One-point extensions of generalized hexagons and octagons

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Abstract

In this note, we prove the uniqueness of the one-point extension $\mathcal{S}$ of a generalized hexagon of order 2 and prove the non-existence of such an extension $\mathcal{S}$ of any other finite generalized hexagon of classical order, different from the one of order 2, and of the known finite generalized octagons provided the following property holds: for any three points x, y and z of $\mathcal{S}$, the graph theoretic distance from y to z in the derived generalized hexagon $\mathcal{S}_x$ is the same as the distance from x to z in $\mathcal{S}_y$.

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1 General Introduction

In 1981, Hölz [6] constructed a family of $2-(q^3+1, q+1, q+2)$-designs whose point set coincides with the point set of the Hermitian unital over the field $\mathsf{GF}(q)$, and with an automorphism group containing $\mathsf{PGU}_3(q)$. Here, q is any odd prime power. Two years later, Thas [9] proved that these designs are one-point extensions of the Ahrens-Szekeres generalized quadrangles $\mathsf{AS}(q)$ of order $(q-1, q+1)$ (see [2]).

Besides this infinite family of one-point extensions of generalized quadrangles, there are only four sporadic examples of one-point extensions of finite thick generalized polygons known.

(1) First there is the one-point extensions of the Fano plane leading to the design of points and planes in the affine geometry $\mathrm{AG}(3,2)$.

(2) The only other one-point extension of a projective plane is the unique Witt design on 22 points related to the Mathieu group M_{22}.

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(3) There exists a unique one-point extension of the unique generalized quadrangle of order 2, and

(4) a one-point extension of the split Cayley generalized hexagon $\mathsf{H}(2)$ of order 2.

The existence of these sporadic examples (different from the Witt design) is due to the fact that the point sets of the corresponding polygons can be identified with the non-zero vectors of some vector space over $\mathsf{GF}(2)$, while the lines can be identified with some *special* 2-spaces. To obtain a one-point extension, one adds the zero vector and all translates of the special 2-spaces. Such extension will be called the *affine extension of* the corresponding polygon.

We will characterize the affine extension $\mathcal{S}$ of the split Cayley hexagon $\mathsf{H}(2)$ using the following combinatorial property (to which we will refer as the *distance property*): for any three points x, y and z, the graph theoretic distance from y to z in the derived generalized hexagon $\mathcal{S}_x$ is the same as from x to z in $\mathcal{S}_y$. From this point on we shall denote the distance in the derived geometry at x by d_x.

Note that in every one of the above described affine extensions the distance property holds.

Theorem 1 *If* Γ *is a one-point extension of a generalized hexagon of classical order* (s,t) *satisfying the distance property, then* Γ *is isomorphic to the affine extension of the classical generalized hexagon* $\mathsf{H}(2)$.

The previous theorem can be extended to the known generalized octagons, for which we prove the following result:

Theorem 2 *There exists no one-point extension of known generalized octagons satisfying the distance property.*

The above results, together with knowledge of the maximal subgroups of the automorphism groups of the classical generalized polygons, imply the following characterization of the affine extension of $\mathsf{H}(2)$.

Theorem 3 *Suppose* Γ *is a one-point extension of a classical generalized hexagon or octagon admitting a flag (i.e. incident point-block) transitive automorphism group, then* Γ *is isomorphic to the affine extension of the classical generalized hexagon* $\mathsf{H}(2)$.

2 Preliminaries

2.1 Generalized n-gons

Let Γ be a point-line geometry with $\mathcal{P}$ the point set and $\mathcal{L}$ the line set. We will usually denote the (symmetric assumed) incidence relation by $\mathtt{I}$. The *incidence* *graph* of Γ is the (bipartite) graph with set of vertices $\mathcal{P} \cup \mathcal{L}$ and adjacency given by incidence. A *generalized n-gon* (or *generalized polygon* if n is unspecified) Γ (of order (s,t)) is a point-line geometry the incidence graph of which has diameter n and girth $2n$ (and every line is incident with $s+1$ points; every point incident with $t+1$ lines). Defining the *dual* of a point-line geometry Γ by the geometry obtained from γ by interchanging the point set with the line line set, immediately implies that the *dual* of a generalized polygon (of order (s,t)) is a generalized polygon (of order (t,s)).

Let Γ be a generalized polygon. The definition implies that, given any two elements a,b of $\mathcal{P}\cup\mathcal{L}$, either these elements are at distance n from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path (in the incidence graph) from a to b. In the latter case, let $\gamma = (a, \ldots, b'_a, b)$ be this shortest path, then the element b'_a is called the *projection* $\mathrm{proj}_a b$ *of* a *onto* b. We define $\Gamma_m(x)$ as the set of objects at distance m from x and denote $\Gamma_i(u) \cap \Gamma_{n-i}(w)$ by $u^w_{[i]}$. Furthermore it is convenient to write u^w instead of $u^w_{[2]}$. Whenever two points u and v are at distance $d < n$, we use the convention of denoting the unique point collinear to u at distance $d-2$ from w by u_w. Finally, the set $a^\perp$ is defined to be the set of all points collinear with a.

A path $(a,b,c,d,\ldots)$ shall occasionally be denoted by $a \mathtt{~I~} b \mathtt{~I~} c \mathtt{~I~} d \mathtt{~I~} \cdots$. Also, we denote collinearity by $\perp$, and then that path shall also be denoted by $a \perp c \perp \cdots$, if a is a point.

If $s=1$ or $t=1$, the n-gon is called *weak*, if both parameters are equal to 1 it is said to be *thin*, while for $s,t>1$ it is called *thick*.

Let (s,t) be the order of a generalized n-gon Γ with v points and b lines. The $2-(v+1, s+2, t+1)$ design $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ is said to be a *one-point extension* of Γ (or briefly *extension*) if for any point x of $\mathcal{P}$ the derived structure of $\mathcal{S}$ in x is a generalized n-gon of order (s,t), and for at least one point it is isomorphic to Γ. Recall that the derived structure of $\mathcal{S}$ in x is the 1-design $\mathcal{S}_x = (\mathcal{P}_x, \mathcal{B}_x, \mathtt{I}_x)$ with $\mathcal{P}_x = \mathcal{P} \setminus \{x\}$, $\mathcal{B}_x$ the set of all blocks of $\mathcal{B}$ incident with x in which we remove the point x, and $\mathtt{I}_x$ the incidence induced by $\mathtt{I}$.

We remark that in the literature one sometimes only considers one-point extensions in which all derived geometries are isomorphic. We do not require that.

2.2 Generalized hexagons

The only known thick finite generalized hexagons and octagons are so-called classical, i.e. they arise in a standard way from certain classes of Chevalley groups. For the generalized hexagons, these groups are $\mathsf{G}_2(q)$ and ${}^3\mathsf{D}_4(q)$. We provide some more details for each of these cases.

The generalized hexagons related to $\mathsf{G}_2(q)$ are denoted by $\mathsf{H}(q)$ and $\mathsf{H}(q)^D$ (for dual). Tits [10] constructs the *split Cayley hexagon* $\mathsf{H}(q)$ geometrically as follows. Consider a non-degenerate quadric $\mathsf{Q}(6,q)$ in the projective space $\mathsf{PG}(6,q)$. Choose coordinates in $\mathsf{PG}(6,q)$ in such a way that $\mathsf{Q}(6,q)$ has equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$, and let the points of $\mathsf{H}(q)$ be all points of $\mathsf{Q}(6,q)$. The lines of $\mathsf{H}(q)$ are the lines on $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{06}, p_{12}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$.

