The uniqueness of the SDPS-set of the symplectic dual polar space $DW(4n-1, q)$, $n \geq 2$

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Abstract

SDPS-sets are very nice sets of points in dual polar spaces which themselves carry the structure of dual polar spaces. They were introduced in [8] because they gave rise to new valuations and hyperplanes of dual polar spaces. In the present paper, we show that the symplectic dual polar space $DW(4n-1, q)$, $n \geq 2$, has up to isomorphisms a unique SDPS-set.

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1 Introduction

1.1 Basic definitions and properties

Let Π be a non-degenerate polar space of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. We call Δ a *dual polar space*. By Shult and Yanushka [17] and Cameron [3] (see also De Bruyn

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[4]), Δ is a near polygon which means that for every point p and every line L, there exists a unique point on L nearest to p. The distance $\mathrm{d}(x,y)$ between two points x and y of Δ is measured in the point or collinearity graph of Δ. For every point x, for every nonempty subset X of the point-set P of Δ and for every $i \in \mathbb{N}$, we define $\Delta_i(x) := \{y \in P \,|\, \mathrm{d}(x,y) = i\}$, $\Delta_i^*(x) := \{y \in P \,|\, \mathrm{d}(x,y) \leq i\}$, $x^\perp := \Delta_1^*(x)$, $\mathrm{d}(x,X) = \min\{\mathrm{d}(x,y) \,|\, y \in X\}$, $\Delta_i(X) = \{y \in P \,|\, \mathrm{d}(y,X) = i\}$ and $\Delta_i^*(X) = \{y \in P \,|\, \mathrm{d}(y,X) \leq i\}$. If X_1 and X_2 are two nonempty sets of points, then we define $\mathrm{d}(X_1,X_2) := \min\{\mathrm{d}(x_1,x_2) \,|\, x_1 \in X_1 \text{ and } x_2 \in X_2\}$.

We will denote a dual polar space by putting a "D" in front of the name of the corresponding polar space. The dual polar spaces we will meet in this paper are the symplectic dual polar space $DW(2n-1,q)$ related to a symplectic polarity of the projective space $\mathrm{PG}(2n-1,q)$, the hermitian dual polar space $DH(k,q^2)$ related to a non-singular hermitian variety in $\mathrm{PG}(k,q^2)$ and the orthogonal dual polar space $DQ^-(2n+1,q)$ related to a non-singular elliptic quadric in $\mathrm{PG}(2n+1,q)$.

There exists a bijective correspondence between the nonempty convex subspaces of a dual polar space Δ of rank $n \geq 2$ and the possibly empty singular subspaces of the associated polar space Π: if α is a singular subspace of Π, then the set of all maximal singular subspaces containing α is a convex subspace of Δ. Conversely, every convex subspace of Δ is obtained in this way. The maximal distance between two points of a convex subspace A is called the *diameter* of A and is denoted by $\mathrm{diam}(A)$. The convex subspaces of diameter 0 and 1 are the points and lines of Δ. The convex subspaces of diameter 2, 3, respectively $n-1$, are called the *quads*, *hexes*, respectively *maxes*, of Δ. The convex subspaces through a given point x of Δ determine a projective space of dimension $n-1$. If x and y are two points of Δ, then $\langle x,y\rangle$ denotes the smallest convex subspace containing x and y, i.e. $\langle x,y\rangle$ is the intersection of all convex subspaces containing x and y. More generally, we will use the notation $\langle *_1, *_2, \ldots, *_k\rangle$ to denote the smallest convex subspace containing the objects $*_1, *_2, \ldots, *_k$ (which can be points, lines, quads, etc.). If x is a point and A is a nonempty convex subspace of Δ, then A contains a unique point $\pi_A(x)$ nearest to x and $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_A(x)) + \mathrm{d}(\pi_A(x),y)$ for every point y of A. We call $\pi_A(x)$ the *projection* of x onto A. If F_1 and F_2 are two convex subspaces of Δ of respective diameters δ_1 and δ_2, then either $F_1 \cap F_2 = \emptyset$ or ($F_1 \cap F_2 \neq \emptyset$ and $\mathrm{diam}(F_1 \cap F_2) \geq \delta_1 + \delta_2 - n$).

A *hyperplane* of a dual polar space Δ is a proper subspace meeting each line (necessarily in a unique point or the whole line). Since Δ is a near polygon, the set H_x of points at non-maximal distance from a given point x is a hyperplane of Δ, called the *singular hyperplane with deepest point x*.

A function f from the point-set of a dual polar space Δ to $\mathbb{N}$ is called a *valuation* of Δ if it satisfies the following properties (we call $f(x)$ the value of x):

(V1) there exists at least one point with value 0;

(V2) every line L of Δ contains a unique point x_L with smallest value and $f(x) = f(x_L) + 1$ for every point x of L different from x_L;

(V3) every point x of Δ is contained in a necessarily unique convex subspace F_x such that the following properties are satisfied for every $y \in F_x$: (i) $f(y) \leq f(x)$; (ii) if z is a point collinear with y such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations were introduced in De Bruyn and Vandecasteele [7] in the context of general near polygons. They are important structures for the following purposes: (1) classification of near polygons; (2) study of isometric embeddings between near polygons; (3) construction of hyperplanes of near polygons, in particular of dual polar spaces; (4) characterizations of classes of hyperplanes of dual polar spaces. The construction of hyperplanes from valuations is explained in the following proposition:

Proposition 1.1 (Proposition 2 of [8]) *Let f be a valuation of a dual polar space Δ and let M denote the maximal value attained by f. Then the set of points with value at most $M-1$ is a hyperplane H_f of Δ.*

Let Δ be a thick dual polar space of rank $2n$, $n \geq 0$. (We take the following convention: a dual polar space of rank 0 is a point and a dual polar space of rank 1 is a line.) A set X of points of Δ is called an *SDPS-set* (SDPS = sub dual polar space) of Δ if it satisfies the following properties:

(1) No two points of X are collinear in Δ.

(2) If $x, y \in X$ such that $\mathrm{d}(x, y) = 2$, then $X \cap \langle x, y \rangle$ is an ovoid of the quad $\langle x, y \rangle$.

(3) The point-line geometry $\widetilde{\Delta}$ whose points are the elements of X and whose lines are the quads of Δ containing at least two points of X (natural incidence) is a dual polar space of rank n.

(4) For all $x, y \in X$, $\mathrm{d}(x, y) = 2 \cdot \delta(x, y)$. Here, $\mathrm{d}(x, y)$ and $\delta(x, y)$ denote the distances between x and y in the respective dual polar spaces Δ and $\widetilde{\Delta}$.

(5) If $x \in X$ and if L is a line of Δ through x, then L is contained in a quad of Δ which contains at least two points of X.

SDPS-sets in thick dual polar spaces of rank $2n$ were introduced by De Bruyn and Vandecasteele [8] for general n, and independently (although not using this terminology) by Pralle and Shpectorov [15] for $n = 2$. Note that condition (5) is only implicitly in [8]. In [8], we only considered finite dual polar spaces and the possibilities for $(\Delta, \widetilde{\Delta})$ listed there force condition (5) to hold. All the proofs mentioned in [8] are still valid in the infinite case (after a slight modification) if one assumes that the extra condition (5) holds, see Sections 5.8, 5.9 and 5.10 of De Bruyn [4].

Proposition 1.2 (Theorem 4 of De Bruyn and Vandecasteele [8]) *Let X be an SDPS-set of a thick dual polar space Δ of rank $2n \geq 0$. For every point x of Δ, we define $f(x) := d(x, X)$. Then f is a valuation of Δ.*

By Propositions 1.1 and 1.2, with every SDPS-set of a thick dual polar space Δ, there is associated a hyperplane of Δ. For a characterization of these hyperplanes, we refer to De Bruyn [5].

