ovoids and partial spreads to ovoids and spreads when the finite classical polar spaces have ovoids and spreads. We refer to [16, 32].

Recently, attention was also paid to the problem of the cardinality of the smallest maximal partial ovoids and the smallest maximal partial spreads in finite classical generalized quadrangles and polar spaces. We mention in particular [1, 2, 9, 11, 14, 22, 24]. In particular, [21, 24] addressed these problems for the classical generalized quadrangles.

We now present a large number of results on the smallest maximal partial ovoids and spreads, and on large maximal partial ovoids and spreads, for the finite classical orthogonal and symplectic polar spaces. For the analogous results for the classical hermitian polar spaces, we refer to [10]. We conclude the article with tables containing the present status on these problems.

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2 Glynn's techniques for quadrics

One of the first lower bounds on the size of partial spreads is by Glynn [15]. He shows that a maximal partial spread of $\mathrm{PG}(3, q)$ has at least $2q$ lines. Under the Klein-correspondence, this result translates into a result on maximal partial ovoids of $Q^+(5, q)$. In fact, one finds in the literature three different results on partial structures that all use the same technique.

Result 2.1 (a) [15] *A maximal partial spread of* PG$(3,q)$ *has at least* $2q$ *lines.*

(b) [14] *A maximal partial spread of* $H(3,q^2)$ *has at least* $2q+1$ *lines, and if* $q \geq 4$*, then it has at least* $2q+2$ *lines.*

(c) [9] *A maximal partial spread of* $W(3,q)$*,* q *odd, has at least* $1.419q$ *lines.*

Using the Klein-correspondence, each of these results translates into a result on maximal partial ovoids of the polar space $Q^+(5,q)$, $Q^-(5,q)$ or $Q(4,q)$. In fact, when looking at the translation of the proofs to the polar spaces, one sees immediately that all proofs are the same and can be moreover generalized to all quadrics; we explain below why $Q(2n,q)$, q even, does not occur.

Theorem 2.2 (a) *A maximal partial ovoid of* $Q^+(5,q)$ *has at least* $2q$ *points. A maximal partial ovoid of* $Q^+(2n+1,q)$*,* $n \geq 3$*, has at least* $2q+1$ *points.*

(b) *A maximal partial ovoid of* $Q^-(5,q)$ *has at least* $2q+1$ *points. If* $q \geq 4$*, it has at least* $2q+2$ *points. A maximal partial ovoid of* $Q^-(2n+1,q)$*,* $n \geq 3$*, has at least* $2q+1$ *points.*

(c) *A maximal partial ovoid of* $Q(4,q)$*,* q *odd, has at least* $1.419q$ *points. A maximal partial ovoid of* $Q(2n,q)$*,* $n=3$*, has at least* $2q$ *points if* $q \in \{3,5,7\}$*, and at least* $2q-1$ *points for odd* $q \geq 9$*. A maximal partial ovoid of* $Q(2n,q)$*,* $n \geq 4$*,* q *odd, has at least* $2q+1$ *points, except for* $n=4$ *and* $q=3$ *when the bound is only* $2q$*.*

Proof. (a) Let $\mathcal{O}$ be a maximal partial ovoid of $Q^+(2n+1,q)$. Let $w = |\mathcal{O}|$, and denote by n_i the number of points of $Q^+(2n+1,q) \setminus \mathcal{O}$ that are joined to exactly i points of $\mathcal{O}$ by lines of $Q^+(2n+1,q)$. Then we have

$$\sum_i n_i = |Q^+(2n+1,q)| - w, \tag{1}$$

$$\sum_i n_i i = wq|Q^+(2n-1,q)|, \tag{2}$$

$$\sum_i n_i i(i-1) = w(w-1)|Q^+(2n-1,q)|, \qquad (3)$$

$$\sum_i n_i i(i-1)(i-2) = w(w-1)(w-2)|Q(2n-2,q)|. \qquad (4)$$

The first equation just states that every point of the hyperbolic quadric outside $\mathcal{O}$ is counted. The second equation is obtained by counting pairs (u,v), with $u \in Q^+(2n+1,q) \setminus \mathcal{O}$ and $v \in \mathcal{O}$, such that uv is a line of the quadric. The third equation is obtained by counting triples (u, v_1, v_2), with $u \in Q^+(2n+1,q) \setminus \mathcal{O}$ and $v_1, v_2 \in \mathcal{O}$, such that $v_1 \neq v_2$ and uv_1 and uv_2 are lines of the quadric. The last equation is obtained by counting 4-tuples (u, v_1, v_2, v_3), with $u \in Q^+(2n+1,q) \setminus \mathcal{O}$ and $v_1, v_2, v_3 \in \mathcal{O}$, such that the points v_i are three distinct points of $\mathcal{O}$ and all lines uv_i belong to the quadric. Note that the three points v_i span a plane, whose polar space meets the quadric in a parabolic quadric $Q(2n-2,q)$. As the partial ovoid is maximal, we have $n_0 = 0$. Hence for each $a \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \sum_i n_i(i-1)(i-a)(i-a-1) \\ &= \sum_i n_i i(i-1)(i-2) - (2a-1)\sum_i n_i i(i-1) + (a^2+a)\sum_i n_i(i-1). \end{aligned}$$

We use $a = 3$. It follows by tedious but straightforward computations that $w > 2q - 1$ for $n = 2$ and $w > 2q$ for $n \geq 3$.

(b) This is very similar to the case $Q^+(2n+1,q)$; in the above formulas for the n_i the values $|Q^+(2n \pm 1, q)|$ have only to be replaced by $|Q^-(2n \pm 1, q)|$.

(c) This is slightly more delicate, here we have

$$\begin{aligned} \sum_i n_i &= |Q(2n,q)| - w, \\ \sum_i n_i i &= wq|Q(2n-2,q)|, \\ \sum_i n_i i(i-1) &= w(w-1)|Q(2n-2,q)|, \\ \sum_i n_i i(i-1)(i-2) &\leq w(w-1)(w-2)|Q^+(2n-3,q)|. \end{aligned}$$

In the last relation we now only have an inequality, since the common perp of three points is an $(2n-3)$-space meeting $Q(2n,q)$ either in an elliptic or hyperbolic non-degenerate quadric $Q^{\pm}(2n-3,q)$, so we can only prove an upper bound. We also use now the inequality

$$
\begin{aligned}
0 &\leq \sum_i n_i(i-1)(i-a)(i-a-1) \\
&= \sum_i n_i i(i-1)(i-2) - (2a-1)\sum_i n_i i(i-1) + (a^2+a)\sum_i n_i(i-1).
\end{aligned}
$$

However, for $n=2$ we use $a=4$ and obtain the result proven by the same technique in [9]. For $n \geq 3$, we use $a=3$. Then after some calculations $w > 2q-2$ if $n=3$, and $w > 2q$ for $n \geq 4$. □

Remark 2.3 The preceding techniques do not work for $Q(2n,q)$, q even, since this quadric has a nucleus N. This means that there are different types of conics on $Q(2n,q)$, q even. First of all, the conics of $Q(2n,q)$, q even, which have N as their nucleus, and secondly, the conics of $Q(2n,q)$, q even, which have a point different from N as nucleus. When wishing to use the same techniques for $Q(2n,q)$, q even, this has to be taken into account.

Fortunately, for $Q(2n,q)$, q even, it is not necessary to have to use the preceding techniques for finding the size of the smallest maximal partial ovoids. These quadrics are projectively equivalent to the symplectic polar spaces $W(2n-1,q)$, q even. For these symplectic polar spaces, the smallest maximal partial ovoids are characterized; they coincide with the hyperbolic lines ℓ of $W(2n-1,q)$ (Theorem 7.1).

