The valuations of the near $2n$-gon $\mathbb{I}_n$

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Abstract

The maximal and next-to-maximal subspaces of a nonsingular parabolic quadric $Q(2n, 2)$, $n \geq 2$, which are not contained in a given hyperbolic quadric $Q^+(2n-1, q) \subset Q(2n, q)$ define a sub near polygon $\mathbb{I}_n$ of the dual polar space $DQ(2n, 2)$. It is known that every valuation of $DQ(2n, 2)$ induces a valuation of $\mathbb{I}_n$. In this paper, we show that also the converse is true: every valuation of $\mathbb{I}_n$ is induced by a valuation of $DQ(2n, 2)$. We will also study the structure of the valuations of $\mathbb{I}_n$.

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1 Introduction

1.1 Basic definitions

Let $\mathcal{S}$ be a dense near $2n$-gon, i.e., $\mathcal{S}$ satisfies the following properties:

(i) For every point p and every line L, there exists a unique point on L nearest to p. Here, distances $\mathrm{d}(\cdot, \cdot)$ are measured in the point graph or collinearity graph of $\mathcal{S}$.

(ii) Every line of $\mathcal{S}$ is incident with at least three points.

(iii) Every two points of $\mathcal{S}$ at distance 2 from each other have at least two common neighbours.

(iv) The maximal distance between two points of $\mathcal{S}$ is equal to n.

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A dense near 0-gon is a point, a dense near 2-gon is a line and a dense near quadrangle is a generalized quadrangle ([9]). By Theorem 4 of [1], every two points x and y of $\mathcal{S}$ at distance $\delta \in \{0, \dots, n\}$ from each other are contained in a unique convex subspace $\langle x, y \rangle$ of diameter δ. These convex subspaces are called *quads*, *hexes*, respectively *maxes*, if $\delta = 2$, $\delta = 3$, respectively $\delta = n - 1$. If X_1 and X_2 are two nonempty sets of points, then we denote by $\mathrm{d}(X_1, X_2)$ the minimal distance between a point of X_1 and a point of X_2. If $X_1, X_2, \dots, X_k$ are nonempty sets of points, then $\langle X_1, X_2, \dots, X_k \rangle$ denotes the smallest convex subspace containing $X_1 \cup X_2 \cup \cdots \cup X_k$, i.e., $\langle X_1, X_2, \dots, X_k \rangle$ is the intersection of all convex subspaces containing $X_1 \cup X_2 \cup \cdots \cup X_k$. A convex subspace F of a dense near polygon $\mathcal{S}$ is called *classical* in $\mathcal{S}$ if for every point x of $\mathcal{S}$, there exists a unique point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. The point $\pi_F(x)$ is called the *projection* of x onto F. We refer to Chapter 2 of [2] for more background information on dense near polygons.

A function f from the point-set $\mathcal{P}$ of $\mathcal{S}$ to $\mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties (we call $f(x)$ the value of x):

(V1) there exists at least one point with value 0;

(V2) every line L of $\mathcal{S}$ contains a unique point x_L with smallest value and $f(x) = f(x_L) + 1$ for every point x of L different from x_L;

(V3) every point x of $\mathcal{S}$ is contained in a convex subspace F_x such that the following properties are satisfied for every $y \in F_x$:

- (i) $f(y) \leq f(x)$;
- (ii) if z is a point collinear with y such that $f(z) = f(y) - 1$, then $z \in F_x$.

One can show, see Proposition 2.5 of [4], that the convex subspace F_x in property $(V3)$ is unique. If f is a valuation of $\mathcal{S}$, then we denote by O_f the set of points with value 0. A quad Q of $\mathcal{S}$ is called *special (with respect to)* f if it contains two distinct points of O_f, or equivalently (see [4]), if it intersects O_f in an ovoid of Q. We denote by G_f the partial linear space with points the elements of O_f and with lines the special quads (natural incidence).

Proposition 1.1 (Proposition 2.12 of [4]) *Let $\mathcal{S}$ be a dense near polygon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a (not necessarily convex) subpolygon of $\mathcal{S}$ for which the following holds:* (1) *F is a dense near polygon;* (2) *F is a subspace of $\mathcal{S}$;* (3) *if x and y are two points of F, then $d_F(x, y) = d_{\mathcal{S}}(x, y)$. Let f denote a valuation of $\mathcal{S}$ and put $m := \min\{f(x) \,|\, x \in \mathcal{P}'\}$. Then the map $f_F : \mathcal{P}' \to \mathbb{N}, x \mapsto f(x) - m$ is a valuation of F.*

Definition. The valuation f_F in Proposition 1.1 is called the valuation of F *induced* by f.

Proposition 1.2 (Proposition 2.4 of [3]) *Let f be a valuation of a dense near polygon, let M denote the maximal value attained by f, and let X denote the set of points with value M. Then $f(x) = M - d(x, X)$ for every point x of $\mathcal{S}$.*

Examples. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon.

(1) For every point x of $\mathcal{S}$, the map $f_x : \mathcal{P} \to \mathbb{N}; y \mapsto \mathrm{d}(x, y)$ is a valuation of $\mathcal{S}$ which we call a *classical valuation*.

(2) Suppose O is an ovoid of $\mathcal{S}$, i.e., a set of points meeting each line in a unique point. For every point x of $\mathcal{S}$, we define $f_O(x) = 0$ if $x \in O$ and $f_O(x) = 1$ otherwise. Then f_O is a valuation of $\mathcal{S}$, which we call an *ovoidal valuation*.

(3) Suppose $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a convex subspace of $\mathcal{S}$ which is classical in $\mathcal{S}$. Suppose that $f' : \mathcal{P}' \to \mathbb{N}$ is a valuation of F. Then the map $f : \mathcal{P} \to \mathbb{N}; x \mapsto f(x) := \mathrm{d}(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $\mathcal{S}$. We call f the extension of f'. If $\mathcal{P}' = \mathcal{P}$, then we say that the extension is *trivial*.

Valuations are a very important tool for classifying dense near polygons, see e.g. [7]. They are also important in the theory of hyperplanes of dense near polygons. With every valuation of a dense near polygon, there is associated a hyperplane, see Proposition 2 of [5]. Also in [5], valuations have been used to construct new hyperplanes of dual polar spaces. In [3], valuations have been used as a tool for classifying hyperplanes.

