Iwasawa decompositions of groups with a root group datum

Tom De Medts Ralf Gramlich* Max Horn

September 24, 2007

1. Introduction

The Iwasawa decomposition of a connected semisimple complex Lie group or a connected semisimple split real Lie group is one of the most fundamental observations of classical Lie theory. It implies that the geometry of a connected semisimple complex resp. split real Lie group G is controlled by any maximal compact subgroup K. Examples are Weyl's unitarian trick in the representation theory of Lie groups, or the transitive action of K on the Tits building G/B. In the case of the connected semisimple split real Lie group of type G_2 the latter implies the existence of an interesting epimorphism from the real building of type G_2, the split Cayley hexagon, onto the real building of type A_2, the real projective plane, by means of the epimorphism $\mathrm{SO}_4(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R})$, cf. [14]. This epimorphism cannot be described using the group of type G_2, because it is quasisimple.

To be able to transfer these ideas to a broader class of groups, we extend the notion of an Iwasawa decomposition in the following way:

Definition 1.1. A group G with a twin BN-pair (cf. Definition 2.2) admits an *Iwasawa decomposition* if there exist an involution $\theta \in \mathrm{Aut}(G)$ such that

(i) $B_+^\theta = B_-$ and

(ii) $G = G_\theta B_+$ where $G_\theta := \mathrm{Fix}_G(\theta)$.

Our interest in the Iwasawa decomposition stems from geometric group theory: the group G_θ acts with a fundamental domain on the flag complex of the building G/B_+. Hence Tits' Lemma [27, Lemma 5], [37, Corollary 1] yields a presentation of G_θ by generators and relations. Since G_θ need not be finitely generated, this presentation is usually formulated as a universal enveloping result of an amalgam.

The following theorem specifies the presentation we have in mind. Refer to Section 2 for a definition of a root group datum and the construction of the twin BN-pair resulting from it.

Theorem 1. *Let G be a group with a root group datum $\{U_\alpha\}_{\alpha\in\Phi}$, and assume that $G = G_\theta B_+$ is an Iwasawa decomposition of G with respect to an involution θ. Furthermore, let Π be a system of fundamental roots of Φ and, for $\{\alpha, \beta\} \subseteq \Pi$, let $X_{\alpha,\beta} := \langle U_\alpha, U_{-\alpha}, U_\beta, U_{-\beta}\rangle$.*

Then θ induces an involution on each $X_{\alpha,\beta}$ and G_θ is the universal enveloping group of the amalgam $((X_{\alpha,\beta})_\theta)_{\{\alpha,\beta\}\subseteq\Pi}$ of fixed point subgroups of the groups $X_{\alpha,\beta}$.

The proof of Theorem 1 can be adapted from what has been done in [12] and [15] for compact real forms of complex Lie groups and of complex Kac-Moody groups: The involution θ induces an involution of each group $X_{\alpha,\beta}$ (cf. Proposition 3.4). By the Iwasawa decomposition the group

*The second author gratefully acknowledges a Heisenberg fellowship by the Deutsche Forschungsgemeinschaft.

G_θ acts with a fundamental domain on the flag complex Δ associated to the building G/B_+. Choose F to be a fundamental domain of Δ stabilized by the standard torus $T := B_+ \cap N$ of G arising from the positive BN-pair of G. The stabilizers of the simplices of F of dimension zero and one with respect to the natural action of G on Δ are exactly the groups $(X_\alpha)_\theta T$ and $(X_{\alpha\beta})_\theta T$, $\alpha, \beta \in \Pi$. By the simple connectedness of building geometries of rank at least three (cf. [6, Theorem IV.5.2] or [36, Theorem 13.32]) plus Tits' Lemma (cf. [27, Lemma 5], [37, Corollary 1]) the group G_θ equals the universal enveloping group of the amalgam $((X_{\alpha\beta})_\theta T)_{\alpha,\beta\in\Pi}$. Finally, the torus T equals the universal enveloping group of the amalgam $(T_{\alpha\beta})_{\alpha,\beta\in\Pi}$, where $T_{\alpha\beta}$ denotes the maximal torus $T \cap X_{\alpha\beta}$ of $X_{\alpha\beta}$, and so by [13, Lemma 29.3] the group G actually equals the universal enveloping group of the amalgam $((X_{\alpha\beta})_\theta)_{\alpha,\beta\in\Pi}$.

From Theorem 1 rises the question which groups actually admit an Iwasawa decomposition. In the literature one can find a lot of information on Iwasawa decompositions for prescribed ground fields, usually over the complex numbers $\mathbb{C}$, the real numbers $\mathbb{R}$, or real closed fields, cf. [3], [4], [17], [18], [22], [28]. In [19] on the other hand, no fixed ground field is chosen. Instead, it is shown that, under the assumption that the ground field is infinite and that each square is a fourth power, the Iwasawa decomposition, the polar decomposition and the KAK-decomposition of the rational points of a connected reductive algebraic group are equivalent. Neither do we choose a fixed field in this paper. Instead, in Theorem 2 below we characterize the fields $\mathbb{F}$ over which a group with an $\mathbb{F}$-locally split root group datum, cf. Definition 2.3, admits an Iwasawa decomposition. We point out that this class of groups contains the class of groups of $\mathbb{F}$-rational points of a connected split semisimple algebraic group defined over $\mathbb{F}$ (cf. [34]) as well as the class of split Kac-Moody groups over $\mathbb{F}$ (cf. [30], [38]). Not every group that admits an Iwasawa decomposition admits a polar or a KAK-decomposition.

Definition 1.2. Let $\mathbb{F}$ be a field, let σ be an automorphism of $\mathbb{F}$ of order one or two and let G be a group with an $\mathbb{F}$-locally split root group datum $\{U_\alpha\}_{\alpha\in\Phi}$. We call an automorphism $\theta \in \mathrm{Aut}(G)$ a *σ-twisted Chevalley involution* of G if it satisfies for all $\alpha \in \Phi$:

(i) $\theta^2 = \mathrm{id}$,

(ii) $U_\alpha^\theta = U_{-\alpha}$, and

(iii) $\theta \circ \sigma$ induces the standard Chevalley involution (resp. its image under the canonical projection) on $X_\alpha := \langle U_\alpha, U_{-\alpha}\rangle \cong (\mathrm{P})\mathrm{SL}_2(\mathbb{F})$.

A σ-twisted Chevalley involution of a split Kac-Moody group can be constructed by taking the product of a sign automorphism and the field automorphism σ, see [8, Section 8.2]. The same is true for all split semisimple algebraic groups. Moreover, by Lemma 3.6 a group with a two-spherical $\mathbb{F}$-locally split root group datum with $|\mathbb{F}| \geq 4$ also admits σ-twisted Chevalley involutions. Hence the following theorem applies to these three classes of groups.

Theorem 2. *Let $\mathbb{F}$ be a field and let G be a group with an $\mathbb{F}$-locally split root group datum. The group G admits an Iwasawa decomposition if and only if $\mathbb{F}$ admits an automorphism σ of order one or two such that*

(i) *-1 is not a norm,*

(ii) (a) *if there exists a rank one subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ of G isomorphic to $\mathrm{SL}_2(\mathbb{F})$, then a sum of norms is a norm, or*

  (b) *if each rank one subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ of G is isomorphic to $\mathrm{PSL}_2(\mathbb{F})$, then a sum of norms is ± 1 times a norm,*

with respect to the norm map $N : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$, and

(iii) *G admits a σ-twisted Chevalley involution.*

We recall that in a finite field $\mathbb{F}_q$ of order $q \equiv 3 \mod 4$ the element -1 is not a square, so that a sum of squares is plus or minus a square, because there exist exactly two square classes, cf. [26, Section VI §62]. Therefore a group with an $\mathbb{F}_q$-locally split root group datum where all rank one subgroups are isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$, such as $\mathrm{PSL}_2(\mathbb{F}_q)$ or arbitrary direct products of copies of $\mathrm{PSL}_2(\mathbb{F}_q)$, admit an Iwasawa decomposition with respect to the standard Chevalley involution. However, we do not know a non-trivial example for a group with an $\mathbb{F}$-locally split root group datum where all rank one subgroups are isomorphic to $\mathrm{PSL}_2(\mathbb{F})$. However, we conjecture that it is possible to construct examples based on the work done in [31].

As mentioned before, among the most prominent groups covered by Theorem 2 probably are the connected split semisimple algebraic groups and the split Kac-Moody groups, so that we explicitly re-state our characterization for these two classes of groups.

Corollary 3. *Let $\mathbb{F}$ be a field and let G be a connected split semisimple algebraic group defined over $\mathbb{F}$ or a split Kac-Moody group over $\mathbb{F}$. The group $G(\mathbb{F})$ admits an Iwasawa decomposition $G(\mathbb{F}) = G_\theta(\mathbb{F})B(\mathbb{F})$ if and only if $\mathbb{F}$ admits an automorphism σ of order one or two such that*

(i) -1 *is not a norm,*

(ii) (a) *if there exists a rank one subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ of G isomorphic to $\mathrm{SL}_2(\mathbb{F})$, then a sum of norms is a norm, or*

(b) *if each rank one subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ of G is isomorphic to $\mathrm{PSL}_2(\mathbb{F})$, then a sum of norms is ± 1 times a norm,*

with respect to the norm map $N : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$.

We would like to mention that Corollary 3 also holds for split reductive algebraic groups, which in the setting we chose are excluded by axiom (RGD0) of Definition 2.3. Simply using a slightly more general notion of a group with a root group datum (such as in [39]) will immediately generalize Corollary 3 to the reductive case.

An application to graph theory

Another consequence of Theorem 2 is a combinatorial local characterization of certain graphs similar to the main result of [2].

Let $\mathbb{F}$ be a field admitting an automorphism σ of order one or two such that

(i) -1 is not a norm and

(ii) a sum of norms is a norm

with respect to the norm map $N : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$. Furthermore, let V be a six-dimensional $\mathbb{F}$-vector space endowed with an anisotropic σ-hermitian sesquilinear form. Define $\mathbf{S}(V)$ to be the graph with the two-dimensional subspaces of V as vertices and adjacency given by orthogonality. Because of this definition it makes sense to use the symbol $\perp$ for adjacency.

There exists a group-theoretical characterization of $\mathbf{S}(V)$ as follows. Let $G \cong \mathrm{SL}_6(\mathbb{F})$, so that G has an $\mathbb{F}$-locally split root group datum $\{U_\alpha\}_{\alpha\in\Phi}$ of type A_5. Let θ be the σ-twisted Chevalley involution of the group G, let $K := \mathrm{Fix}_G(\theta)$ and, for $\alpha \in \Phi$, denote by $(X_\alpha)_\theta$ the fixed point subgroup of the rank one group $X_\alpha = \langle U_\alpha, U_{-\alpha}\rangle$. Then $\mathbf{S}(V)$ is isomorphic to the graph on the K-conjugates of $(X_\alpha)_\theta$ with the commutation relation as adjacency. This setup allows us to define the graph $\mathbf{S}$ for any connected split semisimple algebraic $\mathbb{F}$-group, so that for instance it makes sense to use the symbol $\mathbf{S}(E_6(\mathbb{F}))$. In this notation, we have $\mathbf{S}(V) \cong \mathbf{S}(A_5(\mathbb{F}))$.