We will use the fact that, in $\mathsf{H}(q)$, for any triple of points x, y, z, with x opposite both y, z, one has $x^y = x^z$ whenever $|x^y \cap x^z| \geq 2$. Such a set x^y will be called an *ideal line* (it corresponds to a line of $\mathsf{Q}(6,q)$ not belonging to $\mathsf{H}(q)$). The dual $\mathsf{H}(q)^D$ does not have this property provided q is not divisible by 3 (if 3 divides q, then $\mathsf{H}(q)$ is isomorphic to $\mathsf{H}(q)^D$).

Furthermore, for every triple of elements x, y, z (all points or all lines) with x opposite y, z, one also has $x^y_{[3]} = x^z_{[3]}$ whenever $|x^y_{[3]} \cap x^z_{[3]}| \geq 2$. The sets $x^y_{[3]}$ will be called *reguli*, and this property is called the *regulus property*. In particular, if they consist of lines (points), we shall call them *line (point) reguli*. Every line (point) regulus is determined by two of its elements L, M (u, v) and we denote this line (point) regulus by $\mathcal{R}(L, M)$ $(\mathcal{R}(u, v))$.

For all these properties, see [12], 1.9.17 and 2.4.15.

The generalized hexagons $\mathsf{T}(q^3, q)$ and $\mathsf{T}(q, q^3) = \mathsf{T}(q^3, q)^D$ are constructed from ${}^3\mathsf{D}_4(q)$ and have order (q^3, q) and (q, q^3), respectively. We will not need the actual construction, but only the following properties. Every ordinary heptagon of the hexagon $\mathsf{T}(q, q^3)$ is contained in a (unique) subhexagon isomorphic to $\mathsf{H}(q)^D$. Moreover, both $\mathsf{T}(q^3, q)$ and $\mathsf{T}(q, q^3)$ satisfy the regulus property.

Up to duality, $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$ are the only known finite thick generalized hexagons. Moreover, by a result of Cohen and Tits [5], it is known that any finite thick generalized hexagon of order (s, t) with $s = 2$ is isomorphic to one of the classical hexagons $\mathsf{H}(2)$, $\mathsf{H}(2)^D$ or $\mathsf{T}(2, 8)$.

2.3 Generalized octagons

The only finite thick octagons known to date belong to the family of *Ree-Tits octagons* related to the twisted Chevalley groups of type ${}^2\mathsf{F}_4$ over a finite field $\mathbb{K}$ of even characteristic. In that case s is an odd power of 2 and $t = s^2$. We shall denote this octagon by $\mathsf{O}(s)$ and also call it *classical*. We will need an explicit construction of the smallest member of this family, but with the same effort, we can describe every member.

Let $\mathbb{K}$ be the finite field with 2^{2e-1} elements, for some positive natural number e. We denote the field automorphism $x \mapsto x^{2^e}$ by σ. For $k = (k_0, k_1) \in \mathbb{K}^2$, we set $\mathrm{Tr}(k) = k_0^{\sigma+1} + k_1$ and also $\mathrm{N}(k) = k_0^{\sigma+2} + k_0k_1 + k_1^{\sigma}$. Define a multiplication $a \otimes k = a \otimes (k_0, k_1) = (ak_0, a^{\sigma+1}k_1)$ for $a \in \mathbb{K}$ and $k \in \mathbb{K}^2$, and an addition $(k_0, k_1) \oplus (l_0, l_1) = (k_0 + l_0, k_1 + l_1 + l_0k_0^{\sigma})$, for $k, l \in \mathbb{K}^2$. Following Tits [11] we denote the group parameterized by the pairs $(k_0, k_1) \in \mathbb{K} \times \mathbb{K}$ with the previous addition as operation law by $\mathbb{K}_\sigma^{(2)}$. Also write $(k_0,k_1)^\sigma$ for (k_0^σ,k_1^σ). We denote the set of pairs of elements of $\mathbb{K}$ with the above structure by $\mathbb{K}_{(\sigma)}$.

Then the point set of $\mathsf{O}(q)$, $q=2^{2e-1}$, is the union of the sets $\{(\infty)\}$, $\mathbb{K}$, $\mathbb{K}_{(\sigma)}\times\mathbb{K}$, $\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}$, $\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}$, $\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}$, $\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}$ and $\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}$, and we write the elements with round parentheses. The line set is similarly the union of the sets $\{[\infty]\}$, $\mathbb{K}_{(\sigma)}$, $\mathbb{K}\times\mathbb{K}_{(\sigma)}$, $\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}$, $\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}$, $\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}$, $\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}$ and $\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}\times\mathbb{K}\times\mathbb{K}_{(\sigma)}$, and we write the elements with square brackets.

Incidence is given as follows. For every $a,a',a'',a''',b,b',b''\in\mathbb{K}$, and every k,k',k'', k''', l, l', $l''\in\mathbb{K}_{(\sigma)}$, we have

$$(a,l,a',l',a'',l'',a''')\ \mathtt{I}\ [a,l,a',l',a'',l'']\ \mathtt{I}\ (a,l,a',l',a'')\ \mathtt{I}\ [a,l,a',l']\ \mathtt{I}\ (a,l,a')\ \mathtt{I}\ [a,l]\ \mathtt{I}\ (a)\ \mathtt{I}\ [\infty]\ \mathtt{I}\ (\infty)\ \mathtt{I}\ [k]\ \mathtt{I}\ (k,b)\ \mathtt{I}\ [k,b,k']\ \mathtt{I}\ (k,b,k',b')\ \mathtt{I}\ [k,b,k',b',k'']\ \mathtt{I}\ (k,b,k',b',k'',b'')\ \mathtt{I}\ [k,b,k',b',k'',b'',k''']$$

and $(a,l,a',l',a'',l'',a''')\ \mathtt{I}\ [k,b,k',b',k'',b'',k''']$ if and only if

$$\begin{aligned}
k''' &= (l_0,l_1)\ \oplus\ a\otimes(k_0,k_1)\ \oplus\ (0,al_0'+a^\sigma l_0'')\\
b'' &= a'+a^{\sigma+1}\mathrm{N}(k)+k_0(al_0'+a^\sigma l_0''+\mathrm{Tr}(l))\\
&\quad +a^\sigma(a'''+l_0k_1)+al_0''^\sigma+l_0l_0'\\
k'' &= a^\sigma\otimes(k_1,\mathrm{Tr}(k)\mathrm{N}(k))\ \oplus\ k_0\otimes(l_0,l_1)^\sigma\\
&\quad \oplus\ (0,\mathrm{Tr}(k)\mathrm{N}(l)+a^{\sigma+1}l_0\mathrm{N}(k)^\sigma\\
&\quad +\mathrm{Tr}(k)(aa'+a^\sigma l_0l_0''+a^{\sigma+1}a''')\\
&\quad +\mathrm{Tr}(l)(k_1^\sigma a+a''')+k_1^\sigma a^{\sigma+1}l_0''+k_0^{\sigma+1}a^2l_0''^\sigma\\
&\quad +k_0(a'+al_0''^\sigma+k_1a^\sigma l_0+a^\sigma a''')^\sigma\\
&\quad +k_0^\sigma l_0(a'+al_0''^\sigma+k_1a^\sigma l_0+a^\sigma a''')\\
&\quad +a(l_1''+a'''^\sigma l_0+a'''l_0')\\
&\quad +l_0''(a'+a^\sigma a''')+a''l_0+l_0l_0'l_0'')\\
&\quad \oplus\ (l_0',l_1')\\
b' &= a''+a^{\sigma+1}\mathrm{N}(k)^\sigma+a(k_0l_0''+l_0k_1+a''')^\sigma\\
&\quad +\mathrm{Tr}(k)(l_1+a^\sigma l_0'')+k_0^\sigma(a'+a^\sigma a''')+l_0'l_0''+l_0^\sigma a'''\\
k' &= (l_0'',l_1'')\ \oplus\ a\otimes(\mathrm{Tr}(k),k_0\mathrm{N}(k)^\sigma)\ \oplus\ l_0\otimes(k_0,k_1)^\sigma\\
&\quad \oplus\ (0,\mathrm{N}(k)(a^\sigma l_0''+l_1)+k_0(a''+l_0'l_0''+aa'''^\sigma+l_0^\sigma a''')\\
&\quad +k_1(k_1l_0a^\sigma+a'+al_0''^\sigma+a^\sigma a''')\\
&\quad +k_0k_1^\sigma al_0^\sigma+a'''^\sigma l_0+a'''l_0')\\
b &= a'''+a\mathrm{N}(k)+l_0k_1+k_0l_0''
\end{aligned}$$

and no other incidences occur.