1.2 The dual polar space $DQ(2n, 2)$

Let $Q(2n, 2)$ denote a nonsingular (parabolic) quadric in $\mathrm{PG}(2n, 2)$, $n \geq 1$. The dual polar space $DQ(2n, 2)$ is the point-line geometry with points, respectively lines, the maximal, respectively next-to-maximal, subspaces of $Q(2n, 2)$ with reverse containment as incidence relation. $DQ(2n, 2)$ is a dense near $2n$-gon. By convention, $DQ(0, 2)$ is a point. If α is a subspace of $Q(2n, 2)$, then the set of all generators through α defines a convex subspace of $DQ(2n, 2)$. Conversely, every convex subspace of $DQ(2n, 2)$ is obtained in this way.

The dual polar space $DQ(4n, 2)$, $n \in \mathbb{N}$, admits so-called SDPS-sets. An *SDPS-set* of $DQ(4n, 2)$ is a set X of points satisfying: (i) no two points of X are collinear; (ii) if $x, y \in X$ such that $\mathrm{d}(x, y) = 2$, then $X \cap \langle x, y \rangle$ is an ovoid of the quad $\langle x, y \rangle$; (iii) the point-line geometry $\mathcal{A}$ whose points are the elements of X and whose lines are the quads of $DQ(4n, 2)$ containing at least two points of X (natural incidence) is isomorphic to $DQ(2n, 4)$; (iv) if x and y are two points of X, then the distance between x and y in $DQ(4n, 2)$ is twice the distance between x and y in the geometry $\mathcal{A}$. If X is an SDPS-set of $DQ(4n, 2)$, then every line of $DQ(4n, 2)$ through a point of X is contained in a unique quad which intersects X in an ovoid.

If X is an SDPS-set of $DQ(4n, 2)$, then by Theorem 4 of [5], the map $f : DQ(4n, 2) \to \mathbb{N}; x \mapsto \mathrm{d}(x, X)$ is a valuation of $DQ(4n, 2)$, a so-called *SDPS-valuation*. The following proposition describes the structure of the valuations of $DQ(2n, 2)$.

Proposition 1.3 (Corollary 2.6 of [8]) *If f is a valuation of $DQ(2n, 2)$, $n \geq 0$, then f is the possibly trivial extension of an SDPS-valuation in a convex subspace (of even diameter) of $DQ(2n, 2)$.*

Remark. SDPS-sets and SDPS-valuations can be defined for general thick dual polar spaces, see De Bruyn and Vandecasteele [5] or Chapter 5 of De Bruyn [2]. SDPS-sets in thick dual polar spaces of rank 4 were also studied by Pralle and Shpectorov [10].

1.3 The near $2n$-gon $\mathbb{I}_n$ and the results

Again, let $Q(2n, 2)$, $n \geq 2$, be a nonsingular parabolic quadric of $\mathrm{PG}(2n, 2)$ and let Π be a hyperplane of $\mathrm{PG}(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1, 2)$. The maximal subspaces of $Q(2n, 2)$ which are not contained in Π form a hyperplane of $DQ(2n, 2)$, i.e., a proper subspace meeting each line of $DQ(2n, 2)$. The geometry induced on this hyperplane is a dense near $2n$-gon which we will denote by $\mathbb{I}_n$. Every point of $\mathbb{I}_n$ is contained in a unique line of $DQ(2n, 2)$ which is not contained in $\mathbb{I}_n$. The generalized quadrangle $\mathbb{I}_2$ is isomorphic to the (3×3)-grid.

Let α be a subspace of $Q(2n, 2)$ which is not contained in $Q^+(2n-1, 2)$ if $\delta := \dim(\alpha) \in \{n-2, n-1\}$. Then the set of generators through α not contained in $Q^+(2n-1, 2)$ is a convex subspace A_α of $\mathbb{I}_n$. Conversely, every convex subspace is obtained in this way. If $\delta \leq n-3$ and $\alpha \subset Q^+(2n-1, 2)$, then $A_\alpha \cong \mathbb{I}_{n-1-\delta}$. If $\delta \leq n-3$ and $\alpha \not\subset Q^+(2n-1, 2)$, then $A_\alpha \cong DQ(2\delta, 2)$.

The embedding of $\mathbb{I}_n$ in $DQ(2n, 2)$ is an isometric one. So, by Proposition 1.1, every valuation of $DQ(2n, 2)$ induces a valuation of $\mathbb{I}_n$. In this paper, we will prove that also the converse is true.

Theorem 1.4 (Section 2) *Every valuation f of $\mathbb{I}_n$, $n \geq 2$, is induced by a valuation f' of $DQ(2n, 2)$. If $n \geq 3$, then f' is uniquely determined by f.*

Theorem 1.4 has already been proved in [6, Section 8.4] for the case $n = 3$ and in [8] for the case $n = 4$. Theorem 1.4 is easy to prove if $n = 2$, but the uniqueness for f' is not necessarily true. If f is an ovoidal valuation of $\mathbb{I}_2$, then f is induced by a unique classical and a unique ovoidal valuation of $DQ(4,2)$.

In the present paper, we will also determine the structure of the valuations of $\mathbb{I}_n$. We will show in Proposition 3.5 that f is the (generalized) extension of a valuation in the convex subspace $\langle O_f \rangle$ of $\mathbb{I}_n$. We will also determine the structure of G_f. We will prove the following result.

Theorem 1.5 (Propositions 3.5 and 4.1) *If f is a valuation of $\mathbb{I}_n$, $n \geq 3$, then the incidence structure G_f is isomorphic to one of the following geometries:*

- *a point;*
- *the projective space* $\mathrm{PG}(n-1,2)$*;*
- *the dual polar space $DQ(2m,4)$ for some m satisfying $1 \leq m \leq \frac{n-1}{2}$;*
- *the partial linear space $DQ'(2m,4)$ for some m satisfying $1 \leq m \leq \frac{n}{2}$.*

In Theorem 1.5, $DQ(2m,4)$ is the dual polar space associated with the nonsingular parabolic quadric $Q(2m,4)$ of $\mathrm{PG}(2m,4)$ and $DQ'(2m,4)$ is the subgeometry of $DQ(2m,4)$ induced on the set of all generators which are not contained in a given hyperbolic quadric $Q^-(2m-1,4) \subseteq Q(2m,4)$.