The proof of [2, Theorem 4.1.2], which deals with the special case of $\mathbb{F} = \mathbb{C}$ and σ complex conjugation, applies verbatim to the setting just introduced, so that we obtain the following result. Recall that if Δ is a graph, one says that a graph Γ is *locally* Δ if for each vertex $x \in \Gamma$ the subgraph of Γ induced on the neighbors of x is isomorphic to Δ.

Theorem 4. *Let $\mathbb{F}$ be a field admitting an automorphism σ of order one or two such that -1 is not a norm and a sum of norms is a norm with respect to the norm map $N : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$. Let Γ be a connected locally $\mathbf{S}(A_5(\mathbb{F}))$ graph satisfying that, for every chain $x \perp w \perp y \perp z \perp x \perp y$ consisting of four distinct vertices of Γ, the vertices w, x, z have a unique common neighbor if and only if the vertices w, y, z have a unique common neighbor.*

Then Γ is a quotient of $\mathbf{S}(A_7(\mathbb{F}))$ or of $\mathbf{S}(E_6(\mathbb{F}))$.

An application to group theory

Finally, as explained in [16], Theorem 4 implies the following group-theoretic statement via a standard argument. All twisted groups are understood to be anisotropic forms of connected split semisimple algebraic groups with respect to the Chevalley involution composed with the indicated field automorphism.

Theorem 5. *Let $\mathbb{F}$ be a field admitting an automorphism σ of order one or two such that -1 is not a norm and a sum of norms is a norm with respect to the norm map $N : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$. Moreover, let G be a group containing an involution x and a subgroup $K \trianglelefteq C_G(x)$ such that*

(i) *$K \cong \mathrm{SU}_6(\mathbb{F}, \sigma)$;*

(ii) *$C_G(K)$ contains a subgroup $X \cong \mathrm{SU}_2(\mathbb{F}, \sigma)$ with $x = Z(X)$;*

(iii) *there exists an involution $g \in G$ such that $Y := gXg$ is contained in K;*

(iv) *if V is a natural module for K, then the commutator $[Y, V] := \{yv - v \in V \mid y \in Y, v \in V\}$ has $\mathbb{F}$-dimension two;*

(v) *$G = \langle K, gKg\rangle$; moreover, there exists $z \in K \cap gKg$ which is a gKg-conjugate of x and a K-conjugate of gxg.*

Then

$$G/Z(G) \cong \mathrm{PSU}_8(\mathbb{F}, \sigma) \text{ or } G/Z(G) \cong {}^2E_6(\mathbb{F}, \sigma).$$

Organization of the article

In Section 2 we quickly recall the definitions of a (twin) BN-pair and a root group datum. Section 3 provides some basic facts about flips of groups with a root group datum. In Section 4 we thoroughly study flips of (P)$\mathrm{SL}_2(\mathbb{F})$ and, finally, in Section 5 we use a local-to-global argument to prove Theorem 2 based on our findings in Section 4. In Appendix A we use Moufang sets in order to deduce some results about flips of (P)$\mathrm{SL}_2(\mathbb{D})$, where $\mathbb{D}$ is an arbitrary division ring.

Acknowledgements

The authors thank Bernhard Mühlherr for several discussions on the topic of this article. Moreover, the authors express their gratitude to Andreas Mars for a very careful proof-reading and very helpful comments on an earlier version of this paper.

2. Twin BN-pairs and root group data

Definition 2.1. We call the tuple (G, B, N, S) consisting of a group G with subgroups B and N and of a subset S of the coset space $N/(B \cap N)$, a *Tits system* or a *BN-pair* if the following conditions are satisfied:

(i) $G = \langle B, N\rangle$;

(ii) $T := B \cap N$ is normal in N;

(iii) the elements of S have order two and generate the group $W := N/T$, called *Weyl group*;

(iv) $BwBsB \subset BwsB \cup BwB$ for all $w \in W$, $s \in S$;

(v) $sBs \not\subset B$ for all $s \in S$.

Definition 2.2. Let (G, B_+, N, S) and (G, B_-, N, S) be Tits systems such that $B_+ \cap N = B_- \cap N$, i.e. with equal Weyl group W. Then (G, B_+, B_-, N, S) is called a *twin Tits system* or *twin BN-pair* if the following conditions are satisfied, cf. [39]:

(i) $B_\varepsilon w B_{-\varepsilon} s B_{-\varepsilon} = B_\varepsilon ws B_{-\varepsilon}$ for $\varepsilon = \pm$ and all $w \in W$, $s \in S$ such that $l(ws) < l(w)$;

(ii) $B_+ s \cap B_- = \emptyset$ for all $s \in S$.

A group G admitting a BN-pair satisfies $G = \bigsqcup_{w \in W} BwB$, the *Bruhat decomposition* of G. For each $s \in S$ the set $P_s := B \cup BsB$ is a subgroup of G. A Tits system (G, B, N, S) leads to a *building* whose set of chambers equals G/B and whose distance function $\delta : G/B \times G/B \to W$ is given by $\delta(gB, hB) = w$ if and only if $Bh^{-1}gB = BwB$.

A group G with a twin BN-pair hence yields two buildings G/B_+ and G/B_- with distance functions δ_+ and δ_-. Furthermore, it admits the *Birkhoff decomposition* $G = \bigsqcup_{w \in W} B_+ w B_-$ from which we can define the codistance function $\delta_* : (G/B_- \times G/B_+) \cup (G/B_+ \times G/B_-) \to W$ via $\delta_*(gB_-, hB_+) = w$ if and only if $B_+ h^{-1} g B_- = B_+ w B_-$ and $\delta_*(hB_+, gB_-) := (\delta_*(gB_-, hB_+))^{-1}$. The tuple $((G/B_+, \delta_+), (G/B_-, \delta_-), \delta_*)$ is called the *twin building* of G.

In the present paper we are only interested in (twin) buildings coming from a group with a (twin) BN-pair. For detailed treatments of the theory of buildings we refer to [6], [33], [36], [40]. Twin buildings are treated in [30], [39].

Let (W, S) be a Coxeter system and let Φ be the set of its roots. Following [8] a *root* is a set of the form $w\alpha_s$, where $w \in W$ and $\alpha_s = \{x \in W \mid l(sx) = l(x) + 1\}$ with length function l of W with respect to the generating set S. Moreover, let Π be a system of fundamental roots of Φ and, for $\varepsilon = \pm$, let Φ_ε denote the set of positive, resp. negative roots of Φ with respect to Π. For a root $\alpha \in \Phi$, denote by s_α the reflection of W which permutes α and $-\alpha$. For each $w \in W$, define $\Phi_w := \{\alpha \in \Phi_+ \mid w\alpha \in \Phi_-\}$. A pair $\{\alpha, \beta\}$ of roots is called *prenilpotent* if $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ are both non-empty. In that case denote by $[\alpha, \beta]$ the set of all roots γ of Φ such that $\alpha \cap \beta \subseteq \gamma$ and $(-\alpha) \cap (-\beta) \subseteq -\gamma$, and set $]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}$.

The following definition of a root group datum is taken from [7].

Definition 2.3. A *root group datum* of type (W, S) for a group G is a family $\{U_\alpha\}_{\alpha \in \Phi}$ of subgroups (the *root subgroups*) of G satisfying the following axioms, where $U_+ := \langle U_\alpha \mid \alpha \in \Phi_+ \rangle$ and $U_- := \langle U_\alpha \mid \alpha \in \Phi_- \rangle$:

(RGD0) For each $\alpha \in \Phi$, we have $U_\alpha \neq 1$; moreover, $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.

(RGD1) For each $\alpha \in \Pi$ we have $U_\alpha \not\subseteq U_-$.

(RGD2) For each $\alpha \in \Pi$ and $u \in U_\alpha \setminus \{1\}$, there exist elements u', u'' of $U_{-\alpha}$ such that the product $\mu(u) := u'uu''$ conjugates U_β onto $U_{s_\alpha(\beta)}$ for each $\beta \in \Phi$.

(RGD3) For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, we have $[U_\alpha, U_\beta] \subset \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle$.

(RGD4) For each $\alpha \in \Pi$ there exists $\alpha' \in \Phi_{s_\alpha}$ such that $U_\beta \subseteq U_{\alpha'}$ for each $\beta \in \Phi_{s_\alpha}$.

Defining

$$\begin{aligned} T &:= \langle \mu(u)\mu(v) \mid u, v \in U_\alpha \setminus \{1\}, \alpha \in \Pi \rangle, \\ N &:= \langle \mu(u) \mid u \in U_\alpha \setminus \{1\}, \alpha \in \Pi \rangle, \\ B_+ &:= T.U_+, \\ B_- &:= T.U_-, \end{aligned}$$

we obtain a twin BN-pair of G, which by the above leads to a twin building on which G acts. We set $X_\alpha := \langle U_\alpha, U_{-\alpha} \rangle$ and $X_{\alpha,\beta} := \langle X_\alpha, X_\beta \rangle$. A root group datum is called *locally split* if the group

T is abelian and if for each $\alpha \in \Phi$ there is a field $\mathbb{F}_\alpha$ such that the root group datum $\{U_\alpha, U_{-\alpha}\}$ of X_α of type A_1 is isomorphic to the natural root group datum of $\mathrm{SL}_2(\mathbb{F}_\alpha)$ or $\mathrm{PSL}_2(\mathbb{F}_\alpha)$. A locally split root group datum is called $\mathbb{F}$-*locally split* if $\mathbb{F}_\alpha \cong \mathbb{F}$ for all $\alpha \in \Phi$.

3. Flips of groups with a BN-pair

Definition 3.1. Let G be a group with a twin BN-pair B_+, B_-, N. An automorphism θ of G is called a BN-*flip* if the following holds:

(i) $\theta^2 = \mathrm{id}$;

(ii) $B_+^\theta = B_-$;

(iii) θ centralizes the Weyl group N/T.

We now give the definition of a flip of a twin building and describe the correspondence between a BN-flip and a flip of the twin building induced by the twin BN-pair. The concept of a flip of a twin building has been introduced in [5]. For the purpose of the present paper it suffices to work with twin buildings obtained from a group with a twin BN-pair.

Definition 3.2. Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*) = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ be a twin building. A *building flip* is an involutory permutation θ of $\mathcal{C}_+ \cup \mathcal{C}_-$ with the following properties:

(i) $\mathcal{C}_+^\theta = \mathcal{C}_-$;

(ii) θ flips the distances, i.e., for $\varepsilon = \pm$ and for all $x, y \in \mathcal{C}_\varepsilon$ we have $\delta_\varepsilon(x, y) = \delta_{-\varepsilon}(x^\theta, y^\theta)$; and

(iii) θ preserves the codistance, i.e., for $\varepsilon = \pm$ and for all $x \in \mathcal{C}_\varepsilon$, $y \in \mathcal{C}_{-\varepsilon}$ we have $\delta_*(x, y) = \delta_*(x^\theta, y^\theta)$.

If additionally

(iv) there exists a chamber $c \in \mathcal{C}_\pm$ such that $\delta_*(c, c^\theta) = 1_W$,

the building flip θ is called a *Phan involution*.

