A recursive construction for the dual polar spaces $DQ(2n, 2)$

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Abstract

In [10], Sahoo gave new combinatorial constructions for the near hexagons $\mathbb{I}_3$ and $DQ(6, 2)$ in terms of ordered pairs of collinear points of the generalized quadrangle $W(2)$. Replacing $W(2)$ by an arbitrary dual polar space of type $DQ(2n, 2)$, $n \geq 2$, we obtain a generalization of these constructions. By using a construction alluded to in [5] we show that these generalized constructions give rise to near $2n$-gons which are isomorphic to $\mathbb{I}_n$ and $DQ(2n, 2)$. In this way, we obtain a recursive construction for the dual polar spaces $DQ(2n, 2)$, $n \geq 2$, different from the one given in [4].

Keywords: dual polar space, near polygon, generalized quadrangle
MSC2000: 51A50, 51E12, 05B25

1 Introduction

1.1 Elementary definitions

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point $\pi_L(p)$ on L nearest to p. Here, distances $\mathrm{d}(\cdot, \cdot)$ are measured in the point graph or collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then

*Postdoctoral Fellow of the Research Foundation - Flanders (Belgium) the near polygon $\mathcal{S}$ is called a *near $2d$-gon*. A near 0-gon is a point and a near 2-gon is a line. The class of the near quadrangles coincides with the class of the so-called generalized quadrangles. A good source for information on near polygons is the recent book [6] of the author. For more background information on generalized quadrangles, we refer to the book of Payne and Thas [9].

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. If x and y are two points of $\mathcal{S}$, then we write $x \sim y$ if $\mathrm{d}(x, y) = 1$ and $x \not\sim y$ if $\mathrm{d}(x, y) \neq 1$. If X_1 and X_2 are two non-empty sets of points of $\mathcal{S}$, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If X_1 is a singleton $\{x_1\}$, we will also write $\mathrm{d}(x_1, X_2)$ instead of $\mathrm{d}(\{x_1\}, X_2)$. For every $i \in \mathbb{Z}$ and every non-empty set X of points of $\mathcal{S}$, $\Gamma_i(X)$ denotes the set of all points y for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$. We define $x^\perp := \Gamma_0(x) \cup \Gamma_1(x)$ for every point x of $\mathcal{S}$. If X is a set of points, then we define $X^\perp := \bigcap_{x \in X} x^\perp$ (with the convention that $X^\perp = \mathcal{P}$ if $X = \emptyset$) and $X^{\perp\perp} := (X^\perp)^\perp$.

If L_1 and L_2 are two lines of a near polygon $\mathcal{S}$, then one of the following two cases occurs (see e.g. Theorem 1.3 of [6]): (i) every point of L_1 has distance $\mathrm{d}(L_1, L_2)$ from L_2 and every point of L_2 has distance $\mathrm{d}(L_1, L_2)$ from L_1; (ii) there exist unique points $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, x_1) + \mathrm{d}(x_1, x_2) + \mathrm{d}(x_2, y)$ for any $x \in L_1$ and any $y \in L_2$. If case (i) occurs, then we say that L_1 and L_2 are *parallel* (notation: $L_1 \| L_2$).

A near polygon is called *slim* if every line is incident with precisely 3 points. A near polygon is called *dense* if every line is incident with at least 3 points and if every two points at distance 2 have at least 2 common neighbours. By Theorem 4 of Brouwer and Wilbrink [2], every two points of a dense near $2n$-gon at distance $\delta \in \{0, \ldots, n\}$ from each other are contained in a unique convex sub-(near-)2δ-gon. These convex subpolygons are called *quads* if $\delta = 2$, *hexes* if $\delta = 3$ and *maxes* if $\delta = n - 1$. The maximal distance between two points of a convex subpolygon F is called the *diameter* of F and is denoted as $\mathrm{diam}(F)$. If $X_1, X_2, \ldots, X_k$ are $k \geq 1$ objects of a dense near polygon $\mathcal{S}$ (like points or sets of points), then $\langle X_1, X_2, \ldots, X_k \rangle$ denotes the smallest convex subspace of $\mathcal{S}$ containing $X_1, X_2, \ldots, X_k$.

Let F be a convex subspace of a dense near polygon $\mathcal{S}$. F is called *big* in $\mathcal{S}$ if $F \neq \mathcal{S}$ and if every point of $\mathcal{S}$ not contained in F is collinear with a (necessarily unique) point of F. A point x of $\mathcal{S}$ is called *classical* with respect to F, if there exists a unique point $x' \in F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, x') + \mathrm{d}(x', y)$ for every point y of F. We will denote the point x' also by $\pi_F(x)$ and call it the *projection* from x on F. Every point of $\Gamma_1(F)$ is classical with respect to F. If X is a set of points of $\mathcal{S}$ which are classical with respect to F, then we define $\pi_F(X) := \{\pi_F(x) \,|\, x \in X\}$. F is called *classical* in $\mathcal{S}$ if every point of $\mathcal{S}$ is classical with respect to F. Every big subpolygon of $\mathcal{S}$ is classical in $\mathcal{S}$.

If F_1 and F_2 are two convex subspaces of a dense near $2d$-gon $\mathcal{S}$ with respective diameters d_1 and d_2 such that $F_1 \cap F_2 \neq \emptyset$ and F_1 is classical in $\mathcal{S}$, then the convex subspace $F_1 \cap F_2$ of $\mathcal{S}$ has diameter at least $d_1 + d_2 - d$ by Theorem 2.32 of [6].

Suppose F is a convex subpolygon of a slim dense near polygon $\mathcal{S}$. For every point x of F, we define $\mathcal{R}_F(x) := x$. If x is a point of $\mathcal{S}$ not contained in F, then we put $\mathcal{R}_F(x)$ equal to the unique point of the line $x\pi_F(x)$ different from x and $\pi_F(x)$. By Theorem 1.11 of [6], $\mathcal{R}_F$ is an automorphism of $\mathcal{S}$. $\mathcal{R}_F$ is called the *reflection about* F.

Let Q be a quad of a dense near polygon $\mathcal{S}$ and let x be a point of $\mathcal{S}$ at distance δ from Q. By Shult and Yanushka [11, Proposition 2.6], there are two possibilities. Either $\Gamma_\delta(x) \cap Q$ is a point of Q or $\Gamma_\delta(x) \cap Q$ is an *ovoid* of Q, i.e. a set of points of Q intersecting each line of Q in a unique point. In the former case, x is necessarily classical with respect to Q and we write $x \in \Gamma_{\delta,C}(Q)$. In the latter case, x is called *ovoidal with respect to* Q and we write $x \in \Gamma_{\delta,O}(Q)$.