It is our hope that Theorems 1.4 and 1.5 will contribute to the project of classifying all dense near polygons with three points per line. During classifications of dense near polygons, valuations play a very important role, see e.g. [7].

2 Proof of Theorem 1.4

We will make use of the following lemma.

Lemma 2.1 *Let x be a point of a dense near $2n$-gon, $n \geq 3$. Let Γ be the graph with vertices the hexes through x, two distinct hexes being adjacent whenever they intersect in a quad. Then Γ is connected.*

Proof. Let H_1 and H_2 be two hexes through x.

(a) If $H_1 = H_2$ or $H_1 \cap H_2$ is a quad, then H_1 and H_2 are connected by a path.

(b) Suppose $H_1 \cap H_2$ is a line L. Let L_i, $i \in \{1,2\}$, denote a line of H_i through x distinct from L. Then the hex $\langle L, L_1, L_2 \rangle$ is a common neighbour of H_1 and H_2 in the graph Γ.

(c) Suppose $H_1 \cap H_2$ is a point x. Let L_i, $i \in \{1,2\}$, denote a line through x contained in H_i, and let H_3 be a hex through L_1 and L_2. By (a)+(b), we know that there exists a path in Γ connecting H_3 and H_i, $i \in \{1,2\}$. Hence, also H_1 and H_2 are connected by a path. $\blacksquare$

We will prove Theorem 1.4 by induction on n.

Suppose first that $n = 2$. Then the embedding of $\mathbb{I}_2$ in $DQ(4,2)$ is just the embedding of the (3×3)-grid in the generalized quadrangle $W(2)$. Every valuation of a generalized quadrangle is either classical or ovoidal by Corollary 2.11 of [4]. Every classical valuation f of $\mathbb{I}_2$ is induced by a unique valuation f' of $DQ(4,2)$. The valuation f' is classical and $O_{f'} = O_f$. Every ovoidal valuation g of $\mathbb{I}_2$ is induced by a unique classical valuation g_1 of $DQ(4,2)$ and a unique ovoidal valuation g_2 of $DQ(4,2)$. The point in O_{g_1} is the unique point of $DQ(4,2) \setminus \mathbb{I}_2$ collinear with all points of O_g, and the ovoid O_{g_2} is the unique ovoid of $DQ(4,2)$ containing O_g.

The main theorem has already been proved in [6, Section 8.4] for the case $n = 3$ and in [8] for the case $n = 4$.

Suppose now that $n \geq 5$ and that the main theorem holds for every near $2m$-gon $\mathbb{I}_m$ with $2 \leq m \leq n-1$. Let f be a valuation of $\mathbb{I}_n$. We will regard $\mathbb{I}_n$ as a sub-near-polygon of $DQ(2n,2)$. The embedding of $\mathbb{I}_n$ in $DQ(2n,2)$ is an isometric one. Convex subspaces of diameter 2, respectively 3, of $\mathbb{I}_n$ will be called *quads* and *hexes*, respectively. Convex subspaces of diameter 2, respectively 3, of $DQ(2n,2)$ will be called QUADS and HEXES, respectively.

Definition. Let F denote a convex subspace of $DQ(2n,2)$ and suppose that the diameter δ of F satisfies $3 \leq \delta \leq n-1$. By the induction hypothesis, there exists a unique function $\overline{f}_F$ from F to $\mathbb{Z}$ satisfying the following properties:

(i) $\overline{f}_F(y) = f(y)$ for every point y of $F \cap \mathbb{I}_n$;

(ii) if ϵ is the minimal value attained by $\overline{f}_F$, then the map $F \to \mathbb{N}; y \mapsto \overline{f}_F(y) - \epsilon$ is a valuation of F.

Lemma 2.2 *Let F_1 and F_2 denote two convex subspaces of $DQ(2n,2)$ such that $F_1 \subseteq F_2$. Let δ_i, $i \in \{1,2\}$, denote the diameter of F_i and suppose that $3 \leq \delta_1 \leq \delta_2 \leq n-1$. Then $\overline{f}_{F_1}(x) = \overline{f}_{F_2}(x)$ for every point x of F_1.*

Proof. Let ϵ_2 denote the minimal value of $\overline{f}_{F_2}$ and let f_{F_2} denote the valuation of F_2 mapping each point x of F_2 to $\overline{f}_{F_2}(x) - \epsilon_2$. Put $\epsilon_1 :=$

$\min\{f_{F_2}(x) \,|\, x \in F_1\}$. By Proposition 1.1, the map $f_{F_1} : F_1 \to \mathbb{N}; x \mapsto f_{F_2}(x) - \epsilon_1$ is a valuation of F_1. Let f'_{F_1} denote the map $F_1 \to \mathbb{N}; x \mapsto f_{F_1}(x) + \epsilon_1 + \epsilon_2$. Then for every $x \in F_1 \cap \mathbb{I}_n$,

$$f'_{F_1}(x) = f_{F_1}(x) + \epsilon_1 + \epsilon_2 = f_{F_2}(x) + \epsilon_2 = \overline{f}_{F_2}(x) = f(x).$$

Since f_{F_1} is a valuation of F_1, the minimal value of f'_{F_1} is equal to $\epsilon_1 + \epsilon_2$. It readily follows that $\overline{f}_{F_1} = f'_{F_1}$. Now, for every $x \in F_1$,

$$\overline{f}_{F_1}(x) = f'_{F_1}(x) = f_{F_1}(x) + \epsilon_1 + \epsilon_2 = f_{F_2}(x) + \epsilon_2 = \overline{f}_{F_2}(x).$$

This proves the lemma. ∎

Lemma 2.3 *Let x be a point of $DQ(2n,2) \setminus \mathbb{I}_n$ and let H_1 and H_2 be two HEXES of $DQ(2n,2)$ through x. Then $\overline{f}_{H_1}(x) = \overline{f}_{H_2}(x)$.*

Proof. By Lemma 2.1, it suffices to prove the lemma in the case that H_1 and H_2 intersect in a quad. Let F denote the sub near octagon $\langle H_1, H_2 \rangle$ of $DQ(2n,2)$. By Lemma 2.2, $\overline{f}_{H_1}(x) = \overline{f}_F(x) = \overline{f}_{H_2}(x)$. (Recall that $n \geq 5$.) ∎

Define now the following map $\overline{f}$ from the point-set of $DQ(2n,2)$ to $\mathbb{Z}$:

- if $x \in \mathbb{I}_n$, then $\overline{f}(x) = f(x)$;
- if $x \in DQ(2n,2) \setminus \mathbb{I}_n$, then $\overline{f}(x) = \overline{f}_H(x)$, where H is any hex through x.

