Combinatorial Bounds for the Number of Rational Points on a Hypersurface*

By Koen Thas†
Ghent University, Department of Pure Mathematics and Computer Algebra, Galglaan 2, B-9000 Ghent, Belgium, E-mail: kthas@cage.rug.ac.be

Abstract

In this paper, we first survey combinatorial bounds for the number of points on a hypersurface in finite projective space, with sketches of proofs. Then we obtain new such bounds for certain types of hypersurfaces, as a corollary of results on more general combinatorial structures. In particular, the results apply to irreducible hypersurfaces.

1 Introduction

A set of points $\mathcal{K}$ of $\mathbf{PG}(2,q)$ is called a $\{k;n\}$*-arc* if its number of points is k and if each line meets $\mathcal{K}$ in at most n points. Suppose $\mathcal{K}$ is as such, and let x be a point of $\mathcal{K}$. As each line through x meets $\mathcal{K}$ in at most $n-1$ points different from x, we obtain

$$|\mathcal{K}| \leq 1+(q+1)(n-1) = qn+n-q.$$

If equality holds, then we call $\mathcal{K}$ a *maximal arc*.

Let $\mathcal{C}$ be a non-singular curve of degree d in $\mathbf{PG}(2,q)$. Then with R the number of $\mathbf{GF}(q)$-rational points on $\mathcal{S}$, we have the following.

Theorem 1.1 (H. Hasse [3, 4]; A. Weil [14, 15])

$$|R-(q+1)| \leq (d-1)(d-2)\sqrt{q}.$$

■

*__Mathematics Subject Classification (2000)__: 14J70, 05B25, 51E20, 51E21.

†__Acknowledgment.__ The author is a Postdoctoral Fellow of the Fund for Scientific Research — Flanders (Belgium).

We refer to Theorem 1.1 as being the "Hasse-Weil bound".

Now let $\mathcal{C}$ be a plane algebraic curve of degree d in $\mathbf{PG}(2,q)$, and suppose $\mathcal{C}$ does not contain any line over $\mathbf{GF}(q)$, i.e., $\mathcal{C}$ has no *regular linear components*. Then each line meets $\mathcal{C}$ in at most d points, whence with R the number of rational points of $\mathcal{C}$, we have that $\mathcal{C}$ is an $\{R;d'\}$-arc for some $d' \leq d$. Hence

$$|R| \leq qd' + d' - q \leq qd + d - q.$$

We compare the latter bound with the Hasse-Weil bound. Define the "defect"

$$f(d,q) = (qd + d - q) - (q + 1 + (d-1)(d-2)\sqrt{q}).$$

Then if $q \geq 2$ and $d \geq \sqrt{q} + 2$, one observes that $f(d,q) < 0$, implying that the trivial combinatorial bound of above is better than the Hasse-Weil bound under the aforementioned assumptions. This was first noted explicitly in J. A. Thas [10].

In this paper, we are interested in "similar" combinatorial bounds for the number of rational points on a hypersurface in finite projective space.

Note. These notes grew out of ideas of a talk by the author presented at the conference "Arithmetic, Geometry and Coding Theory" (AGCT-9), CIRM-Luminy, Marseilles, May 19-23, 2003.

2 Some Terminology and Notation

We will use the following terminology throughout.

- $\mathbf{PG}(n,q)$ is the n-dimensional projective space over the Galois field $\mathbf{GF}(q)$ with q elements.

- $p_n = \frac{q^{n+1}-1}{q-1}$ is the number of points of $\mathbf{PG}(n,q)$; in particular, $p_{-1} = 0$. By convention, $p_k = 0$ whenever $k \leq -1$.

- $\Phi(X_0, X_1, \ldots, X_n) = \Phi$ is a homogeneous nonzero polynomial of degree $d \leq q+1$, with coefficients in $\mathbf{GF}(q)$ (so Φ defines a *hypersurface* in $\mathbf{PG}(n,q)$; in particular, for $n = 2$, Φ defines an *algebraic curve* in $\mathbf{PG}(2,q)$, or a *plane algebraic curve*), and S is the set of $\mathbf{GF}(q)$-rational points of Φ. If Φ is irreducible (over $\mathbf{GF}(q)$), then we call the associated hypersurface also *irreducible*.