The problem on the existence of ovoids of $Q^+(2n+1,q)$, $n > 3$, has only been solved for $q=2$ and $q=3$ [5, 19, 29]. The hyperbolic quadric $Q^+(7,q)$ has ovoids when q is even, q is an odd prime, or $q \equiv 0$ or $2 \pmod 3$. The Klein quadric $Q^+(5,q)$ has ovoids.

The study of the extendability of partial ovoids $\mathcal{O}$ to ovoids was performed in particular for partial ovoids $\mathcal{O}$ of size $q^2+1-\delta$, δ small, on the Klein quadric $Q^+(5,q)$. A partial ovoid of $Q^+(5,q)$ corresponds via the Klein-correspondence to a partial spread of $\mathrm{PG}(3,q)$, so extendability results on partial ovoids of $Q^+(5,q)$ are equivalent to extendability results on partial spreads in $\mathrm{PG}(3,q)$ [23].

Regarding partial ovoids of $Q^+(7,q)$, there are extendability results of partial ovoids $\mathcal{O}$ of size $q^3+1-\delta$, δ small, to ovoids [16].

The problem of finding a similar extendability result on partial ovoids of $Q^+(2n+1,q)$, $n>3$, has not yet been addressed. We now present such a result. In case ovoids of $Q^+(2n+1,q)$ would exist, therefore an extendability result is obtained. Otherwise, an upper bound on the size of partial ovoids of $Q^+(2n+1,q)$ is obtained.

Theorem 2.4 *A maximal partial ovoid $\mathcal{O}$ of $Q^+(2n+1,q)$, that is not an ovoid, has at most $q^n - q^{(n-1)/2}$ points.*

Proof. Let $\mathcal{O}$ be a maximal partial ovoid and denote its number of points by $w := q^n + 1 - \delta$. Consider a point P not in the partial ovoid. As the partial ovoid is maximal, some point R of $\mathcal{O}$ lies in the perp $P^\perp$ of P. Let π be a generator on P and R. Every point $X \in \mathcal{O}$, with $X \neq R$, gives rise to the hyperplane $X^\perp \cap \pi$ of π. Different points X give different hyperplanes of π, since an $(n-1)$-subspace of π lies only in two generators, one of which is π. As $R \in \mathcal{O}$, then the hyperplanes $X^\perp \cap \pi$ do not contain R. Since π has $q^n - q^{n-1}$ hyperplanes not containing P and not containing R, we see that at least $|\mathcal{O}| - 1 - (q^n - q^{n-1}) = q^{n-1} - \delta$ of the points $X \in \mathcal{O} \setminus \{R\}$ produce hyperplanes $X^\perp \cap \pi$ containing P. This shows that $|P^\perp \cap \mathcal{O}| \geq q^{n-1} + 1 - \delta$.

Denote by n_i the number of points of $Q^+(2n+1,q) \setminus \mathcal{O}$ that are joined to exactly i points of $\mathcal{O}$ by lines of $Q^+(2n+1,q)$. Then $n_i = 0$ for $i < q^{n-1}+1-\delta$ and $i > q^{n-1}+1$, and hence

$$0 \leq \sum_i n_i(i - q^{n-1} - 1)(i - q^{n-1})(i - q^{n-1} - 1 + \delta).$$

We can calculate the right hand side using the equations (1) to (4) of the previous proof, which also hold in our situation. The result is

$$((d-1)^2 - q^{n-1})d(q^{n-1}+1)q^{n-1}.$$

As this is non-negative, it follows that $d = 0$ or $(d-1)^2 \geq q^{n-1}$. □

Remark 2.5 Translating this result via the Klein-correspondence to $\mathrm{PG}(3,q)$, we find that a maximal partial line spread of $\mathrm{PG}(3,q)$, that is not a spread, has at most $q^2 - \sqrt{q}$ lines. This might indicate that we can not expect stronger results from a counting argument.

3 Inductive bounds

If it is known that ovoids do not exist in a particular polar space $\mathcal{P}_r$ of rank r, then this implies the non-existence of ovoids in higher rank polar spaces $\mathcal{P}_{r+r'}$, of the same type of rank $r + r'$, $r' > 0$. As particular example, we mention that the non-existence of ovoids in $Q^+(9, 2)$ [19] implies immediately the non-existence of ovoids in $Q^+(2n + 1, 2)$, $n > 4$.

This property makes it possible to formulate inductive bounds on the deficiencies of partial ovoids in two finite classical polar spaces $\mathcal{P}_r$ and $\mathcal{P}_{r+1}$ of the same type having ranks r and $r + 1$, if $\mathcal{P}_r$ does not have ovoids, and we know a deficiency result on partial ovoids of $\mathcal{P}_r$.

We first present a general bound, which works for all classical finite polar spaces. To simplify the proof, we give it for $Q^+(2n + 1, q)$.

Theorem 3.1 *Let $\mathcal{P}_r$ and $\mathcal{P}_{r+1}$ be two finite classical polar spaces of the same type, having rank r and $r + 1$, naturally embedded in a finite projective space of order q. Assume that partial ovoids of $\mathcal{P}_r$ always have at least deficiency ϵ_r, then partial ovoids of $\mathcal{P}_{r+1}$ have at least deficiency $q\epsilon_r$.*

Proof. Consider a singular line l of $\mathcal{P}_{r+1} = Q^+(2r + 3, q)$ on a point $P \in \mathcal{O}$. Then every other point of $\mathcal{O}$ is perpendicular to exactly one of the other q points of l. It follows that $q^{r+1} - \epsilon_{r+1} = |\mathcal{O}| - 1 \leq q(q^r - \epsilon_r)$. □

The following bound holds for maximal partial ovoids in the classical symplectic polar spaces.

Theorem 3.2 *If $x_{n,q}$ denotes the cardinality of a largest size of a partial ovoid of $W(2n + 1, q)$, then*

$$x_{n,q} \leq 2 + (q - 1)x_{n-1,q}.$$

Proof. Consider a partial ovoid $\mathcal{O}$ of $W(2n + 1, q)$, $n \geq 2$. Choose two points P, P' of $\mathcal{O}$ and consider the line PP' of the ambient projective space $\mathrm{PG}(2n+1, q)$ on P and P'. Every point of $\mathcal{O} \setminus PP'$ is perpendicular to exactly one point of $PP' \setminus \mathcal{O}$. On the other hand, if $R \in PP' \setminus \mathcal{O}$, then the points of $R^\perp \cap \mathcal{O}$ induce a partial ovoid in the $W(2n - 1, q)$ seen in the quotient geometry on R. Thus R is perpendicular to at most $x_{n-1,q}$ points of $\mathcal{O}$.

Assume that the line PP' contains s points of $\mathcal{O}$, then $|\mathcal{O}| \leq s + (q+1-s)x_{n-1,q} \leq 2 + (q-1)x_{n-1,q}$. □

If q is even, then $x_1 = q^2+1$, since $W(3,q)$ then is isomorphic to $Q(4,q)$ and has ovoids $Q^-(3,q)$. Then this formula already excludes ovoids in $W(5,q)$ and improves on Thas' upper bound q^3-q+2 for the size of partial ovoids in $W(5,q)$ [32]. If q is odd, then it is known that $x_1 \leq q^2-q+1$ [30] (for a different proof, see [21]). However, the inductive bound obtained from the lemma starting with x_1 is not the best we can do, as we will show in Section 6, where we will deduce a better upper bound on the size of partial ovoids in $W(5,q)$ (Theorem 6.1).

The following inductive bound of Klein and Thas holds for the elliptic polar spaces [20, 32].