We will now show that the map $\overline{f}$ satisfies the properties (V2) and (V3) in the definition of valuation.

Lemma 2.4 *The map $\overline{f}$ satisfies property $(V2)$.*

Proof. Let L denote an arbitrary line of $DQ(2n,2)$. If L is a line of $\mathbb{I}_n$, then there exists a unique point x_L on L such that $f(x_L) = f(y) - 1$ for every $y \in L \setminus \{x_L\}$. Hence, $\overline{f}(x_L) = \overline{f}(y) - 1$ for every $y \in L \setminus \{x_L\}$.

Suppose L is not a line of $\mathbb{I}_n$ and let H denote an arbitrary HEX through L. Then there exists a constant ϵ such that the map $x \mapsto \overline{f}_H(x) + \epsilon$ is a valuation of H. Hence, there exists a unique point x_L on L such that $\overline{f}_H(x_L) = \overline{f}_H(y) - 1$ for every $y \in L \setminus \{x_L\}$. It follows that $\overline{f}(x_L) = \overline{f}_H(x_L) = \overline{f}_H(y) - 1 = \overline{f}(y) - 1$ for every $y \in L \setminus \{x_L\}$. ∎

For every point x of $DQ(2n,2)$, let $\mathcal{L}_x$ denote the linear space with points, respectively lines, the lines, respectively quads, through x. Then $\mathcal{L}_x$ is isomorphic to the point-line system of $\mathrm{PG}(n-1,2)$. For every point x of $DQ(2n,2)$, let S_x denote the set of lines through x containing a point with value $\overline{f}(x) - 1$.

Lemma 2.5 *The set S_x is a subspace of $\mathcal{L}_x$.*

Proof. Suppose L_1 and L_2 are two distinct lines through x belonging to S_x and that L_3 is a line of the quad $\langle L_1, L_2 \rangle$ through x. Let H denote an arbitrary HEX through $\langle L_1, L_2 \rangle$. Let G_x denote the convex subspace of H through x which satisfies property (V3) with respect to the function $\overline{f}_H$. Since L_1 and L_2 contain points with $\overline{f}_H$-value $\overline{f}_H(x) - 1$, $L_1, L_2 \subseteq G_x$. Hence also $L_3 \subseteq G_x$. So, L_3 contains a unique point with $\overline{f}_H$-value $\overline{f}_H(x) - 1$. The lemma now readily follows. ∎

For every point x of $DQ(2n, 2)$, let F_x denote the unique convex subspace of $DQ(2n, 2)$ through x such that the lines of F_x through x are precisely the lines of S_x.

We will now show that $\overline{f}$ satisfies property (V3) with respect to the convex subspaces F_x.

Lemma 2.6 *Suppose x is a point of $DQ(2n, 2)$ such that $F_x = DQ(2n, 2)$. Then F_x satisfies property $(V3)$.*

Proof. Let X denote the set of points with $\overline{f}$-value at most $\overline{f}(x)$. We must show that X coincides with the whole point set of $DQ(2n, 2)$. By Lemma 2.4, X is a subspace of $DQ(2n, 2)$. Let F denote an arbitrary convex subspace of diameter $n - 1$ through x. Then there exists a constant ϵ such that the map $F \to \mathbb{Z}; y \mapsto \overline{f}_F(y) + \epsilon$ is a valuation of F. Let F'_x denote the convex subspace through x which satisfies property (V3) with respect to this valuation. Every line of F through x contains a point with $\overline{f}_F$-value $\overline{f}_F(x) - 1$. Hence, $F'_x = F$. It follows that $\overline{f}(y) = \overline{f}_F(y) \leq \overline{f}_F(x) = \overline{f}(x)$ for every point y of F. So, every point of $DQ(2n, 2)$ at distance at most $n - 1$ from x belongs to X, i.e., $H_x \subseteq X$. Here, H_x denotes the so-called singular hyperplane with deepest point x which consists of all points of $DQ(2n, 2)$ at non-maximal distance from x.

Now, let L denote an arbitrary line through x. Then L contains a unique point with value $\overline{f}(x) - 1$. Let x' denote the third point on that line. By Lemma 2.4, $\overline{f}(x') = \overline{f}(x)$. Let L' denote an arbitrary line through x'. Every point of L' has distance at most 2 from x and hence has value at most $\overline{f}(x) = \overline{f}(x')$ by the previous paragraph. By Lemma 2.4, L' contains a unique point with value $\overline{f}(x') - 1$. It follows that $F_{x'} = DQ(2n, 2)$. As before, we can conclude that the singular hyperplane $H_{x'}$ with deepest point x' is contained in X.