- $|S| = N$.

3 The Bounds of J. -P. Serre

In J.-P. Serre's *Lettre à M. Tsfasman* [7], an interesting bound for the maximal number of points on a hypersurface of the n-dimensional projective space $\mathbf{PG}(n,q)$ over $\mathbf{GF}(q)$ is given. This answered affirmatively a question posed by M. Tsfasman. With the notation of the previous section, J.-P. Serre proves the following

Theorem 3.1 (J.-P. Serre [7]) *We have that*

$$N \leq dq^{n-1} + p_{n-2}.$$

Sketch of Proof. We do not consider the case $d = q+1$, as in that case $p_n = dq^{n-1} + p_{n-2}$. So $d \leq q$. The proof is by induction on n. We assume that $n \geq 2$ as the case $n = 1$ is easy. Let $F_1, F_2, \ldots, F_r$ be the different linear factors of Φ over $\mathbf{GF}(q)$, and suppose that $\Pi_1, \Pi_2, \ldots, \Pi_r$ are the hyperplanes of $\mathbf{PG}(n,q)$ which correspond to $F_1, F_2, \ldots, F_r$, respectively. Then $U = \bigcup_{i=1}^{r} \Pi_i$ is contained in S. We assume that $S \neq \emptyset$, as that case is trivial. We distinguish two cases.

(i) $U = S$ AND $r \geq 1$. Then the theorem follows from observing that for $1 \leq m \leq r$ $(\leq d)$,

$$|\Pi_1 \cup \Pi_2 \cup \ldots \cup \Pi_m| \leq mq^{n-1} + p_{n-2}.$$

(ii) $U \neq S$ OR Φ HAS NO LINEAR FACTORS OVER $\mathbf{GF}(q)$. Let p be a point of S which is not in U. If then Π is a hyperplane of $\mathbf{PG}(n,q)$ containing p, then the restriction of Φ to Π is not identically zero. Hence we can apply the Induction Hypothesis on $S \cap \Pi$ to obtain that

$$|S \cap \Pi| \leq dq^{n-2} + p_{n-3}.$$

Now we count in two ways the number θ of point-hyperplane pairs (p', Π') for which p' is a point of $S \setminus \{p\}$ and Π' is a hyperplane containing p and p'. Clearly we have that

$$\theta = (N-1)p_{n-2}.$$

Now fix a hyperplane Π'' containing p; then there are at most $dq^{n-2} + p_{n-3} - 1$ points contained in $(S \cap \Pi'') \setminus \{p\}$. Hence

$$\theta \leq p_{n-1}(dq^{n-2} + p_{n-3} - 1).$$

Thus we obtain that $N \leq 1 + \frac{p_{n-1}}{p_{n-2}}[dq^{n-2} + p_{n-3} - 1]$, and the result follows. ■

Further on, an alternative proof of Theorem 3.1 will be obtained.

4 The Bounds of K. Thas

Using essentially the same combinatorial technique as in [7], in [12] we provided a bound which is relative to the maximal dimension of a subspace of $\mathbf{PG}(n,q)$ which is completely contained in the hypersurface. The lower that dimension, the better the bound. Next, by using a different argument, we derived a bound which is again relative to the maximal dimension of a subspace of $\mathbf{PG}(n,q)$ which is completely contained in the hypersurface. If that dimension increases for the latter case, the bound gets better.
As such, the bounds are complementary.

4.1 Bounds relative to the maximal dimension of a subspace contained in the hypersurface

Consider the projective space $\mathbf{PG}(n,q)$, and define $F(n,k,q)$, $0 \leq k \leq n$, $k \in \mathbb{N}$, by

$$F(n,k,q) = \sum_{i=k}^{n-2} q^i \frac{p_{n-1}}{p_i p_{i+1}}$$

if $k \in \{1,2,\ldots,n-2\}$, and $F(n,k,q) = 0$ if $k = 0$ or $k \geq n-1$.