Let $Q(2n, 2)$, $n \geq 2$, be a nonsingular parabolic quadric of $\mathrm{PG}(2n, 2)$. Let $DQ(2n, 2)$ denote the point-line geometry whose points are the generators (= subspaces of maximal dimension $n-1$) of $Q(2n, 2)$ and whose lines are the $(n-2)$-dimensional subspaces of $Q(2n, 2)$, with incidence given by reverse containment. $DQ(2n, 2)$ is a so-called *dual polar space* (Cameron [3]). $DQ(2n, 2)$ is a slim dense near $2n$-gon. If α is a totally singular subspace of dimension $n-1-k$, $k \in \{0, \ldots, n\}$, of $Q(2n, 2)$, then the set of all generators of $Q(2n, 2)$ containing α is a convex sub-$2k$-gon of $DQ(2n, 2)$. Conversely, every convex sub-$2k$-gon of $DQ(2n, 2)$ is obtained in this way. Every convex subpolygon of $DQ(2n, 2)$ is classical in $DQ(2n, 2)$. The quads of $DQ(2n, 2)$ are isomorphic to the generalized quadrangle $W(2)$, which is the (up to isomorphisms) unique slim generalized quadrangle with three lines through each point. If x and y are two points of $DQ(2n, 2)$ at distance 2 from each other, then $\{x, y\}^{\perp\perp}$ is a set $\{x, y, z\}$ of 3 points which is contained in the quad $\langle x, y\rangle$. We call $\{x, y\}^{\perp\perp} = \{x, y, z\}$ the *hyperbolic line* of $DQ(2n, 2)$ through the points x and y. If a and b are two distinct points of $\{x, y\}^\perp$, then $\{x, y\}^\perp = \{a, b\}^{\perp\perp}$. We say that the hyperbolic lines $\{x, y\}^\perp$ and $\{x, y\}^{\perp\perp}$

of $DQ(2n,2)$ are *orthogonal*.

Consider now a hyperplane of $\mathrm{PG}(2n,2)$ which intersects $Q(2n,2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1,2)$. The set of generators of $Q(2n,2)$ not contained in $Q^+(2n-1,2)$ is a subspace of $DQ(2n,2)$. By Brouwer et al. [1, p. 352–353], the point-line geometry induced on that subspace is a slim dense near $2n$-gon. Following the terminology of [6], we denote this near $2n$-gon by $\mathbb{I}_n$. The generalized quadrangle $\mathbb{I}_2$ is isomorphic to the (3×3)-grid. The convex subspaces of $\mathbb{I}_n$ have been studied in [6, Section 6.4]. If π is a subspace of dimension $n-1-k$, $k\in\{0,\ldots,n\}$, on $Q(2n,2)$ which is not contained in $Q^+(2n-1,2)$ if $k\in\{0,1\}$, then the set X_π of all generators of $Q(2n,2)$ through π which are not contained in $Q^+(2n-1,2)$ is a convex sub-$2k$-gon of $\mathbb{I}_n$. Conversely, every convex sub-$2k$-gon of $\mathbb{I}_n$ is obtained in this way. If $k\geq 2$ and π is not contained in $Q^+(2n-1,2)$, then (the point-line geometry induced on) X_π is isomorphic to $DQ(2k,2)$. If $k\geq 2$ and π is contained in $Q^+(2n-1,2)$, then X_π is isomorphic to $\mathbb{I}_k$. So, every quad of $\mathbb{I}_n$ is isomorphic to either $DQ(4,2)\cong W(2)$ or the (3×3)-grid $\mathbb{I}_2$. One readily sees that every line of $\mathbb{I}_n$ is contained in a unique grid-quad. If π is a point of $Q(2n,2)\setminus Q^+(2n-1,2)$, then $X_\pi\cong DQ(2n-2,2)$ is big in $\mathbb{I}_n$. Conversely, every big max of $\mathbb{I}_n$ is of the form X_π for some point $\pi\in Q(2n,2)\setminus Q^+(2n-1,2)$. If π is a generator of $Q^+(2n-1,2)$, then the set of generators of $Q(2n,2)$ not contained in $Q^+(2n-1,2)$ intersecting π in a subspace of dimension $n-2$ is called a *projective set* of $\mathbb{I}_n$. If X is a projective set of $\mathbb{I}_n$, then by De Bruyn and Vandecasteele [7, Section 8] the following holds for all $x_1,x_2\in X$ with $x_1\neq x_2$: (i) $\mathrm{d}(x_1,x_2)=2$; (ii) $\langle x_1,x_2\rangle$ is a grid-quad; (iii) $\langle x_1,x_2\rangle\cap X$ is an ovoid of $\langle x_1,x_2\rangle$.

1.2 The point-line geometry $\mathcal{S}_1(n)$

With the dual polar space $DQ(2n-2,2)$, $n\geq 3$, there is associated a point-line geometry $\mathcal{S}_1(n)$ in the following way. The points of $\mathcal{S}_1(n)$ are all the ordered pairs (x,y) of points of $DQ(2n-2,2)$ satisfying $y\in x^\perp$. There are 4 types of lines in $\mathcal{S}_1(n)$.

(a) *Lines of Type I* of $\mathcal{S}_1(n)$ are of the form $\{(x,x),(y,y),(z,z)\}$, where $\{x,y,z\}$ is an arbitrary line of $DQ(2n-2,2)$.

(b) *Lines of Type II* of $\mathcal{S}_1(n)$ are of the form $\{(x,x),(x,y),(x,z)\}$ where $\{x,y,z\}$ is an arbitrary line of $DQ(2n-2,2)$.

(c) *Lines of Type III* of $\mathcal{S}_1(n)$ are of the form $\{(x,y),(y,z),(z,x)\}$ where $\{x,y,z\}$ is an arbitrary line of $DQ(2n-2,2)$.

(d) *Lines of Type IV* of $\mathcal{S}_1(n)$ are of the form $\{(x,x'),(y,y'),(z,z')\}$ where x, y, z, x', y' and z' are mutually distinct points of $DQ(2n-2,2)$ satisfying: (i) $\{x,y,z\}$ is a line of $DQ(2n-2,2)$; (ii) $\mathrm{d}(x,x') = \mathrm{d}(y,y') = \mathrm{d}(z,z') = 1$; (iii) x, y, z, x', y' and z' are contained in a $W(2)$-quad of $DQ(2n-2,2)$ but not in a (3×3)-subgrid; (iv) x', y' and z' are mutually noncollinear.

Incidence is containment. Notice that with every line $\{x,y,z\}$ of $DQ(2n-2,2)$, there corresponds a unique line of Type I of $\mathcal{S}_1(n)$, three lines of Type II of $\mathcal{S}_1(n)$ and two lines of Type III of $\mathcal{S}_1(n)$.

The above construction for the point-line geometry $\mathcal{S}_1(n)$ is a straightforward generalization of a construction given in De Bruyn [5]. If $n=3$, then the dual polar space $DQ(2n-2,2)$ is isomorphic to the generalized quadrangle $W(2)$ and the construction reduces to the one given in [5, p. 51].

1.3 The point-line geometry $\mathcal{S}_2(n)$

With the dual polar space $DQ(2n-2,2)$, $n\geq 3$, there is associated a point-line geometry $\mathcal{S}_2(n)$ in the following way. The points of $\mathcal{S}_2(n)$ are all the pairs (x,y), where x and y are points of $DQ(2n-2,2)$ satisfying $y\in x^{\perp}$. The lines of $\mathcal{S}_2(n)$ are all the triples $\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$, where $\{x_1,x_2,x_3\}$ is either a line or a hyperbolic line of $DQ(2n-2,2)$ and $\{y_1,y_2,y_3\}=\{x_1,x_2,x_3\}^{\perp}$. Incidence is containment.

The above construction for the point-line geometry $\mathcal{S}_2(n)$ is a straightforward generalization of a construction given in Sahoo [10]. If $n=3$, then the dual polar space $DQ(2n-2,2)$ is isomorphic to the generalized quadrangle $W(2)$ and the construction reduces to the one given in [10, Section 2.1].

1.4 The point-line geometry $\mathcal{S}_3(n)$

With the dual polar space $DQ(2n-2,2)$, $n\geq 3$, there is associated a point-line geometry $\mathcal{S}_3(n)$ in the following way. There are 3 types of points in $\mathcal{S}_3(n)$.