Now, by Lemma 6.1 of Shult [11], the singular hyperplanes H_x and $H_{x'}$ are maximal subspaces of $DQ(2n, 2)$. Since $H_x \cup H_{x'} \subseteq X$ and since X is a subspace, it follows that X coincides with the whole point-set of $DQ(2n, 2)$. ∎

Lemma 2.7 *Suppose x is a point of $DQ(2n,2)$ such that F_x is a proper convex subspace of $DQ(2n,2)$, then $\overline{f}(y) \leq \overline{f}(x)$ for every point y of F_x.*

Proof. Let F denote a convex subspace of diameter $n-1$ through F_x. There exists a constant ϵ such that the map $F \to \mathbb{Z}; y \mapsto \overline{f}_F(y) + \epsilon$ is a valuation of F. Let F'_x denote the convex subspace of F through x which satisfies property (V3) with respect to this valuation. The lines of F through x containing a point with $\overline{f}_F$-value $\overline{f}_F(x) - 1$ are precisely the lines of F_x through x. It follows that $F'_x = F_x$. Hence, $\overline{f}(y) = \overline{f}_F(y) \leq \overline{f}_F(x) = \overline{f}(x)$ for every point y of F_x. ■

Lemma 2.8 *Suppose x is a point of $DQ(2n,2)$ such that F_x is a proper convex subspace of $DQ(2n,2)$. Let y and z be points of $DQ(2n,2)$ such that $y \in F_x$, $d(y,z) = 1$ and $\overline{f}(z) = \overline{f}(y) - 1$. Then $z \in F_x$.*

Proof. Put $k := \overline{f}(x) - \overline{f}(y)$. By Proposition 1.2 applied to the valuation of F_x induced by $\overline{f}$, there exists a path $u_0, u_1, \ldots, u_k$ of length k between a point $u_0 \in F_x$ with $\overline{f}$-value $\overline{f}(x)$ and the point $u_k = y$.

By Lemmas 2.4 and 2.7, every line of F_x through u_0 contains a point with $\overline{f}$-value $\overline{f}(x) - 1 = \overline{f}(u_0) - 1$. It follows that $F_x \subseteq F_{u_0}$. Since $x \in F_{u_0}$ with $\overline{f}(x) = \overline{f}(u_0)$, we can apply the same reasoning again (use also Lemma 2.6 for the case $F_{u_0} = DQ(2n,2)$) and we find that $F_{u_0} \subseteq F_x$. So, $F_x = F_{u_0}$.

Suppose z were not contained in F_x. Define inductively the following path $v_0, v_1, \ldots, v_k$ of points:

- $v_k = z$;
- v_i, $i \in \{0, \ldots, k-1\}$, is a common neighbour of u_i and v_{i+1} different from u_{i+1}.

One readily verifies by induction that $v_i \notin F_x$ for every $i \in \{0, \ldots, k\}$. In particular, $v_0 \notin F_x$. By Lemma 2.4 and the fact that $F_{u_0} = F_x$, $\overline{f}(v_0) = \overline{f}(u_0)+1$. Hence, $\overline{f}(v_0) - \overline{f}(v_k) = \overline{f}(u_0)+1-\overline{f}(z) = (\overline{f}(x)+1)-(\overline{f}(y)-1) = k+2$. On the other hand, $\overline{f}(v_0) - \overline{f}(v_k) = (\overline{f}(v_0) - \overline{f}(v_1)) + (\overline{f}(v_1) - \overline{f}(v_2)) + \cdots + (\overline{f}(v_{k-1}) - \overline{f}(v_k)) \leq k$ by Lemma 2.4. So, our assumption $z \notin F_x$ was wrong. This proves the lemma. ■

By Lemmas 2.6, 2.7 and 2.8, we obtain:

Corollary 2.9 *The function $\overline{f}$ satisfies property $(V3)$ with respect to the convex subspaces F_x.*

Now, let $\epsilon \in \{-1, 0\}$ denote the minimal value attained by $\overline{f}$. For every point x of $DQ(2n,2)$, we define $f'(x) = \overline{f}(x) - \epsilon$. Then f' satisfies properties

(V1), (V2), (V3) and hence is a valuation of $DQ(2n,2)$. Obviously, f is induced by f'. It is also clear from the construction that f' is the unique valuation of $DQ(2n,2)$ inducing f. This proves Theorem 1.4.

3 Extensions of valuations

Consider the near $2n$-gon $\mathbb{I}_n$, $n \geq 2$. Suppose as in Section 2 that $\mathbb{I}_n$ is isometrically embedded in $DQ(2n,2)$.

Definition. A *projective set* of $\mathbb{I}_n$ is a nonempty set X of points satisfying the following properties:

(i) if x_1 and x_2 are two points of X, then $\mathrm{d}(x_1,x_2)=2$ and the quad $\langle x_1,x_2\rangle$ of $\mathbb{I}_n$ containing x_1 and x_2 intersects X in an ovoid;

(ii) the incidence structure with points the elements of X and with lines the quads of $\mathbb{I}_n$ containing three points of X is isomorphic to the point-line system of $\mathrm{PG}(n-1,2)$.

If x is a point of $DQ(2n,2)\setminus\mathbb{I}_n$, then $x^\perp\cap\mathbb{I}_n$ is a projective set. Conversely, if X is a projective set of $\mathbb{I}_n$, then there exists a unique point $x\in DQ(2n,2)\setminus\mathbb{I}_n$ such that $X=x^\perp\cap\mathbb{I}_n$. We refer to Section 8.2 of [6] for more details on projective sets.

Lemma 3.1 *Let x be a point of $DQ(2n,2)\setminus\mathbb{I}_n$ and let X be the projective set $x^\perp\cap\mathbb{I}_n$ of $\mathbb{I}_n$. Then for every point y of $\mathbb{I}_n$, $d(X,y)=d(x,y)-1$.*

Proof. Since $\mathrm{d}(x,X)=1$, $\mathrm{d}(X,y)\geq\mathrm{d}(x,y)-1$. We will now show that $\mathrm{d}(x,y)-1\geq\mathrm{d}(X,y)$ for every point y of $\mathbb{I}_n$. Let F denote a convex subspace of $\mathbb{I}_n$ through y isomorphic to $DQ(2n-2,2)$. Then $\mathrm{d}(x,y)-1=\mathrm{d}(\pi_F(x),y)$. Since $\pi_F(x)\in X$, we necessarily have $\mathrm{d}(x,y)-1\geq\mathrm{d}(X,y)$. This proves the lemma. ■

Let f be a valuation of $\mathbb{I}_n$. If $n\geq 3$ or ($n=2$ and f classical), then by Theorem 1.4 there exists a unique map $\overline{f}$ from the point-set of $DQ(2n,2)$ to $\mathbb{Z}$ satisfying the following properties:

(i) $\overline{f}(x)=f(x)$ for every point x of $\mathbb{I}_n$;

(ii) there exists a constant $\epsilon\in\{0,1\}$ such that the map $x\mapsto\overline{f}(x)+\epsilon$ is a valuation of $DQ(2n,2)$.