Theorem 4.1 (K. Thas [12])

$$N \leq dq^{n-1} + p_{n-2} + (d-(q+1))F(n,k,q),$$

where $k \leq n-1$ is the maximal dimension of a $\mathbf{PG}(k,q) \subset \mathbf{PG}(n,q)$ which is completely contained in S.

Sketch of Proof. As in the proof of Theorem 3.1, we do not consider the case $d = q+1$, so that $d \leq q$. The proof is by the same induction argument on n as in Theorem 3.1. We assume that $n \geq 2$. Let $F_1, F_2, \ldots, F_r$ be the different linear factors of Φ over $\mathbf{GF}(q)$, and suppose that $\Pi_1, \Pi_2, \ldots, \Pi_r$ are the hyperplanes of $\mathbf{PG}(n,q)$ which correspond to $F_1, F_2, \ldots, F_r$, respectively. Then $U = \bigcup_{i=1}^r \Pi_i$ is contained in S. We assume that $S \neq \emptyset$ as before. We distinguish again two cases.

(i) $U = S$ AND $r \geq 1$. In that case, $k = n-1$, so $F(n,k,q) = 0$. See Theorem 3.1 for this case.

(ii) $U \neq S$ OR Φ HAS NO LINEAR FACTORS OVER $\mathbf{GF}(q)$. Let p be a point of S which is not in U. If Π is a hyperplane of $\mathbf{PG}(n,q)$ containing p, then the restriction of Φ to Π is not identically zero. Hence we can apply the Induction Hypothesis on $S \cap \Pi$ to obtain that

$$|S \cap \Pi| \leq dq^{n-2} + p_{n-3} + (d-(q+1))F(n-1,k',q),$$

where k' is the maximal dimension of a $\mathbf{PG}(k', q)$ which is completely contained in $S \cap \Pi$. We now count in two ways the number θ of point-hyperplane pairs (p', Π') for which p' is a point of $S \setminus \{p\}$ and Π' is a hyperplane containing p and p'. We have that

$$\theta = (N - 1)p_{n-2}.$$

Now fix a hyperplane Π'' containing p; then there are at most $dq^{n-2} + p_{n-3} + (d - (q + 1))F(n-1, k_m, q) - 1$ points contained in $(S \cap \Pi'') \setminus \{p\}$, where k_m is the maximal k^* for which there is hyperplane Π^* of $\mathbf{PG}(n, q)$ containing p, so that $S \cap \Pi^*$ contains a $\mathbf{PG}(k^*, q)$ (note that $r \leq r'$ implies that $F(n, r, q) \geq F(n, r', q)$). Hence

$$\theta \leq p_{n-1}(dq^{n-2} + p_{n-3} + (d - (q + 1))F(n - 1, k_m, q) - 1).$$

We obtain that

$$N \leq 1 + \frac{p_{n-1}}{p_{n-2}}[dq^{n-2} + p_{n-3} + (d - (q + 1))F(n - 1, k_m, q) - 1],$$

and direct computations lead to

$$N \leq dq^{n-1} + p_{n-2} + (d - (q + 1))F(n, k_m, q).$$

The theorem follows. ■

The lower k, the better the result.
Note that

$$F(n, k, q) = \sum_{i=k}^{n-2} q^i \frac{p_{n-1}}{p_i p_{i+1}} = \sum_{i=k}^{n-2} (q^{n-2-i} + \frac{p_{n-3-i}}{p_{i+1}})(1 - \frac{p_{i-1}}{p_i}) > q^{n-k-2} - \frac{1}{q},$$

for $1 \leq k \leq n - 2$.