Result 3.3 *If $x_{n,q}$ denotes the cardinality of a largest size partial ovoid of $Q^-(2n+1,q)$, then*

$$x_{n,q} \leq 2 + \frac{q^n+1}{q^{n-1}+1}(x_{n-1,q}-2).$$

Proof. See [20, Theorem 1]. □

The following inductive bound holds for the hyperbolic polar spaces.

Theorem 3.4 *If $x_{n,q}$ denotes the cardinality of a largest size partial ovoid of $Q^+(2n+1,q)$, then*

$$x_{n,q} \leq 2 + \frac{q^n-1}{q^{n-1}-1}(x_{n-1,q}-2).$$

Proof. This is proven in the same way as Result 3.3. □

Application 3.5 Suppose for a certain n and q, we know that $Q^+(2n+1,q)$ has no ovoid. Then, by Theorem 2.4, the partial ovoids of $Q^+(2n+1,q)$ have at most $q^n - q^{(n-1)/2}$ points. Then Theorem 3.1 shows that the partial ovoids of $Q^+(2m+1,q)$ have at most $q^m+1-q^{m-n}(q^{(n-1)/2}+1)$ points for $m \geq n$. This is better than the result of Theorem 2.4.

Remark 3.6 We now compare the inductive bounds of 3.1-3.4.

- For the symplectic polar spaces, the bound of Theorem 3.2 is better than the one of Theorem 3.1.
- For the elliptic polar spaces, the bound of Result 3.3 is better than the one of Theorem 3.1.
- For the hyperbolic polar spaces, the bound of Theorem 3.4 is better than the one of Theorem 3.1.

Thas [32] has shown that a partial ovoid of $Q^-(5,q)$ has at most q^3-q^2+q+1 points and that a partial ovoid of $Q^-(2n+1,q)$, $n \geq 3$, has at most $q^{n+1}-q^2+2$ points. We improved this result in the following theorem of [10].

Theorem 3.7 *A partial ovoid of* $Q^-(5,q)$ *has at most* $(q^3+q+2)/2$ *points.*

The inductive bound of Lemma 3.3 now leads to the following general upper bound.

Corollary 3.8 *A partial ovoid of* $Q^-(2n+1,q)$*,* $n \geq 2$*, has at most*

$$2 + \frac{1}{2} \cdot \frac{q^n - 1}{q+1} \cdot (q^2 + q + 2)$$

points.

4 Upper bounds on the size of partial ovoids in parabolic quadrics

4.1 The non-prime case

A nice feature of Theorem 2.4 is that it now can be used to obtain a similar result on the deficiency of maximal partial ovoids of positive deficiency on $Q(2n,q)$, q odd, q not a prime.

Lemma 4.1 *Consider $Q(2n,q) \subseteq Q^+(2n+1,q)$, $n \geq 3$, q odd, q not a prime, and suppose that $Q^+(2n+1,q)$ has an ovoid $\mathcal{O}$ with $q^n+1-\delta$, $\delta > 0$, points in $Q(2n,q)$. Then $\delta \geq 2(q^{n-2}+q^{n-3}+\cdots+1)+1$.*

Proof. Assume on the contrary that $0 < \delta \leq 2(q^{n-2}+\cdots+q+1)$. Let $\mathcal{M} = \{P_1,\ldots,P_\delta\}$ be the points of $\mathcal{O}$ not in $Q(2n,q)$. Let us call a generator of $Q(2n,q)$ a *free* generator if it does not meet $\mathcal{O}$. Every free generator of $Q(2n,q)$ lies in two generators of $Q^+(2n+1,q)$ and these meet $\mathcal{M}$. In other words, for every free generator π, the subspace $\pi^\perp$ meets $\mathcal{M}$ in two points.

On the other hand, every point P_i gives rise to the quadric $P_i^\perp \cap Q(2n,q) = Q^+(2n-1,q)$, and the $2(q+1)(q^2+1)\cdots(q^{n-1}+1)$ generators of this $Q^+(2n-1,q)$ are free generators of $Q(2n,q)$. If two points P_i and P_j give rise to different quadrics $Q^+(2n-1,q)$, then these two $Q^+(2n-1,q)$ share either no generator if they intersect in a parabolic quadric $Q(2n-2,q)$, or exactly $2(q+1)\cdots(q^{n-2}+1)$ generators of $Q(2n,q)$ if they intersect in a tangent cone. As each free generator in one of the hyperbolic quadrics $Q^+(2n-1,q)$ occurs in exactly two of these hyperbolic quadrics $Q^+(2n-1,q)$ and as $\delta < q^{n-1}+1$, we see that each $Q^+(2n-1,q)$ that comes from a point P_i must arise from two points P_i and P_j. Hence δ is even, and the free generators of $Q(2n,q)$ are the generators of the $\delta/2$ different quadrics $P_i^\perp \cap Q(2n,q) = Q^+(2n-1,q)$.

We now reduce the problem to a problem on partial ovoids on $Q(6,q)$. The free generators of $Q(2n,q)$ to $\mathcal{O}$ belong to $\delta/2$ hyperbolic quadrics $Q^+(2n-1,q)$. Let $Q_1,\ldots,Q_{\delta/2}$ be the distinct hyperbolic quadrics $Q^+(2n-1,q)$ of $Q(2n,q)$ completely consisting of free generators to $\mathcal{O}$. Assume that Q_i corresponds to the points P_{2i-1} and P_{2i} of $\mathcal{M}$. These hyperbolic quadrics Q_i and Q_j pairwise intersect in a $(2n-2)$-dimensional parabolic quadric $Q(2n-2,q)$. Namely, if two of them, for instance, Q_i and Q_j, intersect in a tangent cone, then these hyperbolic quadrics share free generators. Let π be one of the free generators in $Q_i \cap Q_j$. Then π would lie in precisely two generators of $Q^+(2n+1,q)$ containing the points $P_{2i-1}, P_{2i}, P_{2j-1}, P_{2j}$ of $\mathcal{O}$. This is impossible.

Consider the hyperbolic quadric Q_1. The hyperbolic quadrics $Q_2,\ldots,Q_{\delta/2}$ cover in total $(\delta-2)|Q(2n-2,q)|/2 = (\delta-2)(q^{2n-3}+\cdots+q+1)/2 \leq (q^{n-2}+\cdots+q)(q^{2n-3}+\cdots+q+1)/2$ points of Q_1, counted with multiplicities. Since $|Q_1| = (q^{n-1}+1)(q^n-1)/(q-1)$, this shows that there is a point P of Q_1 lying in at most $q^{n-3}+\cdots+q$ other hyperbolic quadrics $Q_2,\ldots,Q_{\delta/2}$.

In total, this means that P lies in at least one and in at most $q^{n-3}+\cdots+q+1$ hyperbolic quadrics Q_i completely consisting of free generators to $\mathcal{O}$. Assume that P lies in $\delta'/2 \geq 1$ such distinct hyperbolic quadrics. This implies that $|P^{\perp} \cap \mathcal{O}| = q^{n-1} + 1 - \delta'$, where $2 \leq \delta' \leq 2(q^{n-3} + \cdots + q + 1)$. Projecting $P^{\perp} \cap \mathcal{O} \cap Q(2n, q)$ from P onto the base $Q(2n-2, q)$ of the tangent cone $P^{\perp} \cap Q(2n, q)$, a partial ovoid $\mathcal{O}'$ in $Q(2n-2, q)$ of size $q^{n-1} + 1 - \delta'$ is obtained, with $2 \leq \delta' \leq 2(q^{n-3}+\cdots+q+1)$, where there are $\delta'/2$ hyperbolic quadrics $Q^+(2n-3, q)$ of $Q(2n-2, q)$ completely consisting of free generators of $Q(2n-2, q)$ to $\mathcal{O}'$. Moreover, all free generators of $Q(2n-2, q)$ to $\mathcal{O}'$ belong to exactly one of those $\delta'/2$ hyperbolic quadrics.