If $n=2$ and f ovoidal, then there exists a unique ovoidal valuation $\overline{f}$ of $DQ(4,2)$ such that $\overline{f}(x)=f(x)$ for every point x of $\mathbb{I}_2$.

The valuation f is a map from the point-set P of $\mathbb{I}_n$ to $\mathbb{N}$. Let P' denote the set of all projective sets of $\mathbb{I}_n$. We extend f to a map from $P \cup P'$ to $\mathbb{Z}$, mapping $x \in P$ to $f(x)$ and $X \in P'$ to $\overline{f}(x) - 1$, where x is the unique point of $DQ(2n, 2) \setminus \mathbb{I}_n$ for which $x^\perp \cap \mathbb{I}_n = X$. We will denote the extension of f to the set $P \cup P'$ also by f.

Proposition 3.2 *Let x be a point of $\mathbb{I}_n$ and let F denote a convex subspace of $\mathbb{I}_n$ of diameter $\delta \geq 2$. If $F \cong DQ(2\delta, 2)$, then there exists a unique point $\pi_F(x)$ in F nearest to x and $d(x, y) = d(x, \pi_F(x)) + d(\pi_F(x), y)$ for every point y of F. If $F \cong \mathbb{I}_\delta$, then there are two possibilities:*

- (a) *There exists a unique point $\pi_F(x)$ in F nearest to x and $d(x, y) = d(x, \pi_F(x)) + d(\pi_F(x), y)$ for every point y of F.*
- (b) *The points in F nearest to x form a projective set X. For every point y of F, we have $d(x, y) = d(x, X) + d(X, y)$.*

Proof. If $F \cong DQ(2\delta, 2)$, then F is classical in $\mathbb{I}_n$, since F is classical in $DQ(2n, 2)$. Hence, there exists a unique point $\pi_F(x)$ in F nearest to x and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. Suppose now that $F \cong \mathbb{I}_\delta$. Let $\overline{F} \cong DQ(2\delta, 2)$ denote the convex subspace of diameter δ of $DQ(2n, 2)$ containing F. Then there exists a unique point $\pi_{\overline{F}}(x)$ in $\overline{F}$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + \mathrm{d}(\pi_{\overline{F}}(x), y)$ for every point y of $\overline{F}$. If $\pi_{\overline{F}}(x) \in F$, then case (a) of the proposition occurs. Suppose now that $\pi_{\overline{F}}(x) \notin F$. Let X denote the set of points of F collinear with $\pi_{\overline{F}}(x)$. Then X is a projective set of F and is the set of points of F nearest to x. For every point y of F, $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + \mathrm{d}(\pi_{\overline{F}}(x), y) = \mathrm{d}(x, X) + \mathrm{d}(\pi_{\overline{F}}(x), y) - 1 = \mathrm{d}(x, X) + \mathrm{d}(X, y)$ by Lemma 3.1. So, we have case (b) of the proposition. ■

Definition If case (b) of Proposition 3.2 occurs, then we denote the projective set X also by $\pi_F(x)$.

Corollary 3.3 *If x is a point and if F is a convex subspace of $\mathbb{I}_n$, then $d(x, y) = d(x, \pi_F(x)) + d(\pi_F(x), y)$ for every point y of F.*

Proposition 3.4 *Let F denote a convex subspace of $\mathbb{I}_n$ isomorphic to $\mathbb{I}_m$ for some $m \geq 2$, and let f denote a valuation of $\mathbb{I}_m$. Extend f to the set of all projective sets of $\mathbb{I}_m$ as described above. For every point x of $\mathbb{I}_n$, define $f'(x) := d(x, \pi_F(x)) + f(\pi_F(x))$. Then f' is a valuation of $\mathbb{I}_n$.*

Proof. Let $\overline{F} \cong DQ(2m, 2)$ denote the convex subspace of diameter m of $DQ(2n, 2)$ containing F. If $m \geq 3$ or ($m = 2$ and f classical), then let $\overline{f}$ denote the unique map from $\overline{F}$ to $\mathbb{Z}$ satisfying the following properties:

(1) $\overline{f}(x) = f(x)$ for every point x of F;

(2) there exists a constant $\epsilon \in \{0, 1\}$ such that the map $x \mapsto \overline{f}(x) + \epsilon$ is a valuation of $\overline{F}$.

If $m = 2$ and f ovoidal, put $\epsilon = 0$ and let $\overline{f}$ denote the unique ovoidal valuation of $\overline{F}$ such that $\overline{f}(x) = f(x)$ for every point x of $\mathbb{I}_2$.

For every point x of $DQ(2n, 2)$, define $g(x) := \mathrm{d}(x, \pi_{\overline{F}}(x)) + \overline{f}(\pi_{\overline{F}}(x))$. Then the map $x \mapsto g(x) + \epsilon$ is a valuation $\widetilde{g}$ of $DQ(2n, 2)$. Let g' denote the valuation of $\mathbb{I}_n$ induced by $\widetilde{g}$. Then $g'(x) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + \overline{f}(\pi_{\overline{F}}(x))$ for every point x of $\mathbb{I}_n$. [We must show that the minimal value of the function $x \mapsto \mathrm{d}(x, \pi_{\overline{F}}(x)) + \overline{f}(\pi_{\overline{F}}(x))$, $x \in \mathbb{I}_n$, is equal to 0. If $x \in F$, then $\mathrm{d}(x, \pi_{\overline{F}}(x)) + \overline{f}(\pi_{\overline{F}}(x)) = f(x) \geq 0$ with equality if and only if $x \in O_f$. If $x \notin F$, then $\mathrm{d}(x, \pi_{\overline{F}}(x)) + \overline{f}(\pi_{\overline{F}}(x)) \geq 1 + (-1) = 0$.] If $\pi_{\overline{F}}(x) \in F$, then $g'(x) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + \overline{f}(\pi_{\overline{F}}(x)) = \mathrm{d}(x, \pi_F(x)) + f(\pi_F(x)) = f'(x)$. If $\pi_{\overline{F}}(x) \notin F$, then $g'(x) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + \overline{f}(\pi_{\overline{F}}(x)) = [\mathrm{d}(x, \pi_F(x)) - 1] + [f(\pi_F(x)) + 1] = \mathrm{d}(x, \pi_F(x)) + f(\pi_F(x)) = f'(x)$. This proves the proposition. ■

Definitions. (1) The valuation f' in Proposition 3.4 is called the *(generalized) extension* of the valuation f.