Substituting in Theorem 4.1 yields

Theorem 4.2 (K. Thas [12])

$$N \leq dq^{n-1} + p_{n-2} + (d - (q + 1))q^{n-k-2}.$$

(Recall that $N \in \mathbb{N}$, and that $1 \leq d \leq q + 1$.) ■

Notation. We denote by $F(n, q)$ the following:

$$F(n, q) = \frac{p_{n-1}}{p_1} + F(n, 1, q).$$

Clearly, $F(n, q) > \frac{p_{n-1}}{p_1} + q^{n-3} - \frac{1}{q} (> q^{n-2} + q^{n-3} - \frac{1}{q})$.

There is an interesting corollary of the proof of Theorem 4.1, obtained essentially in the same way:

Theorem 4.3 (K. Thas [12]) *Suppose Φ is a homogeneous polynomial of degree $d \leq q+1$ with coefficients in $\mathbf{GF}(q)$. Let S be the set of $\mathbf{GF}(q)$-rational points in $\mathbf{PG}(n,q)$, $n \geq 2$, and suppose the following property is satisfied.*

(L) There is no line in $\mathbf{PG}(n,q)$ which is completely contained in S.

If $N = |S|$, then we have that

$$N \leq dq^{n-1} + p_{n-2} + (d-(q+1))F(n,q).$$

$\blacksquare$

There is a corollary for irreducible hypersurfaces.

Corollary 4.4 *Suppose Φ is an irreducible homogeneous polynomial of degree $d \leq q+1$ with coefficients in $\mathbf{GF}(q)$. Let S and N be as above. If $N = |S|$, then*

$$N \leq dq^{n-1} + p_{n-2} + (d-(q+1))F(n,q).$$

$\blacksquare$

Remark. The function $F(n,q)$ can be adapted in the obvious way to functions $F'(n,q)$ for which $F'(n,q) \geq F(n,q)$ for all n and q, and so that in the previous theorem $F(n,q)$ can be replaced by $F'(n,q)$, if the bound

$$N \leq dq + d - q$$

for the number N of points on a plane algebraic curve in $\mathbf{PG}(2,q)$ of degree d with no linear components (i.e., for $d \leq q$ the case $n = 2$ with the assumption of Property (L)) is improved. See e.g. J. A. Thas [11] for such improvements.

4.2 Bounds relative to the minimal dimension of a subspace contained in the hypersurface

The second main theorem of [12] is the following, which also yields an alternative proof of Theorem 3.1.

Theorem 4.5 (K. Thas [12]) *Suppose Φ is a homogeneous polynomial of degree $d \leq q$ with coefficients in $\mathbf{GF}(q)$. Let S be the set of $\mathbf{GF}(q)$-rational points in $\mathbf{PG}(n,q)$ of Φ, $n \geq 2$, and suppose that Π_{m-1} is a $\mathbf{PG}(m-1,q) \subset \mathbf{PG}(n,q)$ which is contained in S. Suppose that $S \neq \mathbf{PG}(n,q)$, and that no hyperplane of $\mathbf{PG}(n,q)$ which contains Π_{m-1}, is contained in S. Then*

$$N = |S| \leq dq^{n-1} + p_{n-2} + (d-(q+1))q^{m-1}.$$

Sketch of Proof. Suppose that $S \neq \mathbf{PG}(n,q)$, and that Π_{m-1} is a $\mathbf{PG}(m-1,q) \subset \mathbf{PG}(n,q)$ which is contained in S. Suppose that no hyperplane of $\mathbf{PG}(n,q)$ containing Π_{m-1} is contained in S. Then there is a $\mathbf{PG}(m,q) = \Pi_m$ containing Π_{m-1} which is not contained in S. Now count in two ways the number of point-hyperplane pairs (p,Π) for which $p \subset \Pi$, where p is a point of $S \setminus \Pi_m$, $\Pi_{m-1} \subset \Pi$, and where $\Pi_m \not\subset \Pi$. If $\alpha = |\Pi_m \cap S|$ and $\beta = |\Pi_{m-1} \cap S| = p_{m-1}$, then

$$(N-\alpha)q^{n-m-1} \leq q^{n-m}(N'-\beta),$$

with N' the theoretical upper bound for the number of points of S in a hyperplane of $\mathbf{PG}(n,q)$ for which Φ is not identically zero. Applying Theorem 3.1 (and remarking that we are not necessarily in Case (ii) of the proof of that theorem for Π_m), we obtain the desired result. ■