(1) Points of the form (x,y) where x and y are points of $DQ(2n-2,2)$ satisfying $y\in x^{\perp}$.

(2) Points x of $DQ(2n-2,2)$.

(3) Symbols x' where x is a point of $DQ(2n-2,2)$.

There are also 3 types of lines in $\mathcal{S}_3(n)$:

(a) triples $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ where $\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\}$ is a line of $DQ(2n-2, 2)$;

(b) triples $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, where $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are two orthogonal hyperbolic lines of $DQ(2n-2, 2)$;

(c) triples of the form $\{x, (x, y), y'\}$ where x and y are points of $DQ(2n-2, 2)$ satisfying $y \in x^\perp$.

Incidence is containment. Obviously, the set of all points of Type I of $\mathcal{S}_3(n)$ is a hyperplane of $\mathcal{S}_3(n)$, i.e. a proper subspace of $\mathcal{S}_3(n)$ meeting each line. The point-line geometry induced on that hyperplane (by the lines of $\mathcal{S}_3(n)$) is isomorphic to $\mathcal{S}_2(n)$.

The above construction for the point-line geometry $\mathcal{S}_3(n)$ is a straightforward generalization of a construction given in Sahoo [10]. If $n = 3$, then the dual polar space $DQ(2n-2, 2)$ is isomorphic to the generalized quadrangle $W(2)$ and the construction reduces to the one given in [10, Section 2.2].

1.5 The main results

We show that the combinatorial constructions given in Sections 1.2, 1.3 and 1.4 give rise to the near $2n$-gons $\mathbb{I}_n$ and $DQ(2n, 2)$.

Theorem 1.1 (Section 3) *The point-line geometry $\mathcal{S}_1(n)$, $n \geq 3$, is isomorphic to the near $2n$-gon $\mathbb{I}_n$.*

Theorem 1.2 (Section 4) *The point-line geometries $\mathcal{S}_1(n)$ and $\mathcal{S}_2(n)$ are isomorphic for every $n \geq 3$.*

The following is an immediate corollary of Theorems 1.1 and 1.2.

Corollary 1.3 *The incidence structure $\mathcal{S}_2(n)$, $n \geq 3$, is isomorphic to the near $2n$-gon $\mathbb{I}_n$.*

Theorem 1.4 (Section 5) *The incidence structure $\mathcal{S}_3(n)$, $n \geq 3$, is isomorphic to the dual polar space $DQ(2n, 2)$.*

Remarks. (1) Theorem 1.1 is already known if $n = 3$, see De Bruyn [5], where it was shown in a purely combinatorial way that every slim dense near hexagon with parameters $(s, t, T_2) = (2, 5, \{1, 2\})$ is isomorphic to $\mathcal{S}_1(n)$.

(2) Also Theorems 1.2 and 1.4 are known if $n = 3$, see Sahoo [10], where it was shown that $\mathcal{S}_2(3) \cong \mathbb{I}_3$ and $\mathcal{S}_3(3) \cong DQ(6,2)$. The kind of proofs given in [10] seem not to be suitable to deal with the case of general n. Also, in [10] no explicit isomorphisms have been established between the near hexagons $\mathcal{S}_2(3)$ and $\mathbb{I}_3$ and the near hexagons $\mathcal{S}_3(3)$ and $DQ(6,2)$. Structural information on the near hexagons $\mathcal{S}_2(3)$ and $\mathcal{S}_3(3)$ in combination with the classification of all slim dense near hexagons ([1]) gives the desired isomorphisms. Notice also that a classification of all slim dense near $2n$-gons is only available if $n \leq 4$ ([1], [8], [9]).

(3) By Theorem 1.4, the construction given in Section 1.4 allows us to construct an isomorphic copy of the dual polar space $DQ(2n+2,2)$, $n \geq 2$, from the dual polar space $DQ(2n,2)$. So, we obtain a recursive construction for the dual polar spaces $DQ(2n,2)$, $n \geq 2$. A different recursive construction for the dual polar spaces $DQ(2n,2)$, $n \geq 2$, was given in Cooperstein and Shult [4].

2 An equivalence relation

2.1 A few lemmas

Lemma 2.1 *If L_1 and L_2 are two parallel lines of the dual polar space $DQ(2n,2)$, $n \geq 2$, at distance δ from each other, then there exist lines $K_0, K_1, \ldots, K_\delta$ in $DQ(2n,2)$ such that $K_0 = L_1$, $K_\delta = L_2$ and $K_i \| K_{i+1}$, $d(K_i, K_{i+1}) = 1$ for every $i \in \{0, \ldots, \delta - 1\}$.*

Proof. We will prove the lemma by induction on δ. Obviously, the lemma holds if $\delta \in \{0,1\}$. So, suppose $\delta \geq 2$. Let $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x_1, x_2) = \delta$. Let $u \in \Gamma_{\delta-1}(x_1) \cap \Gamma_1(x_2)$. Let F denote the convex sub-$(2\delta+2)$-gon $\langle L_1, L_2 \rangle$, let Q be the quad $\langle u, L_2 \rangle$ and let A be the convex sub-2δ-gon $\langle L_1, u \rangle$. Since A is classical in F, $\mathrm{diam}(Q \cap A) \geq \mathrm{diam}(Q) + \mathrm{diam}(A) - \mathrm{diam}(F) = 2 + \delta - (\delta + 1) = 1$. Hence, $Q \cap A$ is a line M. Since every point of M has distance at most $\delta - 1$ from L_1 (recall that $\mathrm{diam}(A) = \delta$), $M \cap L_2 = \emptyset$. So, M and L_2 are parallel. If L_1 and M were not parallel, then there exist points $y_1 \in L_1$ and $y \in M$ such that $\mathrm{d}(y_1, y) \leq \delta - 2$. If y_2 denotes the unique point of L_2 collinear with y, then $\mathrm{d}(y_1, y_2) \leq \delta - 1$, a contradiction. Hence, also L_1 and M are parallel. By the induction hypothesis, there exist lines $K_0, \ldots, K_{\delta-1}$ such that $K_0 = L_1$, $K_{\delta-1} = M$ and $K_i \| K_{i+1}, \mathrm{d}(K_i, K_{i+1}) = 1$ for every $i \in \{0, \ldots, \delta - 2\}$. If we put $K_\delta = L_2$, then we are done. ■

Remark. If $K_0, K_1, \ldots, K_\delta$ are lines as in Lemma 2.1, then for all $i_1, i_2 \in \{0, \ldots, \delta\}$ with $i_1 \leq i_2$, $\mathrm{d}(K_{i_1}, K_{i_2}) = i_2 - i_1$ and $K_{i_1} \| K_{i_2}$.

Lemma 2.2 *Let Q be a $W(2)$-quad of $\mathbb{I}_n$, $n \geq 3$, and let L_1 and L_2 denote two disjoint lines of Q. Let G_i, $i \in \{1,2\}$, denote the unique grid-quad of $\mathbb{I}_n$ containing L_i. Then $G_1 \subseteq \Gamma_{1,C}(G_2)$ and $G_2 \subseteq \Gamma_{1,C}(G_1)$. Moreover, the map $G_1 \to G_2; x \mapsto \pi_{G_2}(x)$ defines an isomorphism between the grids G_1 and G_2.*

Proof. If x is a point of $G_1 \cap G_2$, then x has distance 1 from a unique point x_1 of L_1 and a unique point x_2 of L_2. Since Q is a convex subspace, it follows that $x \in Q$, regardless of whether $\mathrm{d}(x_1, x_2) = 1$ or $\mathrm{d}(x_1, x_2) = 2$. But this is impossible since $Q \cap G_1 \cap G_2 = (Q \cap G_1) \cap (Q \cap G_2) = L_1 \cap L_2 = \emptyset$. Hence, G_1 and G_2 are disjoint.