(2) A valuation f of a dense near $2n$-gon, $n \geq 0$, is said to have *property (O)* if O_f contains two opposite points, i.e., two points at maximal distance n from each other.

Proposition 3.5 *If f is a valuation of $\mathbb{I}_n$, $n \geq 3$, then precisely one of the following holds:*

(1) *f is a classical valuation;*

(2) *there exists a projective set X in $\mathbb{I}_n$ and $f(x) = d(x, X)$ for every point x of $\mathbb{I}_n$;*

(3) *there exists a convex subspace F of $\mathbb{I}_n$ isomorphic to $DQ(4m, 2)$ for some $m \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ and an SDPS-valuation g of F such that f is the extension of g;*

(4) *there exists a convex subspace F of $\mathbb{I}_n$ isomorphic to $\mathbb{I}_{2m}$ for some $m \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and a valuation g of F satisfying property (O) such that f is the extension of g.*

Proof. By Theorem 1.4, the valuation f is induced by a unique valuation f' of $DQ(2n, 2)$. Let $\overline{F}$ denote the convex subspace $\langle O_{f'} \rangle$ of $DQ(2n, 2)$. By Proposition 1.3, f' is the possibly trivial extension of an SDPS-valuation in $\overline{F}$. So, $\overline{F}$ has even diameter.

(1) Suppose that f' is classical and that the unique point x with f'-value 0 belongs to $\mathbb{I}_n$. Then f is classical and $O_f = \{x\}$. So, we have case (1) of the proposition.

(2) Suppose f' is classical and that the unique point x with f'-value 0 does not belong to $\mathbb{I}_n$. Then $x^\perp \cap \mathbb{I}_n$ is a projective set X and $f(y) = \mathrm{d}(x, y) - 1$ for every point y of $\mathbb{I}_n$. By Lemma 3.1, $f(y) = \mathrm{d}(X, y)$ for every point y of $\mathbb{I}_n$. So, case (2) of the proposition occurs.

(3) Suppose that $\overline{F} \cong DQ(4m, 2)$ is a convex subspace of diameter $2m \geq 2$ contained in $\mathbb{I}_n$. Let f'' denote the valuation of $\overline{F}$ associated with the SDPS-set $O_{f'}$. Then f' is the extension of f'', i.e., for every point x of $DQ(2n, 2)$, $f'(x) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + f''(\pi_{\overline{F}}(x))$. In particular, this equality holds for every point x of $\mathbb{I}_n$. Hence, the valuation f of $\mathbb{I}_n$ is also the extension of the valuation f'' of $\overline{F}$. So, case (3) of the proposition occurs.

(4) Suppose that $\overline{F}$ is a convex subspace of diameter $2m \geq 2$ not contained in $\mathbb{I}_n$. Let f'' denote the valuation of $\overline{F}$ associated with the SDPS-set $O_{f'}$. Then $f'(x) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + f''(\pi_{\overline{F}}(x))$ for every point x of $DQ(2n, 2)$.

Let f''' denote the valuation of $\overline{F} \cap \mathbb{I}_n \cong \mathbb{I}_{2m}$ induced by f''. We will show that f''' satisfies property (O). Let x_1 and x_2 be two points of $O_{f''}$ at distance $2m$ from each other, let y_1 and y_2 be points of $O_{f''}$ satisfying $\mathrm{d}(x_1, y_1) = \mathrm{d}(x_2, y_2) = 2$, $\mathrm{d}(x_1, y_2) = \mathrm{d}(x_2, y_1) = 2m - 2$ and $\mathrm{d}(y_1, y_2) = 2m$. (Such points exist since $G_{f''} \cong DQ(2m, 4)$.) It is easily seen that there exist two points $z_1 \in \langle x_1, y_1 \rangle \cap O_{f''} \cap \mathbb{I}_n$ and $z_2 \in \langle x_2, y_2 \rangle \cap O_{f''} \cap \mathbb{I}_n$ at distance $2m$ from each other. So, f''' satisfies property (O). Now, extend f''' in the natural way to projective sets.

Suppose now that x is a point of $\mathbb{I}_n$ such that $\pi_{\overline{F}}(x) \in \mathbb{I}_n$. Then $f(x) = f'(x) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + f''(\pi_{\overline{F}}(x)) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + f'''(\pi_{\overline{F}}(x))$.

Suppose now that x is a point of $\mathbb{I}_n$ such that $\pi_{\overline{F}}(x) \notin \mathbb{I}_n$. Let X denote the projective set $\pi_{\overline{F}}(x)^\perp \cap (\mathbb{I}_n \cap \overline{F})$ of $\mathbb{I}_n \cap \overline{F}$. Then $f(x) = f'(x) = \mathrm{d}(x, \pi_{\overline{F}}(x)) + f''(\pi_{\overline{F}}(x)) = [\mathrm{d}(x, X) - 1] + [f'''(X) + 1] = \mathrm{d}(x, X) + f'''(X)$. It follows that f is the extension of the valuation f''' of $\mathbb{I}_n \cap \overline{F}$. ■

If case (1) of Proposition 3.5 occurs, then G_f is a point. If case (2) occurs, then $G_f \cong \mathrm{PG}(n-1, 2)$. If case (3) occurs, then $G_f \cong DQ(2m, 4)$ (with the convention that $DQ(2, 4)$ is a line with 5 points). In the following section, we will determine G_f if f is a valuation as in case (4) of Proposition 3.5.

4 Valuations of $\mathbb{I}_{2n}$ satisfying property (O)

Consider in $\mathrm{PG}(2n, 4)$, $n \geq 1$, a nonsingular parabolic quadric $Q(2n, 4)$ and let Π be a hyperplane of $\mathrm{PG}(2n, 4)$ intersecting $Q(2n, 4)$ in a nonsingular hyperbolic quadric $Q^+(2n-1, 4)$. Let $DQ(2n, 4)$ denote the dual polar space associated with $Q(2n, 4)$ and let $DQ'(2n, 4)$ denote the incidence structure whose points, respectively lines, are the $(n-1)$-dimensional, respectively $(n-2)$-dimensional, subspaces of $Q(2n, 4)$ not contained in $Q^+(2n-1, 4)$ (natural incidence). In this section, we will prove the following result.