The following proposition shows that a BN-flip induces a building flip (even a Phan involution) justifying the choice of name. Conversely, as Bernhard Mühlherr pointed out to us, a building flip induces a BN-flip if the group G is perfect.

Proposition 3.3. *Let G be a group with a twin BN-pair inducing the twin building $\mathcal{B}$. Then any BN-flip θ of G induces a Phan involution of $\mathcal{B}$.*

Proof. Recall from Definition 2.2 that $\mathcal{B}$ consists of the buildings G/B_ε with distance functions $\delta_\varepsilon : G/B_\varepsilon \times G/B_\varepsilon \to W$ satisfying $\delta_\varepsilon(gB_\varepsilon, hB_\varepsilon) = w$ if and only if $B_\varepsilon g^{-1} h B_\varepsilon = B_\varepsilon w B_\varepsilon$ for $\varepsilon = \pm$. These buildings are twinned by the codistance function $\delta_* : (G/B_+ \times G/B_-) \cup (G/B_- \times G/B_+) \to W$ satisfying $\delta_*(gB_\varepsilon, hB_{-\varepsilon}) = w$ if and only if $B_\varepsilon g^{-1} h B_{-\varepsilon} = B_\varepsilon w B_{-\varepsilon}$. By definition $\theta^2 = \mathrm{id}$. Moreover, $B_+^\theta = B_-$ implies that θ interchanges the two parts of the twin building. The image of $g^{-1}h \in B_\varepsilon w B_\varepsilon$ under θ satisfies $(g^{-1})^\theta h^\theta \in B_\varepsilon^\theta w^\theta B_\varepsilon^\theta = B_{-\varepsilon} w B_{-\varepsilon}$. Therefore $\delta_\varepsilon(gB_\varepsilon, hB_\varepsilon) = w$ implies $\delta_{-\varepsilon}(g^\theta B_{-\varepsilon}, h^\theta B_{-\varepsilon}) = w$, whence θ flips the distances. For $g^{-1}h \in B_\varepsilon w B_{-\varepsilon}$ we have $(g^{-1})^\theta h^\theta \in B_\varepsilon^\theta w^\theta B_{-\varepsilon}^\theta = B_{-\varepsilon} w B_\varepsilon$, so $\delta_*(gB_\varepsilon, hB_{-\varepsilon}) = w$ implies $\delta_*(g^\theta B_{-\varepsilon}, h^\theta B_\varepsilon) = w$, whence θ preserves the codistance. Finally, the chamber B_+ is mapped onto its opposite chamber B_-. Altogether, this implies that θ induces a Phan involution on the twin building $\mathcal{B}$. □

Proposition 3.4. *Let G be a group with a twin BN-pair B_+, B_-, N, and let θ be an automorphism of G satisfying*

(i) $\theta^2 = \mathrm{id}$*; and*

(ii) $B_+^\theta = B_-$; *moreover, every Borel subgroup of* G *containing* $T = B_+ \cap N = B_- \cap N$ *is mapped to an opposite Borel subgroup.*

Then we also have that

(iii) θ *centralizes the Weyl group* N/T.

In particular, θ *is a* BN*-flip.*

Proof. For each $s \in S$ the set $P_s := B_+ \cup B_+ s B_+$ is a rank one parabolic subgroup of positive sign of G. Let n_s be a representative of s in N. Then $P_s^\theta = B_+^\theta \cup B_+^\theta n_s^\theta B_+^\theta = B_- \cup B_- n_s^\theta B_-$ is a parabolic subgroup of negative sign of G, as it is the image of a subgroup of G under the group automorphism θ and contains B_-. It consists of precisely two Bruhat double cosets, implying it must be a rank one parabolic subgroup. Hence n_s^θ is a representative of some $s'_{n_s} \in S$. As s' is independent of the choice of n_s, the map θ induces a permutation of S by setting $s^\theta := s'_{n_s}$. Since θ maps every Borel group containing T to an opposite one, for every $s \in S$ the chamber sB_+ is opposite to the chamber $s^\theta B_-$ in the associated twin building $\mathcal{B}$. That is, they have codistance 1_W, whence $B_+ s^{-1} s^\theta B_- = B_+ B_-$ which by the uniqueness of the Birkhoff decomposition yields that $s^\theta = s$. Hence θ centralizes $W = \langle S \rangle$. □

Proposition 3.5. *Any* σ*-twisted Chevalley involution* θ *of a group* G *is a* BN*-flip.*

Proof. By definition, θ is an involution. Furthermore, the Borel subgroup B_+ is generated by T and the set of root groups associated to the positive root system $\Phi_+ \subset \Phi$. More precisely, $B_+ = T.\langle U_\alpha \mid \alpha \in \Phi_+ \rangle$. Since $T = \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ by [7, Corollary 4.3.3], the involution θ stabilizes T and maps B_+ to $B_- = T.\langle U_{-\alpha} \mid \alpha \in \Phi_+ \rangle$. Finally, θ acts trivially on $W = N/T$ as each root α of the root lattice of W is mapped onto its negative $-\alpha$, which means that the reflection given by α is mapped onto the reflection given by $-\alpha$, which is identical to the reflection given by α. □

Lemma 3.6. *Let* $\mathbb{F}$ *be a field with at least four elements, let* σ *be an automorphism of* $\mathbb{F}$ *of order one or two, let* G *be a group with a two-spherical* $\mathbb{F}$*-locally split root group datum. Then* G *admits a* σ*-twisted Chevalley involution.*

Proof. By [1] (and also by the unpublished manuscript [25]) the group G is a universal enveloping group of the amalgam $\bigcup_{\alpha,\beta \in \Pi} X_{\alpha,\beta}$ for a system Π of fundamental roots of Φ. This means that any automorphism of $\bigcup_{\alpha,\beta \in \Pi} X_{\alpha,\beta}$ induces an automorphism of G. For each pair $\alpha, \beta \in \Pi$ the σ-twisted Chevalley involution of the split semisimple algebraic group $X_{\alpha,\beta}$ induces σ-twisted Chevalley involutions θ_α on X_α and θ_β on X_β. Therefore there exists an involution of the amalgam $\bigcup_{\alpha,\beta \in \Pi} X_{\alpha,\beta}$ inducing θ_α on X_α. Consequently there exists an involution θ on its universal enveloping group G inducing θ_α on each subgroup X_α. This involution θ of G by construction is a σ-twisted Chevalley involution of G. □

4. Transitive involutions

By Proposition 3.4 it suffices to study rank one groups in order to prove Theorem 2. Since we are interested in groups with a locally split root group datum, we can restrict our attention to flip automorphisms of $\mathrm{SL}_2(\mathbb{F})$ and $\mathrm{PSL}_2(\mathbb{F})$ where $\mathbb{F}$ is an arbitrary field.

In Section 4.1 we classify all suitable flip automorphisms of these two groups. We use that all automorphisms of $\mathrm{PSL}_2(\mathbb{F})$ are induced via projection from automorphisms of $\mathrm{SL}_2(\mathbb{F})$, which follows from the fact that SL_2 is perfect if $|\mathbb{F}| \geq 4$ and is easily verified over the fields of two and three elements. Alternatively one can use the classification of endomorphisms of Steinberg groups or apply the results in [29]. Hence it suffices here to study flips of $\mathrm{SL}_2(\mathbb{F})$. In Section 4.2 we compute the fixed point groups of these flips and give a geometric interpretation for a rank one Iwasawa decomposition. This finally enables us to give a nice sufficient and necessary algebraic criterion for such a local Iwasawa decomposition in Section 4.3.

Furthermore, in Appendix A, we study flips of general Moufang sets which correspond to arbitrary rank one groups. Our aim is to show that it seems feasible to extend the theory to groups beyond $\mathbb{F}$-locally split ones. As a first step we present some results for $\mathrm{SL}_2(\mathbb{D})$ where $\mathbb{D}$ is a division ring.

4.1. Flip automorphisms of $G = \mathbf{SL_2}(\mathbb{F})$

In order to be able to understand flips of $G = \mathrm{SL}_2(\mathbb{F})$ we need to specify a suitable root group datum of G. To this end consider $\mathrm{SL}_2(\mathbb{F})$ as a matrix group acting on its natural module and let T denote the subgroup of diagonal matrices, which is a maximal torus of G. Let U_+ and U_- denote the subgroups of upper resp. lower triangular unipotent matrices, which are the root subgroups with respect to the root system of type A_1 associated to T. The standard Borel subgroups of G then are the groups $B_+ := T.U_+$, $B_- := T.U_-$. Finally, set $N := N_G(T)$ to obtain a BN-pair. Consider a BN-flip θ with respect to this BN-pair, i.e. an involutory automorphism θ of G which interchanges B_+ and B_- and centralizes N/T. It follows that θ stabilizes T and interchanges U_+ and U_-.

Let $K := C_G(\theta)$, the fixed point group of θ. Then θ induces an Iwasawa decomposition $G = KB_+$ if and only if K acts transitively on the projective line $\mathbb{P}_1(\mathbb{F}) = G/B_+$. In this case θ is called *transitive*. Since θ interchanges U_+ and U_- and since the root subgroups are isomorphic to $(\mathbb{F},+)$, there must exist a group automorphism $\phi \in \mathrm{Aut}(\mathbb{F},+)$ such that the equalities

$$\theta\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \begin{pmatrix}1 & 0\\ \phi(x) & 1\end{pmatrix} \qquad \text{and} \qquad \theta\left(\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\right) = \begin{pmatrix}1 & \phi^{-1}(y)\\ 0 & 1\end{pmatrix} \tag{4.1}$$

hold. This weak assumption implies much stronger properties of ϕ, as the next lemma shows.