Finally, we will call the orders of the classical hexagons and octagons themselves *classical*.

3 Proof of the main results

The main goal of this section is to prove Theorem 1 and Theorem 2. We start with a very useful lemma on the parameters of a generalized n-gon, with $n = 6$ or 8, admitting a one-point extension satisfying the distance property. This lemma will be used to show that an extension of a generalized n-gon, with $n = 6$ or 8, of classical order (s, t) that satisfies the distance property, automatically has $s = 2$.

In Section 3.2 we prove Theorem 1. In Section 3.3 we give a second geometric characterization of the extension of $\mathsf{H}(2)$. In Section 3.4 we prove Theorem 2 and finally, in Section 3.5 we prove Theorem 3.

3.1 Preliminary results

Lemma 4 *Suppose $\mathcal{S}$ is a one-point extension of a generalized n-gon Γ of order (s, t), with $n = 6$ or 8. Then*

$$s + 2 \mid 2t(2t - 1)(t + 1)(1 - 2t + 4t^2)$$

when $n = 6$ and

$$s + 2 \mid 2t(4t^2 - 2t + 1)(t + 1)(1 - 2t + 4t^2 - 8t^3)$$

when $n = 8$.

Proof An elementary double counting of the incident point-block pairs within $\mathcal{S}$ yields the result. □

Corollary 5 *There are only finitely many one-point extensions of finite generalized 6 and 8-gons of classical order (s, t), with $s > 1$.*

Proof This follows directly from the divisibility conditions given in Lemma 4. □

From now on, we let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized n-gon, with $n = 6$ or 8 of order (s, t). Also, we denote by $\mathcal{S} = (\mathcal{P}', \mathcal{B}, \mathtt{I}')$ a one-point extension of Γ that satisfies the distance property. Without loss of generality, we may view $\mathcal{P}'$ as the point set of Γ union a new point α. Requiring $\mathcal{S}_\alpha$ to be isomorphic to Γ yields that we may take the points in $\{\alpha\} \cup \Gamma_1(L)$ as the points of a block of $\mathcal{S}$, and this for any line $L \in \mathcal{L}$. This type of blocks, referred to as *Line-blocks*, gives us a first block on any pair of collinear points of Γ. Suppose $\{x, y\}$ is such a collinear pair. Note that the graph theoretic distance in $\mathcal{S}_\alpha$ from a to b is simply given by the distance between these two points within Γ and shall hence be denoted by $d(a,b)$ instead of by $d_\alpha(a,b)$.

For convenience we introduce the following notation $a_0 \dots a_s$ to denote the line $\{a_0, \dots, a_s\}$ of $\mathcal{S}_x$.

If $B = \{x, y, b_0, \dots, b_{s-1}\}$ is a block on $\{x, y\}$ distinct of the Line-block defined by these two points, then inside $\mathcal{S}_x$ the point α is a point off the line $yb_0 \dots b_{s-1}$ for which $d_x(\alpha, y) = 2$. In other words $d_x(\alpha, b_i) = 4$ and hence by the distance property $d(x, b_i) = 4$, for $i \in \{0, \dots, s-1\}$. In the exact same way the situation within $\mathcal{S}_y$ yields $d(y, b_i) = 4$, for $i \in \{0, \dots, s-1\}$. If we now consider the derived generalized n-gon $\mathcal{S}_{b_0}$, then α is a point not on the line $xyb_1 \dots b_{s-1}$ for which $d_{b_0}(\alpha, x) = d_{b_0}(\alpha, y) = 4$. Hence, by definition of a derived generalized n-gon, there exists a unique point b_i collinear to α within $\mathcal{S}_{b_0}$. Consequently $d_{b_0}(\alpha, b_i) = 2$ and $d_{b_0}(\alpha, b_j) = 4$ or equivalently $d(b_0, b_i) = 2$ and $d(b_0, b_j) = 4$, for some $i \in \{1, \dots, s-1\}$ and all $j \in \{1, \dots, s-1\} \setminus \{i\}$. Since b_0 was chosen arbitrary, it is now easy to see that within Γ the $s+2$ points of B are paired off into $\frac{s+2}{2}$ lines through a common point z of Γ.

Note that the configuration of this second and final type of blocks on two collinear points, referred to as *Vee-blocks* with *Vee-point* z, immediately yields s is even and $\frac{s}{2} \leq t$. In any case, a thin generalized n-gon of order $(1,t)$ can never contain such Vee-blocks and is hence non-extendible under the distance property assumption.

Let χ_V be the total number of Vee-blocks. A double count of the incident couple-block pairs $((x,y), V)$, with $d(x,y) = 2$ and V a Vee-block yields

$$|\mathcal{P}|(t+1)st = \chi_V \frac{s+2}{2} 2$$

Let χ be the average number of such Vee-blocks through two points x, y with $d(x,y) = 4$. Then a similar double count yields

$$|\mathcal{P}|(t+1)sts\chi = \chi_V (s+2)s$$

and substituting the above obtained value of χ_V leads to $\chi = 1$. We now claim that no two distinct Vee-blocks share a common pair of points at distance 4. Indeed, let B and B' be two Vee-blocks, with respective Vee-points z and z' and suppose, by way of contradiction, that $\{x, y\} \subseteq B \cap B'$ with $d(x,y) = 4$. Then, obviously, the corresponding Vee-points coincide. So, $B = \{x, y, x_1, y_1\}$ and $B' = \{x, y, x_1', y_1'\}$. In $\mathcal{S}_x$ we see three lines, namely yx_1y_1, $yx_1'y_1'$ and $x_1x_1'z$, which form a triangle, a contradiction.

In other words, there is a unique Vee-block on any two points at distance 4.

Now consider a block $B = \{x, y, c_0, \dots, c_{s-1}\}$ on two points x and y at distance 4 from one another, that is distinct from the unique Vee-block on these two. Within $\mathcal{S}_x$ and $\mathcal{S}_y$ the relative position of the point α and the lines $yc_0 \dots c_{s-1}$ and $xc_0 \dots c_{s-1}$ together with the distance property in $\mathcal{S}$ results into $d(x, c_i) = 6$ and $d(y, c_i)$=6, for all $i \in \{0, \dots, s-1\}$, respectively. Inside $\mathcal{S}_{c_i}$ these distances, again together with the distance property in $\mathcal{S}$, first lead to the existence of a unique point c_j for which $d_{c_i}(\alpha, c_j) = 4$ and secondly to $d_{c_i}(\alpha, c_k) = 6$, for all $k \in \{0, \ldots, s-1\} \setminus \{i, j\}$. If we now let i run through the set $\{0, \ldots, s-1\}$ and carefully take all previously obtained distances into account, one can readily see that we have a subdivision of the points of B into $\frac{s+2}{2}$ pairs $\{a, b\}$ for which $d(a, b) = 4$ and $d(a, c) = d(b, c) = 6$ for all points c in $B \setminus \{a, b\}$.

With these partial results on $\mathcal{S}$, we are ready to state the following lemma.