Let A denote the hex $\langle Q, G_1 \rangle$ of $\mathbb{I}_n$. Since A contains the grid-quad G_1, A is isomorphic to $\mathbb{I}_3$. Hence, the unique grid-quad G_2 through the line $L_2 \subseteq A$ is also contained in A.

Suppose G_2 contains a point u at distance 2 from G_1. Since $\langle u, G_1 \rangle = A$ has diameter 3, $u \in \Gamma_{2,O}(G_1)$, i.e. $\Gamma_2(u) \cap G_1$ is an ovoid of G_1. So, there are precisely 3 quads through u which meet G_1 in a point. If one of these quads, say Q', is isomorphic to $W(2)$, then as Q' is big in A, $\mathrm{diam}(Q' \cap G_1) \geq \mathrm{diam}(Q') + \mathrm{diam}(G_1) - \mathrm{diam}(A) = 1$ and hence $\mathrm{d}(u, G_1) \leq 1$, a contradiction. Hence, the three quads through u meeting G_1 are precisely the 3 grid-quads of $A \cong \mathbb{I}_3$ through u. Since G_2 is a grid-quad through u contained in A, this would imply that $G_1 \cap G_2$ is a point, again a contradiction.

Hence, $G_2 \subseteq \Gamma_1(G_1) = \Gamma_{1,C}(G_1)$. By symmetry, $G_1 \subseteq \Gamma_{1,C}(G_2)$. If L is a line of G_1, then $\pi_{G_2}(L)$ is a line of G_2 (see e.g. [6, Theorem 1.23 (3)]). So, the map $G_1 \to G_2; x \mapsto \pi_{G_2}(x)$ defines an isomorphism between the grids G_1 and G_2. ■

Lemma 2.3 *Let M be a max of $\mathbb{I}_n$, $n \geq 3$, isomorphic to $DQ(2n-2, 2)$. Let L_1 and L_2 be two parallel lines of M at distance δ from each other and let G_i, $i \in \{1,2\}$, denote the unique grid-quad of $\mathbb{I}_n$ containing L_i. Then $G_1 \subseteq \Gamma_{\delta,C}(G_2)$ and $G_2 \subseteq \Gamma_{\delta,C}(G_1)$. Moreover, the map $G_1 \to G_2; x \mapsto \pi_{G_2}(x)$ defines an isomorphism between the grids G_1 and G_2.*

Proof. We will prove the lemma by induction on δ. The lemma holds for $\delta = 1$ by Lemma 2.2 and is trivial for $\delta = 0$. So, suppose $\delta \geq 2$. By Lemma 2.1, there exists a line L_3 in M satisfying $L_1 \parallel L_3 \parallel L_2$, $\mathrm{d}(L_1, L_3) = \delta - 1$ and $\mathrm{d}(L_3, L_2) = 1$. Let G_3 denote the unique grid-quad of $\mathbb{I}_n$ through L_3. Notice that $\langle L_1, L_3\rangle \cong DQ(2\delta, 2)$, $\langle L_1, L_2\rangle \cong DQ(2\delta+2, 2)$, $\langle L_1, L_3, G_1\rangle \cong \mathbb{I}_{\delta+1}$, $\langle L_1, L_2, G_1\rangle \cong \mathbb{I}_{\delta+2}$, $G_3 \subseteq \langle L_1, L_3, G_1\rangle$ and $G_2 \cup G_3 \subseteq \langle L_1, L_2, G_1\rangle$. If $x \in G_2$, then $\mathrm{d}(x, G_3) = 1$ by Lemma 2.2 and hence $\langle G_3, x\rangle = \langle L_3, L_2, G_3\rangle \cong \mathbb{I}_3$. If $x \in \langle L_1, L_3, G_1\rangle$, then $G_2 \subseteq \langle G_3, x\rangle \subseteq \langle L_1, L_3, G_1\rangle$ and hence $\langle L_1, L_2, G_1\rangle \subseteq \langle L_1, L_3, G_1\rangle$, a contradiction, since $\langle L_1, L_3, G_1\rangle \cong \mathbb{I}_{\delta+1}$ and $\langle L_1, L_2, G_1\rangle \cong \mathbb{I}_{\delta+2}$. Hence, $x \notin F := \langle L_1, L_3, G_1\rangle$. Every point x of G_2 has distance 1 from F and hence is classical with respect to F with $\pi_F(x) = \pi_{G_3}(x)$. By the induction hypothesis, $\pi_F(x) \in \Gamma_{\delta-1,C}(G_1)$. Hence, $x \in \Gamma_{\delta,C}(G_1)$ since $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), \pi_{G_1}(\pi_F(x))) + \mathrm{d}(\pi_{G_1}\pi_F(x), y) = \mathrm{d}(x, \pi_{G_1}\pi_F(x)) + \mathrm{d}(\pi_{G_1}\pi_F(x), y)$ for every $y \in G_1$. Since $x \in G_2$ was arbitrary, $G_2 \subseteq \Gamma_{\delta,C}(G_1)$. By symmetry, also $G_1 \subseteq \Gamma_{\delta,C}(G_2)$. If L is a line of G_1, then $\pi_{G_2}(L)$ is a line of G_2 (see e.g. [6, Theorem 1.23 (3)]). So, the map $G_1 \to G_2; x \mapsto \pi_{G_2}(x)$ defines an isomorphism between the grids G_1 and G_2. ■

Definition. The map $G_1 \to G_2; x \mapsto \pi_{G_2}(x)$ defined in Lemma 2.3 is called the *projection* from G_1 onto G_2.

Lemma 2.4 *Let M be a max of $\mathbb{I}_n$, $n \geq 3$, isomorphic to $DQ(2n-2, 2)$, let L_1 and L_2 be two parallel lines of M at distance δ from each other and let Q be a quad of M through L_2 not contained in $\langle L_1, L_2\rangle$. Let G_i, $i \in \{1, 2\}$, denote the unique grid-quad of $\mathbb{I}_n$ containing L_i. Put $F := \langle L_1, L_2, G_2\rangle \cong \mathbb{I}_{\delta+2}$ and $A := \langle Q, G_2\rangle \cong \mathbb{I}_3$. Let x be a point of A.*

(i) If $x \in G_2$, then $x \in \Gamma_{\delta,C}(G_1)$.

(ii) If $x \in \Gamma_1(G_2)$, then $x \in \Gamma_{\delta+1,C}(G_1)$ and $\pi_{G_1}(x) = \pi_{G_1}(\pi_{G_2}(x))$.

(iii) If $x \in \Gamma_2(G_2)$, then $x \in \Gamma_{\delta+2,O}(G_1)$ and $\Gamma_{\delta+2}(x) \cap G_1 = \pi_{G_1}(\Gamma_2(x) \cap G_2)$.

Proof. We will use the following fact.