Lemma 4.1. *The group automorphism $\phi \in \mathrm{Aut}(\mathbb{F},+)$ induces an involution $\theta \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}))$ if and only if $\phi(x) = \varepsilon x^\sigma$ for some field automorphism $\sigma \in \mathrm{Aut}(\mathbb{F})$ of order one or two and some $\varepsilon \in \mathrm{Fix}_{\mathbb{F}}(\sigma)$.*

Proof. The key here is to use the following equation derived from the Steinberg relations for Chevalley groups of rank one for various values $s,t \in \mathbb{F}^\times$, $u \in \mathbb{F}$:

$$\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1}{s} & 1\end{pmatrix}\begin{pmatrix}1 & s-t+u\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{t} & 1\end{pmatrix}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{s}{t} & 0\\ -\frac{u}{st} & \frac{t}{s}\end{pmatrix}. \tag{4.2}$$

Assume θ is an involution induced by some group automorphism ϕ of $(\mathbb{F},+)$ as described in (4.1). In case $s = t$ we can thus apply θ to (4.2) and obtain the equality

$$\begin{pmatrix}1 & 0\\ \phi(t) & 1\end{pmatrix}\begin{pmatrix}1 & -\phi^{-1}(\frac{1}{t})\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \phi(u) & 1\end{pmatrix}\begin{pmatrix}1 & \phi^{-1}(\frac{1}{t})\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\phi(t) & 1\end{pmatrix} = \begin{pmatrix}1 & -\phi^{-1}(\frac{u}{t^2})\\ 0 & 1\end{pmatrix}. \tag{4.3}$$

Reducing (4.3) further by setting $u = t$ and expanding the left side, we arrive at

$$\begin{pmatrix}1-\phi(t)\phi^{-1}(\frac{1}{t})\left(1-\phi(t)\phi^{-1}(\frac{1}{t})\right) & -\phi(t)\phi^{-1}(\frac{1}{t})^2\\ \phi(t)\left(1-\phi(t)\phi^{-1}(\frac{1}{t})\right)^2 & 1+\phi(t)\phi^{-1}(\frac{1}{t})\left(1-\phi(t)\phi^{-1}(\frac{1}{t})\right)\end{pmatrix} = \begin{pmatrix}1 & -\phi^{-1}(\frac{1}{t})\\ 0 & 1\end{pmatrix}$$

which readily implies $\phi^{-1}(t) = \phi(\frac{1}{t})^{-1}$ for all $t \in \mathbb{F}^\times$. Defining $\varepsilon := \phi(1)$ and $\sigma(x) := \frac{\phi(x)}{\varepsilon}$ (note that $\varepsilon \neq 0$ since ϕ is an automorphism of the group $(\mathbb{F},+)$), we again use (4.2), this time for $u = 0$ and arbitrary s and t. We obtain

$$\theta\begin{pmatrix}\frac{s}{t} & 0\\ 0 & \frac{t}{s}\end{pmatrix} = \begin{pmatrix}\frac{\phi(t)}{\phi(s)} & 0\\ 0 & \frac{\phi(s)}{\phi(t)}\end{pmatrix} = \begin{pmatrix}\frac{\sigma(t)}{\sigma(s)} & 0\\ 0 & \frac{\sigma(s)}{\sigma(t)}\end{pmatrix}.$$

We use this equality twice, once substituting (y, xy) for (s,t) and once $(1,x)$ for (s,t) in order to obtain our final equality

$$\begin{pmatrix}\frac{\sigma(xy)}{\sigma(y)} & 0\\ 0 & \frac{\sigma(y)}{\sigma(xy)}\end{pmatrix} = \theta\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix} = \begin{pmatrix}\frac{\sigma(x)}{\sigma(1)} & 0\\ 0 & \frac{\sigma(1)}{\sigma(x)}\end{pmatrix} = \begin{pmatrix}\sigma(x) & 0\\ 0 & \frac{1}{\sigma(x)}\end{pmatrix}.$$

This allows us to conclude that $\sigma(xy) = \sigma(x)\sigma(y)$. We already know that $\sigma(x+y) = \sigma(x)+\sigma(y)$, $\sigma(0) = 0$, $\sigma(1) = 1$, and hence $\sigma \in \mathrm{Aut}(\mathbb{F})$ as required. Furthermore, $\sigma(\varepsilon) = \varepsilon$ since

$$1 = \sigma(\varepsilon\varepsilon^{-1}) = \sigma(\varepsilon)\sigma(\varepsilon^{-1}) = \sigma(\varepsilon)\varepsilon^{-1}\phi(\varepsilon^{-1}) = \varepsilon^{-1}\sigma(\varepsilon)\phi^{-1}(\varepsilon)^{-1} = \varepsilon^{-1}\sigma(\varepsilon).$$

Finally $\sigma^2 = \mathrm{id}$, since

$$\sigma^{-1}(x) = \phi^{-1}(\varepsilon x) = \frac{1}{\phi(\varepsilon^{-1}x^{-1})} = \frac{1}{\varepsilon\sigma(\varepsilon^{-1}x^{-1})} = \frac{1}{\sigma(x^{-1})} = \sigma(x).$$

The converse implication, deriving a group automorphism θ of $\mathrm{SL}_2(\mathbb{F})$ from a given group automorphism ϕ of $(\mathbb{F}, +)$, results from the fact that the following automorphism restricts as required to U_+ resp. U_-:

$$\theta : \mathrm{SL}_2(\mathbb{F}) \to \mathrm{SL}_2(\mathbb{F}) : X \mapsto \theta(X) = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} X^\sigma \begin{pmatrix} 0 & \varepsilon^{-1} \\ 1 & 0 \end{pmatrix}. \qquad \square$$

Definition 4.2. For a field automorphism σ of $\mathbb{F}$ of order one or two and $\delta \in \mathrm{Fix}_{\mathbb{F}}(\sigma)$ define

$$\theta_{\delta,\sigma} : \mathrm{SL}_2(\mathbb{F}) \to \mathrm{SL}_2(\mathbb{F}) : X \mapsto \theta_{\delta,\sigma}(X) = \begin{pmatrix} 0 & 1 \\ -\delta^{-1} & 0 \end{pmatrix} X^\sigma \begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix}.$$

By slight abuse of notation, we will use the same symbol $\theta_{\delta,\sigma}$ to denote the induced flip on $\mathrm{PSL}_2(\mathbb{F})$.

4.2. Centralizers of flips

We now turn our attention to the centralizers of a given flip θ. It is easy to verify that

$$K_{\delta,\sigma} := C_{\mathrm{SL}_2(\mathbb{F})}(\theta_{\delta,\sigma}) = \left\{ \begin{pmatrix} u^\sigma & \delta v^\sigma \\ -v & u \end{pmatrix} \mid uu^\sigma + \delta vv^\sigma = 1 \right\},$$

which is precisely the group preserving the σ-sesquilinear form

$$f(x, y) := x^T \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} y^\sigma$$

on the vector space $\mathbb{F}^2$ and its associated unitary form $q(x) := f(x, x)$. This alternative characterization will turn out to be quite useful.

For $\mathrm{PSL}_2(\mathbb{F})$, the situation is slightly different. Let Z denote the center of $\mathrm{SL}_2(\mathbb{F})$. By definition $\mathrm{PSL}_2(\mathbb{F}) = \mathrm{SL}_2(\mathbb{F})/Z$, so that the centralizer of θ in $\mathrm{PSL}_2(\mathbb{F})$ is $C_{\mathrm{PSL}_2(\mathbb{F})}(\theta) = \{gZ \in \mathrm{PSL}_2(\mathbb{F}) \mid (gZ)^\theta = gZ\}$. We are mainly interested in the action of this centralizer on $\mathbb{P}_1(\mathbb{F})$. Since the action of $\mathrm{PSL}_2(\mathbb{F})$ is induced by that of $\mathrm{SL}_2(\mathbb{F})$, this means studying the preimage of the centralizer in $\mathrm{SL}_2(\mathbb{F})$. This suggests the following definition:

Definition 4.3. Let θ be an automorphism of $\mathrm{SL}_2(\mathbb{F})$. We define the *projective centralizer* of θ in $\mathrm{SL}_2(\mathbb{F})$ as the group $PC_{\mathrm{SL}_2(\mathbb{F})}(\theta) := \{g \in \mathrm{SL}_2(\mathbb{F}) \mid g^\theta \in gZ\}$, which is the preimage of $C_{\mathrm{PSL}_2}(\theta)$ in $\mathrm{SL}_2(\mathbb{F})$ under the canonical projection $\pi : \mathrm{SL}_2 \to \mathrm{PSL}_2$.

We compute

$$PK_{\delta,\sigma} := PC_{\mathrm{SL}_2(\mathbb{F})}(\theta_{\delta,\sigma}) = \left\{ \begin{pmatrix} \varepsilon u^\sigma & \delta\varepsilon v^\sigma \\ -v & u \end{pmatrix} \mid uu^\sigma + \delta vv^\sigma = \varepsilon,\ \varepsilon \in \{-1, +1\} \right\}.$$

While $K_{\delta,\sigma}$ preserves the σ-sesquilinear form $f(x, y)$ and its associated unitary form $q(x)$, the group $PK_{\delta,\sigma}$ preserves these forms up to sign.

4.3. Transitivity of centralizers of flips

The observations of the previous sections allows us to characterize when $\theta_{\delta,\sigma}$ is transitive. Set $N : \mathbb{F} \to \mathbb{F} : a \mapsto aa^\sigma$. Note that $q\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = N(a) + \delta N(b)$. All results are written with $\mathrm{PSL}_2(\mathbb{F})$ in mind. The corresponding results for $\mathrm{SL}_2(\mathbb{F})$ can be obtained by replacing $PK_{\delta,\sigma}$ by $K_{\delta,\sigma}$ and substituting 1 for ε.

Lemma 4.4. *If the involution $\theta_{\delta,\sigma}$ is transitive, then it is conjugate to $\theta_{1,\sigma}$. In case* $\mathrm{char}(\mathbb{F}) = 2$, *the involution $\theta_{\delta,\sigma}$ cannot be transitive.*

Proof. Due to the transitivity of $PK_{\delta,\sigma}$ on $\mathbb{P}_1(\mathbb{F})$, there exists $g \in PK_{\delta,\sigma}$ such that $g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)$ for some $x \neq 0$. Thus

$$1 = q\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \varepsilon q(g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = \varepsilon q\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) = \varepsilon q\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) N(x) = \varepsilon \delta N(x),$$

with $\varepsilon \in \{-1, +1\}$, whence $\delta = \varepsilon N(x^{-1})$. If $\varepsilon = -1$, then q would be isotropic, as $q\left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right) = 0$, contradicting transitivity. (This in particular implies that in case $\mathrm{char}(\mathbb{F}) = 2$, the involution $\theta_{\delta,\sigma}$ cannot be transitive.) Thus $\delta = N(x^{-1})$. Let $Y := \left(\begin{smallmatrix} 1 & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$, and denote by $\mathrm{Inn}_Y(g)$ the inner automorphism of G induced by Y. Then conjugating $\theta_{1,\sigma}$ by $\mathrm{Inn}_Y(g)$ yields $\theta_{\delta,\sigma}$. □

Because of the preceding lemma it remains to determine when exactly $\theta_{1,\sigma}$ is transitive.

Proposition 4.5. *The involution $\theta_{1,\sigma}$ is transitive if and only if -1 is not a norm, and the sum of two norms is ε times a norm, where $\varepsilon \in \{+1, -1\}$.*

Proof. Assume $\theta_{1,\sigma}$ is transitive. Take an arbitrary nonzero vector $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{F}^2$. Due to transitivity, there exists $g \in PK_{1,\sigma}$ such that $g\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ for some nonzero $x \in \mathbb{F}$. Consequently

$$N(a) + N(b) = q\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \varepsilon q(g\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)) = \varepsilon q\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = \varepsilon N(x) \in N(\mathbb{F}),\ \varepsilon \in \{-1, +1\},$$

proving that a sum of two norms is a norm or -1 times a norm. Furthermore, -1 is not a norm as else there would be $x \in \mathbb{F}$ with $N(x) = -1$ and we would obtain the isotropic vector $\left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right)$ contradicting transitivity.