Corollary 4.6 (K. Thas [12]) *Suppose Φ is a homogeneous polynomial of degree $d \leq q$ with coefficients in $\mathbf{GF}(q)$. Let S be its set of $\mathbf{GF}(q)$-rational points in $\mathbf{PG}(n,q)$, $n \geq 2$, and suppose that Π_{m-1} is a subspace of $\mathbf{PG}(n,q)$ of maximal dimension which is contained in S, $m-1 \leq n-2$. Then*

$$N = |S| \leq dq^{n-1} + p_{n-2} + (d-(q+1))q^{m-1}.$$

■

The bound gets better if m increases.

As noted in [12], by using the bound of Theorem 4.1, one can do a little better. Suppose Φ is a homogeneous polynomial of degree $d \leq q$ with coefficients in $\mathbf{GF}(q)$. Let $S \neq \emptyset$ be its set of $\mathbf{GF}(q)$-rational points in $\mathbf{PG}(n,q)$, $n \geq 2$, and suppose that Π_{m-1} is a $\mathbf{PG}(m-1,q) \subset \mathbf{PG}(n,q)$ contained in S, and which is a subspace of $\mathbf{PG}(n,q)$ of maximal dimension which is contained in S, $m-1 \leq n-2$. Then each $\mathbf{PG}(m,q) = \Pi_m$ containing Π_{m-1} is not contained in S. Then applying the proof of Theorem 4.5, and remarking that for each hyperplane Π containing Π_{m-1} and not Π_m, the maximal dimension of its subspaces which are completely contained in S is also $m-1$ (the same holds for Π_m), we have

$$N = |S| \leq qN' - qp_{m-1} + \alpha = q(dq^{n-2} + p_{n-3} + (d-(q+1))F(n-1,m-1,q)) +$$

$$dq^{m-1} + p_{m-2} + (d-(q+1))F(m,m-1,q).$$

It follows that

$$N < dq^{n-1} + p_{n-2} + (d-(q+1))(q^{m-1} + q^{n-m-1} - 1).$$

5 Rational Points on Irreducible Hypersurfaces

In this final section, we will obtain new combinatorial bounds for certain types of hypersurfaces. In particular, these bounds apply to (a.o.) irreducible hypersurfaces in $\mathbf{PG}(n,q)$, $n \geq 4$. We will show that they supply better results than Theorem 4.3. For $n = 3$, the technique of Theorem 4.3 yields a better bound.

5.1 Generalizing $\{k;d\}$-arcs to arbitrary dimension

For the purpose of obtaining the bounds, we introduce *$\{k;d\}$-arcs* $\mathcal{K}$ in $\mathbf{PG}(n,q)$ for any $n \geq 2$ as a set of k points, meeting each line in no more than d points.[1] Also, for reasons of convenience, we demand that $\langle\mathcal{K}\rangle = \mathbf{PG}(n,q)$ (the arc generates the space). Note that $\mathcal{K}$ cannot be empty; the number of points of $\mathcal{K}$ is at least the number of points of a generating set. Sometimes we write *$\{k;d,n\}$-arc* to specify the dimension of the projective space spanned by the arc.

From now on, we suppose that $\mathcal{K}$ is a $\{k;d,n\}$-arc in $\mathbf{PG}(n,q)$, $n \geq 3$ and $d \leq q$.