Claim. *Let $x_1 \in G_1$ and $x_2 \in G_2$ be such that $d(x_1, x_2) = \delta$ and let L be a line of G_2 through x_2. Then $\langle x_1, x_2, L\rangle \cong DQ(2\delta+2, 2)$. As a consequence, $\langle x_1, x_2\rangle \cong DQ(2\delta, 2)$.*

Proof. Let $x_3 \in L \setminus \{x_2\}$ and let x_4 be a point of G_2 at distance 2 from x_2. Then $\mathrm{d}(x_1, x_3) = \delta + 1$, $\mathrm{d}(x_1, x_4) = \delta + 2$, $\langle x_1, x_3\rangle = \langle x_1, x_2, L\rangle$ and $\langle x_1, x_4\rangle = \langle x_1, x_2, G_2\rangle$. The convex sub-$(2\delta+4)$-gon $\langle x_1, x_2, G_2\rangle$ is isomorphic to $\mathbb{I}_{\delta+2}$ since it contains the grid-quad G_2. The convex sub-$(2\delta+2)$-gon $\langle x_1, x_2, L\rangle$ is isomorphic to either $\mathbb{I}_{\delta+1}$ or $DQ(2\delta+2, 2)$. Since $\langle x_1, x_2, G_2\rangle$ is not contained in $\langle x_1, x_2, L\rangle$, the unique grid-quad G_2 through L is not contained in $\langle x_1, x_2, L\rangle$. This implies that $\langle x_1, x_2, L\rangle \cong DQ(2\delta+2, 2)$.

We will now prove Claims (i), (ii) and (iii) of the lemma. Claim (i) follows from Lemma 2.3.

(ii) Suppose $x \in \Gamma_1(G_2)$. Then $x \in \Gamma_1(F)$ and hence x is classical with respect to F with $\pi_F(x) = \pi_{G_2}(x)$. This combined with the fact that $\pi_F(x) \in \Gamma_{\delta,C}(G_1)$ implies that $x \in \Gamma_{\delta+1,C}(G_1)$ and $\pi_{G_1}(x) = \pi_{G_1}(\pi_{G_2}(x))$.

(iii) Suppose $x \in \Gamma_2(G_2)$. Let u_2 be an arbitrary point of $\Gamma_2(x) \cap G_2$ and let v be one of the two neighbours of x and u_2. Put $u_1 := \pi_{G_1}(u_2)$ and let L be an arbitrary line of G_1 through u_1. Then $\langle u_1, u_2, L\rangle \cong DQ(2\delta+2, 2)$ by the above claim and $\langle u_1, u_2, L, v\rangle$ is isomorphic to either $\mathbb{I}_{\delta+2}$ or $DQ(2\delta+4, 2)$ since $v \notin \langle u_1, u_2, L\rangle \subseteq F$. If $G_1 \subseteq \langle u_1, u_2, L, v\rangle$, then as $\mathrm{diam}(\langle u_1, u_2, L, v\rangle) = \mathrm{diam}(\langle u_1, u_2, G_1\rangle) = \delta + 2$, $F = \langle u_1, u_2, G_1\rangle = \langle u_1, u_2, L, v\rangle$, a contradiction, since $v \notin F$. Hence, the unique grid-quad G_1 through L is not contained in $\langle u_1, u_2, L, v\rangle$. This implies that $\langle u_1, u_2, L, v\rangle \cong DQ(2\delta+4, 2)$. It follows that the unique grid-quad $\langle u_2, x\rangle$ through $u_2v \subseteq \langle u_1, u_2, L, v\rangle$ is not contained in $\langle u_1, u_2, L, v\rangle$. So, $\mathrm{d}(x, \langle u_1, u_2, L, v\rangle) = 1$ and x is classical with respect to $\langle u_1, u_2, L, v\rangle$. The unique point of $\langle u_1, u_2, L, v\rangle$ collinear with x is v. Now, $v \in \Gamma_{\delta+1,C}(G_1)$ and $\pi_{G_1}(v) = \pi_{G_1}(\pi_{G_2}(v)) = \pi_{G_1}(u_2) = u_1$. It follows that $\mathrm{d}(x, u_1) = \delta + 2$ and $\mathrm{d}(x, w) = \delta + 3$ for every $w \in L \setminus \{u_1\}$. Since L was an arbitrary line of G_1 through u_1, we have $\mathrm{d}(x, w) = \delta + 3$ for every $w \in (G_1 \cap u_1^{\perp}) \setminus \{u_1\}$. Since u_2 was an arbitrary point of $\Gamma_2(x) \cap G_2$, $\mathrm{d}(x, u) = \delta + 2$ for every $u \in \pi_{G_1}(\Gamma_2(x) \cap G_2)$ and $\mathrm{d}(x, w) = \delta + 3$ for every $w \in G_1 \setminus \pi_{G_1}(\Gamma_2(x) \cap G_2)$. This implies that $x \in \Gamma_{\delta+2,O}(G_1)$ and $\Gamma_{\delta+2}(x) \cap G_1 = \pi_{G_1}(\Gamma_2(x) \cap G_2)$. ∎

Lemma 2.5 *Let M be a max of $\mathbb{I}_n$, $n \geq 3$, isomorphic to $DQ(2n-2, 2)$ and let L_1 and L_2 be two non-parallel lines of M at distance δ from each other. Let x_1 and x_2 be the unique points of L_1 and L_2, respectively, such that $d(x_1, x_2) = \delta$. Let G_i, $i \in \{1, 2\}$, denote the unique grid-quad of $\mathbb{I}_n$ containing L_i. Then*

(i) $\langle G_1, G_2\rangle$ has diameter $\delta + 3$.

(ii) Let $i \in \{1, 2\}$. Then every point $x \in G_i \cap x_i^{\perp}$ is classical with respect to G_{3-i} and $\pi_{G_{3-i}}(x) = x_{3-i}$.

(iii) Let $i \in \{1, 2\}$. Then every point x of $G_i \setminus x_i^{\perp}$ belongs to $\Gamma_{\delta+2,O}(G_{3-i})$ and $\Gamma_{\delta+2}(x) \cap G_{3-i}$ is an ovoid of G_{3-i} containing x_{3-i}.

(iv) The two ovoids O_1, O_1' of G_1 through x_1 and the two ovoids O_2, O_2' of G_2 through x_2 can be chosen in such a way that $d(x, y) = \delta + 2$ for every $(x, y) \in \Big((O_1 \setminus \{x_1\}) \times (O_2 \setminus \{x_2\})\Big) \cup \Big((O_1' \setminus \{x_1\}) \times (O_2' \setminus \{x_2\})\Big)$ and $d(x, y) =$

$\delta+3$ *for every* $(x,y) \in \Big((O_1\setminus\{x_1\})\times(O_2'\setminus\{x_2\})\Big)\cup\Big((O_1'\setminus\{x_1\})\times(O_2\setminus\{x_2\})\Big)$.

Proof. Let L_3 denote a line through x_2 parallel with L_1 (i.e. a line of $\langle L_1, x_2\rangle$ through x_2 not contained in $\langle x_1, x_2\rangle$) and let G_3 denote the unique grid-quad of $\mathbb{I}_n$ through L_3. Since G_2 and G_3 are two different grid-quads through x_2 (they have different intersections with M), $G_2 \cap G_3 = \{x_2\}$. We can apply Lemma 2.4 (with (L_1, L_3) fulfilling the role of (L_1, L_2) and $\langle L_2, L_3\rangle$ the role of Q). By Lemma 2.4 (i)+(ii)+(iii), the maximal distance between a point of G_1 and a point of G_2 is equal to $\delta+3$, proving Claim (i). If $x \in G_2\cap x_2^{\perp}$, then by Lemma 2.4 (i)+(ii), x is classical with respect to G_1 and $\pi_{G_1}(x) = \pi_{G_1}(\pi_{G_3}(x)) = \pi_{G_1}(x_2) = x_1$. This proves Claim (ii) (taking into account a straightforward symmetry). If $x \in G_2\setminus x_2^{\perp}$, then by Lemma 2.4 (iii), $x_2 \in \Gamma_{\delta+2,O}(G_1)$ and the ovoid $\Gamma_{\delta+2}(x)\cap G_1 = \pi_{G_1}(\Gamma_2(x)\cap G_3)$ of G_1 contains the point $\pi_{G_1}(x_2) = x_1$. This proves Claim (iii). If $L = \{\pi_L(x_2), u, v\}$ is a line of G_2 not containing x_2, then $\Gamma_{\delta+2}(u)\cap G_1$ and $\Gamma_{\delta+2}(v)\cap G_1$ are the two ovoids of G_1 through x_1 (see e.g. [6, Theorem 1.23 (7)]). A similar remark holds for lines of G_1 not containing x_1. Claim (iv) now readily follows. ■