Lemma 6 *Suppose $\mathcal{S}$ is a one-point extension of a generalized n-gon, with $n = 6$ or 8, of order (s, t) having the distance property. Then*

- *(a)* $s + 2 \mid 2t(t+1)$.
- *(b)* $t \geq s/2$.
- *(c)* *s is even.*

Proof As noted above, the configuration of the Vee-blocks in $\mathcal{S}$ immediately leads to *(b)* and *(c)*.

To prove part *(a)* of the lemma we consider a fixed point y of Γ and define X as the set of points in $y^\perp$ together with α and the point y itself. A double counting of the incident point-block pairs in X will then complete the proof of the lemma. Indeed, there are three type of points in X: the point α, the point y and any point z in $y^\perp$. Each of these points we claim to be incident with $t+1$ blocks that are entirely contained in X. First of all, since all blocks on α are Line-blocks there are $t+1$ such blocks of $\mathcal{S}$ on α in X. As any two points of $X \setminus \{\alpha\}$ are at most at distance 4 from one another in Γ, we only have to consider Line-blocks and Vee-blocks as possible blocks in X. The point y is, just as α, on $t+1$ Line-blocks in X. A Vee-block on this point would contain points outside X and hence these blocks do not contribute to the counting. Finally, a point z in $y^\perp$ is on a unique Line-block contained in X and since every two points at distance 4 determine a unique Vee-block we have $\frac{ts}{s}$ such blocks in X on z that have y as its Vee-point. The claim now follows.
Counting the incident point-block pairs in X, we obtain $(1+1+(t+1)s)(t+1) = \beta(s+2)$, where β stands for the number of blocks in X. This implies $s+2 \mid (2+st+s)(t+1)$ from which we deduce that

$$s + 2 \mid 2t(t+1).$$

This proves *(a)* and hence the lemma.

□

As a direct consequence of this lemma we find that

Corollary 7 *If there exists a one-point extension of a generalized hexagon of classical order (s, t), with $s \geq 2$, that satisfies the distance property, then $s = 2$.*

Proof By (b) of the above lemma $t \neq \sqrt[3]{s}$. If $t = 1$ or s, then by (a) we have $s + 2 \mid 4$ and $s = 2$. If $t = s^3$, then (a) implies $s + 2 \mid 2^4 \cdot 7$. Using that s is a power of 2 yields the corollary.

□

In other words, there are only four finite generalized hexagons of classical order which hypothetically can have a one-point extension satisfying the distance property, namely the split Cayley hexagon $\mathsf{H}(2)$, its dual, $\mathsf{T}(2, 8)$ and the weak hexagon of order $(2, 1)$.

Regarding finite generalized octagons the lemma implies

Corollary 8 *If there exists a one-point extension of a generalized octagon of classical order* (s, t)*, with* $s \geq 2$*, which satisfies the distance property, then it has order* $(2, 4)$*, order* $(4, 2)$ *or order* $(8, 64)$*.*

Proof By (b) of the above lemma $t = \sqrt{s}$ can only occur for $s = 4$ and the corresponding order $(4, 2)$ satisfies all other restrictions on s and t. By (a) and $t = s^2$ we immediately have $s + 2 \mid 40$ and thus $s = 2$ or $s = 8$. □

Nevertheless, if Γ is a generalized octagon, we are able to give some additional information on the third type of blocks in $\mathcal{S}$ and hereby improve the result in Lemma 6.

We use previous notation. Let $B = \{x, y, c_0, \ldots, c_{s-1}\}$ be a non-Vee-block on $\{x, y\}$, with $d(x, y) = 4$, and suppose B determines the following set of $\frac{s+2}{2}$ pairs of points $\{\{x, y\}, \{c_0, c_1\}, \ldots, \{c_{s-2}, c_{s-1}\}\}$. Remember that within each of these pairs the distance is 4 and all other distances between two points of B are 6. We shall now determine the relative position of a point c_i to the points x and y and claim that every such a point c_i is at distance 4 from the point x_y (=y_x). As $d(c_i, x) = d(c_i, y) = 6$, we immediately have $d(c_i, x_y) = 4$ or $d(c_i, x_y) = 8$. The latter case, however, leads to a contradiction as we shall show. Denote by $B' = \{x, y, b_0, \ldots, b_{s-1}\}$ the unique Vee-block on $\{x, y\}$ and let b_0 and b_1 be the points of B' collinear to x and y, respectively. Within $\mathcal{S}_x$ we now have $d_x(b_0, c_i) = 4$ and hence these two points uniquely determine a Vee-block B'' within $\mathcal{S}_x$. This block contains, next to b_0 and c_i, a point b_j for which $d(b_0, b_j) = 4$ (as x is the unique point of B' that is collinear to b_0). On the other hand, we know that $d(b_0, c_i) = 8$, a contradiction as any block on $\{b_0, b_j\}$ contains points at distance at most 6 from both of them. Hence the claim. If we denote the projection of c_i onto x_y by L_i, then one can prove that $L_i = L_j$, for all $i, j \in \{0, \ldots, s-1\}$. Indeed, first of all $d(c_0, c_1) = 4$ implies $L_0 = L_1$ (as otherwise we obtain a hexagon within Γ). Interchanging the role of $\{x, y\}$ and $\{c_0, c_1\}$ now leads to $d(c_j, x_y) = d(c_j, c_{0_{c_1}}) = 4$ and consequently to $L_j = L_0$, for all $j \in \{2, \ldots, s-1\}$.

In conclusion the block B, a so-called *Wee-block*, determines a unique line, the so-called *Wee-line* of B, at distance 3 from all of its points (distances in Γ). In the exact same way as before a double counting now yields a unique Wee-block on every two points at distance 6 in Γ.

Any other block on $\{x, y\}$, with $d(x, y) = 6$, will contain s additional points, $d_0, \ldots, d_{s-1}$, every one of which is opposite x and y (look within $\mathcal{S}_x$ and $\mathcal{S}_y$ to obtain this result).

We are now ready to state the next lemma.

Lemma 9 *Suppose $\mathcal{S}$ is a one-point extension of a generalized octagon of order (s, t) having the distance property. Then*

$$s + 2 \mid 4t.$$

Proof The proof of this lemma is similar to the proof of Lemma 6.

For some fixed line L of Γ we define the point set X as the union of the set of points on L, in $\Gamma_3(L)$ and in $\{\alpha\}$. Obviously, there are $1 + (s + 1)t$ Line-blocks on α in X. As any two points of $X \setminus \{\alpha\}$ are never opposite, the only blocks within X are Line-, Vee- and Wee-blocks. Next to α, we have two remaining type of points in X. Namely, the ones that are incident with L, and those that are not. A point y on L determines $t + 1$ Line-blocks and $\frac{sts}{s}$ Vee-blocks (with Vee-point on L) in X. In total we obtain $t + 1 + st = 1 + (s + 1)t$ blocks on y in X. Finally consider a point z off L in X. Such a point is contained in a unique Line-block and in $\frac{ts}{s}$ Vee-blocks (with Vee-point L'_z) of $\mathcal{S}$ in X. A Wee-block on z in X has L as its Wee-line and is determined by a single point at distance 6 from it. Hence there are $\frac{sts}{s}$ such blocks on z. In other words, we obtain a total of $1+t+st$ blocks on z. A double counting of incident point-block pairs in X leads to $(1+(s+1)+(s+1)ts)(1+t(s+1)) = \beta(s+2)$, where β stands for the number of blocks in X. From this we deduce that

$$s + 2 \mid 2t(1 - t),$$

which in combination with

$$s + 2 \mid 2t(1 + t)$$

completes the proof of the lemma.