Suppose that Π_{m-1} is a $\mathbf{PG}(m-1,q) \subset \mathbf{PG}(n,q)$ for which $|\Pi_{m-1} \cap \mathcal{K}| = \theta > 0$, where $n-2 \geq m \geq 2$. Let $\mathbf{PG}(m,q) = \Pi_m$ contain Π_{m-1}, and suppose $|\Pi_m \cap \mathcal{K}| = \alpha$. Now count in two ways the number of point-hyperplane pairs (p,Π) for which $p \subset \Pi$, where p is a point of $S \setminus \Pi_m$, $\Pi_{m-1} \subset \Pi$, and where $\Pi_m \not\subset \Pi$. Then

$$(k-\alpha)q^{n-m-1} \leq q^{n-m}(k(d,n-1,q)-\theta),$$

with $k(d,n-1,q)$ the theoretical upper bound for the number of points of a $\{k';d,n-1\}$-arc. We have

$$k \leq qk(d,n-1,q) - q\theta + \alpha.$$

Note. For $d = 2$, this observation is contained in [8].

Define $\ell(\mathcal{K})$ as

$$\ell(\mathcal{K}) = Max\{|L \cap \mathcal{K}| \mid\mid L \quad \text{is a line of} \quad \mathbf{PG}(n,q)\}.$$

In fact, we will use this notation *for any point set of* $\mathbf{PG}(n,q)$. Let $n \geq 3$, and let $\mathcal{K}$ be a $\{k;d\}$-arc in $\mathbf{PG}(n,q)$; then $\mathcal{K}$ is a $\{k;\ell\}$-arc in $\mathbf{PG}(n,q)$. Let L be a line of $\mathbf{PG}(n,q)$ for which $|L \cap \mathcal{K}| = \ell(\mathcal{K})$. Then by considering all the planes of $\mathbf{PG}(n,q)$ that contain L, there must be a plane π for which

$$|\pi \cap \mathcal{K}| \leq \frac{|\mathcal{K}| - \ell}{p_{n-2}} + \ell.$$

[1] Of course, other natural generalizations are possible, but this one is fitted for the present context.

Hence as soon as $n \geq 4$, we have that

$$k \leq qk(\ell, n-1, q) - q\ell + \frac{|\mathcal{K}| - \ell}{p_{n-2}} + \ell,$$

so that

$$k \leq (k(\ell, n-1, q) - \ell) q \frac{p_{n-2}}{p_{n-2} - 1} + \ell.$$

(Note that $\frac{qp_{n-2}}{p_{n-2}-1} = \frac{p_{n-2}}{p_{n-3}}$.) For $n = 3$, we have

Observation 5.1 *Let $\mathcal{K}$ be a $\{k; d, 3\}$-arc in $\mathbf{PG}(3, q)$, where $k > 0$ and $q + 1 > d$. Then*

$$k \leq 1 + \frac{p_2}{p_1}(k(\ell, 2, q) - 1).$$

Proof. Immediately by the proof of Theorem 4.3. ∎

We then have

Theorem 5.2 *Let $\mathcal{K}$ be a $\{k; d, n\}$-arc in $\mathbf{PG}(n, q)$, where $k > 0$, $q + 1 > d$ and $n \geq 4$. Then*

$$|\mathcal{K}| = k \leq [k(\ell, 2, q) - 1] \frac{p_{n-2} p_2}{p_1^2} + (1 - \ell) \frac{p_{n-2}}{p_1} + \ell.$$

Proof. The proof is by induction on n. We use the notation preceding the theorem. First put $n = 4$. Applying the inductional formula obtained prior to Observation 5.1 on the case $n = 3$ from that observation, we have

$$k \leq [1 + \frac{p_2}{p_1}(k(\ell, 2, q) - 1) - \ell] \frac{qp_2}{p_2 - 1} + \ell,$$

which yields the bound of the theorem. For general $n \geq 5$, the same argument leads to the proof. ∎

We are ready to obtain

Theorem 5.3 *Let $\mathcal{K}$ be a $\{k; d, n\}$-arc in $\mathbf{PG}(n, q)$, where $k > 0$, $q + 1 > d$ and $n \geq 3$. Let $\ell(\mathcal{K}) = \ell$ be as before.*