An SDPS-set of a dual polar space of rank 0 consists of the unique point of this dual polar space. An SDPS-set of a thick generalized quadrangle Q is an ovoid of Q. The dual polar spaces $DQ^-(4n+1, q)$ and $DW(4n-1, q)$ admit SDPS-sets for every $n \geq 2$, see De Bruyn and Vandecasteele [8] or Pralle and Shpectorov [15]. The following proposition has been proved in De Bruyn [4, Theorem 5.31], but its proof relies very much on Pralle and Shpectorov [15]:

Proposition 1.3 ([4], [15]) *If X is an SDPS-set of a finite thick dual polar space Δ of rank $2n \geq 4$ and if $\widetilde{\Delta}$ denotes the associated dual polar space of rank n, then one of the following cases occurs:*

(1) $\Delta \cong DW(4n-1, q)$ *and* $\widetilde{\Delta} \cong DW(2n-1, q^2)$ *for some prime power q. If Q is a quad containing two points of X, then $Q \cap X$ is a classical ovoid of Q, i.e. an elliptic quadric $Q^-(3, q)$ in $Q \cong Q(4, q)$.*

(2) $\Delta \cong DQ^-(4n+1, q)$ *and* $\widetilde{\Delta} \cong DH(2n, q^2)$ *for some prime power q. If Q is a quad containing two points of X, then $Q \cap X$ is a classical ovoid of Q, i.e. a unital $H(2, q^2)$ in $Q \cong H(3, q^2)$.*

1.2 The Main Theorem of this paper

SDPS-sets of thick dual polar spaces are important objects because of their connection with valuations and hyperplanes of dual polar spaces. They are also handy to describe isometric embeddings of the symplectic dual polar space $DW(2n-1,q)$ into the hermitian dual polar space $DH(2n-1,q^2)$, see De Bruyn [6]. In the finite case, there are only two possibilities by Proposition 1.3, and although quite much is already known about the structure of the respective SDPS-sets, the uniqueness questions were not yet settled. In this paper, we will prove the uniqueness for one of the two cases.

Main Theorem. *The dual polar space $DW(4n-1,q)$, $n \geq 2$, admits up to isomorphisms a unique SDPS-set.*

We will end this section with a construction of the unique SDPS-set of $DW(4n-1,q)$, $n \geq 2$. Consider the finite field $\mathbb{F}_{q^2}$ with q^2 elements and let $\mathbb{F}_q$ denote the unique subfield of order q of $\mathbb{F}_{q^2}$. Let η denote an arbitrary element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then $\mathbb{F}_{q^2} = \{x_1 + x_2\eta \,|\, x_1, x_2 \in \mathbb{F}_q\}$; define $\tau : \mathbb{F}_{q^2} \to \mathbb{F}_q, x_1 + x_2\eta \mapsto x_1$. Consider the following bijection ϕ between the vector spaces $\mathbb{F}_q^{4n}$ and $\mathbb{F}_{q^2}^{2n}$:

$$\phi(x_1, x_2, \ldots, x_{4n}) = (x_1 + \eta x_2, \ldots, x_{4n-1} + \eta x_{4n}).$$

If $\langle \cdot, \cdot \rangle$ is a non-degenerate symplectic form of $\mathbb{F}_{q^2}^{2n}$, then $\tau(\langle \phi(\cdot), \phi(\cdot) \rangle)$ is a non-degenerate symplectic form in $\mathbb{F}_q^{4n}$. If α is a totally isotropic n-dimensional subspace of $\mathbb{F}_{q^2}^{2n}$, then $\phi^{-1}(\alpha)$ is a $2n$-dimensional totally isotropic subspace of $\mathbb{F}_q^{4n}$. In this way we obtain an "embedding" of $DW(2n-1,q^2)$ in $DW(4n-1,q)$, giving rise to an SDPS-set of $DW(4n-1,q)$.

2 The SDPS-sets of $DW(3,q)$

The conclusion of the Main Theorem does not hold if $n = 1$. The SDPS-sets of $DW(3,q)$ are precisely the ovoids of $Q(4,q)$. Several classes of non-classical ovoids of $Q(4,q)$ exist:

- For each prime power $q = p^h$, p odd prime power and $h \geq 2$, there is a class of non-classical ovoids of $Q(4,q)$ due to Kantor [9].
- For each prime power $q = 2^{2n+1}$, $n \geq 1$, there is a class of non-classical ovoids in $Q(4,q)$ due to Tits [19].
- For each prime power $q = 3^{2n+1}$, $n \geq 1$, there is a class of non-classical ovoids of $Q(4,q)$ due to Kantor [9].

- For each prime power $q = 3^h$, $h \geq 3$, there is a class of non-classical ovoids of $Q(4, q)$ due to Thas and Payne [18].
- The generalized quadrangle $Q(4, 3^5)$ has a class of non-classical ovoids due to Penttila and Williams [14].

For several prime powers q it is known that all ovoids of $Q(4, q)$ are classical.

Proposition 2.1
- *([2], [12]) Every ovoid of* $Q(4, 4)$ *is classical.*
- *([10], [11]) Every ovoid of* $Q(4, 16)$ *is classical.*
- *([1]) Every ovoid of* $Q(4, q)$*,* q *prime, is classical.*

3 Proof of the Main Theorem

3.1 Some lemmas

We first prove some lemmas will be important during the proof of the Main Theorem.

Lemma 3.1 *Let* F *be a convex subspace of a dual polar space* Δ *of rank* $n \geq 2$ *and let* x_1 *and* x_2 *be two collinear points of* Δ*. Then* $d(\pi_F(x_1), \pi_F(x_2)) \leq 1$.

Proof. Straightforward, see e.g. De Bruyn [4, Theorem 1.9]. ∎

Lemma 3.2 *Let* Δ *be a dual polar space of rank* $n \geq 2$ *and let* F_1 *and* F_2 *be two convex subspaces of diameter* $\delta \in \{0, \ldots, n\}$ *of* Δ *which lie at maximal distance* $n - \delta$ *from each other. Then the map* $F_1 \to F_2; x \mapsto \pi_{F_2}(x)$ *defines an isomorphism from* F_1 *to* F_2.

Proof. Straightforward, see e.g. De Bruyn [4, Theorem 1.10]. ∎

Lemma 3.3 *Let* Δ *be a dual polar space of rank* $n \geq 2$*. Let* F *denote a convex subspace of diameter* $\delta \in \{0, \ldots, n\}$ *of* Δ*, let* x_1, x_2 *be points of* F *at maximal distance* δ *from each other and let* F_i*,* $i \in \{1, 2\}$*, denote a convex subspace of diameter* $n - \delta$ *through* x_i *such that* $F \cap F_i = \{x_i\}$*. Then* F_1 *and* F_2 *lie at maximal distance* δ *from each other.*

Proof. Since $\mathrm{diam}(F_1) = \mathrm{diam}(F_2) = n - \delta$, $\mathrm{d}(x, F_2) \leq \delta$ for every point x of F_1. Let $y_1 \in F_1$ and $y_2 \in F_2$ with $\mathrm{d}(y_1, y_2)$ as small as possible. For every $i \in \{1, 2\}$, the point $\pi_F(y_i)$ is contained on a shortest path between y_i and x_i and hence is contained in $F \cap F_i = \{x_i\}$. This proves that $\pi_F(y_1) = x_1$ and $\pi_F(y_2) = x_2$. Considering a shortest path between y_1 and y_2 and applying Lemma 3.1, we obtain $\delta = \mathrm{d}(x_1, x_2) \leq \mathrm{d}(y_1, y_2)$. On the other hand, we already knew that $\mathrm{d}(y_1, y_2) = \mathrm{d}(F_1, F_2) \leq \delta$. This proves that $\mathrm{d}(F_1, F_2) = \delta$. ∎