2.2 The relations R and R'

Consider in the near $2n$-gon $\mathbb{I}_n$, $n \geq 3$, a big max $M \cong DQ(2n-2,2)$. Let $\mathcal{O}$ denote the set of all ovoids in all grid-quads which intersect M in a line. For every $O \in \mathcal{O}$, let G_O denote the unique grid-quad of $\mathbb{I}_n$ containing O and put $L_O := G_O \cap M$. We now define a relation $R \subseteq \mathcal{O}\times\mathcal{O}$. Let $O_1, O_2 \in \mathcal{O}$.

If $L_{O_1} = L_{O_2}$, then $(O_1, O_2) \in R$ if and only if $O_1 = O_2$ or $O_1 \cap O_2 = \emptyset$.

Suppose L_{O_1} and L_{O_2} are non-parallel lines at distance δ from each other. Let x_1 and x_2 be the unique points of respectively L_{O_1} and L_{O_2} such that $\mathrm{d}(x_1, x_2) = \delta$. Let $\widetilde{O_i}$, $i \in \{1,2\}$, denote the unique ovoid of G_{O_i} satisfying $x_i \in \widetilde{O_i}$ and $|O_i \cap \widetilde{O_i}| \in \{0,3\}$. If δ is even, then we say that $(O_1, O_2) \in R$ if and only if every point of $\widetilde{O_1}\setminus\{x_1\}$ has distance $\delta+2$ from every point of $\widetilde{O_2}\setminus\{x_2\}$ (cf. Lemma 2.5 (iv)). If δ is odd, then we say that $(O_1, O_2) \in R$ if and only if every point of $\widetilde{O_1}\setminus\{x_1\}$ has distance $\delta+3$ from every point of $\widetilde{O_2}\setminus\{x_2\}$.

Suppose L_{O_1} and L_{O_2} are parallel lines at distance δ from each other. Let O_1' denote the ovoid $\pi_{G_2}(O_1)$ of G_2. (Recall that $G_2 \subseteq \Gamma_{\delta,C}(G_1)$, see Lemma 2.3). If δ is even, then we say that $(O_1, O_2) \in R$ if and only if $|O_1' \cap O_2| \in \{0,3\}$. If δ is odd, then we say that $(O_1, O_2) \in R$ if and only if

$|O_1' \cap O_2| = 1$.

We now define another relation R' on the set $\mathcal{O}$. If $O_1, O_2 \in \mathcal{O}$, then we say that $(O_1, O_2) \in R'$ if and only if $(O_1, O_2) \in R$ and $\langle L_{O_1}, L_{O_2} \rangle$ is a line or a quad.

The aim of this section is to prove the following proposition.

Proposition 2.6 *The relation R is an equivalence relation with two equivalence classes. Moreover, R is the smallest equivalence relation on the set $\mathcal{O}$ for which $R' \subseteq R$.*

2.3 Proof of Proposition 2.6

Notice that the 6 ovoids of a (3×3)-grid can be divided into 2 classes such that two ovoids belong to a different class (respectively the same class) if they intersect in precisely 1 point (respectively 0 or 3 points). Combining this fact with the definition of the relation R, we can immediately say that

Lemma 2.7 *Let $O_1, O_1', O_2, O_2' \in \mathcal{O}$ such that $G_{O_1} = G_{O_1'}$, $G_{O_2} = G_{O_2'}$ and $(O_1, O_2) \in R$.*
(i) If $|O_1 \cap O_1'|, |O_2 \cap O_2'| \in \{0, 3\}$, then $(O_1', O_2') \in R$.
(ii) If $|O_1 \cap O_1'| = |O_2 \cap O_2'| = 1$, then $(O_1', O_2') \in R$.
(iii) If $|O_1 \cap O_1'| = 1$ and $|O_2 \cap O_2'| \in \{0, 3\}$, then $(O_1', O_2') \notin R$. ■

For every line L of $\mathbb{I}_n$, let G_L denote the unique grid-quad of $\mathbb{I}_n$ containing L. The following lemma is precisely Lemma 3.1 of De Bruyn [5].

Lemma 2.8 ([5]) *Let Q be a $W(2)$-quad of M and let L_1, L_2, L_3 be three lines contained in Q. For every $i \in \{1, 2, 3\}$, let O_i be an ovoid of the grid-quad G_{L_i}. Suppose that $(O_1, O_2) \in R$ and $(O_2, O_3) \in R$. Then also $(O_1, O_3) \in R$.* ■

Lemma 2.9 *Let L_1 and L_2 be two parallel lines of M at distance δ from each other, let Q be a quad of M through L_2 not contained in $\langle L_1, L_2 \rangle$ and let L_3 be a line of Q. For every $i \in \{1, 2, 3\}$, let O_i be an ovoid of the grid-quad $G_i := G_{L_i}$. Suppose $(O_1, O_2) \in R$. Then $(O_1, O_3) \in R$ if and only if $(O_2, O_3) \in R$.*

Proof. Let π denote the projection from G_1 onto G_2. Notice that since $(O_1, O_2) \in R$, we have

(∗) $|O_2 \cap \pi(O_1)| \in \{0, 3\}$ if δ is even and $|O_2 \cap \pi(O_1)| = 1$ if δ is odd.

We will distinguish three cases: (1) $L_2 = L_3$; (2) $L_2 \cap L_3 = \emptyset$; (3) $L_2 \cap L_3$ is a singleton.

Suppose first that $L_2 = L_3$. Then as we have already noticed in Lemma 2.7, $(O_1, O_3) \in R$ if and only if $(O_2, O_3) \in R$.

Suppose next that $L_2 \cap L_3 = \emptyset$. Let π' be the projection from G_2 onto G_3. Then by Lemma 2.2 and Lemma 2.4(ii), $\pi'\pi = \pi' \circ \pi$ equals the projection from G_1 onto G_3. We have $(O_2, O_3) \in R$ if and only if $|\pi'(O_2) \cap O_3| = 1$. By (∗) this happens if and only if $|O_3 \cap \pi'\pi(O_1)| = 1$ if δ is even and $|O_3 \cap \pi'\pi(O_1)| \in \{0, 3\}$ if δ is odd. Since $\mathrm{d}(L_1, L_3) = \delta + 1$ and $L_1 \| L_3$, the latter condition is equivalent with $(O_1, O_3) \in R$.