(i) *If $n = 3$, and q is even, then*

$$|\mathcal{K}| = k \leq 1 + (\ell - 1)p_2.$$

If $n = 3$, and q is odd, then

$$|\mathcal{K}| = k \leq 1 + (\ell - 1)p_2 - \frac{p_2}{p_1}.$$

(ii) *If $n \geq 4$ and q is even, then*

$$|\mathcal{K}| = k \leq (\ell - 1)\frac{p_{n-2}p_2}{p_1} + (1 - \ell)\frac{p_{n-2}}{p_1} + \ell.$$

If $n \geq 4$ and q is odd, then

$$|\mathcal{K}| = k \leq (\ell - 1)\frac{p_{n-2}p_2}{p_1} - \frac{p_{n-2}p_2}{p_1^2} + (1 - \ell)\frac{p_{n-2}}{p_1} + \ell.$$

Proof. As observed in the introduction of this paper, $k \leq k(d, 2, q) \leq \ell q + \ell - q$ for a $\{k; d\}$-arc $\mathcal{K}$ with $\ell = \ell(\mathcal{K})$. If equality holds, then $\mathcal{K}$ is a maximal arc, which in the q odd case does not exist by [2, 1].[2] So when q is odd, $k(d, 2, q) \leq \ell q + \ell - q - 1$. Now substitute in Observation 5.1 and Theorem 5.2. ■

5.2 Applications to hypersurfaces

Suppose $S(d, 2, q)$ is the maximal number of points of a plane algebraic curve of degree d in $\mathbf{PG}(2, q)$ with no regular linear components.

Theorem 5.4 *Suppose Φ is a homogeneous polynomial of degree $d \leq q + 1$ with coefficients in $\mathbf{GF}(q)$. Let S be the set of $\mathbf{GF}(q)$-rational points in $\mathbf{PG}(n, q)$, $n \geq 4$, and suppose Property (L) is satisfied. Then*

$$|S| = N \leq [S(d, 2, q) - 1]\frac{p_{n-2}p_2}{p_1^2} + (1 - \ell)\frac{p_{n-2}}{p_1} + \ell.$$

In particular, the theorem holds for irreducible hypersurfaces.

Proof. By Property (L), S is a $\{k; \ell(S)\}$-arc in $\mathbf{PG}(n, q)$, where $\ell(S) = \ell \leq d$. Now apply Observation 5.1 and Theorem 5.2. ■

We recall a theorem of J. A. Thas [11].

[2] This result solved an old fundamental problem in the theory of projective planes, posed in [9].

Theorem 5.5 (J. A. Thas [11]) *Let R be the number of rational points of a plane algebraic curve of degree d, $d > 2$, with no regular linear components. Then*

$$R \leq qd + d - q - 2.$$

■

If $d = 2$, then $R \in \{1, q+1\}$. For the following result, we only look at the case $d \geq 3$, and leave the case $d = 2$ as an exercise.

Theorem 5.6 *Suppose Φ is a homogeneous polynomial of degree $3 \leq d \leq q+1$ with coefficients in $\mathbf{GF}(q)$. Let S be the set of $\mathbf{GF}(q)$-rational points in $\mathbf{PG}(n, q)$, $n \geq 3$, and suppose Property (L) is satisfied.*

(*i*) *If $n = 3$, then*

$$|S| = N \leq 1 + (\ell - 1)p_2 - 2\frac{p_2}{p_1}.$$

(*ii*) *If $n \geq 4$, then*

$$|S| = N \leq (\ell - 1)\frac{p_{n-2}p_2}{p_1} - 2\frac{p_{n-2}p_2}{p_1^2} + (1 - \ell)\frac{p_{n-2}}{p_1} + \ell.$$

In particular, the theorem holds for irreducible hypersurfaces.