Repeating this construction inductively, a partial ovoid $\mathcal{O}''$ in $Q(6, q)$ of size $q^3 + 1 - \delta''$, where $2 \leq \delta'' \leq 2(q + 1)$, is obtained, and where there are $\delta''/2$ hyperbolic quadrics $Q^+(5, q)$ of $Q(6, q)$ completely consisting of free generators of $Q(6, q)$ to $\mathcal{O}''$. Moreover, all free generators of $Q(6, q)$ to $\mathcal{O}'$ belong to exactly one of those $\delta''/2$ hyperbolic quadrics.

As $(\delta'' - 2)/2 < q + 1$, we find a point P in $Q(6, q)$ that lies in exactly one of the quadrics $Q^+(5, q)$ completely consisting of free generators to $\mathcal{O}''$. The free planes on that point are the $2(q + 1)$ free planes of a degenerate quadric $PQ^+(3, q)$, and exactly $q^2 - 1$ points of $\mathcal{O}''$ lie in $P^{\perp}$. If we go inside of $Q(6, q)$ to the quotient geometry $Q(4, q)$ on P, we see a partial ovoid of size $q^2 - 1$ in which the lines that do not meet the partial ovoid are the lines of a hyperbolic quadric $Q^+(3, q)$.

So we find a maximal partial ovoid of size $q^2 - 1$ on $Q(4, q)$, q odd, q not a prime. We know from [10] that this does not exist if q is different from a prime. □

Corollary 4.2 *The parabolic quadric $Q(6, q)$, q odd, q not a prime, does not have a maximal partial ovoid of size $q^3 + 1 - \delta$ with $0 < \delta < q + 1$.*

Proof. A partial ovoid of $Q(6, q)$, q odd, q not a prime, of size larger than $q^3 - q$, is also a partial ovoid of $Q^+(7, q)$, and can be extended to an ovoid of $Q^+(7, q)$ (Theorem 2.4). But then we have a contradiction in comparison to Lemma 4.1. □

Corollary 4.3 *A partial ovoid of the parabolic quadric $Q(8, q)$, q odd, q not a prime, has at most size $q^4 - q\sqrt{q}$.*

Proof. Gunawardena and Moorhouse proved that $Q(8,q)$, q odd, does not have ovoids [17]. A partial ovoid of $Q(8,q)$, q odd, q not a prime, of size larger than $q^4 - q\sqrt{q}$, is also a partial ovoid of $Q^+(9,q)$, and can be extended to an ovoid of $Q^+(9,q)$ (Theorem 2.4). But then we have a contradiction in comparison to Lemma 4.1. □

We now apply Theorem 3.1.

Corollary 4.4 *The size of a partial ovoid in the parabolic quadric $Q(2n,q)$, $n \geq 4$, q odd, q not a prime, is at most $q^n + 1 - q^{n-4}(q^{3/2} + 1)$.*

4.2 The prime case

We now concentrate on the maximal size for which partial ovoids exist on the parabolic quadrics $Q(2n,q)$, $q > 13$ prime, $n \geq 3$. It is known that every ovoid of $Q(4,q)$, q prime, is an elliptic quadric [3, 4]. This implies that $Q(6,q)$, $q > 3$ prime, has no ovoids [25], so consequently, also $Q(2n,q)$, $n > 3$, $q > 3$ prime, has no ovoids.

To find the upper bound on the size of the partial ovoids on these parabolic quadrics, we rely on the following two results.

Result 4.5 [3, 4] *Every ovoid of $Q(4,q)$, q prime, is an elliptic quadric.*

Result 4.6 [16] *A partial ovoid of $Q(4,q)$ of size q^2 is extendable to an ovoid of $Q(4,q)$.*

We now will exclude the existence of partial ovoids on $Q(6,q)$, $q > 13$ prime, of size $q^3 - 2q + 2$, extending the arguments of [25]. Let $\mathcal{O}$ be a partial ovoid of $Q(6,q)$, $q > 13$ prime, of size $q^3 + 1 - \delta$, with $\delta \leq 2q - 1$. A *free* generator of $\mathcal{O}$ is a generator not containing a point of $\mathcal{O}$.

Lemma 4.7 *Let l be a line of $Q(6,q)$ external to $\mathcal{O}$ with $|l^\perp \cap \mathcal{O}| = q + 1$. Then $l^\perp \cap \mathcal{O}$ is a conic.*

Proof. Let $l^\perp \cap \mathcal{O} = \{x_1, \ldots, x_{q+1}\}$. Let π be a generator of $Q(6,q)$ through l and let $\pi \cap \mathcal{O} = \{R\}$.

The tangent hyperplanes $S^\perp$ of the $q^3 - \delta$ points S of $\mathcal{O} \setminus \{R\}$ intersect π in lines not passing through R. Every line of π lies in $q+1$ generators of $Q(6,q)$, π included, so this shows that there are at most $q^2 \cdot q - (q^3 - \delta) = \delta$ lines of π, not passing through R, which lie in a free generator of $\mathcal{O}$.

The line l does not lie in a free generator. Since $\delta \le 2q-1$, there are at least two points z_1 and z_2 on l for which $|z_1^\perp \cap \mathcal{O}|, |z_2^\perp \cap \mathcal{O}| \ge q^2$.

The point sets $z_i^\perp \cap \mathcal{O}$ are projected from the points z_i onto elliptic quadrics $Q^-(3,q)_i$ if $|z_i^\perp \cap \mathcal{O}| = q^2 + 1$, and are projected from the points z_i onto elliptic quadrics $Q^-(3,q)_i$ minus one point if $|z_i^\perp \cap \mathcal{O}| = q^2$ (Results 4.5 and 4.6).

So $z_i^\perp \cap \mathcal{O}$ lies in a cone C_{z_i} with vertex z_i and base an elliptic quadric $Q^-(3,q)_i$, $i = 1, 2$. Let C_{z_i} lie in the 4-dimensional space π_i.

Now $\pi_i \neq l^\perp$ as π_i contains points of $\mathcal{O}$ not in $l^\perp$; so $\pi_i \cap l^\perp$ is a 3-dimensional space Σ_i on z_i. Hence, $C_{z_i} \cap l^\perp$ is a quadratic cone K_i in Σ_i.

Since $x_1, \ldots, x_{q+1}$ are the only points of $\mathcal{O}$ in $l^\perp$, necessarily K_i is the quadratic cone consisting of the lines $z_i x_1, \ldots, z_i x_{q+1}$.

Now $\Sigma_1 \neq \Sigma_2$, or else $z_1 z_2 = l \subset \Sigma_i \subset \pi_i$, implying that $l \cap \mathcal{O} \neq \emptyset$, as Σ_i meets $Q(6,q)$ in a quadratic cone K_i having on every one of its lines a point of $\mathcal{O}$.

Now Σ_1 and Σ_2 are two 3-dimensional spaces in the 4-dimensional space $l^\perp$, so they intersect in a plane containing the $q+1$ points of $l^\perp \cap \mathcal{O}$. Since the points of $l^\perp \cap \mathcal{O}$ are pairwise non-collinear, this plane intersects $Q(6,q)$ in a conic. □

Lemma 4.8 *Let P be a point of $Q(6,q) \setminus \mathcal{O}$ for which $|P^\perp \cap \mathcal{O}| \ge q^2$. Then $P^\perp \cap \mathcal{O}$ is a 3-dimensional elliptic quadric, or a 3-dimensional elliptic quadric minus one point.*

Proof. Consider the tangent hyperplane $P^\perp$ of P to $Q(6,q)$. This tangent hyperplane intersects $Q(6,q)$ into a cone with vertex P and base a parabolic quadric $Q = Q(4,q)$.