□

As a direct consequence of this lemma we find that

Corollary 10 *If there exists a one-point extension of a generalized octagon of classical order (s, t), with $s \geq 2$, that satisfies the distance property, then $s = 2$ and $t = 4$.*

Proof By Lemma 6 we already know that such a generalized octagon Γ has to have order $(2, 4)$, $(4, 2)$ or $(8, 64)$. Obviously, Lemma 9 immediately rules out both orders $(4, 2)$ and $(8, 64)$, leaving us with $s = 2$ and $t = 4$ as the only possible parameters of Γ.

□

3.2 Proof of Theorem 1

Say Γ is a generalized hexagon of order $(2, t)$. We shall now try to construct an extension, $\mathcal{S}$, of Γ only using the distance property. First of all, the blocks constructed above for a generalized n-gon, with $n = 6$ or 8, determine three type of blocks in $\mathcal{S}$. Namely, Line-blocks, Vee-blocks and a third type of blocks that still has to be further analyzed. Since $s = 2$ a Vee-block consists of four points in the symmetrical difference of any two intersecting lines of Γ. If $B = \{x, y, c_0, c_1\}$ is a non-Vee-block on $\{x, y\}$, with $d(x, y) = 4$, then $d(c_0, c_1) = 4$ and all other distances between two points of B are 6. An easy counting argument shows that any two opposite points of Γ are in $t + 1$ of these blocks. Hence this type of blocks is in fact the final type of blocks in $\mathcal{S}$ and moreover the only type of blocks on any two opposite points.

Suppose that $B = \{a, b, c, d\}$ is a block of $\mathcal{S}$, with a and b opposite points in Γ. We will now determine the relative position of c to $a = a_0$ and of d to $b = a_3$ under the assumption that $d(a, c) = d(b, d) = 4$.

Denote the projection of b onto the line az (with $z = a_c$) by a_1 and b_{a_1} by a_2. Finally, denote the third point on any line $a_i a_j$ by a_{ij}, for i and j elements of $\{0, 1, 2, 3\}$.

With this notation, let us first assume that z differs from a_1 and call p_{cz} the third point on the line cz (from now on we shall always use this convention).

Within $\mathcal{S}_b$ we will show that this situation can never occur. First of all, we find $a_1 a_{12} a_{23}$ as a line in $\mathcal{S}_b$ corresponding to a Vee-block through b. On this line the point a_{23} is collinear to α, as it is collinear to b in Γ. On the other hand, we defined B as a block of $\mathcal{S}$ and thus find a, c and d also to be the points of a line in $\mathcal{S}_b$. We now claim that the point d is collinear to a_1 in $\mathcal{S}_b$. Indeed, in $\mathcal{S}_a$ the point c is both collinear to b and d (by definition of B) and to a_1 and p_{cz} (Vee-block). Therefore we find $d_a(b, a_1) = 4$ and thus $d_b(a_1, a) = 4$. In a similar way (by using the distance property in $\mathcal{S}_c$) one finds that $d_b(a_1, c) = 4$, which leads to the claim.

Now, as $d(b, d) = 4$, we obtain $d_b(d, \alpha) = 4$. This together with the fact that d is collinear to a_1 in $\mathcal{S}_b$, implies that d has to be the point a_{12}, contradicting the fact that d is opposite a in Γ.

Hence c has to be collinear to the projection from b onto a certain line through a. Note that this already implies $t \neq 1$.

Before starting to determine the actual structure of such a block we first show that, next to B being a block, a point p collinear to a_1 but distinct of c can never be in a block with a and b. Indeed, otherwise we obtain within $\mathcal{S}_a$ a triangle pbc when p is on the line $a_1 c$ or a quadrangle $bca_{01}p$ if p is on another line (distinct from aa_1, $a_1 c$ and $a_1 a_2$) through a_1 (note that this situation only occurs when $t > 2$).

This simple result implies that for every one of the paths from a to b there exists a block containing these two points and a point with the same relative position to a and b as c. In other words, for every path

$$a \mathtt{I} L \mathtt{I} a_1 \mathtt{I} M \mathtt{I} a_2 \mathtt{I} N \mathtt{I} b$$

exactly one of the points collinear to a_1, at distance 4 from a and opposite b, is in a block with a and b.

We now include this path $(a = a_0, a_1, a_2, a_3 = b)$ into an ordinary hexagon $(a_0, \ldots, a_6 = a_0)$ and suppose that d is, in the same way as c is to a_1, collinear to a_4. As we mentioned above, c and d are opposite points. We now have one of three situations (noting that one can interchange the role of $[a, a_1]$ with $[b_{a_4}]$ and c with d) each of which will be shown to be contradictory: first of all, p_{ca_1} can be collinear to p_{da_4} (this is the only possibility for $t = 2$); secondly, p_{ca_1} can be at distance 4 from a_4 as is a_1 to d; and finally, a_1 and a_4 are at distance 4 from d and c, respectively.

In the first case, we look inside $\mathcal{S}_a$ and obtain either a triangle, a quadrangle or a pentagon in this derived hexagon, as we shall show. First of all bcd is a line of $\mathcal{S}_a$. Furthermore we have that a, d and a_5' (a point collinear to a_5) belong to a block of this last type. As do a, b and a_5'' (collinear to a_5). Now, if a_5' equals a_5'' we get a triangle; if they are collinear we obtain a quadrangle and otherwise we obtain a pentagon where in both latter cases we use the fact that every point that is collinear to a_5 is in a Vee block with a and a_{05}.

Both the second and third situation do not occur when $t = 2$. Nevertheless, when $t > 2$ these situations as well as the previous one lead to the following contradictions.

If p_{ca_1} projects onto a_4 as does a_1 onto d, we obtain a pentagon inside $\mathcal{S}_d$: first of all the point b is in a Vee-block with d and p_{da_4}, hence inside $\mathcal{S}_d$ these two points, b and p_{da_4}, are collinear. As p_{ca_1} projects onto a_4 and a_1 projects onto d, the point c has to project onto the point p_{da_4}. Therefore c is in a block with d and a point x collinear to p_{da_4}. In other words, c is collinear to x in $\mathcal{S}_d$ and as x is also in a Vee-block with d and a_4 we obtain a pentagon $(bp_{da_4}a_4xc)$ inside what ought to be a generalized hexagon, a contradiction.

Finally, when a_1 and a_4 project onto d and c, respectively, there exists a quadrangle inside $\mathcal{S}_d$, a contradiction as we shall show. Just as in the latter case b is collinear to p_{da_4} in this particular derived hexagon. However, since there exists a path from c to d passing through the point a_4, the point c is in a block with d and a point collinear to a_4. This last point is also in a Vee-block with d and p_{da_4}, hence obtaining a quadrangle in $\mathcal{S}_d$.

Conclusion: if B should be a block of $\mathcal{S}$, then c has to be collinear to a_1 and d has to be collinear to a_2 and this for some path $a \perp a_1 \perp a_2 \perp b$ from a to b.

For Γ isomorphic to $\mathsf{H}(2)$ we shall show that c can not be on the ideal line through a and a_2, with notations as described above. On the other hand, for Γ isomorphic to the dual of $\mathsf{H}(2)$, or later on also for Γ isomorphic to $\mathsf{T}(2, 8)$, the following will lead to a contradiction and we will be able to conclude that these geometries are non-extendible under the given assumption.

Let us first assume Γ to be isomorphic to $\mathsf{H}(2)$ and suppose, by way of contradiction, that the point c does belong to the ideal line aa_2 and denote $\{a, b, c, d\}$ by B. Let $a_1 \perp a_2 \perp d \perp x \perp y \perp a \perp a_1$ be the points of an ordinary hexagon. The points x and y depend on the choice of line through a. As c is on an ideal line with a and

a_2, every such a line L through a determines a line regulus with a_2d with as third line, N, a line incident with c. Call z the point on N that is collinear to x.