Lemma 3.4 *Let M_1 and M_2 be two maxes of a dual polar space Δ of rank $n \geq 2$ which meet each other. Then there exists a max M_3 of Δ which is disjoint from M_1 and M_2.*

Proof. Let $y \in M_1 \cap M_2$ and let $x \in \Delta_1(y)$ not contained in $M_1 \cup M_2$. Then any max through x not containing the line xy is disjoint from M_1 and M_2. ∎

Lemma 3.5 *Let Γ denote the graph on the line set of $DW(2n-1, q)$, $n \geq 2$, with two vertices of Γ adjacent whenever the corresponding lines are disjoint and contained in a quad of $DW(2n-1, q)$. Then Γ is connected.*

Proof. Let L_1 and L_2 denote two arbitrary lines of $DW(2n-1, q)$. We will prove by induction on $\mathrm{d}(L_1, L_2)$ that there exists a path in Γ connecting L_1 and L_2.

Suppose first that $\mathrm{d}(L_1, L_2) = 0$. If $L_1 = L_2$, then we are done. So, suppose $L_1 \neq L_2$. Then L_1 and L_2 are contained in a quad $Q \cong Q(4, q)$. If L_3 is a line of Q disjoint from L_1 and L_2, then (L_1, L_3, L_2) is a path in Γ.

Suppose next that $\mathrm{d}(L_1, L_2) \geq 1$ and let $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x_1, x_2) = \mathrm{d}(L_1, L_2)$. Let x_3 be a point of $\Delta_1(x_2)$ at distance $\mathrm{d}(L_1, L_2) - 1$ from x_1. Let L_3 denote a line through x_3 contained in the quad $\langle x_3, L_2 \rangle$, but different from x_3x_2. Then L_2 and L_3 are adjacent in Γ. By the induction hypothesis, there exists a path in Γ connecting L_1 and L_3. Hence, there also exists a path in Γ connecting L_1 and L_2. ∎

Lemma 3.6 *Let M_1 and M_2 be two disjoint maxes of the dual polar space $DW(2n-1, q)$, $n \geq 2$. Then there exists a unique set $\{M_1, M_2, \ldots, M_{q+1}\}$ of mutually disjoint maxes with the property that every line meeting M_1 and M_2 also meets M_i, $i \in \{3, \ldots, q+1\}$. If z is a point of $DW(2n-1, q)$ not contained in $M_1 \cup M_2 \cup \cdots \cup M_{q+1}$, then there exists a unique quad through z which intersects each M_i, $i \in \{1, \ldots, q+1\}$, in a line.*

Proof. Let x_i, $i \in \{1, 2\}$, denote the point of the polar space $W(2n-1, q)$ corresponding with M_i. Since M_1 and M_2 are disjoint, the points x_1 and x_2 are contained in a hyperbolic line $\{x_1, x_2, \ldots, x_{q+1}\}$. Let M_i, $i \in \{3, \ldots, q+1\}$, denote the max of $DW(2n-1, q)$ corresponding with x_i. Then $\{M_1, M_2, \ldots, M_{q+1}\}$ is a set of mutually disjoint maxes with the property that every line meeting M_1 and M_2 also meets M_i, $i \in \{3, \ldots, q+1\}$.

Let u and v be two opposite points of M_1 and put $L_1 = \langle u, \pi_{M_2}(u)\rangle$ and $L_2 = \langle v, \pi_{M_2}(v)\rangle$. Then by Lemma 3.3, L_1 and L_2 lie at maximal distance $n-1$ from each other. There are now at most $q+1$ maxes which meet every line connecting a point of M_1 with a point of M_2, namely the $q+1$ maxes $\langle w, \pi_{L_2}(w)\rangle$, where $w \in L_1$. This proves the uniqueness of the set $\{M_1, M_2, \ldots, M_{q+1}\}$.

Now, let z denote a point of $DW(2n-1, q)$ not contained in $M_1 \cup M_2 \cup \cdots \cup M_{q+1}$. Let z_1 denote the unique point of M_1 collinear with z and let z_2 denote the unique point of M_2 collinear with z_1. Since z is not contained in $M_1 \cup M_2 \cup \cdots \cup M_{q+1}$, $zz_1 \neq z_1z_2$. Obviously, the quad $\langle zz_1, z_1z_2\rangle$ is the unique quad through z meeting M_1 and M_2 (necessarily in lines). This quad also meets M_i, $i \in \{3, \ldots, q+1\}$ in a line since $z_1z_2 \cap M_i \neq \emptyset$. ∎

Lemma 3.7 *A $(q+1) \times (q+1)$-subgrid G of the generalized quadrangle $Q(4, q)$ is a maximal subspace of $Q(4, q)$. In other words, the graph on the set $Q(4, q) \setminus G$ induced by the collinearity graph of $Q(4, q)$ is connected.*

Proof. By Payne and Thas [13, 2.3.1], any subspace of $Q(4, q)$ containing G induces a subquadrangle of $Q(4, q)$. Now, the only proper subquadrangle of $Q(4, q)$ containing G is G itself, proving the lemma. ∎

Definition. Let X be an SDPS-set of $\Delta \cong DW(4n-1, q)$, $n \geq 1$, and let $\widetilde{\Delta}$ denote the dual polar space isomorphic to $DW(2n-1, q^2)$ associated with X. A convex subspace F of Δ is called *X-special* if it is of the form $\langle x_1, x_2\rangle$ for two points x_1 and x_2 of X. Since $\mathrm{d}(x_1, x_2) = 2 \cdot \delta(x_1, x_2)$, where $\delta(\cdot, \cdot)$ denotes the distance function in $\widetilde{\Delta}$, every X-special convex subspace of Δ has even diameter. If F is an X-special convex subspace, then by De Bruyn and Vandecasteele [8, Lemma 7], $F \cap X$ is an SDPS-set of F and is a convex subspace of $\widetilde{\Delta}$ whose diameter is half the diameter of F regarded as convex subspace of Δ. Also, by De Bruyn and Vandecasteele [8, Lemma 4] no two distinct X-special quads can intersect in a line.

Lemma 3.8 *Let X be an SDPS-set of $\Delta \cong DW(4n-1,q)$, $n \geq 1$, let x be a point of X and let F be an X-special convex subspace of Δ. Then $\pi_F(x) \in X$.*

Proof. Let $\widetilde{\Delta}$ denote the dual polar space isomorphic to $DW(2n-1,q^2)$ associated with X. Let $2d$ denote the diameter of F in Δ. Let $\mathrm{d}(\cdot,\cdot)$, respectively $\delta(\cdot,\cdot)$, denote the distance function in Δ, respectively $\widetilde{\Delta}$. Since $F \cap X$ is a convex subspace of $\widetilde{\Delta}$, there exists a unique point y in $F \cap X$ nearest to x (in $\widetilde{\Delta}$). Let z denote a point of $F \cap X$ such that $\delta(y,z) = d$. Then $\mathrm{d}(y,z) = 2d$. So, y and z are opposite points of F. Since $\delta(x,z) = \delta(x,y) + \delta(y,z)$, we have that $\mathrm{d}(x,z) = \mathrm{d}(x,y) + \mathrm{d}(y,z)$. Since $\mathrm{d}(y,z)$ attains its maximal value $2d$, we necessarily have $\pi_F(x) = y \in X$. This proves the lemma. ∎