For the converse implication assume that the sum of two norms is a norm or -1 times a norm, and -1 is not a norm. It suffices to show that there exists an element of $PK_{1,\sigma}$ that maps an arbitrary non-trivial vector $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{F}^2$ onto some non-trivial vector $\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$. Choose x such that $\varepsilon N(x) = N(a) + N(b)$ for $\varepsilon \in \{-1, +1\}$, and note that $x \neq 0$, as else $N(a) = -N(b)$ contradicting that -1 is not a norm. Thus the equation

$$\begin{pmatrix} \varepsilon \left(\frac{a}{x}\right)^\sigma & \varepsilon \left(\frac{b}{x}\right)^\sigma \\ -\frac{b}{x} & \frac{a}{x} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

finishes the proof, since the given matrix is clearly in $PK_{1,\sigma}$. □

Corollary 4.6. *The group* (P)$\mathrm{SL}_2(\mathbb{F})$ *admits an Iwasawa decomposition if and only if* $\mathbb{F}$ *admits an automorphism σ of order one or two such that*

(i) -1 *is not a norm, and*

(ii) (a) *either a sum of norms is a norm (in the* $\mathrm{SL}_2(\mathbb{F})$ *case)*

(b) *or a sum of norms is ε times a norm, where $\varepsilon \in \{+1, -1\}$ (in the* $\mathrm{PSL}_2(\mathbb{F})$ *case),*

with respect to the norm map $N : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$.

Proof. Assume we have an Iwasawa decomposition of G. Then we in particular have a transitive involution θ which interchanges U_+ and U_- and satisfies $G = G_\theta B_+$. Then by Lemmas 4.1 and 4.4 plus Proposition 4.5 the claim for $\mathbb{F}$ follows. If on the other hand $\mathbb{F}$ is as described, then again by Proposition 4.5 the map $\theta_{1,\sigma}$ induces an Iwasawa decomposition. □

4.4. Fields permitting Iwasawa decompositions

Besides the real closed fields and the field of complex numbers there exist lots of fields admitting automorphisms that satisfy the conditions of Corollary 4.6. For instance any pythagorean formally real field $\mathbb{F}$ satisfies the conditions of Corollary 4.6 with respect to the identity automorphism as does $\mathbb{F}[\sqrt{-1}]$ with respect to the non-trivial Galois automorphism. In the $\mathrm{PSL}_2(\mathbb{F})$ case the finite fields $\mathbb{F}_q$ with $q \equiv 3 \mod 4$ yield additional examples.

Remark 4.7. Fields that are pythagorean and formally real have been studied very thoroughly, cf. [23], [24], [32]. In this remark we collect some of their most prominent properties in order to get an idea what kind of fields we are dealing with in this paper.

(i) A field is formally real pythagorean if and only if its Witt group is torsionfree, cf. [24, Theorem VIII.4.1].

(ii) A field is formally real pythagorean if and only if it is the intersection of a non-empty family of euclidean subfields of its algebraic closure, cf. [24, Theorem VIII.4.4].

(iii) If a field $\mathbb{F}$ is formally real and pythagorean, then so is the field $\mathbb{F}((t))$ of formal Laurent series, cf. [32, Theorem 18.9].

(iv) If a field $\mathbb{F}$ is real closed, then the field $\mathbb{F}((t_1))\cdots((t_n))$ is pythagorean and has 2^{n+1} square classes, cf. [32, Theorem 18.9].

(v) If $\mathbb{F}$ is pythagorean but not formally real, then $\mathbb{F}$ is quadratically closed, cf. [32, Theorem 16.4]. In particular, the intersection of the real numbers with the field of the numbers which are constructible with straightedge and compass, is pythagorean and formally real.

(vi) If $\mathbb{F}$ is a field in which -1 is not a square, then it is pythagorean (and hence formally real) if and only if $\mathbb{F}$ does not admit any cyclic extension of order four, cf. [11].

Inspired by classical Lie theory and the passage from complex Lie groups to their split real forms, the question arises when an Iwasawa decomposition $G = G_\theta B_+$ of a group G with an $\mathbb{F}$-locally split root group datum with respect to an involution θ involving a non-trivial field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ implies the existence of an Iwasawa decomposition over the field $\mathrm{Fix}_{\mathbb{F}}(\sigma)$ with respect to an involution involving the trivial field automorphism on $\mathrm{Fix}_{\mathbb{F}}(\sigma)$. The following example shows that this in general is not the case.

Example 4.8. Let $\mathbb{F}$ be a formally real field which is not pythagorean and admits four square classes. Such fields exist, see for example [35]. This means exactly two square classes contain absolutely positive elements, so that there exists a unique ordering. Choose a positive non-square element $w \in \mathbb{F}$. Set $\alpha := \sqrt{-w}$ and $\tilde{\mathbb{F}} := \mathbb{F}[\alpha]$. Then

$$N(x_0 + \alpha x_1) + N(y_0 + \alpha y_1) = x_0^2 + wx_1^2 + y_0^2 + wy_1^2$$

which is a non-negative number, hence either a square or a square multiple of w. Hence there exist z_0 and z_1 in $\mathbb{F}$ such that

$$N(x_0 + \alpha x_1) + N(y_0 + \alpha y_1) = x_0^2 + wx_1^2 + y_0^2 + wy_1^2 = z_0^2 + wz_1^2 = N(z_0 + \alpha z_1)$$

and thus the field $\tilde{\mathbb{F}}$ together with the non-trivial Galois automorphism satisfies the conditions of Corollary 4.6, while $\mathbb{F}$ together with the identity does not, because $\mathbb{F}$ is not pythagorean.

5. A local criterion for Iwasawa decompositions

It is well-known that an adjacency-preserving action of a group G on a connected chamber system $\mathcal{C}$ over I is transitive if and only if there exists a chamber $c \in \mathcal{C}$ such that for each $i \in I$ the normalizer $N_G([c]_i)$ acts transitively on the i-panel $[c]_i$ of $\mathcal{C}$ containing c. It is implied by the following observation concerning permutation groups.

Proposition 5.1. *Let X be a set endowed with a family of equivalence relations $(\sim_i)_{i\in I} \subseteq X \times X$ such that the transitive hull of $\bigcup_{i\in I} \sim_i$ equals $X \times X$. Moreover, let G be a group acting on X as permutations preserving each equivalence relation. If there exists a point $p \in X$ such that the normalizer $G_i := G_{[p]_i}$ of $[p]_i$ in G acts transitively on $[p]_i$, then G acts transitively on X.*

Proof. It suffices to show that there exists $g \in G$ mapping p to an arbitrary point q. Since the transitive hull of $\bigcup_{i\in I} \sim_i$ equals $X\times X$, we can find a path $p = p_0 \sim_{i_1} p_1 \sim_{i_2} p_2 \cdots p_{n-1} \sim_{i_n} p_n = q$ from p to q. We prove this proposition by induction on n. For $n = 0$ nothing has to be shown, so that we can assume that there exists $g_0 \in G$ mapping p to p_{n-1}. Let $q' := g_0^{-1}q$ be the inverse image of q under this map. Then $p \sim_{i_n} q'$, and by the transitivity of G_{i_n} on $[p]_{i_n}$ there exists $g_1 \in G_{i_n}$ mapping p to q'. Hence $g := g_0g_1$ maps p to q, as $g.p = (g_0g_1).p = g_0.(g_1.p) = g_0.q' = q$. □

Corollary 5.2. *Let $\mathcal{C}$ be a connected chamber system and let G be a group of automorphisms of $\mathcal{C}$. The group G acts transitively on $\mathcal{C}$ if and only if there exists a chamber $c \in \mathcal{C}$, such that for each panel P of $\mathcal{C}$ containing c the stabilizer G_P acts transitively on P.*

Corollary 5.3. *Let $\mathcal{B}$ be a twin building obtained from a group G with a twin BN-pair. Let θ be a BN-flip of G, let $K := G_\theta$ be the group of all elements of G centralized by θ. The group K acts transitively on the positive/negative half of $\mathcal{B}$ if and only if there exists a chamber c opposite c^θ such that for each panel P at c the stabilizer K_P of that panel in K acts transitively on P.*

We finally have assembled all tools required to prove our main result.

Proof of Theorem 2. Assume the existence of an Iwasawa decomposition of G, so by definition there exists an involution θ of G such that $G = G_\theta B_+$. Hence any Borel subgroup of G is mapped onto an opposite one, so that by Proposition 3.4 the involution θ centralizes the Weyl group N/T. Therefore for any simple root α, the involution θ normalizes the group $X_\alpha := \langle U_\alpha, U_{-\alpha}\rangle$ which by $\mathbb{F}$-local splitness is isomorphic to (P)SL$_2(\mathbb{F})$. In particular the restriction $\theta_{|X_\alpha}$ of θ to X_α is a BN-flip, so that by Corollary 5.3 it induces an Iwasawa decomposition of X_α. By Corollary 4.6 the field $\mathbb{F}$ admits an automorphism σ with the required properties.

For the converse implication, let θ be the σ-twisted Chevalley involution of G. As a simple consequence of its definition, θ induces a BN-flip θ_α on each X_α; see also Proposition 3.5, where we see that θ is in fact a BN-flip of G. By Proposition 4.5, these induced flips are transitive. Hence by Corollary 5.3, we have $G = G_\theta B_+$, proving that G admits an Iwasawa decomposition. □

A. Moufang sets

We now turn our attention to the study of Moufang sets. The motivation for this appendix is that by Proposition 3.4 we need to understand flips of arbitrary rank one subgroups in order to understand flips of arbitrary non-split groups with a root group datum. Moufang sets are essentially equivalent to these and seem to be the natural setting to study flips.

Maybe Moufang sets, or rather Moufang sets with transitive flips (cf. Sections A.2 and A.3), can be used for a better understanding of certain anisotropic forms of reductive algebraic groups. Indeed, the machinery developed in this paper implies that an anisotropic form K of a reductive algebraic group G acts transitively on the building $\mathcal{B}$ of G if and only if there exists a chamber c of $\mathcal{B}$ such that, for each i-panel P containing c, the stabilizer K_P acts transitively on P, cf. Corollary 5.3. Therefore any result about Moufang sets with transitive flips has immediate consequences for certain classes of anisotropic algebraic groups (those which act transitively on a building).

A.1. Moufang sets and pointed Moufang sets

In order to be consistent with the standard notation used in the theory of Moufang sets we will always denote the action of a permutation on a set on the right, i.e. we will write $a\varphi$ rather than $\varphi(a)$.

Definition A.1. A *Moufang set* is a set X together with a collection of subgroups $(U_x)_{x\in X}$, such that each U_x is a subgroup of $\mathrm{Sym}(X)$ fixing x and acting regularly (i.e. sharply transitively) on $X \setminus \{x\}$, and such that each U_x permutes the set $\{U_y \mid y \in X\}$ by conjugation. The group $G := \langle U_x \mid x \in X\rangle$ is called the *little projective group* of the Moufang set; the groups U_x are called *root groups*.

Our approach to Moufang sets is taken from [9]. Let $\mathbb{M} = (X, (U_x)_{x\in X})$ be an arbitrary Moufang set, and assume that two of the elements of X are called 0 and ∞. Let $U := X \setminus \{\infty\}$. Each $\alpha \in U_\infty$ is uniquely determined by the image of 0 under α. If $0\alpha = a$, we write $\alpha =: \alpha_a$. Hence $U_\infty = \{\alpha_a \mid a \in U\}$. We make U into a (not necessarily abelian) group with composition + and identity 0, by setting

$$a + b := a\alpha_b\,. \tag{A.1}$$

Clearly, $U \cong U_\infty$. Now let τ be an element of G interchanging 0 and ∞. (Such an element always exists, since G is doubly transitive on X.) By the definition of a Moufang set, we have

$$U_0 = U_\infty^\tau \text{ and } U_a = U_0^{\alpha_a} \tag{A.2}$$

for all $a \in U$. In particular, the Moufang set $\mathbb{M}$ is completely determined by the group U and the permutation τ; we will denote it by $\mathbb{M} = \mathbb{M}(U, \tau)$.