Within $\mathcal{S}_d$ we obtain the following path (with u/v meaning either u or v)

$$abc \mathtt{I} a \mathtt{I} ay'(z/p_{xz}) \mathtt{I} z/p_{xz} \mathtt{I} (z/p_{xz})(p_{xz}/z)p_{dx} \mathtt{I} p_{dx} \mathtt{I} p_{dx}yp_{xy}$$

where y' is a point collinear to y and the first and second line of this path correspond with blocks of the third type, while the third and fourth correspond to Vee-blocks. Now, as c and d also define a block with either y or p_{xy} we obtain a pentagon in $\mathcal{S}_d$, a contradiction and we are done.

To complete the construction of the extension of Γ we used the regulus $\mathcal{R}(L, a_2d)$ to exclude the point on N collinear to a_1 from this type of blocks. However, in $\mathsf{H}(2)^D$, opposed to the situation in $\mathsf{H}(2)$, that point depends on the choice of L and therefore none of the points collinear to a_1 can be in a block with a and b. In other words there does not exist a one-point extension of $\mathsf{H}(2)^D$ under the assumption of the distance property, while in $\mathsf{H}(2)$ the extension is unique.

When dealing with $\mathsf{T}(2,8)$ we have to be a bit more careful. Let $\{a, d, x', y'\}$ be a block of the extension, where x' and y' are points collinear to x and y, respectively. Applying the exact same technique as above we rule out the point m_0 collinear to a_1 that is collinear to x' (if $d(a_1, x') = 4$) or is at distance 4 from x' (if $d(a_1, x') = 6$). Indeed, replace (z/p_{xz}) by x' and (p_{xz}/z) by $p_{xx'}$ in the above path to obtain a pentagon in $\mathcal{S}_d$. Now, consider m_i a point collinear to a_1 and incident with one of the regulus lines (of $\mathcal{R}(a_2d, ay)$) distinct of the one through m_0 and suppose this point is in a block with a, b and d.

Inside $\mathcal{S}_d$ we then obtain an n-gon, with $n \leq 5$

$$abm_i \mathtt{I} a \mathtt{I} ax' \mathtt{I} x' \mathtt{I} p_{dx}x' \mathtt{I} p_{dx} \mathtt{I} p_{dx}x''$$

where x'' is in a block of the third type with d and m_i. Hence none of the points m_i, $i = 0, \ldots, 6$, collinear to a_1 and on a line of $\mathcal{R}(a_2d, ay) \setminus \{a_2d, ay\}$, can be in such a block. However, for every point $l_i \neq a_1, m_i$, $i = 0, \ldots, 6$, on a_1m_i there exists a unique subhexagon of order 2 in $\mathsf{T}(2,8)$ (which consequently is isomorphic to $\mathsf{H}(2)^D$) containing l_i in addition to that fixed ordinary hexagon. This means that there is a unique third line M through a (namely within that subhexagon) that determines a line regulus with a_2d having a regulus line incident with l_i.

Summarizing all of this, we can exclude all points collinear to a_1 from a block through a and b, which is in contradiction with previous findings and we are done.

3.3 Some consequences

In this section we reprove an older result of the first author using the results of the previous section. The motivation to do so is twofold. We first want to tell what kind of characterization was previously known. Secondly, we want to show that the current one is slightly more general and entails the one mentioned below.

Lemma 11 *If in a one-point extension $\mathcal{S}$ of a generalized hexagon Γ of order $(2,t)$ every pair of meeting lines in every derivation defines a Vee-block (by symmetric difference), then it satisfies the distance property.*

Proof To prove the lemma we have to show that for all x, y, z in $\mathcal{S}$

$$d_x(y,z) = d_y(x,z).$$

Without loss of generality, we may assume x to be the point α of $\mathcal{S}$. It now suffices to prove that if y and z have distance $d \in \{2,4\}$ in Γ, then α and z have distance d in $\mathcal{S}_y$ (and consequently the case $d = 6$ follows). First, suppose y and z are collinear points of Γ. Then α, y and z are in a line block of $\mathcal{S}$ and hence α and z are collinear in $\mathcal{S}_y$. If, on the other hand, y and z are at distance 4 from one another, then they belong to a unique Vee-block $yuzv$ of $\mathcal{S}$. Within the derived generalized hexagon $\mathcal{S}_y$ we now have a path

$$z \mathtt{~I~} zuv \mathtt{~I~} u \mathtt{~I~} u\alpha \mathtt{~I~} \alpha$$

of length 4 from z to α and we are done.

□

The following theorem is an immediate consequence of Theorem 1.

Theorem 12 *There exists a unique one-point extension of the split Cayley hexagon of order 2 under the assumption that this extension satisfies the following block property: for every two blocks B and B' with $|B \cap B'| = 2$ the set $(B \cup B') \setminus (B \cap B')$ is the point set of a block of the extension.*

Proof Indeed, starting with this block property we can deduce that, considering two intersecting lines in the hexagon, such an extension has to contain all Vee-blocks, as defined above. However, by Lemma 11 such a one-point extension consequently satisfies the distance property and hence Theorem 12 is a direct result from Theorem 1.

□

3.4 Proof of Theorem 2

Let Γ be a generalized octagon of order $(2,4)$. In this section we will prove that there exists no one-point extension $\mathcal{S}$ of Γ satisfying the distance property. To prove this, we determine the consistence of all type of blocks in such a hypothetical extension and finally encounter a contradiction. By previous findings we already know the exact composition of three types of blocks in $\mathcal{S}$. Namely, $\mathcal{S}$ contains Line-, Vee- and Wee-blocks, as described above. Moreover, since $s = 2$ a Vee-block is here, just as in Section 3.2, the symmetrical difference of any two intersecting lines. We shall now describe the final type of blocks in $\mathcal{S}$, i.e. a non-Wee-block on two points at distance 6.

Say $B = \{x, y, b_0, b_1\}$ is a non-Wee-block on $\{x, y\}$, with $d(x, y) = 6$. Then, as we already mentioned above, considering $\mathcal{S}_x$ and $\mathcal{S}_y$ yields $d(x, b_i) = d(y, b_i) = 8$, for $i = 0, 1$. Within $\mathcal{S}_{b_0}$ we now have the line xyb_1 and the point α that is opposite both x and y. Hence $d_{b_0}(\alpha, b_1) = 6$ and consequently $d(b_0, b_1) = 6$. The distances within this type of blocks are thereby similar to the ones within Vee- and Wee-blocks. Here the distance within a pair is 6 and otherwise 8 instead of 2 and 4, and 4 and 6, respectively.

An easy counting argument now shows that there are $t + 1$ of these blocks on any two opposite points of Γ. Suppose $B = \{a, b, c, d\}$ is such a block of $\mathcal{S}$, with a and b opposite points of Γ and $d(a, c) = d(b, d) = 6$. Using the same convention of notations as before, we will prove that there has to be a path $\gamma = (a, a_1, a_2, a_3, b)$ from a to b such that the points c and d are collinear to the respective points a_{12} and a_{23}. To prove this we will proceed in a number of consecutive steps.