Lemma 3.9 *Let M denote a max of the dual polar space $\Delta = DH(2n-1,q^2)$, $n \geq 2$. Then there exists a group G of automorphisms of Δ satisfying the following properties:*

(*i*) *every element of G fixes M point-wise and every line meeting M set-wise;*

(*ii*) *if L is a line meeting M in a unique point x, then G acts regularly on $L \setminus \{x\}$.*

Proof. Let V denote a $2n$-dimensional vector space over $\mathbb{F}_{q^2}$ equipped with a non-degenerate hermitian form $(\cdot,\cdot)$ which is linear in the first argument and semi-linear in the second. Let $H(2n-1,q^2)$ and $DH(2n-1,q^2)$ denote the corresponding polar and dual polar space. Let $\langle \bar{x}_M \rangle$ denote the point of $H(2n-1,q^2)$ corresponding with the max M. For every $k \in \mathbb{F}_{q^2}$ satisfying $k^q + k = 0$, the linear map $\bar{y} \mapsto \bar{y} - k(\bar{y},\bar{x}_M)\bar{x}_M$ defines an automorphism of $H(2n-1,q^2)$. The corresponding automorphism θ_k of $DH(2n-1,q^2)$ fixes M point-wise and every line meeting M set-wise. It is straightforward to verify that $G := \{\theta_k \,|\, k \in \mathbb{F}_{q^2} \text{ with } k^q + k = 0\}$ is a group of automorphisms of $DH(2n-1,q^2)$ acting regularly on each set $L \setminus \{x\}$, where L is a line of Δ meeting M in a unique point x. ∎

3.2 Upper bound for the number of SDPS-sets

Definition. An SDPS-set X of $DW(4n-1,q)$, $n \geq 1$, is called *classical* if $Q \cap X$ is a classical ovoid of Q for every X-special quad Q. By Proposition 1.3, every SDPS-set of $DW(4n-1,q)$ is classical if $n \geq 2$. Let $\lambda(n)$, $n \geq 1$, denote the number of classical SDPS-sets of $DW(4n-1,q)$. The number of classical ovoids of the generalized quadrangle $Q(4,q)$ is equal to

$$\lambda(1) \;=\; \frac{q^2(q^2-1)}{2}.$$

Lemma 3.10 *The number of classical SDPS-sets containing two given opposite points of $DW(4n-1,q)$, $n \geq 1$, is equal to*

$$\frac{\lambda(n)}{(q+1)(q^3+1)\cdots(q^{2n-1}+1)\cdot q^{n^2}}.$$

Proof. The automorphism group of $DW(4n-1,q)$ acts transitively on the set of pairs of opposite points of $DW(4n-1,q)$. Hence, there exists a constant $\lambda'(n)$ such that every two opposite points of $DW(4n-1,q)$ are contained in precisely $\lambda'(n)$ classical SDPS-sets. Counting in two different ways the number of triples (X,x_1,x_2), where x_1 and x_2 are two opposite points of $DW(4n-1,q)$ and where X is a classical SDPS-set containing the points x_1 and x_2 gives

$$(q+1)(q^2+1)\cdots(q^{2n}+1)\cdot q^{1+2+\ldots+2n}\cdot\lambda'(n)$$
$$=\lambda(n)\cdot(q^2+1)(q^4+1)\cdots(q^{2n}+1)\cdot q^{2+4+\cdots+2n}.$$

(The dual polar space $DW(4n-1,q)$ contains $(q+1)(q^2+1)\cdots(q^{2n}+1)$ points and there are $q^{1+2+\cdots+2n}$ points in $DW(4n-1,q)$ which are opposite to a given point of $DW(4n-1,q)$. Recall also that X carries the structure of a dual polar space $DW(2n-1,q^2)$.) The lemma now readily follows. ■

Lemma 3.11 *Let Q_1 and Q_2 be two quads of $DW(7,q)$ at maximal distance 2 from each other, let $x \in Q_1$ and put $Q_3 := \langle x, \pi_{Q_2}(x)\rangle$. Let Q_4 denote a quad through x such that $Q_4 \cap Q_1 = Q_4 \cap Q_3 = \{x\}$. Let O_1 be a classical ovoid of Q_1 containing the point x. Then there exists at most one (classical) SDPS-set X of $DW(7,q)$ satisfying:*

(1) *$Q_1 \cap X = O_1$;*

(2) *the quads Q_2, Q_3 and Q_4 are X-special.*

Proof. Let X_1 and X_2 denote two SDPS-sets of $DW(7,q)$ satisfying the above conditions. Let S_i, $i \in \{1,2\}$, be the generalized quadrangle isomorphic to $DW(3,q^2) \cong Q(4,q^2)$ defined on the set X_i by the X_i-special quads of $DW(7,q)$. By Lemma 3.3 applied to the triple $(F, F_1, F_2) = (Q_3, Q_2, Q_4)$, Q_2 and Q_4 lie at maximal distance 2 from each other. By Lemma 3.2, $O_2 := \pi_{Q_2}(O_1)$ is a classical ovoid of Q_2 and $O_4 := \pi_{Q_4}(O_2)$ is a classical ovoid of Q_4. By Lemma 3.8, $O_2 \cup O_4 \subseteq X_1 \cap X_2$. Now, let $\mathcal{V}$ denote the set of quads of $DW(7,q)$ which intersect Q_1 in a point of O_1 and Q_2 in a point of O_2. Then $Q_3 \in \mathcal{V}$ and $\mathrm{d}(Q, Q_4) = 2$ for every $Q \in \mathcal{V} \setminus \{Q_3\}$ by Lemma 3.3 applied to the triple $(F, F_1, F_2) = (Q_1, Q_4, Q)$. As before we can conclude that $O_Q := \pi_Q(O_4)$ is a classical ovoid of Q contained in $X_1 \cap X_2$. Now, let Q_5 be an arbitrary quad of $\mathcal{V} \setminus \{Q_3\}$. Then $\mathrm{d}(Q_5, Q_3) = 2$ by Lemma 3.3 applied to the triple $(F, F_1, F_2) = (Q_1, Q_3, Q_5)$. So, $O_{Q_3} := \pi_{Q_3}(O_{Q_5})$ is a classical ovoid of Q_3 which is contained in $X_1 \cap X_2$. Now, put $Y := \bigcup_{Q \in \mathcal{V}} O_Q$. Then Y defines a $(q^2+1) \times (q^2+1)$-subgrid in both the generalized quadrangle $S_1 \cong Q(4, q^2)$ and $S_2 \cong Q(4, q^2)$. Let y denote an arbitrary point of $O_4 \setminus \{x\}$ and let R and R' be two distinct elements of $\mathcal{V}$. Since $y \in X_1 \cap X_2$, $\pi_R(y) \in X_1 \cap X_2$ by Lemma 3.8. Applying Lemma 3.3 to the triple $(F, F_1, F_2) = (Q_1, R, R')$, we find $\mathrm{d}(R, R') = 2$. So, there exists a unique quad R'' through $\pi_R(y)$ intersecting R' in a unique point. Since Y defines a $(q^2+1) \times (q^2+1)$-subgrid of S_i, $i \in \{1, 2\}$, this quad is X_i-special. Since no two special X_i-quads can intersect in a line, $R'' \cap \langle y, \pi_R(y) \rangle = \{\pi_R(y)\}$. By Lemma 3.3 applied to the triple $(F, F_1, F_2) = (R'', \langle y, \pi_R(y) \rangle, R')$, $\mathrm{d}(R', \langle y, \pi_R(y) \rangle) = 2$. As before, it follows that $\pi_{\langle y, \pi_R(y) \rangle}(O_{R'})$ is a classical ovoid of $\langle y, \pi_R(y) \rangle$ which is contained in $X_1 \cap X_2$. The set $\pi_{\langle y, \pi_R(y) \rangle}(O_{R'})$ corresponds with a line of S_i, $i \in \{1, 2\}$, meeting the $(q^2+1) \times (q^2+1)$-subgrid Y. By Lemma 3.7, it now readily follows that $X_1 = X_2$. This proves the lemma. ■