The point set $P^\perp \cap \mathcal{O}$ is projected from P onto an elliptic quadric Q_3, or elliptic quadric Q_3 minus one point, contained in Q (Results 4.5 and 4.6).

Let R be the polar point of Q_3 with respect to Q. The polar points with respect to Q of a bisecant l_1 of Q_3 form a conic C in a plane π through R.

A conic plane through l_1 to Q_3 has a polar line with respect to Q which is a line in π through R which is either external or bisecant to C. So R lies on $(q+1)/2$ bisecants to C in π, that is, R is an interior point of C.

At least $(q-1)/2$ planes through l_1 intersect Q_3 in a conic containing $q+1$ projected points of $P^\perp \cap \mathcal{O}$ and correspond under the polarity of Q to bisecants to C passing through R. If such a bisecant intersects C in the points P_1 and P_2, then PP_1 is a line of $Q(6,q)$ for which $|(PP_1)^\perp \cap \mathcal{O}| = q+1$. So $(PP_1)^\perp \cap \mathcal{O}$ is a conic (Lemma 4.7).

The preceding paragraph shows that every two points of $P^\perp \cap \mathcal{O}$ lie in at least $(q-1)/2$ conics completely contained in $P^\perp \cap \mathcal{O}$. We now show that $P^\perp \cap \mathcal{O}$ is either an elliptic quadric, or an elliptic quadric minus one point.

Consider two points R_1 and R_2 of $P^\perp \cap \mathcal{O}$. The line R_1R_2 lies in at least $(q-1)/2$ conics completely contained in $\mathcal{O}$. Let C_1 be one of those conics. Let R_3 be an other point of $P^\perp \cap \mathcal{O}$, not lying in C_1. Then also the line R_1R_3 lies in at least $(q-1)/2$ conics completely contained in $\mathcal{O}$. The fact that these conics all are projected from P onto the elliptic quadric Q_3 shows that at least $(q-3)/2$ of those conics through R_1R_3 share a point with $C_1 \setminus \{R_1\}$. So the 3-dimensional space $\langle C_1, R_3\rangle$ contains at least $2+(q-3)(q-1)/2+(q-(q-3)/2) = (q^2-3q+10)/2$ points of $P^\perp \cap \mathcal{O}$. Here, we first counted the points R_1 and R_3, then the points in the $(q-3)/2$ conics through R_1R_3 intersecting C_1 in a second point, and then the remaining points of C_1.

This shows that every two points of $P^\perp \cap \mathcal{O}$ lie in a 3-dimensional elliptic quadric $\mathcal{E}$ containing at least $(q^2-3q+10)/2$ points of $\mathcal{O}$.

Assume that not all the points of $P^\perp \cap \mathcal{O}$ lie in a 3-dimensional elliptic quadric. Then there exist at least two elliptic quadrics $\mathcal{E}_1$ and $\mathcal{E}_2$ containing at least $(q^2-3q+10)/2$ points of $P^\perp \cap \mathcal{O}$. Let $S_1 \in \mathcal{E}_1 \setminus \mathcal{E}_2$ and let $S_2 \in \mathcal{E}_2 \setminus \mathcal{E}_1$, $S_1, S_2 \in \mathcal{O}$, then S_1S_2 lies in a third elliptic quadric containing at least $(q^2-3q+10)/2$ points of $P^\perp \cap \mathcal{O}$. Using the fact that two distinct elliptic quadrics share at most $q+1$ points, this implies that $P^\perp \cap \mathcal{O}$ contains at least $3(q^2-3q+10)/2 - 3(q+1) > q^2+1$ points. This is false. $\square$

Theorem 4.9 *Every partial ovoid of* $Q(6,q)$, $q > 13$ *prime, contains at most* q^3-2q+1 *points.*

Proof. Let $\mathcal{O}$ be a partial ovoid of $Q(6,q)$ of size $q^3+1-\delta$, where $\delta \leq 2q-1$. In the proof of Lemma 4.7, the existence of a point P for which $|P^\perp \cap \mathcal{O}| \geq q^2$

was proven. Then the preceding lemma shows that there is a 3-dimensional elliptic quadric $\mathcal{E}$ completely contained, except for at most one point, in $\mathcal{O}$.

Let R be a point of $\mathcal{O}\setminus\mathcal{E}$. Then the 4-space $\langle \mathcal{E}, R\rangle$ intersects $Q(6,q)$ in a cone with base $\mathcal{E}$, or in a non-singular 4-dimensional parabolic quadric $Q(4,q)$. In the second case, since an ovoid of $Q(4,q)$ contains q^2+1 points, necessarily $\mathcal{E}$ contains q^2 points of $\mathcal{O}$. But then q generators of $Q(4,q)$ through R intersect $\mathcal{E}$ in a point of $\mathcal{O}$, so we find collinear points in $\mathcal{O}$, which is impossible. In the first case, this would imply that R is not the vertex of this cone, or else two points of $\mathcal{O}$ are collinear. This also implies that $\mathcal{E}$ contains exactly q^2 points of $\mathcal{O}$, and that R lies on the unique line of this quadratic cone passing through the unique point S of $\mathcal{E}$ not in $\mathcal{O}$.

But then S is collinear with all $|\mathcal{O} \setminus \mathcal{E}| > q^3 - q^2 - 2q + 1$ points of $\mathcal{O} \setminus \mathcal{E}$. This contradicts $|\mathcal{O} \cap S^\perp| \leq q^2 + 1$.

So, in both cases, we find a contradiction. No partial ovoids exist in $Q(6,q)$, $q > 13$ prime, of size larger than $q^3 - 2q + 1$. □

We now apply Theorem 3.1.

Corollary 4.10 *Every partial ovoid of $Q(2n,q)$, $q > 13$ prime, $n \geq 3$, contains at most $q^n - 2q^{n-2} + 1$ points.*

5 Maximal partial spreads on $Q(6,q)$ and $Q^+(7,q)$

Theorem 2.4 implies several other results on maximal partial ovoids and maximal partial spreads. Note that $Q(6,q)$ and $Q^+(7,q)$ have spreads when q is even, q is an odd prime, or $q \equiv 0$ or $2 \pmod 3$ [18, Table A VI.2].

Theorem 5.1 *The hyperbolic quadric $Q^+(7,q)$ does not have maximal partial spreads of size $q^3 + 1 - \delta$, for $0 < \delta < q + 1$.*

Proof. This follows from Theorem 2.4, by applying the triality principle. Under this principle, a maximal partial ovoid of $Q^+(7,q)$ corresponds to a maximal partial spread of $Q^+(7,q)$. □

Theorem 5.2 *The parabolic quadric $Q(6,q)$ does not have maximal partial spreads of size $q^3 + 1 - \delta$, for $0 < \delta < q + 1$.*

Proof. Let $\mathcal{S}$ be a partial spread of $Q(6,q)$ of size $q^3+1-\delta$, for $0<\delta<q+1$. Embed $Q(6,q)$ into $Q^+(7,q)$. The planes of $\mathcal{S}$ lie in two generators of $Q^+(7,q)$. Distinct generators of the same equivalence class of $Q^+(7,q)$ either are disjoint or intersect in a line.