Proof. Combine Theorem 5.4 with Theorem 5.5. ■

5.3 Comparison of the bounds

We compare the obtained bounds for hypersurfaces with Property (L) with the bounds of Theorem 4.3. For the case $n = 3$, the inductional step which is applied in Theorem 5.6(i) is the same as that of Theorem 4.3, so we only look at the case $n \geq 4$. Of course, comparison is only possible when replacing 'd' by 'ℓ' in Theorem 4.3 (and its corollary). Also, since the inductive funtion $F(n, q)$ of Theorem 4.3 was constructed based on $S(d, 2, q) \leq q\ell + \ell - q \leq qd + d - q$, we also use this value for the bounds just obtained (recall the remark following Corollary 4.4). We note that $F(n, q) \ll p_{n-2}$ before starting the calculation. Then

$$|S| = N \leq [S(d, 2, q) - 1]\frac{p_{n-2}p_2}{p_1^2} + (1 - \ell)\frac{p_{n-2}}{p_1} + \ell = (\ell - 1)\frac{p_{n-2}p_2}{p_1} + (1 - \ell)\frac{p_{n-2}}{p_1} + \ell$$

$$\leq \ell q^{n-1} + p_{n-2} + (\ell - q - 1)F(n, q) \Leftrightarrow (q + 1 - \ell)F(n, q) + (\ell - 2)p_{n-2} + 1 \leq q^{n-1}$$

$$\Leftrightarrow F(n, q) \leq p_{n-2}.$$

References

[1] S. Ball and A. Blokhuis. An easier proof of the maximal arcs conjecture, *Proc. Amer. Math. Soc.* **126** (1998), 3377–3380.

[2] S. Ball, A. Blokhuis and F. Mazzocca. Maximal arcs in Desarguesian planes of odd order do not exist, *Combinatorica* **17** (1997), 31–41.

[3] H. Hasse. Abstrakte Begründung der Komplexen Multiplication und Riemannsche Vermutung in Functionenkörpern, *Abh. Math. Sem. Univ. Hamburg* **10** (1934), 325–348.

[4] H. Hasse. Punti razionali sopra curva algebriche a congruenze, *Reale Accademia d' Italia, Fondazzione Allessandro Volta, Rome, Atti dei Convegni* **9** (1943), 85–140.

[5] J. -P. Serre. Nombre de points des courbes algébriques sur F_q, *Séminaire de Théorie des Nombres de Bordeaux (Talence 1982/1983), Université de Bordeaux I, Talence*, exposé no. 22 (1983), 8 pp.

[6] J. -P. Serre. Sur le nombre de points rationnels d'une courbe algébrique sur un corps fini, *C. R. Acad. Sci. Paris Sér. I* **296** **1983**, 397–402.

[7] J. -P. Serre. Lettre à M. Tsfasman, *Astérisque* **198-199-200** (1991), 351–353.

[8] L. Storme, J. A. Thas and S. K. J. Vereecke. New upper bounds for the sizes of caps in finite projective spaces, *J. Geom.* **73** (2002), 176–193.

[9] J. A. Thas. Some results concerning $\{(q+1)(n-1); n\}$-arcs and $\{(q+1)\ (n-1)+1; n\}$-arcs in finite projective planes of order q, *J. Combin. Theory Ser. A* **19** (1975), 228–232.

[10] J. A. Thas. Elementary proofs of two fundamental theorems of B. Segre without using the Hasse-Weil theorem, *J. Combin. Theory Ser. A* **34** (1983), 381–384.

[11] J. A. Thas. Complete arcs and algebraic curves in $\mathrm{PG}(2,q)$, *J. Algebra* **106** (1987), 451–464.

[12] K. Thas. On the number of points of a hypersurface in finite projective space (after J.-P. Serre), 11 pp., Submitted to *J. Algebraic Geom.*

[13] K. Thas. Topics in Finite and Algebraic Geometry, 72 pp., Submitted to *Bull. London Math. Soc.*

[14] A. Weil. *Sur les Courbes Algébriques et les Variétés qui s'en Déduisent*, Hermann, Paris, 1948.

[15] A. Weil. *Variétés Abéliennes et Courbes Algébriques*, Hermann, Paris, 1948.