Remark A.2. In view of equation (A.1), it makes sense to use the convention that $a + \infty = \infty + a = \infty$ for all $a \in U$.

Definition A.3. For each $a \in U$, we define $\gamma_a := \alpha_a^\tau$, i.e. $x\gamma_a = (x\tau^{-1} + a)\tau$ for all $x \in X$. Consequently, $U_0 = \{\gamma_a \mid a \in U\}$.

Definition A.4. For each $a \in U^* = U\backslash\{0\}$, we define a *Hua map* to be

$$h_a := \tau\alpha_a\tau^{-1}\alpha_{-(a\tau^{-1})}\tau\alpha_{-(-(a\tau^{-1}))\tau} \in \mathrm{Sym}(X);$$

if we use the convention of Remark A.2, then we can write this explicitly as $h_a : X \to X : x \mapsto \big((x\tau + a)\tau^{-1} - a\tau^{-1}\big)\tau - \big(-(a\tau^{-1})\big)\tau$. Observe that each h_a fixes the elements 0 and ∞. We define the *Hua subgroup* of $\mathbb{M}$ as $H := \langle h_a \mid a \in U^*\rangle$. By [9, Theorem 3.1], the group H equals $G_{0,\infty} := \mathrm{Stab}_G(0, \infty)$, and by [9, Theorem 3.2], the restriction of each Hua map to U is additive, i.e. $H \le \mathrm{Aut}(U)$.

Definition A.5. For each $a \in U^*$, we define a *μ-map* $\mu_a := \tau^{-1}h_a$. Then μ_a is the unique element in the set $U_0^*\alpha_aU_0^*$ interchanging 0 and ∞. In particular, $\mu_a^{-1} = \mu_{-a}$.

Definition A.6. Let $(X, (U_x)_{x\in X})$ and $(Y, (V_y)_{y\in Y})$ be two Moufang sets. A bijection β from X to Y is called an *isomorphism* of Moufang sets, if the induced map $\chi_\beta : \mathrm{Sym}(X) \to \mathrm{Sym}(Y) : g \mapsto \beta^{-1}g\beta$ maps each root group U_x isomorphically onto the corresponding root group $V_{x\beta}$. An *automorphism* of $\mathbb{M} = (X, (U_x)_{x\in X})$ is an isomorphism from $\mathbb{M}$ to itself. The group of all automorphisms of $\mathbb{M}$ will be denoted by $\mathrm{Aut}(\mathbb{M})$.

Now we introduce pointed Moufang sets, which will be Moufang sets with a fixed identity element. We will then, in analogy with the theory of Jordan algebras, introduce the notions of an isotope of a pointed Moufang set, and we will define Jordan isomorphisms between Moufang sets.

Definition A.7. A *pointed Moufang set* is a pair $(\mathbb{M}, e)$, where $\mathbb{M} = \mathbb{M}(U, \rho)$ is a Moufang set and e is an arbitrary element of U^*. The *τ-map* of this pointed Moufang set is $\tau := \mu_{-e} = \mu_e^{-1}$, and the *Hua maps* are the maps $h_a := \tau\mu_a = \mu_{-e}\mu_a$ for all $a \in U^*$. We also define the *opposite Hua maps* $g_a := \tau^{-1}\mu_a = \mu_e\mu_a$ for all $a \in U^*$. Clearly, $\mathbb{M} = \mathbb{M}(U, \tau) = \mathbb{M}(U, \tau^{-1})$.

Note that, in contrast with Moufang sets which are not pointed, the maps τ, h_a and g_a are completely determined by the data $(\mathbb{M}, e)$. On the other hand, there can be many different elements f for which $(\mathbb{M}, e) = (\mathbb{M}, f)$, namely all those for which $\mu_e = \mu_f$.

Definition A.8. Let $(\mathbb{M}, e)$ and $(\mathbb{M}', f)$ be two pointed Moufang sets, with $\mathbb{M} = \mathbb{M}(U, \rho)$ and $\mathbb{M}' = \mathbb{M}(U', \rho')$. A *pointed isomorphism* from $(\mathbb{M}, e)$ to $(\mathbb{M}', f)$ is an isomorphism from U to U' mapping e to f and extending to a Moufang set isomorphism from $\mathbb{M}$ to $\mathbb{M}'$ (by mapping ∞ to ∞'). A pointed isomorphism from $(\mathbb{M}, e)$ to itself is called a *pointed automorphism* of $(\mathbb{M}, e)$, and the group of all pointed automorphisms is denoted by $\mathrm{Aut}(\mathbb{M}, e)$.

Observe that $G \cap \mathrm{Aut}(\mathbb{M}, e) = C_H(e)$.

Definition A.9. Let $(\mathbb{M}, e)$ be a pointed Moufang set, and let $a \in U^*$ be arbitrary. Then $(\mathbb{M}, a)$ is called the *a-isotope* of $(\mathbb{M}, e)$, or simply an *isotope* if one does not want to specify the element a. The τ-map and the Hua maps of $(\mathbb{M}, a)$ will be denoted by $\tau^{(a)}$ and $h_b^{(a)}$, respectively. Observe that

$$\tau^{(a)} = \mu_{-a} \quad \text{and} \quad h_b^{(a)} = \mu_{-a}\mu_b = h_a^{-1} h_b \tag{A.3}$$

for all $a, b \in U^*$.

Our notion of an a-isotope is, in a certain sense, the inverse of the usual notion of an a-isotope in (quadratic) Jordan algebras, where our a-isotope would be called the a^{-1}-isotope (where a^{-1} denotes the inverse in the Jordan algebra) and where $h_b^{(a)} := h_a h_b$. It is, in the general context of Moufang sets, not natural to try to be compatible with this convention, because h_a^{-1} is in general not of the form h_b for some $b \in U^*$. In fact, we have $h_a^{-1} = g_{a\tau}$ for all $a \in U^*$; see [9, Lemma 3.8(i)].

Definition A.10. Let $(\mathbb{M}, e)$ and $(\mathbb{M}', f)$ be two pointed Moufang sets with $\mathbb{M} = \mathbb{M}(U, \rho)$ and $\mathbb{M}' = \mathbb{M}(U', \rho')$, and with Hua maps h_a and k_a, respectively. An isomorphism φ from U to U' is called a *Jordan isomorphism* if $(bh_a)\varphi = (b\varphi)k_{a\varphi}$ for all $a, b \in U^*$. If $(\mathbb{M}', f)$ is an isotope $(\mathbb{M}, a)$ of $(\mathbb{M}, e)$, then a Jordan isomorphism from $(\mathbb{M}, e)$ to $(\mathbb{M}, a)$ is called an *isotopy* from $(\mathbb{M}, e)$ to its a-isotope. Explicitly, a map $\varphi \in \mathrm{Aut}(U)$ is an isotopy if and only if

$$h_a \varphi = \varphi h_{a\varphi}^{(e\varphi)} \tag{A.4}$$

for all $a \in U^*$. The group of all isotopies from $(\mathbb{M}, e)$ to an isotope is called the *structure group* of $(\mathbb{M}, e)$, and is denoted by $\mathrm{Str}(\mathbb{M}, e)$. Note that it is not clear whether $\mathrm{Str}(\mathbb{M}, e) \leq \mathrm{Aut}(\mathbb{M})$. Also observe that $G \cap \mathrm{Str}(\mathbb{M}, e) = H$; we call H the *inner structure group* of $(\mathbb{M}, e)$.

A.2. Flips of Moufang sets

Our goal in this section is to determine all involutions $\theta \in \mathrm{Aut}(G)$ interchanging U_∞ and U_0. Such an involution θ maps each α_a to some $\gamma_{a\varphi}$ and each γ_b to some $\alpha_{b\psi}$. Since $\theta \in \mathrm{Aut}(G)$, we have $\varphi, \psi \in \mathrm{Aut}(U)$. Moreover, $\theta^2 = \mathrm{id}$ implies $\psi = \varphi^{-1}$. In particular, θ is competely determined by φ. More precisely, for each $\varphi \in \mathrm{Aut}(U)$, we define

$$\theta_\varphi : U_\infty \cup U_0 \to U_0 \cup U_\infty : \begin{cases} \alpha_a \mapsto \gamma_{a\varphi} \\ \gamma_a \mapsto \alpha_{a\varphi^{-1}} \end{cases} ;$$

the question is when θ_φ extends to an automorphism of G. Observe that if θ_φ extends, then this extension is unique and is involutory, since θ is involutory on $U_\infty \cup U_0$ and $G = \langle U_\infty \cup U_0 \rangle$.

Proposition A.11. *Let $\varphi \in \mathrm{Aut}(U)$. Then θ_φ extends to an (involutory) automorphism of G if and only if $(\varphi\tau)^2 = \mathrm{id}$. Moreover, if this is the case, then $\varphi \in \mathrm{Aut}(\mathbb{M})$.*

Proof. Let $\theta := \theta_\varphi$ and $\beta := \varphi\tau$. Assume first that θ extends to an automorphism χ of G. Then

$$\chi(U_a) = \chi(U_0^{\alpha_a}) = \chi(U_0)^{\chi(\alpha_a)} = U_\infty^{\gamma_{a\varphi}} = U_{a\varphi\tau} = U_{a\beta} \tag{A.5}$$

for all $a \in U$. Since θ^2 is the identity on $U_\infty \cup U_0$ and since $G = \langle U_\infty, U_0 \rangle$, this implies that $\chi^2 = 1$ and hence $\beta^2 = 1$.

Conversely, assume that $\beta^2 = 1$, and let χ_β be as in Definition A.6. Then for all $a \in U$,

$$\chi_\beta(\alpha_a) = \alpha_a^{\varphi\tau} = \alpha_{a\varphi}^{\tau} = \gamma_{a\varphi}\,,$$
$$\chi_\beta(\gamma_a) = \gamma_a^{\varphi\tau} = \gamma_a^{\tau^{-1}\varphi^{-1}} = \alpha_a^{\varphi^{-1}} = \alpha_{a\varphi^{-1}}\,;$$

hence χ_β and θ coincide on $U_\infty \cup U_0$. Note that χ_β is an (inner) automorphism of $\mathrm{Sym}(X)$, and hence the same calculation as in equation (A.5) (with χ_β in place of χ) shows that $\beta \in \mathrm{Aut}(\mathbb{M})$. Hence the restriction of χ_β to G is an automorphism of G; this is the (unique) extension of θ to an element of $\mathrm{Aut}(G)$.

Finally, since we have just shown that $\beta \in \mathrm{Aut}(\mathbb{M})$ and since obviously $\tau \in \mathrm{Aut}(\mathbb{M})$, we conclude that $\varphi \in \mathrm{Aut}(\mathbb{M})$ as well. □

Definition A.12. An automorphism $\varphi \in \mathrm{Aut}(U)$ with the property that $(\varphi\tau)^2 = 1$ will be called a *flip automorphism* of $\mathbb{M}$.

The following theorem gives important information about such flip automorphisms.