So, if we consider the generators of the same equivalence class of $Q^+(7,q)$ containing a plane of $\mathcal{S}$, then a partial spread $\mathcal{S}'$ of $Q^+(7,q)$ is obtained. By the previous theorem, this partial spread $\mathcal{S}'$ is extendable to a spread $\mathcal{S}'^*$ of $Q^+(7,q)$. The intersections of the solids of $\mathcal{S}'^*$ with the parabolic quadric $Q(6,q)$ containing $\mathcal{S}$ form a spread $\mathcal{S}^*$ of $Q(6,q)$; so the partial spread $\mathcal{S}$ is extendable to a spread $\mathcal{S}^*$ of $Q(6,q)$. □

Associated to the 6-dimensional parabolic quadric $Q(6,q)$ is the *split Cayley hexagon* $H(q)$. As indicated in [16, Section 6], results on maximal partial spreads of $Q(6,q)$ imply results on maximal partial ovoids of the hexagon $H(q)$. The proof of [16, Corollary 6.1] can be used to prove the following result. For a brief description of the generalized hexagon $H(q)$, we refer to [16, Section 6].

Theorem 5.3 *Let $\mathcal{O}$ be a maximal partial ovoid of the generalized hexagon $H(q)$ of size $q^3+1-\delta$, where $0<\delta<q+1$. Then δ is even.*

Proof. The proof of [16, Corollary 6.1] proceeds in the following way. The partial ovoid $\mathcal{O}$ defines a partial spread $\mathcal{S}$, of the same size, of $Q(6,q)$. This partial spread $\mathcal{S}$ is extendable to a spread $\mathcal{S}^*$. Let π be a plane of $\mathcal{S}^* \setminus \mathcal{S}$. Either π consists of a point P and all the points at distance two of P in $H(q)$, or π consists of q^2+q+1 points of $H(q)$ which are pairwise at distance four. In the first case, P extends $\mathcal{O}$ to a larger partial ovoid, which is impossible. In the second case, there corresponds to π a second plane π^* also consisting of q^2+q+1 points at distance four. It is shown in [16, Corollary 6.1] that also π^* belongs to $\mathcal{S}^* \setminus \mathcal{S}$. So we can partitioning the planes of $\mathcal{S}^* \setminus \mathcal{S}$ into pairs. This shows that the deficiency δ is even. □

6 Upper bounds on the sizes of partial ovoids in symplectic polar spaces

Thas [32] has shown that a maximal partial ovoid of $W(2n+1,q)$, $n \geq 2$, has at most $q^{n+1}-q+2$ points. We improve this result in the following theorem.

Theorem 6.1 *A partial ovoid of $W(5,q)$ has at most*

$$1+\frac{q}{2}\left(\sqrt{5q^4+6q^3+7q^2+6q+1}-q^2-q-1\right)$$

points.

Proof. Let $\mathcal{O}$ be a partial ovoid of $W(5,q)$ with $s := |\mathcal{O}| = q^3+1-\delta$ points. We call a plane of $W(5,q)$ *free* if it is missing $\mathcal{O}$. The number of free planes is $E := \delta\theta_3$. We count the number of tuples (P_1,P_2,X,π), where P_1 and P_2 are different points of $\mathcal{O}$, where π is a free plane, and where X is a point of π that is collinear with P_1 and P_2 in $W(5,q)$.

For a free plane π, every point of $\mathcal{O}$ is perpendicular to $q+1$ points of π. It follows that the number of triples (P_1,P_2,X) of different points $P_1,P_2 \in \mathcal{O}$ and points $X \in \pi$ perpendicular to P_1 and P_2 is at least $A(A-1)(q^2+q+1)$ with $A := s(q+1)/(q^2+q+1)$.

Thus the total number of tuples (P_1,P_2,X,π) is at least $\delta\theta_3 A(A-1)(q^2+q+1)$. It follows that there exist two different points $P_1,P_2 \in \mathcal{O}$ that occur in at least

$$i := \frac{\delta\theta_3 A(A-1)(q^2+q+1)}{s(s-1)} = \frac{\delta\theta_3(q+1)(A-1)}{s-1}$$

of these 4-tuples. Let l be the secant line on P_1 and P_2 of $\mathrm{PG}(5,q^2)$. Then $l^\perp$ is a 3-space meeting $W(5,q)$ in a symplectic polar space $W(3,q)$. No point of $\mathcal{O}$ lies in $l^\perp$. Denote by k the number of points of $\mathcal{O}$ on l.

For this line l, we now count the number of pairs (X,Y), with X in $W(3,q) = W(5,q)\cap l^\perp$ and Y a point of $\mathcal{O}$ but not on l, $X \in Y^\perp$.

Starting with Y, we find that this number is $(s-k)(q^2+q+1)$, since $Y \notin l$ implies that $Y^\perp \cap l^\perp$ is a plane.

Starting with X, we see in the quotient space $X^\perp$ of X a polar space $W(3,q)$, and $X^\perp \cap \mathcal{O}$ induces a partial ovoid in $X^\perp$. If i_X is the number of free planes on X, then $|X^\perp \cap \mathcal{O}| = q^2+1-\frac{i_X}{q+1}$. Exactly k of these points lie on the line l. As $\sum i_X \geq i$, we find the upper bound

$$\theta_3(q^2+1-k) - \frac{i}{q+1}$$

for the number of pairs (X,Y). It follows that

$$(s-k)(q^2+q+1) \leq \theta_3(q^2+1-k) - \frac{\delta\theta_3(q+1)(A-1)}{(s-1)(q+1)}.$$

As $k \geq 2$, this remains true when k is replaced by two. Doing this and replacing $A = s(q+1)/(q^2+q+1)$ and $s = q^3+1-\delta$, we find

$$\frac{q^2(-q^6+q^3+d(3q^3+q^2+q)-d^2)}{(q^2+q+1)(q^3-d)} \geq 0.$$

Hence

$$-q^6+q^3+d(3q^3+q^2+q)-d^2 \geq 0.$$

Solving for δ gives

$$\delta \geq \frac{q}{2}\left(3q^2+q+1-\sqrt{5q^4+6q^3+7q^2+6q+1}\right)$$

and the assertion follows. □

Remark 6.2 For $q=2$, the known results of Dye [13] are better than the bound of Theorem 6.1. For $q=3$, a better bound is obtained from the inductive bound of Theorem 3.2 using that a partial ovoid of $W(3,q)$ has at most q^2+1-q points [30]. For larger q however, the bound in theorem 6.1 is better. We give a table with the bound for $q \leq 11$.

$q =$	2	3	4	5	7	8	9	11
$s \leq$	7	16	43	83	222	329	466	845

Corollary 6.3 *A partial ovoid of $W(2n+1,q)$, $q \neq 2$, $n \geq 2$, has at most*

$$2\frac{(q-1)^{n-2}-1}{q-2} + (q-1)^{n-2}x_2$$

points, where $x_2 = 1 + \frac{q}{2}\left(\sqrt{5q^4+6q^3+7q^2+6q+1} - q^2 - q - 1\right)$.

Proof. The statement follows from the preceding theorem and the inductive bound of Theorem 3.2. □

7 Lower bounds on the size of partial ovoids in symplectic polar spaces

After discussing upper bounds on the size of partial ovoids in symplectic polar spaces, we discuss lower bounds on the size of maximal partial ovoids of symplectic polar spaces. This follows greatly the results of Cimráková, De Winter, Fack, and Storme, who investigated this problem for the symplectic generalized quadrangle $W(3, q)$ [9].

Theorem 7.1 (a) *A maximal partial ovoid $\mathcal{O}$ of $W(2n+1, q)$ is a minimal blocking set with respect to the hyperplanes of* $\mathrm{PG}(2n+1, q)$.