Lemma 3.12 *Let x be a point of $DW(4n-1, q)$, $n \geq 2$. Let Q_1 and Q_2 denote two quads through x and let F denote a convex subspace of diameter $2n-2$ through x such that $Q_1 \cap Q_2 = Q_1 \cap F = Q_2 \cap F = \{x\}$. Let y denote a point of $DW(4n-1, q)$ at distance $2n$ from x and let Y denote a classical SDPS-set of F such that x, $\pi_F(y) \in Y$ and $\langle Q_1, Q_2 \rangle \cap F$ is a Y-special quad. Then there exists at most one (classical) SDPS-set X in $DW(4n-1, q)$ satisfying:*

(i) $x, y \in X$;

(ii) Q_1 *and* Q_2 *are* X*-classical;*

(iii) $X \cap F = Y$.

Proof. Let Ω denote the set of all SDPS-sets which satisfy the above conditions. Let X^* be an arbitrary element of Ω and let $\mathcal{I}$ be the intersection of all SDPS-sets of Ω. Then $\mathcal{I} \subseteq X^*$. We will now also show that $X^* \subseteq \mathcal{I}$, i.e. X^* is contained in each SDPS-set X of Ω. Since both X^* and X carry the structure of a dual polar space isomorphic to $DW(2n-1, q^2)$, we then necessarily have that $X^* = X$.

By Lemma 3.8 the point $\pi_{Q_2}(y)$ belongs to $\mathcal{I}$. Since $\mathrm{d}(x, y) = 2n$, $\mathrm{d}(y, \pi_{Q_2}(y)) = 2n-2$. The convex subspace $\langle \pi_{Q_2}(y), y \rangle$ is X-special for every $X \in \Omega$. Since $\langle y, \pi_{Q_2}(y) \rangle \cap Q_2 = \{\pi_{Q_2}(y)\}$, $F_2 := \langle y, \pi_{Q_2}(y) \rangle$ and $F_1 := F$ lie at maximal distance 2 from each other by Lemma 3.3. By Lemmas 3.2 and 3.8, $Y_2 := \pi_{F_2}(Y)$ is an SDPS-set of F_2 which is contained in $\mathcal{I}$, in other words $X \cap F_2 = Y_2$ for every $X \in \Omega$. With a similar reasoning as in the proof of Lemma 3.3, we know that every point of Q_1 lies at maximal distance 2 from F_2. By Lemma 3.8, $Q_1' := \pi_{F_2}(Q_1)$ is Y_2-special. So, $Q_1' \cap Y_2$ is a classical ovoid of Q_1'. Put $O_1 := \pi_{Q_1}(Q_1' \cap Y_2)$. By Lemma 3.8, $O_1 \subseteq \mathcal{I}$. By Lemma 3.3, Q_1 and $\langle y, \pi_F(y) \rangle$ lie at maximal distance $2n-2$ from each other and by Lemma 3.8, $O' := \pi_{\langle y, \pi_F(y) \rangle}(O_1)$ is a classical ovoid of $\langle y, \pi_F(y) \rangle$ which is contained in $\mathcal{I}$. Similarly, Q_2 and $\langle y, \pi_F(y) \rangle$ lie at maximal distance $2n-2$ from each other and $O_2 := \pi_{Q_2}(O')$ is a classical ovoid of Q_2 contained in $\mathcal{I}$.

Now, let $\mathcal{V}$ denote the set of all convex subspaces of diameter $2n-2$ containing a point of O_2 and a point of O'. Then every convex subspace of $\mathcal{V}$ is X-special for every $X \in \Omega$. Notice also that by Lemma 3.3, every two distinct elements of $\mathcal{V}$ lie at maximal distance 2 from each other.

Now, for every $G \in \mathcal{V}$, put $Y_G := \pi_G(Y)$. Also put $Z := \bigcup_{G \in \mathcal{V}} Y_G$. By Lemma 3.8, $Z \subseteq \mathcal{I}$.

We will now show that $X^* \subseteq \mathcal{I}$. Let Δ^* denote the dual polar space isomorphic to $DW(2n-1, q^2)$ defined on the set X^* by the X^*-special quads. The convex subspaces of $\mathcal{V}$ define a set $\mathcal{V}'$ of q^2+1 maxes of Δ^* in the sense of Lemma 3.6, i.e. every line of Δ^* meeting two distinct maxes of $\mathcal{V}'$ meets every max of $\mathcal{V}'$. In order to show that $X^* \subseteq \mathcal{I}$, we must show that every quad of Δ^* meeting every max of $\mathcal{V}'$ in a line is contained in $\mathcal{I}$. By Lemma 3.5, the following two steps are sufficient to prove this claim:

- **Step 1:** We show that $Q^* \subseteq \mathcal{I}$ for a particular quad Q^* of Δ^* which meets every max of $\mathcal{V}'$ in a line.

- **Step 2:** Suppose Q_1 and Q_2 are two mutually disjoint quads of Δ^*

which are contained in a hex of Δ^* and which meet every max of $\mathcal{V}'$ in a line. We show that if $Q_1 \subseteq \mathcal{I}$, then also $Q_2 \subseteq \mathcal{I}$.

We first prove Step 1. For every $X \in \Omega$, Q_1 and Q_2 are X-special and hence also $\langle Q_1, Q_2 \rangle$ is X-special since Q_1 and Q_2 intersect in a unique point (of X). Since the convex subspace $\langle Q_1, Q_2 \rangle$ is X^*-special and meets F_1 and F_2 in X^*-special quads, it follows that $Q^* := \langle Q_1, Q_2 \rangle \cap X^*$ is a quad of Δ^* which meets every element of $\mathcal{V}'$ in a line. For every $X \in \Omega$, the quads $Q_1, Q_2, \langle Q_1, Q_2 \rangle \cap F_1$ and $\langle Q_1, Q_2 \rangle \cap F_2$ of $DW(4n-1, q)$ are X-special. Moreover, $\langle Q_1, Q_2 \rangle \cap F_1 \cap X = \langle Q_1, Q_2 \rangle \cap F_1 \cap Y = \langle Q_1, Q_2 \rangle \cap F_1 \cap X^*$. By Lemma 3.11, $Q^* = \langle Q_1, Q_2 \rangle \cap X^* = \langle Q_1, Q_2 \rangle \cap X$. Hence, $Q^* \subseteq \mathcal{I}$.