Theorem A.13. *Let $\mathbb{M}$ be a Moufang set, and let φ be a flip automorphism of $\mathbb{M}$. Then*

$$g_{a\varphi} = \varphi \cdot h_a \cdot \varphi$$

for all $a \in U^*$. Moreover, if e is an identity element of $\mathbb{M}$, i.e. $\tau = \mu_{-e}$, then $\varphi \in \mathrm{Str}(\mathbb{M}, e) \cap \mathrm{Aut}(\mathbb{M})$.

Proof. For each $a \in U^*$, the map g_a is the Hua map of a with τ replaced by τ^{-1}, and hence $g_{a\varphi} = \tau^{-1}\alpha_{a\varphi}\tau\alpha_{-a\varphi\tau}\tau^{-1}\alpha_{-(-(-a\varphi\tau))\tau^{-1}}$ for all $a \in U^*$. Using the facts that $\alpha_a^{\varphi} = \alpha_{a\varphi}$, $\varphi\tau = \tau^{-1}\varphi^{-1}$ and $(-a)\varphi = -a\varphi$ several times, we get $\varphi^{-1}g_{a\varphi} = \tau\alpha_a\tau^{-1}\alpha_{-a\tau^{-1}}\tau\alpha_{-(-(-a\tau^{-1}))\tau}\varphi = h_a\varphi$. In particular, if e is an identity element of $\mathbb{M}$, then $h_e = 1$ and hence $\varphi^{-1}g_{e\varphi} = \varphi$. It follows that $\varphi g_{e\varphi}^{-1}g_{a\varphi} = h_a\varphi$ for all $a \in U^*$. However, $g_{e\varphi}^{-1}g_{a\varphi} = (\mu_e\mu_{e\varphi})^{-1}(\mu_e\mu_{a\varphi}) = (\mu_{-e}\mu_{e\varphi})^{-1}(\mu_{-e}\mu_{a\varphi}) = h_{e\varphi}^{-1}h_{a\varphi} = h_{a\varphi}^{(e\varphi)}$ and hence $h_a\varphi = \varphi h_{a\varphi}^{(e\varphi)}$ for all $a \in U^*$, proving that $\varphi \in \mathrm{Str}(\mathbb{M}, e)$. The fact that $\varphi \in \mathrm{Aut}(\mathbb{M})$ was shown in Proposition A.11 above. □

We will now illustrate the strength of Theorem A.13 by explicitly determining all flips of $\mathrm{PSL}_2(\mathbb{D})$, where $\mathbb{D}$ is a field or a skew field. This can be considered as a natural extension of the results from Lemma 4.1 to the non-commutative case.

Proposition A.14. *Let $\mathbb{D}$ be an arbitrary field or skew field, and let $\mathbb{M} = \mathbb{M}(\mathbb{D})$ be the corresponding Moufang set, i.e. the Moufang set $\mathbb{M} = \mathbb{M}(U, \tau)$ where $U := (\mathbb{D}, +)$ and $\tau : \mathbb{D}^* \to \mathbb{D}^* : x \mapsto -x^{-1}$.*

(i) *Let φ be a flip automorphism of $\mathbb{M}$. Then there exists an automorphism or anti-automorphism σ of $\mathbb{D}$ and an element $\varepsilon \in \mathrm{Fix}_{\mathbb{D}}(\sigma)$ such that $x\varphi = \varepsilon\sigma(x)$ for all $x \in \mathbb{D}$. If σ is an automorphism, then $\sigma^2(x) = \varepsilon^{-1}x\varepsilon$ for all $x \in \mathbb{D}$; if σ is an anti-automorphism, then $\sigma^2 = 1$.*

(ii) *Conversely, suppose that either σ is an anti-automorphism of order 2 and $\varepsilon \in \mathrm{Fix}_{\mathbb{D}}(\sigma)$ is arbitrary, or σ is an automorphism such that $\sigma^2(x) = \varepsilon^{-1}x\varepsilon$ for some $\varepsilon \in \mathrm{Fix}_{\mathbb{D}}(\sigma)$. Then the map $\varphi : \mathbb{D} \to \mathbb{D} : x \mapsto \varepsilon\sigma(x)$ is a flip automorphism of $\mathbb{M}$.*

Proof. (i) Observe that $1 \in \mathbb{D}^*$ is an identity element of $\mathbb{M}$; also note that $\tau^2 = \mathrm{id}$. For all $a, b \in U^*$, we have $bh_a = aba$. The condition $(\varphi\tau)^2 = 1$ translates to

$$(a^{-1})\varphi = (a\varphi^{-1})^{-1} \tag{A.6}$$

for all $a \in \mathbb{D}^*$. Let $\varepsilon := 1\varphi$; then $bh_a^{(1\varphi)} = bh_{1\varphi}^{-1}h_a = a\varepsilon^{-1}b\varepsilon^{-1}a$ for all $a, b \in U^*$. By Theorem A.13, $\varphi \in \mathrm{Str}(\mathbb{M}, e)$, which means that $bh_a\varphi = b\varphi h_{a\varphi}^{(1\varphi)}$ for all $a, b \in U^*$, or explicitly, $(aba)\varphi = a\varphi \cdot \varepsilon^{-1} \cdot b\varphi \cdot \varepsilon^{-1} \cdot a\varphi$ for all $a, b \in \mathbb{D}^*$. Now let $\sigma(a) := \varepsilon^{-1} \cdot a\varphi$ for all $a \in \mathbb{D}$. Then $\sigma \in \mathrm{Aut}(U)$, and the previous equation can be rewritten as $\sigma(aba) =$

$\sigma(a)\sigma(b)\sigma(a)$ for all $a, b \in \mathbb{D}$, i.e. σ is a Jordan automorphism of $\mathbb{D}$. It is a well known result by Jacobson and Rickart [21] (see also [20, page 2]), which simply amounts to calculating that $\big(\sigma(ab)-\sigma(a)\sigma(b)\big)\cdot\big(\sigma(ab)-\sigma(b)\sigma(a)\big) = 0$, that σ is either an automorphism or an anti-automorphism of $\mathbb{D}$. Now by equation (A.6), we have $(\varepsilon^{-1})\varphi = (\varepsilon\varphi^{-1})^{-1} = 1^{-1} = 1$, and hence $\sigma(\varepsilon^{-1}) = \varepsilon^{-1}$; since σ is an automorphism or anti-automorphism, it follows that $\sigma(\varepsilon) = \varepsilon$. Finally, again by equation (A.6), we obtain $\sigma(\varepsilon\sigma(a)) = \sigma(a\varphi) = \sigma\big(((a^{-1})\varphi^{-1})^{-1}\big) = \sigma((a^{-1})\varphi^{-1})^{-1} = (\varepsilon^{-1}a^{-1})^{-1} = a\varepsilon$ for all $a \in \mathbb{D}^*$. If σ is an automorphism, then this can be rewritten as $\varepsilon\sigma^2(a) = a\varepsilon$, or $\sigma^2(a) = \varepsilon^{-1}a\varepsilon$; if σ is an anti-automorphism, we get $\sigma^2(a)\varepsilon = a\varepsilon$, i.e. $\sigma^2 = 1$.

(ii) It suffices to check that equation (A.6) holds. This amounts to checking that $\varepsilon\sigma(a^{-1}) = (\sigma^{-1}(\varepsilon^{-1}a))^{-1}$ for all $a \in \mathbb{D}$. It is straightforward to check that this is valid in both cases. □

By [29] the flips of $\mathrm{SL}_2(\mathbb{D})$ are just the lifts of the flips of $\mathrm{PSL}_2(\mathbb{D})$.

A.3. Transitivity of the obvious flip

Definition A.15. If $\tau^2 = \mathrm{id}$, then $\varphi = 1$ is a flip automorphism. We will call the corresponding automorphism θ_1 of G (as defined in the beginning of this section) the *obvious flip*. Observe that θ_1 is just conjugation by τ.

Definition A.16. A flip automorphism $\varphi \in \mathrm{Aut}(U)$ is called *transitive* if the group $\mathrm{Fix}_G(\theta_\varphi)$ is transitive on X.

Let $\mathbb{M} = \mathbb{M}(U, \tau)$ be a Moufang set with $\tau^2 = \mathrm{id}$. Then the obvious flip θ_1 is transitive if and only if $C_G(\tau)$ is transitive on X, because $\mathrm{Fix}_G(\theta_1) = C_G(\tau)$.

Lemma A.17. *Let $\mathbb{M} = \mathbb{M}(U, \tau)$ be a Moufang set with $\tau^2 = \mathrm{id}$, and assume that the obvious flip is transitive. Then τ has no fixed points.*

Proof. Assume that $a\tau = a$ for some $a \in U^*$. Let $g \in C_G(\tau)$ be such that $0g = a$. Then $\infty g = 0\tau g = 0g\tau = a\tau = a = 0g$ and hence $\infty = 0$, a contradiction. □

We now examine the transitivity of the obvious flip for $\mathbb{M}(\mathbb{D})$ where $\mathbb{D}$ is an arbitrary skew field.

Definition A.18. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{GL}_2(\mathbb{D})$, then the Dieudonné determinant $\det(g) \in \mathbb{D}^*/[\mathbb{D}^*, \mathbb{D}^*]$ is defined as

$$\det(g) := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := \begin{cases} ad - aca^{-1}b & \text{if } a \neq 0\,; \\ -cb & \text{if } a = 0\,; \end{cases}$$

see [10]. Then $\mathrm{SL}_2(\mathbb{D})$ is precisely the kernel of the Dieudonné determinant, i.e. a matrix $g \in \mathsf{GL}_2(\mathbb{D})$ lies in $\mathrm{SL}_2(\mathbb{D})$ if and only if $\det(g) \in [\mathbb{D}^*, \mathbb{D}^*]$. Also observe that $\det(\lambda g) \equiv \det(g\lambda) \equiv \lambda^2 \det(g) \mod [\mathbb{D}^*, \mathbb{D}^*]$ for all $\lambda \in \mathbb{D}^*$.

Lemma A.19. *Let $G = \mathrm{SL}_2(\mathbb{D})$ and let $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G$. Then*

$$C_G(\tau) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; \begin{matrix} a^2 + aba^{-1}b \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a \neq 0 \\ b^2 \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a = 0 \end{matrix} \right\};$$

$$PC_G(\tau) = \left\{ \begin{pmatrix} \epsilon a & \epsilon b \\ -b & a \end{pmatrix} \;\middle|\; \begin{matrix} \epsilon\cdot(a^2 + aba^{-1}b) \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a \neq 0 \\ \epsilon\cdot b^2 \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a = 0 \end{matrix} \quad \text{where } \epsilon = \pm 1 \right\}.$$

Proof. This is a straightforward calculation. □

Proposition A.20. *Let $G = \mathrm{SL}_2(\mathbb{D})$ and let $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G$. Let X be the projective line over $\mathbb{D}$, i.e. $X = \{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\mathbb{D} \neq 0 \mid a, b \in \mathbb{D}\}$. Then the following are equivalent:*

(i) *$C_G(\tau)$ is transitive on X;*

(ii) $a^2 + aba^{-1}b \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ *for all* $a, b \in \mathbb{D}^*$;

(iii) $1 + a^2 \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ *for all* $a \in \mathbb{D}^*$.