Proposition 4.1 *If f is a valuation of $\mathbb{I}_{2n}$, $n \geq 1$, satisfying property (O), then $G_f \cong DQ'(2n, 4)$.*

So, let f be a valuation of $\mathbb{I}_{2n}$, $n \geq 2$, satisfying property (O). By Theorem 1.4, the valuation f is induced by a valuation f' of $DQ(4n, 2)$. By (the proof of) Proposition 3.5, the valuation f' also satisfies property (O). By Proposition 1.3, the valuation f' arises from an SDPS-set of $DQ(4n, 2)$. So, $G_{f'} \cong DQ(2n, 4)$. In the sequel, we will regard the set $O_{f'}$ as the set of all generators of the quadric $Q(2n, 4)$. Then $O_f \subseteq O_{f'}$ is a certain set of generators of $Q(2n, 4)$.

Lemma 4.2 *The set $O_{f'} \setminus O_f$ is a convex set of points of $DQ(4n, 2)$. (But it is not a subspace!)*

Proof. Suppose the contrary. Then there exist points x_1, x_2 and x_3 such that $x_1, x_2 \in O_{f'} \setminus O_f$, $x_3 \in O_f$, $\mathrm{d}(x_1, x_3) = \mathrm{d}(x_1, x_2) - 2$ and $\mathrm{d}(x_2, x_3) = 2$. The convex subspaces $\langle x_1, x_3 \rangle$ and $\langle x_3, x_2 \rangle$ of $DQ(4n, 2)$ only intersect in the point x_3. Let L denote the unique line of $DQ(4n, 2)$ through x_3 not contained in $\mathbb{I}_{2n}$. Since $x_1 \in O_{f'} \setminus O_f$, $\langle x_1, x_3 \rangle$ is not a convex subspace of $\mathbb{I}_{2n}$ and it follows that $L \subseteq \langle x_1, x_3 \rangle$. Similarly, because $x_2 \in O_{f'} \setminus O_f$, L must be contained in $\langle x_2, x_3 \rangle$. A contradiction follows. ■

Lemma 4.3 *The number $|O_{f'} \setminus O_f|$ is equal to the number of generators of the hyperbolic quadric $Q^+(2n-1, 4)$.*

Proof. Let F denote a convex subspace of diameter $2n-1$ of $\mathbb{I}_{2n}$ isomorphic to $DQ(4n-2, 2)$. Then by Lemma 8 of [5], $F \cap O_{f'}$ is an SDPS-set in a convex subspace F' of F isomorphic to $DQ(4n-4, 2)$. Hence, the number $|F' \cap O_{f'}|$ is equal to the number of generators of $Q(2n-2, 4)$. Now, every point y of $O_{f'} \setminus F'$ has distance 2 from a unique point y' of $O_{f'} \cap F'$ (since $G_{f'} \cong DQ(2n, 4)$) and every point y' of $O_{f'} \cap F'$ is contained in a unique special QUAD (with respect to f') which is not a quad of $\mathbb{I}_{2n}$, namely the unique special QUAD containing the unique line through y' not contained in $\mathbb{I}_{2n}$. Since every such QUAD contains exactly two points of $O_{f'} \setminus O_f$, $|O_{f'} \setminus O_f|$ is twice the number of generators of $Q(2n-2, 4)$. This number equals the number of generators of the hyperbolic quadric $Q^+(2n-1, 4)$. ■

Lemma 4.4 *There exist two points in $O_{f'} \setminus O_f$ at maximal distance $2n$ from each other.*

Proof. Let F denote a convex subspace of diameter $2n-1$ of $\mathbb{I}_{2n}$ isomorphic to $DQ(4n-2,4)$. Then $F \cap O_{f'}$ is an SDPS-set in a convex subspace F' of F isomorphic to $DQ(4n-4,4)$. Let x_1 and x_2 be two points of $F' \cap O_{f'}$ at maximal distance $2n-2$ from each other. Let Q_i, $i \in \{1,2\}$, denote the unique special QUAD through x_i which is not a quad of $\mathbb{I}_{2n}$. Let u_i and v_i denote the two points of $Q_i \cap (O_{f'} \setminus O_f)$. Since $G_{f'} \cong DQ(2n,4)$, every point of $Q_1 \cap O_{f'}$ has distance $2n-2$ from a unique point of $Q_2 \cap O_{f'}$. Hence, $\mathrm{d}(u_1,v_1) = 2n$ or $\mathrm{d}(u_1,v_2) = 2n$. This proves the lemma. ■

We are now ready to prove Proposition 4.1. Let π_1 and π_2 be two points of $O_{f'} \setminus O_f$ at distance $2n$ from each other. Then π_1 and π_2 can be regarded as two disjoint generators of $Q(2n,4)$. The space $\langle \pi_1, \pi_2 \rangle$ intersects $Q(2n,4)$ in a nonsingular hyperbolic quadric $Q^+(2n-1,4)$. The set of generators of $Q(2n,4)$ contained in $Q^+(2n-1,4)$ is a convex set of points of $DQ(2n,4)$. The smallest convex set of points of $DQ(2n,4)$ containing π_1 and π_2 coincides with the set of generators of $Q^+(2n-1,4)$. [For, let $DQ^+(2n-1,4)$ denote the dual polar space associated with $Q^+(2n-1,4)$. Since every line of $DQ^+(2n-1,4)$ contains precisely two points, every convex set of points of $DQ^+(2n-1,4)$ is also a convex subspace of $DQ^+(2n-1,4)$. So, every convex set of points containing the opposite points π_1 and π_2 must coincide with the whole set of points of $DQ^+(2n-1,4)$.] By Lemma 4.2, the set of generators of $Q^+(2n-1,4)$ is contained in $O_{f'} \setminus O_f$. By Lemma 4.3, it then follows that both sets coincide. Proposition 4.1 now readily follows.

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