Suppose finally that $L_2 \cap L_3$ is a singleton $\{x\}$. Put $x' = \pi_{G_1}(x)$. Let O_1' denote the unique ovoid of G_1 through x' such that $|O_1 \cap O_1'| \in \{0, 3\}$ and let O_2' denote the unique ovoid of G_2 through x such that $|O_2 \cap O_2'| \in \{0, 3\}$. By Lemma 2.7, $(O_1', O_2') \in R$. Let O_3' denote the unique ovoid of G_3 through x such that $|O_3 \cap O_3'| \in \{0, 3\}$. Then $(O_1, O_3) \in R$ if and only if $(O_1', O_3') \in R$ and $(O_2, O_3) \in R$ if and only if $(O_2', O_3') \in R$. Now, $(O_2', O_3') \in R$ if and only if every point of $O_3' \setminus \{x\}$ has distance 2 from every point of $O_2' \setminus \{x\}$. By Lemma 2.4 (iii), this precisely happens when every point of $O_3' \setminus \{x\}$ has distance $\delta + 2$ from every point of $\pi^{-1}(O_2') \setminus \{x'\}$. Since $(O_1', O_2') \in R$, $\pi^{-1}(O_2') = O_1'$ if δ is even. If δ is odd, then $\pi^{-1}(O_2')$ is the other ovoid of G_1 through x'. So, $(O_2', O_3') \in R$ if and only if $(O_1', O_3') \in R$ finishing the proof of the lemma. ∎

Lemma 2.10 *Let $O, O' \in \mathcal{O}$ with $(O, O') \in R$. Then there exist elements $O_1, O_2, \ldots, O_k \in \mathcal{O}$ (for some $k \geq 1$) such that $O_1 = O$, $O_k = O'$ and $(O_i, O_{i+1}) \in R'$ for every $i \in \{1, \ldots, k-1\}$.*

Proof. Put $G = G_O$, $G' = G_{O'}$, $L = L_O$ and $L' = L_{O'}$. We will consider two cases: (1) the lines L and L' are parallel; (2) the lines L and L' are not parallel.

(1) Suppose L and L' are parallel. If $\mathrm{d}(L, L') \leq 1$, then $(O, O') \in R$ implies $(O, O') \in R'$ and we are done.

Suppose therefore that $\mathrm{d}(L, L') \geq 2$. Let L'' be a line of M such that $\mathrm{d}(L, L'') = \mathrm{d}(L, L') - 1$, $\mathrm{d}(L', L'') = 1$ and $L \parallel L'' \parallel L'$ (cf. Lemma 2.1) and put $G'' := G_{L''}$. Let Q be the quad $\langle L'', L' \rangle$. Then Q is not contained in $\langle L, L'' \rangle$. So we can apply Lemma 2.9. Let O'' be an ovoid of G'' such that $(O, O'') \in R$. Then by Lemma 2.9 and the fact that $(O, O') \in R$, $(O'', O') \in R$, i.e. $(O'', O') \in R'$. By the induction hypothesis, there exist $O_1, O_2, \dots, O_{k'} \in \mathcal{O}$ such that $O_1 = O$, $O_{k'} = O''$ and $(O_i, O_{i+1}) \in R'$ for every $i \in \{1, \dots, k'-1\}$. Now, $(O'', O') \in R'$. So, if we put $O_{k'+1} = O'$, then we are done.

(2) Suppose L and L' are not parallel. Again, we will prove the claim by induction on $\mathrm{d}(L, L')$.

Suppose first that $\mathrm{d}(L, L') = 0$. Then $(O, O') \in R$ implies $(O, O') \in R'$ and we are done.

Suppose next that $\delta := \mathrm{d}(L, L') \geq 1$. Let x and x' be the unique points of L and L', respectively, such that $\mathrm{d}(x, x') = \delta$. Let L'' be a line of M through x' parallel with L, i.e. a line through x' contained in $\langle x', L \rangle$, but not in $\langle x, x' \rangle$. Let O'' be an ovoid of $G'' := G_{L''}$ such that $(O, O'') \in R$. Now, put $Q := \langle L'', L' \rangle$. Then the quad Q is not contained in $\langle L, L'' \rangle$. So, as before we can apply Lemma 2.9 and conclude that $(O'', O') \in R$. Now, by (1) there exist elements $O_1, O_2, \dots, O_{k'} \in \mathcal{O}$ such that $O_1 = O$, $O_{k'} = O''$ and $(O_i, O_{i+1}) \in R'$ for every $i \in \{1, \dots, k'-1\}$. Since $(O'', O') \in R'$, we can take $O_{k'+1} = O'$ and we are done. ∎

Lemma 2.11 *Let $O_1, O_2, O_3 \in \mathcal{O}$ such that $(O_1, O_2) \in R$ and $(O_2, O_3) \in R'$. Then $(O_1, O_3) \in R$.*

Proof. Fix O_1 and put $L_1 := L_{O_1}$. If L_2 and L_3 are two lines of M such that $\mathrm{diam}(\langle L_2, L_3 \rangle) \in \{1, 2\}$, then we say that Property $P(L_2, L_3)$ is satisfied if the conclusion of the lemma holds for each triple $(O_1', O_2', O_3') \in \mathcal{O} \times \mathcal{O} \times \mathcal{O}$ for which $O_1' = O_1$, $L_{O_2'} = L_2$ and $L_{O_3'} = L_3$.

Claim 1. *$P(L, L)$ is satisfied for every line L of M.*
Proof. This follows from Lemma 2.7.

Claim 2. *If L_2 and L_3 are lines of M such that Property $P(L_2, L_3)$ is satisfied, then also Property $P(L_3, L_2)$ is satisfied.*
Proof. Let O_3' and O_2' be elements of $\mathcal{O}$ such that $(O_1, O_3') \in R$, $(O_3', O_2') \in R'$, $L_{O_3'} = L_3$ and $L_{O_2'} = L_2$. We need to show that $(O_1, O_2') \in R$. Let O_2'' and O_3'' be elements of $\mathcal{O}$ such that $(O_1, O_2'') \in R$, $(O_2'', O_3'') \in R'$, $L_{O_2''} = L_2$ and $L_{O_3''} = L_3$. By Property $P(L_2, L_3)$, $(O_1, O_3'') \in R$. Since also $(O_1, O_3') \in R$, we necessarily have $(O_3', O_3'') \in R$ by Lemma 2.7. This combined with the

facts that $(O'_2, O'_3) \in R$ and $(O''_2, O''_3) \in R$ yields $(O''_2, O'_2) \in R$ by Lemma 2.7. Applying Lemma 2.7 to the facts that $(O_1, O''_2) \in R$ and $(O''_2, O'_2) \in R$ yields $(O_1, O'_2) \in R$.

Claim 3. *Let Q be a quad of M and let L_2, L_3, L_4 be three lines of Q. If Properties $P(L_2, L_3)$ and $P(L_3, L_4)$ are satisfied, then also Property $P(L_2, L_4)$ is satisfied.*
PROOF. Let O'_2 and O'_4 be elements of $\mathcal{O}$ such that $(O_1, O'_2) \in R$, $(O'_2, O'_4) \in R'$, $L_{O'_2} = L_2$ and $L_{O'_4} = L_4$. We need to show that $(O_1, O'_4) \in R$. Let O'_3 be an element of $\mathcal{O}$ such that $(O'_2, O'_3) \in R'$ and $L_{O'_3} = L_3$. Then by Lemma 2.8, also $(O'_3, O'_4) \in R'$. By Property $P(L_2, L_3)$ and the facts that $(O_1, O'_2) \in R$ and $(O'_2, O'_3) \in R'$, we have that $(O_1, O'_3) \in R$. By Property $P(L_3, L_4)$ and the facts that $(O_1, O'_3) \in R$ and $(O'_3, O'_4) \in R'$, we have $(O_1, O'_4) \in R$.