We prove Step 2. Let Q_1 and Q_2 be two mutually disjoint quads of Δ^* which are contained in a hex of Δ^* and which meet every max of $\mathcal{V}'$ in a line. Let $\mathrm{d}(\cdot, \cdot)$, respectively $\delta(\cdot, \cdot)$, denote the distance function in $DW(4n-1, q)$, respectively Δ^*. Let x_1 and y_1 be two points of Q_1 such that $\delta(x_1, y_1) = 2$. Let x_2 and y_2 be the unique points of Q_2 such that $\delta(x_1, x_2) = \delta(y_1, y_2) = 1$. Then $\delta(x_2, y_2) = 2$ and $\delta(x_1, y_2) = \delta(x_2, y_1) = 3$. Hence, $\mathrm{d}(x_1, y_1) = \mathrm{d}(x_2, y_2) = 4$, $\mathrm{d}(x_1, x_2) = \mathrm{d}(y_1, y_2) = 2$ and $\mathrm{d}(x_1, y_2) = \mathrm{d}(y_1, x_2) = 6$. If $\langle x_1, x_2 \rangle$ and $\langle x_1, y_1 \rangle$ meet in a line L, then $\mathrm{d}(x_2, y_1) \leq \mathrm{d}(x_2, \pi_L(x_2)) + \mathrm{d}(\pi_L(x_2), y_1) \leq 1 + 4 = 5$, a contradiction. Hence, $\langle x_1, x_2 \rangle$ and $G_1 := \langle x_1, y_1 \rangle$ intersect in the singleton $\{x_1\}$. Similarly, $\langle x_1, x_2 \rangle$ and $G_2 := \langle x_2, y_2 \rangle$ intersect in the singleton $\{x_2\}$. By Lemma 3.3, G_1 and G_2 lie at maximal distance 2 from each other. If $Q_1 \subseteq \mathcal{I}$, then we also have that $\pi_{G_2}(Q_1) = Q_2 \subseteq \mathcal{I}$ by Lemma 3.8.

This proves the lemma. ∎

Lemma 3.13 *For every $n \geq 2$, $\lambda(n) \leq q^{4n-2}(q^{4n-2} - 1) \cdot \lambda(n-1)$.*

Proof. We count in two different ways the number of tuples $(x, y, Q_1, Q_2, F, Y, X)$ which satisfy the conditions of Lemma 3.12.

Step 1: *There are $\alpha_1 = (q+1)(q^2+1) \cdots (q^{2n}+1)$ possibilities for x.*
PROOF. This is precisely the number of points of $\Delta = DW(4n-1, q)$.

Step 2: *For given x, there are $\alpha_2 = q^{1+2+\cdots+2n}$ possibilities for y.*
PROOF. This is precisely the number of points of $\Delta = DW(4n-1, q)$ opposite to x.

Step 3: *For given x and y, there are $\alpha_3 = \frac{(q^{2n}-1)(q^{2n}-q)}{(q^2-1)(q^2-q)}$ possibilities for F.*
PROOF. Recall that $Res_\Delta(x)$ is isomorphic to $\mathrm{PG}(2n-1, q)$. The convex

subspace F corresponds with a $(2n-3)$-dimensional subspace of $Res_\Delta(x)$. So, there are precisely $\alpha_3 = \frac{(q^{2n}-1)(q^{2n}-q)}{(q^2-1)(q^2-q)}$ possibilities for F.

Step 4: *For given x, y and F, there are $\alpha_4 = q^{4n-4}$ possibilities for Q_1.*
Proof. Reason again in the projective space $Res_\Delta(x) \cong \mathrm{PG}(2n-1,q)$. The number of possibilities for Q_1 is equal to the number of lines of $Res_\Delta(x)$ disjoint with a given $(2n-3)$-dimensional subspace. This number is equal to q^{4n-4}.

Step 5: *For given x, y, F and Q_1, there are*

$$\alpha_5 \cdot \lambda(n-1) := \frac{\lambda(n-1)}{(q+1)(q^3+1)\cdots(q^{2n-3}+1)q^{(n-1)^2}}$$

possibilities for Y.
Proof. The SDPS-set Y of F must contain x and $\pi_F(y)$. By Lemma 3.10, there are

$$\frac{\lambda(n-1)}{(q+1)(q^3+1)\cdots(q^{2n-3}+1)q^{(n-1)^2}}$$

possibilities for Y.

Step 6: *For given x, y, F, Q_1 and Y, there are $\alpha_6 = \frac{q^{2n-2}-1}{q^2-1}(q^2-1)(q^2-q)$ possibilities for Q_2.*
Proof. Since Q_1 and Q_2 are X-special quads and intersect in a point of X, the convex suboctagon $\langle Q_1, Q_2\rangle$ is also X-special and hence intersects F in an X-special quad. This quad is necessarily Y-special. The set of Y-special quads of F through y is equal to the number of lines of $DW(2n-3,q^2)$ through a given point of $DW(2n-3,q^2)$, i.e. equal to $\frac{q^{2n-2}-1}{q^2-1}$. If $\langle Q_1,Q_2\rangle \cap F$ is known, then also $\langle Q_1, Q_2\rangle$ is known, since $\langle Q_1,Q_2\rangle = \langle Q_1, \langle Q_1,Q_2\rangle \cap F\rangle$.

Now, suppose U is a convex suboctagon through Q_1 intersecting F in a quad. We count the number of quads of U through x which have no line in common with Q_1 and $F \cap U$. This number is equal to the number of lines of $\mathrm{PG}(3,q)$ which are disjoint with the union of two given mutually disjoint lines of $\mathrm{PG}(3,q)$. This number equals $(q^2-1)(q^2-q)$.

In conclusion, we can say that there are precisely $\alpha_6 = \frac{q^{2n-2}-1}{q^2-1}(q^2-1)(q^2-q)$ possibilities for Q_2.

Step 7: *For given x, y, F, Q_1, Y and Q_2, there is at most one possibility for X.*
Proof. This is precisely Lemma 3.12.

By Steps 1 till 7, there are at most

$$\lambda(n-1)\cdot\prod_{i=1}^{6}\alpha_i$$

possible tuples $(x, y, Q_1, Q_2, F, Y, X)$. Via a second counting we will calculate the precise number of such tuples.

Step 8: *There are $\lambda(n)$ possibilities for X.*
PROOF. By definition of $\lambda(n)$.

Step 9: *For given X, there are $\alpha_7 = (q^2+1)(q^4+1)\cdots(q^{2n}+1)$ possibilities for x.*
PROOF. This is precisely the number of points of $DW(2n-1, q^2)$.

Step 10: *For given X and x, there are $\alpha_8 = q^{2+4+\cdots+2n}$ possibilities for y.*
PROOF. This is precisely the number of points of $DW(2n-1, q^2)$ which are opposite to a given point of $DW(2n-1, q^2)$.

Step 11: *For given X, x and y, there are precisely $\alpha_9 = \frac{q^{2n}-1}{q^2-1}\cdot q^{2n-2}\cdot(q^{2n-2}-1)$ possibilities for (F, Q_1, Q_2).*
PROOF. Let $DW(2n-1, q^2)$ denote the dual polar space associated with the SDPS-set X. The convex subspaces of $DW(2n-1, q^2)$ through x define a projective space isomorphic to $\mathrm{PG}(n-1, q^2)$. F corresponds with a hyperplane of this projective space and Q_1 and Q_2 correspond with two distinct points of this projective space not contained in that hyperplane. It follows that there are $\alpha_9 = \frac{q^{2n}-1}{q^2-1}\cdot q^{2n-2}\cdot(q^{2n-2}-1)$ possibilities for (F, Q_1, Q_2).

Step 12: *For given X, x, y, F, Q_1 and Q_2, there is only one possibility for Y.*
PROOF. This follows from the fact that $Y = F \cap X$.