(b) *The smallest maximal partial ovoids of $W(2n+1, q)$ are equal to hyperbolic lines ℓ of $W(2n+1, q)$.*

Proof. (a) If $\mathcal{O}$ would be skew to some hyperplane Π of $\mathrm{PG}(2n+1, q)$, then $\Pi = R^{\perp}$ for some point R, and then R extends $\mathcal{O}$ to a larger partial ovoid.

Assume that the point R is not necessary in $\mathcal{O}$ in order for $\mathcal{O}$ to be a blocking set. Then every hyperplane through R contains a second point of $\mathcal{O}$. In particular, $R^{\perp}$ contains a second point R' of $\mathcal{O}$. But then there is at least one generator containing R and R', contradicting the definition of partial ovoid.

(b) The theorem of Bose and Burton [6] now implies that lines are the smallest candidates for maximal partial ovoids of $W(2n+1, q)$. Effectively, the hyperbolic lines of $W(2n+1, q)$ are maximal partial ovoids of $W(2n+1, q)$, and every line of $\mathrm{PG}(2n+1, q)$ that is a maximal partial ovoid of $W(2n+1, q)$ is a hyperbolic line of $W(2n+1, q)$. □

We now focus on the problem of finding the second smallest maximal partial ovoids of $W(2n+1, q)$.

Lemma 7.2 *Let $\mathcal{O}$ be a maximal partial ovoid in $W(2n'+1, q)$. Then $\mathcal{O}$ induces a maximal partial ovoid of the same size in $W(2n+1, q)$, $n \geq n'$.*

Proof. Consider $W(2n+1, q)$ in its canonical bilinear form $F = (X_0Y_1 - X_1Y_0) + (X_2Y_3 - X_3Y_2) + \cdots + (X_{2n}Y_{2n+1} - X_{2n+1}Y_{2n})$.

Construct the maximal partial ovoid $\mathcal{O}$ in the symplectic space $W(2n'+1,q)$: $X_{2n'+2} = \cdots = X_{2n+1} = 0$. Then $\mathcal{O}$ is a maximal partial ovoid of $W(2n+1,q)$.

Namely, for every point R in $W(2n+1,q)$, $R^\perp$ intersects $W(2n'+1,q)$ in at least a hyperplane. This intersection contains at least one point of $\mathcal{O}$. Hence, R does not extend $\mathcal{O}$ to a larger partial ovoid. □

In [9], the following example of a maximal partial ovoid of $W(3,q)$ was given.

Consider a hyperbolic line ℓ and let $P \in \ell$. Consider in $P^\perp$ on every totally isotropic line ℓ_i, $i = 1,\ldots,q+1$, through P exactly one point $P_i \notin \ell^\perp$. Then the set $(\ell \setminus \{P\}) \cup \{P_1,\ldots,P_{q+1}\}$ is a maximal partial ovoid of size $2q+1$ in $W(3,q)$, and consequently, it defines a maximal partial ovoid of size $2q+1$ in $W(2n+1,q)$.

Computer searches performed by Cimráková in $W(3,q)$, q small, suggest that this example is the second smallest maximal partial ovoid in $W(3,q)$ [8].

The smallest minimal blocking sets, different from lines, have been characterized for q square, and $q = p^3$, $p = p_0^h$, p_0 prime, $h \geq 1$. They are respectively Baer subplanes in $PG(2,q)$, q square, [7], and planar blocking sets of size p^3+p^2+1 and p^3+p^2+p+1 in $PG(2,p^3)$ equal to projected subgeometries $PG(3,p)$, and subgeometries $PG(3,p)$ naturally embedded in $PG(3,p^3)$ [26, 27, 28]. They were excluded as maximal partial ovoids of $W(3,q)$ in [9, 12]. We now exclude them as maximal partial ovoids in $W(2n+1,q)$, $n \geq 2$.

Theorem 7.3 *A Baer subplane $B = \mathrm{PG}(2,q)$ cannot be a partial ovoid of $W(2n+1,q^2)$, $n \geq 2$.*

Proof. Let Π be the plane $\mathrm{PG}(2,q^2)$ containing B. If $\Pi \subset P^\perp$ for some point P of Π, then there is a totally isotropic line through P, lying in Π, containing $q+1$ points of B, so then B is not a partial ovoid.

So, from now on, we assume that for every point P of Π, we have $\Pi \not\subset P^\perp$. Then $\Pi \cap \Pi^\perp = \emptyset$. Here, $\dim \Pi^\perp = 2n-2 \geq 2$.

Project from a point $R \in \Pi^\perp$ onto its quotient geometry $W(2n-1,q)$ in $R^\perp$. Then B can be considered as a Baer subplane which is a partial ovoid in this quotient geometry $W(2n-1,q)$.

By induction, we can reduce the problem to that of a Baer subplane that is a partial ovoid in $W(3,q^2)$. This was excluded in [9]. □

Theorem 7.4 *A subgeometry* $\mathrm{PG}(3,p)$ *or a projected subgeometry* $\mathrm{PG}(3,p)$ *cannot be a partial ovoid in* $W(2n+1,p^3)$*,* $n \geq 2$*.*

Proof. The projected subgeometries $\mathrm{PG}(3,p)$, which are planar blocking sets of size $p^3+p^2(+p)+1$, are eliminated by the same arguments as the Baer subplanes.

Let $\mathrm{PG}(3,p)$ be a subgeometry naturally embedded in $\Pi_3 = \mathrm{PG}(3,p^3)$. The existence of $\mathrm{PG}(3,p)$ as a partial ovoid of $W(3,p^3)$ was eliminated in [12]. Assume by induction on n that $\mathrm{PG}(3,p)$ is not a partial ovoid of $W(2n'+1,p^3)$, for $n' < n$.

If $\Pi_3 \cap \Pi_3^\perp = \emptyset$, then we can project from a point $P \in \Pi_3^\perp$ to its quotient geometry in $P^\perp$; to get $\mathrm{PG}(3,p)$ projected into a partial ovoid of $W(2n-1,p^3)$. By induction, this was excluded.

So $\Pi_3 \cap \Pi_3^\perp \neq \emptyset$. Let $P \in \Pi_3 \cap \Pi_3^\perp$, then P will project the subgeometry $\mathrm{PG}(3,p)$ onto a planar blocking set of size p^3+p^2+p+1 of $W(2n-1,p^3)$. This however is excluded in the same way as the Baer subplane was excluded as a partial ovoid of a symplectic polar space. □

As in [9, 12], this leads to the following corollaries.

Corollary 7.5 (1) *The second smallest maximal partial ovoids of* $W(2n+1,p^2)$*,* $n \geq 1$*,* $p > 2$ *prime, have at least size* $3(p^2+1)/2+1$*.*

(2) *The second smallest maximal partial ovoids of* $W(2n+1,p^3)$*,* $n \geq 1$*,* $p \geq 7$ *prime, have at least size* $3(p^3+1)/2$*.*

8 Small maximal partial spreads in polar spaces

Recently, in the case of the generalized quadrangles, particular attention was paid to small maximal partial spreads [1, 2, 9, 14]. Research on small maximal partial spreads in arbitrary classical polar spaces has not yet been performed. The following lower bound can be seen as the trivial lower bound on the size of maximal partial spreads in classical polar spaces.

Theorem 8.1 *Let $\mathcal{P}$ be a classical polar space. Let $\mathcal{P}'$ be the corresponding classical polar space of the same type of rank 2, i.e., which is a generalized quadrangle. Assume that $\mathcal{P}'$ has order (s,t).*

Then every maximal partial spread of $\mathcal{P}$ has at least size $t+1$.