Step 1: $\mathbf{d(b, a_c) = 6}$

As b and c are (by definition) opposite points there is a unique point closest to b on every line through c. More in particular, the point b is at distance 6 from either $p_{c_a c}$ or c_a. This latter case immediately implies $d(b, a_c) = 6$ by Step 2 (namely, by interchanging the roles of a and c) which we shall prove later on. On the other hand, assume b to be at distance 6 from $p_{c_a c}$ (further on denoted by q to simplify notation). Suppose moreover, by way of contradiction, that b and a_c are opposite. Inside the derived geometry at c one finds a and b to be collinear (by definition). In the mean time, we will show that within $\mathcal{S}_c$ the point q is at distance 4 from a, while it is opposite b, a contradiction and Step 1 is proved. Since $d(a, q) = 6$, there is a unique Wee-block on these two points containing a point, say x, collinear with $c_a = q_a$. Within $\mathcal{S}_q$ we thus obtain

$$a \perp x \perp c$$

and hence find $d_q(a, c) = d_c(a, q) = 4$. The distance from b to c, again inside $\mathcal{S}_q$, is obtained by looking at the following path within $\mathcal{S}_q$

$$b \perp y \perp p_{q_b q} \perp \alpha \perp c$$

corresponding to the unique Wee-block on b an q, a Vee-block on y and q, with $d(y, q_b) = 2$, and two Line-blocks on q.
Hence b and c 'inside $\mathcal{S}_q$' and consequently b and q 'inside $\mathcal{S}_c$' are opposite points and we are done.

Step 2: $\mathbf{d(b, c_a) = 6}$

If not, then Γ containing no heptagons together with b being opposite to c yields $d(b, q) = 6$ (q as defined above). However, since we only made use of the path from

a to c and the one from b to c to come to a contradiction in Step 1 (and these paths contained the same points as the ones now), the exact same arguments used above prove Step 2.

This step, together with Step 1, now forces $d(b, p_{c_a a_c}) = 4$ and hence we may assume $\gamma = (a, a_1, a_2, a_3, b)$ to be the path from a to b such that c is collinear to a_{12}. We now prove

Step 3: $\neg(\exists \mathbf{c}' | \mathbf{d}(\mathbf{a_{12}}, \mathbf{c}') = \mathbf{2}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}'\} \subset \mathbf{B}' \neq \mathbf{B})$

If there exists such a point c', then the situation within $\mathcal{S}_a$ forces the set $\{c, c', d\}$ to be contained in a block of $\mathcal{S}$ (as these points determine a Vee-block within $\mathcal{S}_a$). However, the points c and d are opposite, as where c and c' are at most at distance 4 from one another, a contradiction.

A direct consequence of the previous step is that for every path $\gamma = (a, a_1, a_2, a_3, b)$ from a to b there has to be a block of this type containing a, b and a point collinear to a_{12} and hence (interchange the role of a and b) also a (distinct or same) block through a, b and a point collinear to a_{23}.

With γ and c as before, we come to the final Step in which we delete the 'distinct or' of the previous sentence.

Step 4: $\mathbf{d}(\mathbf{a_{23}}, \mathbf{d}) = \mathbf{2}$

To prove this final step we include the path γ into an ordinary octagon ($a = a_0, \ldots, a_4 = b, \ldots, a_8 = a$), suppose that d is, in the same way as c is to a_{12}, collinear to a_{56} and bring to mind that c and d are opposite points. We now have one of two situation – namely c can be at distance 6 from a_{56} or at distance 6 from $p_{a_{56}d}$ – each of which will be shown to be contradictory.

Indeed, suppose $d(c, a_{56}) = 6$ and denote the path from c to a_{56} by $(c = c_0, c_1, c_2, c_3 = a_{56})$. Let x be the point collinear to c_{23} that is in a block of the yet-to-be-described-type with c and d. As b and d are at distance 6 and d_b equals a_{56} we know that these points determine a Wee-block containing a point y collinear to a_{56}. This point y, on its turn, is in a Vee-block with d and $p_{a_{56}d}$. Also, x and d are in a Wee-block with a point y' collinear to $a_{56}(= d_x)$ and y' is in a Vee-block with d and $p_{a_{56}d}$. All of this together with the fact that b, c and d are in the given block B leads to a contradiction within $\mathcal{S}_d$, namely

$$c \perp x \perp y' \perp p_{da_{56}} \perp y \perp b \perp c.$$

In this path the point y can be equal to (respectively collinear to or at distance 4 from) y', in which case we obtain a quadrangle (respectively pentagon or hexagon).

If, on the other hand, the point $p_{a_{56}d}$ is at distance 6 from c, then one obtains a heptagon inside $\mathcal{S}_d$, as we shall show. Let x be the point at distance 4 from $p_{a_{56}d}$ that is in a block of the yet-to-be-described-type with c and d; y be the point collinear to $p_{a_{56}d}$ that is in a Wee-block with x and d; and z be the point collinear to a_{56} that is in a Wee-block with b and d. Then all of these blocks on d (note that d is in a Vee-block with z and $p_{a_{56}d}$) give following closed path

$$c \perp x \perp y \perp a_{56} \perp p_{a_{56}d} \perp z \perp b \perp c$$

inside $\mathcal{S}_d$, a contradiction and we are done.

Conclusion: if B should be a block of $\mathcal{S}$, then c has to be collinear to a_{12} and d has to be collinear to a_{23} and this for some path $a \perp a_1 \perp a_2 \perp a_3 \perp b$ from a to b. Note that a_2 is the unique point of Γ that is at distance 4 from all points in B and this point is uniquely determined by any one of the two couples in B that are at distance 6 from one another.

To complete the proof of this theorem we consider such a block $B = \{a, b, c, d\}$ of this final type, which will be referred to as *Xee-blocks*, with c and d collinear to a_{12} and a_{23}, respectively, and include γ into an ordinary octagon $(a = a_0, \ldots, b = a_4, \ldots a_8 = a)$.

Suppose the point $c = (a_{12})_{a_{56}}$ is in a block with a and b. Then c is also in a Xee-block with b and a point y collinear to a_6. This point y, on its turn, is in a Wee-block with b and a point z collinear to a_5. In the mean time, a is in a Xee-block with b and u (collinear to a_{56}), which is in a Wee-block with b and a point, say z', collinear to a_5. It is now easy to see that if z and z' coincide, are collinear or are at distance 4 from one another, we respectively obtain a pentagon, a hexagon or at most a heptagon inside $\mathcal{S}_b$, a contradiction. Hence c can not be this particular point.

Say c equals the second point on the line $a_{12}(a_{12})_{a_{56}}$ and d' is the unique point that is both in a block with $\{a, b\}$ and is collinear to a_{56}. If $B' = \{a, b, c', d'\}$ denotes this block on $\{a, b, d'\}$, then $\{c, c', d, d'\}$ has to be a block of $\mathcal{S}$ as well (within $\mathcal{S}_a$ or $\mathcal{S}_b$ these points are on a Vee-block). More in particular, as $d(c, d) = 8$ the block B' has to be a Xee-block of $\mathcal{S}$. We now have one of two situations, either the distance from c to d' is 6 or it is 8. We claim that both these situations lead to a contradiction.
If $d(c, d') = 6$, then both d and c' have to be at distance 4 from the point p_{xy}, with $x = c_{d'} = p_{a_{12}c}$ and $y = d'_c = p_{a_{56}d'}$. However, as $d(x, d) = 8$ the distance from p_{xy} to d should be at least 6.
Suppose, on the other hand, that c is opposite d' and denote the path from c to a_{56} by $(c = c_0, c_1, c_2, c_3 = a_{56})$. As c is also opposite d, it has to be at distance 6 from c'. This point c' is collinear to a_{67}, which is at distance 6 from c. Hence $c' = p_{a_{67}u}$ with $u = (a_{67})_c$. In the mean time, by definition of a Xee-block, the unique point x that is at distance 4 from both c and c', is also at distance 4 from d', a contradiction as we obtain a heptagon $(xua_{67}a_6a_{56}d'x_{d'})$ within Γ. Hence the claim is proved.