Summarizing, we can say that there are $\lambda(n)\cdot\prod_{i=7}^{9}\alpha_i$ possible tuples (x, y, Q_1, Q_2, F, X). By the first discussion, we know that

$$\lambda(n)\cdot\prod_{i=7}^{9}\alpha_i \leq \lambda(n-1)\cdot\prod_{i=1}^{6}\alpha_i.$$

It follows that

$$\lambda(n) \leq \frac{\prod_{i=1}^{6}\alpha_i}{\prod_{i=7}^{9}\alpha_i}\cdot\lambda(n-1) = q^{4n-2}\cdot(q^{4n-2}-1)\cdot\lambda(n-1).$$

■

Since $\lambda(1) = \frac{q^2(q^2-1)}{2}$, we have:

Corollary 3.14 *There are at most $\frac{1}{2}q^{2n^2}(q^2-1)(q^6-1)\cdots(q^{4n-2}-1)$ classical SDPS-sets in the dual polar space $DW(4n-1,q)$, $n \geq 1$.*

3.3 Subtended SDPS-sets

We mention the following two propositions which we take from De Bruyn [6] (Theorems 1.5 and 1.6).

Proposition 3.15 ([6]) *Up to isomorphism, there exists a unique isometric embedding of $DW(2n-1,q)$ into $DH(2n-1,q^2)$ $(n \geq 2)$.*

Proposition 3.16 ([6]) *Let Δ be a dual polar space isomorphic to $DW(2n-1,q)$, $n \geq 2$, which is isometrically embedded into the dual polar space $\Delta' = DH(2n-1,q^2)$. Then the following holds:*

(i) $\max\{d(x,\Delta) \mid x \in \Delta'\} = \lfloor \frac{n}{2} \rfloor$;

(ii) if $d(x,\Delta) = \delta$, then $\Delta_\delta(x) \cap \Delta$ is an SDPS-set in a convex subspace of diameter 2δ of Δ;

(iii) if n is even, then the set of points of Δ' at distance at most $\frac{n}{2}-1$ from Δ is a hyperplane of Δ';

(iv) if n is even, then the complement of the hyperplane defined in (iii) has $q^{\frac{n^2}{2}}(q^2-1)(q^6-1)\cdots(q^{2n-2}-1)$ points.

Now, let the dual polar space $\Delta = DW(4n-1,q)$, $n \geq 1$, be isometrically embedded into $DH(4n-1,q^2)$ and let H be the hyperplane of $DH(4n-1,q^2)$ which consists of all points of $DH(4n-1,q^2)$ at distance at most $n-1$ from Δ. If x belongs to the complement $\overline{H}$ of H, then $\Delta_n(x) \cap \Delta$ is an SDPS-set of $DW(4n-1,q)$ by Proposition 3.16 (ii). We call any SDPS-set which can be obtained in this way a *subtended SDPS-set*.

Lemma 3.17 *Any two subtended SDPS-sets of $\Delta = DW(4n-1,q)$, $n \geq 1$, are isomorphic.*

Proof. By Shult [16, Lemma 6.1], the complement $\overline{H}$ of the hyperplane H is connected. Hence, it suffices to show the following: if x_1 and x_2 are two collinear points of $\overline{H}$, then $\Delta_n(x_1) \cap \Delta$ and $\Delta_n(x_2) \cap \Delta$ are isomorphic SDPS-sets of Δ. Let y denote the unique point of the line x_1x_2 at distance $n-1$ from $\Delta = DW(4n-1,q)$. Then $\Delta_{n-1}(y) \cap \Delta$ is an SDPS-set in a convex subspace F of diameter $2n-2$ of Δ. Let M denote an arbitrary max of Δ containing F. Let $\overline{F}$ (respectively $\overline{M}$) denote the unique convex subspace of diameter $2n-2$ (respectively $2n-1$) of $DH(4n-1,q^2)$ containing F (respectively M). Notice that $y \in \overline{F}$ since by Proposition 3.16 (ii) y is contained on a shortest path between two points of $\Delta_{n-1}(y) \cap \Delta$ at maximal distance $2n-2$ from each other. Now, the embedding of M into $\overline{M}$ is isometric. By Proposition 3.16 (i), it follows that the maximal distance from a point of M to $\overline{M}$ is equal to $n-1$. This implies that $x_1x_2 \cap \overline{M} = y$. By Lemma 3.9, there exists an automorphism θ of $DH(2n-1,q^2)$ satisfying the following properties:

(1) θ fixes $\overline{M}$ point-wise and every line meeting $\overline{M}$ set-wise;

(2) $\theta(x_1) = x_2$.

Now, since M is a max of $DW(4n-1,q)$, there exist a collection of lines of $DW(4n-1,q)$ meeting M which cover the whole point-set of $DW(4n-1,q)$. Each line of this collection is fixed by θ. Hence, $\theta(\Delta) = \Delta$. It follows that $\theta(\Delta_n(x_1) \cap \Delta) = \Delta_n(x_2) \cap \Delta$. This is precisely what we needed to show. ■

The question which one can ask now is whether that there exist two points x_1 and x_2 in $\overline{H}$ such that $\Delta_n(x_1) \cap \Delta = \Delta_n(x_2) \cap \Delta$. The answer is affirmative for the case $n = 1$. If one looks to the case of an isometric embedding of $Q(4,q)$ into $Q(5,q)$, then every classical ovoid of $Q(4,q)$ is subtended by precisely two points of $Q(5,q) \setminus Q(4,q)$. We will prove that a similar property holds for every $n \geq 2$. In Section 3.4, we prove the following:

Lemma 3.18 (Section 3.4) *If X is an SDPS-set of $\Delta = DW(4n-1,q)$, $n \geq 1$, then there are at most 2 points $x \in \overline{H}$ such that $\Delta_n(x) \cap \Delta = X$.*

Corollary 3.19 *There are at least $\frac{|\overline{H}|}{2} = \frac{1}{2}q^{2n^2}(q^2-1)(q^6-1)\cdots(q^{4n-2}-1)$ subtended SDPS-sets in $DW(4n-1,q)$, $n \geq 1$.*

Combining this with Corollary 3.14, we find that

Theorem 3.20 (1) *There are precisely $\frac{1}{2}q^{2n^2}(q^2-1)(q^6-1)\cdots(q^{4n-2}-1)$ classical SDPS-sets in $DW(4n-1,q)$, $n \geq 1$.*

(2) *Every classical SDPS-set of $DW(4n-1,q)$, $n \geq 1$, is subtended.*

(3) *All classical SDPS-sets of $DW(4n-1,q)$, $n \geq 1$, are isomorphic.*

3.4 Proof of Lemma 3.18

Lemma 3.21 *Let $DW(4n-1,q)$, $n \geq 1$, be isometrically embedded into $DH(4n-1,q^2)$. Let x be a point of $DH(4n-1,q^2)$ at distance n from $DW(4n-1,q)$ and let X be the SDPS-set $\Delta_n(x) \cap DW(4n-1,q)$ of $DW(4n-1,q)$. For every line L of $DH(4n-1,q^2)$ through x, let y_L denote the unique point of L at distance $n-1$ from $DW(4n-1,q)$ and let F_L denote the unique convex subspace of diameter $2n-2$ of $DW(4n-1,q)$ containing all points of $\Delta_{n-1}(y_L) \cap DW(4n-1,q)$. Then the map $L \mapsto F_L$ is a bijection between the set of lines of $DH(4n-1,q^2)$ through x and the set of X-special convex subspaces of diameter $2n-2$ of $DW(4n-1,q)$.*

Proof. Since there are as many lines in $DH(4n-1,q^2)$ through x as there are X-special convex subspaces of diameter $2n-2$ in $DW(4n-1,q)$, namely $1+q^2+\ldots+q^{4n-2} = (1+q^{2n})(1+q^2+\cdots+q^{2n-2})$, it suffices to show injectivity.