Proof. Let $\mathcal{P} = \mathcal{P}_n$ be naturally embedded in the projective space of dimension n. Let $\mathcal{S} = \{\pi_1, \ldots, \pi_x\}$ be a maximal partial spread of $\mathcal{P}$. Let R be a hole.

Then $R^\perp$ intersects the generators $\pi_i, i = 1, \ldots, x$, into hyperplanes $\pi'_1, \ldots, \pi'_x$ of these generators. These hyperplane intersections $\pi'_1, \ldots, \pi'_x$ are projected from R onto generators $\pi^*_1, \ldots, \pi^*_x$ of the quotient polar space $\mathcal{P}^* = \mathcal{P}_{n-2}$ of R in $R^\perp$, forming a set of generators $\mathcal{S}^*$ in the polar space $\mathcal{P}_{n-2}$. Every generator of $\mathcal{P}_{n-2}$ must intersect at least one generator of $\mathcal{S}^*$, or else, if π is a generator of $\mathcal{P}_{n-2}$ skew to $\mathcal{S}^*$, then $\langle R, \pi\rangle$ is a generator of $\mathcal{P}_n$ skew to the maximal partial spread $\mathcal{S}$.

Repeating this argument, it is possible to obtain a set of generators $\mathcal{S}^{**}$ in the generalized quadrangle $\mathcal{P}'$ of order (s,t) such that every line of $\mathcal{P}'$ intersects at least one line in $\mathcal{S}^{**}$. This implies that $|\mathcal{S}| \geq |\mathcal{S}^{**}| \geq t+1$. □

For the hyperbolic quadrics $Q^+(2n+1,q)$, this would imply the lower bound $t = 2$. In case of the hyperbolic quadrics $Q^+(4n+3,q)$, this can be improved to the lower bound $q+1$.

Theorem 8.2 *A maximal partial spread of $Q^+(4n+3,q)$ has at least size $q+1$.*

Proof. Let $\mathcal{S}$ be a maximal partial spread of $Q^+(4n+3,q)$. The generators in $\mathcal{S}$ belong to the same equivalence class of generators of $Q^+(4n+3,q)$. The inductive argument of the proof of the preceding theorem leads to a set of generators of the hyperbolic quadric $Q^+(3,q)$, lying in the same equivalence class. These lines then lie in the same regulus of $Q^+(3,q)$, and intersect every line of $Q^+(3,q)$. Then $\mathcal{S}^{**}$ contains all lines of a given regulus of $Q^+(3,q)$. □

9 Tables

To present an overview of the current results on small or large maximal partial ovoids and maximal partial spreads in symplectic and orthogonal polar spaces, we collect the results in two tables. The similar results for the hermitian polar spaces are presented in [10]. We present the results for the classical symplectic and orthogonal polar spaces of rank $r \geq 3$. The corresponding results for rank $r = 2$, i.e., the classical finite generalized quadrangles are presented in [24].

polar space	lower bounds	sharp	upper bounds
$W(2n+1,q)$, $q>2$	$q+1$	yes	$2\frac{(q-1)^{n-2}-1}{q-2}+(q-1)^{n-2}\cdot$ $q^3+1-q(\sqrt{q}-1)(q-\sqrt{q}+1)$
$Q(2n+2,q)$, q even	$q+1$	yes	$2\frac{(q-1)^{n-2}-1}{q-2}+(q-1)^{n-2}\cdot$ $q^3+1-q(\sqrt{q}-1)(q-\sqrt{q}+1)$
$Q(6,q)$, q odd, q not prime	$2q-1$		q^3+1
$Q(2n,q)$, $n\geq 4$, q odd, q not prime	$2q+1$		$q^n+1-q^{n-4}(q\sqrt{q}+1)$
$Q(2n,q)$, $n\geq 3$, $q>13$ odd prime	$2q+1$		$q^n-2q^{n-2}+1$
$Q^-(2n+1,q)$, $n\geq 3$	$2q+1$		$2+\frac{1}{2}\cdot\frac{q^n-1}{q+1}\cdot(q^2+q+2)$
$Q^+(2n+1,q)$, $n\geq 3$, $q>3$	$2q+1$		q^n+1

Table 1: Bounds on maximal partial ovoids

In Table 1, the results for $W(2n+1,q)$ and for $Q(2n+2,q)$, q even, arise from Corollary 6.3, for $Q(2n,q)$, q odd, from Corollaries 4.4 and 4.10, for $Q^-(2n+1,q)$ from Corollary 3.8. Since the existence problem on ovoids of $Q^+(2n+1,q)$, $n \geq 3$, $q > 3$, is still open, we state the size of an ovoid as upper bound.

Remark 9.1 Next to the upper bounds on the size of maximal partial ovoids in symplectic and orthogonal polar spaces, presented in Table 1, there are the important bounds of Blokhuis and Moorhouse [5]. For large values of n, these upper bounds of Blokhuis and Moorhouse are better than the bounds of Table 1. It is however difficult to make an exact comparison between the bounds of Table 1 and those of Blokhuis and Moorhouse. For this reason, we refer to [5] for the bounds of Blokhuis and Moorhouse.

polar space	lower bounds	upper bounds	sharp
$W(2n-1,q)$	$q+1$	q^n+1	yes
$Q(2n,q)$, q even	$q+1$	q^n+1	yes
$Q(6,q)$, q odd, with q prime or $q \equiv 0$ or $2 \pmod 3$	$q+1$	q^3+1	yes
$Q(6,q)$, q odd, with q not prime, $q \equiv 1 \pmod 3$	$q+1$	q^3+1	
$Q(4n,q)$, $n \geq 2$, q odd	$q+1$	$q^n+1-\delta$	
$Q(4n+2,q)$, q odd, $n \geq 2$	$q+1$	q^n+1	
$Q^+(7,q)$, q odd, with q prime or $q \equiv 0$ or $2 \pmod 3$	$q+1$	q^3+1	
$Q^+(7,q)$, q odd, with q not prime, $q \equiv 1 \pmod 3$	$q+1$	q^3+1	
$Q^+(4n+3,q)$, $n > 1$, q odd	$q+1$	q^n+1	
$Q^+(4n+3,q)$, $n > 1$, q even	$q+1$	q^n+1	yes
$Q^+(4n+1,q)$, $n \geq 1$	2	2	yes
$Q^-(2n+1,q)$, $n \geq 3$, q even	q^2+1	$q^{n+1}+1$	yes
$Q^-(2n+1,q)$, $n \geq 3$, q odd	q^2+1	$q^{n+1}+1$	

Table 2: Bounds on maximal partial spreads

In Table 2, only the lower bound 2 on the size of small maximal partial spreads of $Q^+(4n+1,q)$ is known to be sharp. The last column indicates whether the upper bounds on the size of the largest maximal partial spreads are sharp.

The lower bound on the size of maximal partial spreads arises from Theorems 8.1 and 8.2. The results on the upper bounds on the size of maximal partial spreads arise from [18, Table AVI.2] where the list of the known results on the existence problem of spreads in the finite classical polar spaces is given.

The non-existence of spreads in $Q(4n,q)$, q odd, was proven in [31, 32]. In [16], the extendability problem of partial spreads, having small positive deficiency δ, to spreads, was discussed. In case of the non-existence of spreads, this result implies an upper bound on the size of a partial spread. This upper bound is related to the problem of the classification of the blocking sets in $\mathrm{PG}(2,q)$. In the table entry for $Q(4n,q)$, $n \geq 2$, q odd, there always holds that $\delta \geq \epsilon$ where $q+1+\epsilon$ is the size of the smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$. In cases that the smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$

are characterized, larger values of δ are allowed. For instance, for q an odd square, $q > 16$, the results of [16] imply that $\delta \geq q^{5/8}/\sqrt{2} + 1$.

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