Proof. Since $a^2 + aba^{-1}b = a^2(1 + a^{-1}ba^{-1}b)$, we have $a^2 + aba^{-1}b \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ if and only if $1 + a^{-1}ba^{-1}b \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$. Equivalence between (ii) and (iii) follows by replacing $a^{-1}b$ by a in the latter term.

Assume now that (ii) holds. Let $a, b \in \mathbb{D}^*$ be arbitrary; we want to show that there exists some $g \in C_G(\tau)$ mapping $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ for some $z \in \mathbb{D}^*$. By (ii), we know that there is some $c \in \mathbb{D}^*$ such that $b^{-2} + b^{-1}a^{-1}ba^{-1} \equiv c^{-2} \mod [\mathbb{D}^*, \mathbb{D}^*]$. Let $g := \left(\begin{smallmatrix} cb^{-1} & ca^{-1} \\ -ca^{-1} & cb^{-1} \end{smallmatrix}\right)$. Then $\det(g) \equiv c^2(b^{-2} + b^{-1}a^{-1}ba^{-1}) \equiv 1$, i.e. $g \in G$. Moreover, $g\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ for $z = c(b^{-1}a + a^{-1}b)$, proving that $C_G(\tau)$ acts transitively on X.

Conversely, assume that $C_G(\tau)$ acts transitively on X. Let $a, b \in \mathbb{D}^*$ be arbitrary; then there exists some $g \in C_G(\tau)$ mapping $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\mathbb{D}$ to $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\mathbb{D}$, i.e. there is some $z \in \mathbb{D}^*$ such that g maps $\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. By Lemma A.19, we know that g has the form $g = \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$ with $x^2 + xyx^{-1}y \in [\mathbb{D}^*, \mathbb{D}^*]$. Then $g\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} xz \\ -yz \end{smallmatrix}\right)$, and hence $a = xz$ and $b = -yz$. Hence $a^2 + aba^{-1}b = xzxz + xzyx^{-1}yz = xzx^{-1} \cdot (x^2 + xyx^{-1}y) \cdot z$, and since $x^2 + xyx^{-1}y \in [\mathbb{D}^*, \mathbb{D}^*]$, this implies $a^2 + aba^{-1}b \equiv xzx^{-1}z \equiv z^2 \mod [\mathbb{D}^*, \mathbb{D}^*]$. Since $a, b \in \mathbb{D}^*$ were arbitrary, this proves (ii). ☐

Proposition A.21. *Let* $G = \mathrm{PSL}_2(\mathbb{D})$, *let* $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{D})$, *and let* $\tilde{\tau}$ *be the image of* τ *in* G. *Let* X *be the projective line over* $\mathbb{D}$, *i.e.* $X = \{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\mathbb{D} \mid a, b \in \mathbb{D}, \text{not both zero}\}$. *Then the following are equivalent:*

(i) $C_G(\tilde{\tau})$ *is transitive on* X;

(ii) $PC_G(\tau)$ *is transitive on* X;

(iii) $a^2 + aba^{-1}b \in \{\pm 1\} \cdot (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ *for all* $a, b \in \mathbb{D}^*$;

(iv) $1 + a^2 \in \{\pm 1\} \cdot (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ *for all* $a \in \mathbb{D}^*$.

Proof. The equivalence between (i) and (ii) follows immediately from the definition of the projective centralizer $PC_G(\tau)$. The other equivalences are shown exactly as in the proof of Proposition A.20 above. ☐

Corollary A.22. (i) *Let* $G = \mathrm{SL}_2(\mathbb{D})$, *and assume that for all* $a \in \mathbb{D}^*$, *we have* $1 + h_a \in H$. *Then* $C_G(\tau)$ *acts transitively on* X.

(ii) *Let* $G = \mathrm{PSL}_2(\mathbb{D})$, *and assume that for all* $a \in \mathbb{D}^*$, *we have* $1 + h_a \in \{\pm 1\} \cdot H$. *Then* $C_G(\tilde{\tau})$ *acts transitively on* X.

Proof. We only show (i). The proof of (ii) is completely similar. So let $a \in \mathbb{D}^*$ be arbitrary, and assume that $1 + h_a = h \in H$. Then $1 + 1h_a = 1h$, i.e. $1 + a^2 = 1h$. Write $h = h_{x_1} \cdots h_{x_n}$ with $x_1, \ldots, x_n \in \mathbb{D}^*$. Then $1h = x_n \cdots x_1 \cdot 1 \cdot x_1 \cdots x_n \equiv (x_1 \cdots x_n)^2 \mod [\mathbb{D}^*, \mathbb{D}^*]$, and hence $1 + a^2 = 1h \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$. So (iii) of Proposition A.20 holds, and therefore the group $C_G(\tau)$ acts transitively on X. ☐

A natural extension of the study of the obvious flip would be to study its close relatives, the *semi-obvious* flips, which are obtained by composing the obvious flip with a field (anti-)automorphism.

References

[1] P. Abramenko, B. Mühlherr, Présentation des certaines BN-paires jumelées comme sommes amalgamées, *C. R. Acad. Sci. Paris Sér. I Math.* **325** (1997), 701–706.

[2] K. Altmann, *Centralisers of fundamental subgroups*, PhD thesis, TU Darmstadt 2007, `http://www.mathematik.tu-darmstadt.de/~gramlich/docs/kristinadiss.pdf`.

[3] V. Balan, J. Dorfmeister, Birkhoff decompositions and Iwasawa decompositions for loop groups, *Tohoku Math. J.* **53** (2001), 593–615.

[4] D. Beltita, Iwasawa decompositions of some infinite-dimensional Lie groups, arXiv:math/0701404v1.

[5] C. Bennett, R. Gramlich, C. Hoffman, S. Shpectorov, Curtis-Phan-Tits theory. In: *Groups, Combinatorics and Geometry*, Proceedings of the Durham Conference on Groups and Geometry 2001 (edited by Ivanov, Liebeck, Saxl), World Scientific, New Jersey 2003, 13–29.

[6] K.S. Brown, *Buildings*, Springer, Berlin 1989.

[7] P.-E. Caprace, *Groups with a root group datum*, Lecture Notes, University of Oxford 2007, `http://homepages.ulb.ac.be/~pcaprace/papers_pdf/root_data.pdf`.

[8] P.-E. Caprace, B. Mühlherr, Isomorphisms of Kac-Moody groups, *Invent. Math.* **161** (2005), 361–388.

[9] T. De Medts, R. M. Weiss, Moufang sets and Jordan division algebras, *Math. Ann.* **335** (2006), 415–433.

[10] J. Dieudonné, Les déterminant sur un corps non-commutatif, *Bull. Soc. Math. France* **71** (1943), 171–180.

[11] J. Diller, A. Dress, Zur Galoistheorie pythagoräischer Körper, *Arch. Math.* **16** (1965), 148–152.

[12] H. Glöckner, R. Gramlich, T. Hartnick, Final group topologies, Phan systems, and Pontryagin duality, submitted, arXiv:math/0603537v2.

[13] D. Gorenstein, R. Lyons, R. Solomon, *The classification of the finite simple groups. Number 2. Part I. Chapter G. General group theory*, American Mathematical Society, Providence 1995.

[14] R. Gramlich, *Homomorphisms of generalized hexagons*, Master's Thesis, Universität Würzburg 1998.

[15] R. Gramlich, Defining amalgams of compact Lie groups, *J. Lie Theory* **16** (2006), 1–18.

[16] R. Gramlich, Developments in Phan theory, submitted, `http://www.mathematik.tu-darmstadt.de/~gramlich/docs/survey2.pdf`.

[17] F. Grosshans, Semi-simple algebraic groups defined over a real closed field, *Am. J. Math.* **94** (1972), 473–485.

[18] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York 1978.

[19] A. G. Helminck, Shu Ping Wang, On rationality properties of involutions of reductive groups, *Adv. Math.* **99** (1993), 26–96.

[20] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society, Providence 1968.

[21] N. Jacobson, C.E. Rickart, Jordan homomorphisms of rings, *Trans. Amer. Math. Soc.* **69** (1950), 479–502.

[22] B. Krötz, A novel characterization of the Iwasawa decomposition of a simple Lie group, arXiv:0705.1279v1.

[23] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin, Reading 1973.

[24] T. Y. Lam, *Introduction to quadratic forms over fields*, American Mathematical Society, Providence 2005.

[25] B. Mühlherr, On the simple connectedness of a chamber system associated to a twin building, unpublished note, 1996.

[26] O.T. O'Meara, *Introduction to quadractic forms*, Springer, Berlin 1973.

[27] A. Pasini, Some remarks on covers and apartments, in: *Finite geometries* (edited by C.A. Baker, L.M. Batten), Dekker, New York 1985, 223–250.

[28] K.C. Pati, D. Parashar, Iwasawa decompositions of affine Kac-Moody algebras using Satake diagrams, *J. Math. Phys.* **39** (1998), 5015–5023.

[29] Ren Hong Shu, Wan Zhe Xian, Wu Xiao Long, Automorphisms of $PSL(2, K)$ over skew fields, *Acta Mathematica Sinica (N.S.)* **3** (1987), no.1, 45–53.

[30] B. Rémy, Groupes de Kac-Moody déployés et presque déployés, *Astérisque* **277** (2002).

[31] B. Rémy, M. Ronan, Topological groups of Kac-Moody type, right-angled twinnings and their lattices, *Comment. Math. Helv.* **81** (2006), no. 1, 191–219.

[32] A. R. Rijwade, *Squares*, Cambridge University Press, Cambridge 1993.

[33] M. Ronan, *Lectures on buildings*, Academic Press, Boston 1989.

[34] T. A. Springer, *Linear algebraic groups*, Birkhäuser, Basel 1998.

[35] K. Szymiczek, Quadratic forms over fields with finite square class number, *Acta Arith.* **28** (1975), 195–221.

[36] J. Tits, *Buildings of spherical type and finite BN-pairs*, Springer, Berlin 1974.

[37] J. Tits, Ordonnés, immeubles et sommes amalgamées, *Bull. Soc. Math. Belg. Sér. A* **38** (1986), 367–387.

[38] J. Tits, Uniqueness and presentation of Kac-Moody groups over fields, *J. Algebra* **105** (1987), 542–573.

[39] J. Tits, Twin buildings and groups of Kac-Moody type, in: *Groups, Combinatorics and Geometry* (edited by M.W. Liebeck, J. Saxl), Cambridge 1992, 249 – 286.

[40] R. Weiss, *The structure of spherical buildings*, Princeton University Press, Princeton 2003.

Authors' addresses:

Tom De Medts
Department of Pure Mathematics and Computer Algebra
Ghent University
Krijgslaan 281, S22
B-9000 Gent
Belgium
e-mail: `tdemedts@cage.UGent.be`

Ralf Gramlich, Max Horn
TU Darmstadt

Fachbereich Mathematik / AG 5
Schloßgartenstraße 7
64289 Darmstadt
Germany
e-mail: gramlich@mathematik.tu-darmstadt.de
mhorn@mathematik.tu-darmstadt.de

Second author's alternative address:
University of Birmingham
School of Mathematics
Edgbaston
Birmingham
B15 2TT
United Kingdom
e-mail: ralfg@maths.bham.ac.uk