For every X-special convex subspace F of diameter $2n-2$ of $DW(4n-1,q)$, the unique convex subspace $\overline{F}$ of diameter $2n-2$ of $DH(4n-1,q^2)$ containing F only contains points at distance at most $n-1$ from F (and hence also from $DW(4n-1,q)$) by Proposition 3.16 (i). Hence, there exists at most one line through x meeting $\overline{F}$. If L is a line through x such that $F_L = F$, then by Proposition 3.16 (ii), y_L is contained on a shortest path between two points of $F \cap \Delta_{n-1}(y_L)$ at maximal distance $2n-2$ from each other. It follows that $y_L \in \overline{F}$, i.e. L meets $\overline{F}$. The injectivity now readily follows. ∎

Lemma 3.22 *Let $DW(4n-1,q)$, $n \geq 1$, be isometrically embedded into the dual polar space $DH(4n-1,q^2)$ and let X be an SDPS-set of $DW(4n-1,q)$. Let F_1 and F_2 be two X-special convex subspaces of diameter $2n-2$ of $DW(4n-1,q)$ such that $(F_1 \cap X) \cap (F_2 \cap X) = \emptyset$. Let $\overline{F_i}$, $i \in \{1,2\}$, denote the unique convex subspace of diameter $2n-2$ of $DH(4n-1,q^2)$ containing F_i. Then $\overline{F_1}$ and $\overline{F_2}$ lie at maximal distance 2 from each other.*

Proof. Let $DW(2n-1,q^2)$ denote the dual polar space defined on the set X by the X-special quads and let $\mathrm{d}(\cdot,\cdot)$, respectively $\delta(\cdot,\cdot)$, denote the distance function in $DH(4n-1,q^2)$, respectively $DW(2n-1,q^2)$. Let x_1 and y_1 be two points of $F_1 \cap X$ such that $\delta(x_1,y_1) = n-1$ and let x_2 and y_2 be the unique points of $F_2 \cap X$ such that $\delta(x_1,x_2) = \delta(y_1,y_2) = 1$. Then $\delta(x_2,y_2) = n-1$, $\delta(x_1,y_2) = n$ and $\delta(x_2,y_1) = n$. It follows that $\mathrm{d}(x_1,y_1) = \mathrm{d}(x_2,y_2) = 2n-2$,

$\mathrm{d}(x_1,x_2)=\mathrm{d}(y_1,y_2)=2$ and $\mathrm{d}(x_1,y_2)=\mathrm{d}(x_2,y_1)=2n$. If $\langle x_1,x_2\rangle\cap\overline{F_2}$ is a line L, then $\mathrm{d}(x_1,y_2)\leq\mathrm{d}(x_1,\pi_L(x_1))+\mathrm{d}(\pi_L(x_1),y_2)\leq 1+(2n-2)=2n-1$, a contradiction. Hence, $\langle x_1,x_2\rangle\cap\overline{F_2}=\{x_2\}$. Similarly, $\langle x_1,x_2\rangle\cap\overline{F_1}=\{x_1\}$. So, the triple $(\langle x_1,x_2\rangle,\overline{F_1},\overline{F_2})$ satisfies the conditions of Lemma 3.3. It follows that $\overline{F_1}$ and $\overline{F_2}$ lie at maximal distance 2 from each other. ∎

The following lemma is precisely Lemma 3.18.

Lemma 3.23 *Let $DW(4n-1,q)$, $n\geq 1$, be isometrically embedded in $DH(4n-1,q^2)$ and let X be a classical SDPS-set of $DW(4n-1,q)$. Then there exist at most two points x in $DH(4n-1,q^2)$ at distance n from $DW(4n-1,q)$ such that $\Delta_n(x)\cap DW(4n-1,q)=X$.*

Proof. We will prove this by induction on n. As already remarked above the lemma holds for $n=1$ since every classical ovoid of $Q(4,q)$ is subtended by precisely two points of $Q(5,q)\setminus Q(4,q)$. So, suppose $n\geq 2$ and that the lemma holds for smaller values of n.

Let F^* denote a given X-special convex subspace of diameter $2n-2$ of $DW(4n-1,q)$ and let $\overline{F^*}$ denote the unique convex subspace of diameter $2n-2$ of $DH(4n-1,q^2)$ containing F^*. Then by the induction hypothesis, there exists at most two and hence precisely two (see the end of Section 3.3) points x_1 and x_2 in $\overline{F^*}$ at distance $n-1$ from F^* such that $\Delta_{n-1}(x_1)\cap F^*=\Delta_{n-1}(x_2)\cap F^*=F^*\cap X$. By Lemma 3.21, if x is a point of $DH(4n-1,q^2)$ at distance n from $DW(4n-1,q)$ such that $\Delta_n(x)\cap DW(4n-1,q)=X$, then x is collinear with either x_1 or x_2. So, it suffices to show the following:

> $(*)$ there exists at most one point x in $DH(4n-1,q^2)$ at distance n from $DW(4n-1,q)$ such that $\Delta_n(x)\cap DW(4n-1,q)=X$ and x is collinear with x_1.

Now, for every X-special convex subspace F of diameter $2n-2$ of $DW(4n-1,q)$, we construct a point x_F of the unique convex subspace $\overline{F}$ of diameter $2n-2$ of $DH(4n-1,q^2)$ containing F. If $F=F^*$, then we define $x_F=x_1$. If F is disjoint from F^*, then x_F denotes the unique point of $\overline{F}$ at distance 2 from x_1 (see Lemma 3.22). If $F\neq F^*$ and F^* meet, then take an X-special convex subspace F' of diameter $2n-2$ of $DW(4n-1,q)$ disjoint from F and F^* (cf. Lemma 3.4) and let x_F denote the unique point of $\overline{F}$ at distance 2 from $x_{F'}$ (see Lemma 3.22). Let U denote the set of all points x_F, where F is an X-special convex subspace of diameter $2n-2$ of $DW(4n-1,q)$.

Suppose x is a point of $DH(4n-1,q^2)$ at distance n from $DW(4n-1,q)$ such that $\Delta_n(x) \cap DW(4n-1,q) = X$ and x is collinear with x_1. If F is an X-special convex subspace of diameter $2n-2$ of $DW(4n-1,q)$ disjoint from F^*, then the unique point of $\overline{F}$ collinear with x has distance 2 from x_1 and hence coincides with x_F. Now, let $F \neq F^*$ be an X-special convex subspace of diameter $2n-2$ of $DW(4n-1,q)$ meeting F^* and take F' as above. We already know that $x_{F'}$ is collinear with x. Now, the unique point of $\overline{F}$ collinear with x (see Lemma 3.21) has distance 2 from $x_{F'}$ and hence coincides with x_F.

Hence, we can say the following: if x is a point of $DH(4n-1,q^2)$ at distance n from $DW(4n-1,q)$ such that $\Delta_n(x) \cap DW(4n-1,q) = X$ and x is collinear with x_1, then x is collinear with every point of U. So, in order to establish $(*)$, it suffices to show that there is at most one point at distance n from $DW(4n-1,q)$ which is collinear with all points of U. Notice that if such a point exists, then U consists of points at mutual distance 2 from each other. Let u_1 and u_2 be two arbitrary distinct points of U. Then we may suppose that $\mathrm{d}(u_1,u_2) = 2$. Let Q be a quad of $\langle u_1, u_2 \rangle$. Since $|U|$ is equal to the number of lines of $DH(2n-1,q^2)$ through x, i.e. $1+q^2+\ldots+q^{4n-2}$, not all points of U are contained in Q. Let u_3 be a point of $U \setminus Q$. Then x (if it exists) necessarily coincides with the unique point of Q collinear with u_3. [Notice that x must be contained in Q since it is collinear with u_1 and u_2.] ■

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