In conclusion, the point c has to be opposite a_{56} and this for every ordinary octagon $O = (a = a_0, \ldots, b = a_4, \ldots, a_8 = a)$ containing the fixed path γ. For every such an octagon, c can now either be closest to a_5 or to a_6. In the former case, however, we obtain a contradiction within $\mathcal{S}_b$, as we shall show. First of all, c is in a Xee-block with b and a point x collinear to b_{12}, where $(b = b_0, b_1 = a_5, \ldots, b_4 = c)$ is the path from b to c containing the point a_5. The point x is in a Wee-block with b and a point y collinear to a_5. On the other hand the point a is, next to being in a Xee-block with $\{b, c\}$, also in a Xee-block on $\{z, b\}$, where z is a point collinear to a_{56}. This point z is now in a Wee-block with b and a point y' that is collinear to a_5. Both y and y' are either in the same or in a distinct Vee-block with b and a_{45}. The former case yields a pentagon or a hexagon inside $\mathcal{S}_b$, while in the latter case yields a heptagon within this derived octagon, a contradiction.

Hence, we may conclude that for every such an octagon O on γ, the point c has to be at distance 6 from the point a_6.

We can now choose coordinates in $\mathsf{O}(2)$ in such a way that $a = (0, [0,0], 0)$, $b = (1, [0,0], 0, [0,0], 0)$, $a_{12} = (\infty)$ and hence $c = (K, B)$. After some tedious calculations we find that the point a_6 (of $(\Gamma_4(a) \cap \Gamma_4(b)) \setminus \{(1)\}$) is one of the following four points

$$v_{ij} = (0, (0,0), 0, (i,j), 0, (0,j), i), \forall i, j \in \mathsf{GF}(2).$$

If $d(c, v_{ij}) = 6$, then the point c belongs to the trace $(\infty)^{v_{ij}}$. However

$$(\infty)^{(a,l,a',l',a'',l'',a''')} = \{(a)\} \cup \{(k, a''' + aN(k) + l_0 k_1 + l''_0 k_0) | k \in \mathsf{GF}(2)^2\}$$

and hence

$$(\infty)^{v_{00}} = (\infty)^{v_{01}} = \{(0)\} \cup \{(k, 0) | k \in \mathsf{GF}(2)^2\}$$

while

$$(\infty)^{v_{10}} = (\infty)^{v_{11}} = \{(1)\} \cup \{(k, 1) | k \in \mathsf{GF}(2)^2\}.$$

Obviously these two traces share no points and hence there exists no such a point c in $\mathsf{O}(2)$ and we are done.

3.5 Flag-transitive one-point extensions

Suppose $\mathcal{S}$ is a one-point extension of a classical generalized hexagon or octagon of finite order (s, t) admitting a *flag-transitive* automorphism group G. That is an automorphism group which is transitive on the set of all incident point-block pairs of $\mathcal{S}$.

Let x be a point of $\mathcal{S}$. Then by G_x we denote the subgroup of G stabilizing x. Clearly, G_x induces an action on the set of points and the set of lines of the generalized polygon $\mathcal{S}_x$. As the group G is flag-transitive, the group G_x is transitive on the set of lines of $\mathcal{S}_x$.

A *point-orbital* of G_x is an orbit of G_x in its induced action on $\mathcal{P}_x \times \mathcal{P}_x$. A point-orbital $\mathcal{O}$ is called *self-paired* if for all $(y, z) \in \mathcal{O}$ we also have $(z, y) \in \mathcal{O}$.

Lemma 13 *A point-orbital $\mathcal{O}$ of G_x is self-paired if and only if for each $(y, z) \in \mathcal{O}$ there is a $t \in G_x$ with $y^t = z$ and $z^t = y$.*

Proof Obvious. □

Lemma 14 *Suppose p is a point of $\mathcal{S}$. If all point-orbitals of G_p are self-paired, then $\mathcal{S}$ has the distance property.*

Proof Suppose x, y, z are three points of $\mathcal{S}$. Since G is transitive on the points, we may assume that for p equal to x, y or z, all G_p point-orbitals are self-paired. This implies that there are t_x, t_y and t_z in G with

$$x^{t_x} = x, y^{t_x} = z, z^{t_x} = y;$$
$$y^{t_y} = y, z^{t_y} = x, x^{t_y} = z;$$
$$z^{t_z} = z, x^{t_z} = y, y^{t_z} = x.$$

The subgroup $\langle t_x, t_y, t_z\rangle$ of G induces the full symmetric group on $\{x, y, z\}$. As a consequence we have

$$d_x(y, z) = d_y(z, x) = d_z(y, x).$$

So, $\mathcal{S}$ has the distance property. □

Theorem 15 *Suppose $\mathcal{S}$ is a one-point extension of a classical generalized hexagon or octagon of order (s, t). If $\mathcal{S}$ admits a flag-transitive automorphism group G, then $\mathcal{S}$ is isomorphic to the unique affine extension of $\mathsf{H}(2)$ and G is isomorphic to $2^6 : \mathsf{G}_2(2)'$ or $2^6 : \mathsf{G}_2(2)$.*

Proof Let x be a point of $\mathcal{S}$. Then the group G_x induces a line-transitive action on the classical generalized hexagon or octagon $\mathcal{S}_x$.

The automorphism group of $\Gamma = \mathcal{S}_x$ is contained in the automorphism group of one of the following groups $\mathsf{G}_2(q)'$ (in case Γ is $\mathsf{H}(q)$ or $\mathsf{H}(2)^D$), ${}^3\mathsf{D}_4(q)$ (in case Γ is $\mathsf{T}(q^3, q)$ or its dual) or ${}^2\mathsf{F}_4(q)'$ (in case Γ is $\mathsf{O}(q)$ or its dual), where q is equal to either s or t. By the classification of the maximal subgroups of these groups as given in [1, 7, 8], it is easily seen that G_x in its action on $\mathcal{S}_x$ has to contain the simple group $\mathsf{G}_2(q)'$, ${}^3\mathsf{D}_4(q)$ or ${}^2\mathsf{F}_4(q)'$, respectively. As each of these groups, except for $\mathsf{G}_2(2)'$ and ${}^2\mathsf{F}_4(2)'$ acts distance transitively on the points and lines of the corresponding classical polygon, see [12, Section 4.8], we find, except possibly in the two exceptional cases, all G_x point-orbitals to be self-paired. So in all cases, except when G_x induces $\mathsf{G}_2(2)'$ or or ${}^2\mathsf{F}_4(2)'$ on $\mathcal{S}_x$, we find that $\mathcal{S}$ satisfies the distance property. If G_x induces $\mathsf{G}_2(2)'$ or ${}^2\mathsf{F}_4(2)'$, then there are only two non-self-paired orbitals, both consisting of pairs of points (y, z) at mutual distance 6, or 8, respectively, in $\mathcal{S}_x$. Indeed, from the information in [3], G_x acts as a rank 5 group or rank 6 group, respectively (on the point set of the dual of $\mathsf{H}(2)$ or the dual of $\mathsf{O}(2)$, respectively), and since the full automorphism group of the generalized polygon acts as a rank 4 group or a rank 5 group, respectively, then the non-self-paired orbitals have the same size. Moreover they must occur with points at maximal distance as otherwise there would be more non-self-paired orbitals with points at larger distance. So the only situation to rule out is the case where $d_x(y,z) = 6$ or 8, respectively, and $d_y(x,z) < 6$ or 8, respectively. If we consider the group $G_{x,y}$, then in $\mathcal{S}_x$, the length of the orbit of z under $G_{x,y}$ is a power of 2 (being half of the number of points of $\mathcal{S}_x$ opposite y), while this is certainly not true for the size of the orbit of z under $G_{x,y}$ in $\mathcal{S}_y$. This contradiction shows that the distance property is also satisfied in these exceptional cases.

The theorem follows easily from Theorem 1 and Theorem 2. □

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