If Q is a quad of M, then by De Bruyn [6, Theorem 1.23], either $\pi_Q(L_1)$ is a point or a line. In the former case, no line of Q is parallel with L_1. In the latter case, $L_1 \subseteq \Gamma_{\delta,C}(Q)$ where $\delta := \mathrm{d}(L_1, Q)$. Lemma 2.11 now follows from Claims 4 and 5 below.

Claim 4. *If Q is a quad of M such that $L'_1 := \pi_Q(L_1)$ is a line of Q, then Property $P(L_2, L_3)$ is satisfied for any two lines L_2 and L_3 of Q.*
PROOF. Let O'_2 and O'_3 be elements of $\mathcal{O}$ such that $(O_1, O'_2) \in R$, $(O'_2, O'_3) \in R'$, $L_{O'_2} = L_2$ and $L_{O'_3} = L_3$. We need to show that $(O_1, O'_3) \in R$. The line L'_1 is parallel with L_1 and the quad Q is not contained in $\langle L_1, L'_1 \rangle$. Let O'_1 denote an ovoid of $G_{L'_1}$ such that $(O_1, O'_1) \in R$. Since also $(O, O'_2) \in R$, $(O'_1, O'_2) \in R$ by Lemma 2.9. This in combination with $(O'_2, O'_3) \in R$ and Lemma 2.8 gives $(O'_1, O'_3) \in R$. By Lemma 2.9 and the facts that $(O_1, O'_1) \in R$ and $(O'_1, O'_3) \in R$, we have $(O_1, O'_3) \in R$.

Claim 5. *If Q is a quad of M such that $\pi_Q(L_1)$ is a singleton $\{x_2\}$, then Property $P(L_2, L_3)$ is satisfied for any two lines L_2 and L_3 of Q.*
PROOF. In view of Claims 1, 2 and 3, it suffices to prove this if L_2 and L_3 are two disjoint lines of Q such that $x_2 \in L_2$. Suppose $O_1, O_2 \in \mathcal{O}$ such that $(O_1, O_2) \in R$, $(O_2, O_3) \in R'$, $L_{O_2} = L_2$ and $L_{O_3} = L_3$. Put $\delta := \mathrm{d}(L_1, Q)$. Recall that no line of Q is parallel with L_1. Let x_3 denote the unique point of L_3 collinear with x_2 and let x_1 denote the unique point of L_1 such that $\mathrm{d}(x_1, x_2) = \delta$ and $\mathrm{d}(x_1, x_3) = \delta + 1$. Let K_2 denote a line through x_2 parallel with L_1 and let K_3 be a line through x_3 different from x_2x_3 and contained in the quad $\langle x_3, K_2 \rangle$. Then $\mathrm{d}(K_2, L_1) = \delta$, $\mathrm{d}(K_3, L_1) = \delta + 1$ and $K_3 \| L_1$.

Put $Q_i := \langle K_i, L_i \rangle$, $i \in \{2,3\}$. Since L_i, $i \in \{2,3\}$, contains a point a point at distance $\delta - 1 + i$ from L_1, Q_i is not contained in $\langle L_1, K_i \rangle$. Now, let O_i', $i \in \{1,2,3\}$, denote the unique element of $\mathcal{O}$ such that $L_{O_i'} = L_i$, $x_i \in O_i'$ and $|O_i \cap O_i'| \in \{0,3\}$. Since $(O_1, O_2) \in R$ and $(O_2, O_3) \in R'$, $(O_1', O_2') \in R$ and $(O_2', O_3') \in R'$ by Lemma 2.7. Now, let O_2'' denote the unique element of $\mathcal{O}$ such that $L_{O_2''} = K_2$, $x_2 \in O_2''$ and $(O_1', O_2'') \in R$. By Lemma 2.4 (iii) and the fact that $(O_1', O_2') \in R$, every point of $O_2' \setminus \{x_2\}$ has distance 2 from every point of $O_2'' \setminus \{x_2\}$. By Lemma 2.4 (iii) and the fact that $(O_2', O_3') \in R'$, every point of $O_2'' \setminus \{x_2\}$ has distance 4 from every point of $O_3' \setminus \{x_3\}$. Now, let O_3'' be the unique element of $\mathcal{O}$ such that $L_{O_3''} = K_3$, $x_3 \in O_3''$ and $(O_2'', O_3'') \in R'$. Then by Lemma 2.4 (iii) and the fact that every point of $O_3' \setminus \{x_3\}$ has distance 4 from every point of $O_2'' \setminus \{x_2\}$, it follows that every point of $O_3' \setminus \{x_3\}$ has distance 2 from every point of $O_3'' \setminus \{x_3\}$. Since $(O_1', O_2'') \in R$ and $(O_2'', O_3'') \in R'$, it follows that $(O_1', O_3'') \in R$ by Claim 4. This together with the fact that every point of $O_3' \setminus \{x_3\}$ has distance 2 from every point of $O_3'' \setminus \{x_3\}$ implies that $(O_1', O_3') \in R$ (recall again Lemma 2.4 (iii)). So, $(O_1, O_3) \in R$ and Property $P(L_2, L_3)$ is satisfied. ■

From Lemmas 2.10 and 2.11, it now follows that R is the smallest equivalence relation on the set $\mathcal{O}$ satisfying $R' \subseteq R$. By Lemma 2.7 there are precisely two equivalence classes. This proves Proposition 2.6.

3 Proof of Theorem 1.1

Consider in the near $2n$-gon $\mathbb{I}_n$, $n \geq 3$, a big max $M \cong DQ(2n-2, 2)$. Let $\mathcal{O}$ denote the set of all ovoids in all grid-quads which intersect M in a line. Then by Proposition 2.6 an equivalence relation R can be defined on the set $\mathcal{O}$. Put $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are the two equivalence classes of R. We now define a map θ between the point-set of $\mathbb{I}_n$ and the point-set of $\mathcal{S}_1(n)$.

- If $x \in M$, then we define $\theta(x) := (x, x)$.
- If $x \in \mathbb{I}_n \setminus M$, then let L_x denote the unique line through x meeting M in a point and let G_x denote the unique grid-quad of $\mathbb{I}_n$ containing L_x. Notice that $G_x \cap M$ is a line since M is big in $\mathbb{I}_n$. Now, there exists a unique ovoid $O \in \mathcal{O}_1$ such that $x \in O \subseteq G_x$. Put $L_x \cap M = \{x_1\}$ and $O \cap M = \{x_2\}$. Then we define $\theta(x) := (x_1, x_2)$.

Lemma 3.1 *θ is a bijection between the set of points of $\mathbb{I}_n$ and the set of points of $\mathcal{S}_1(n)$.*

Proof. Let (x_1, x_2) be an arbitrary point of $\mathcal{S}_1(n)$ and consider the equation $\theta(x) = (x_1, x_2)$.

If $x_1 = x_2$, then $x = x_1$ is the unique solution of that equation.

Suppose therefore that $x_1 \neq x_2$. Let G denote the unique grid-quad of $\mathbb{I}_n$ containing x_1x_2 and let L denote the unique line of G through x_1 different from x_1x_2. There exists a unique $O \in \mathcal{O}_1$ such that $x_2 \in O \subseteq G$. Put $O \cap L = \{u\}$. Then $x = u$ is the unique solution of the equation $\theta(x) = (x_1, x_2)$. ■

We now divide the set of lines of $\mathbb{I}_n$ into 4 classes.

A line of $\mathbb{I}_n$ is said to be of *Type I* if it is contained in M.

A line of $\mathbb{I}_n$ is said to be of *Type II* if it intersects M in a unique point.

A line L of $\mathbb{I}_n$ is said to be of *Type III* if it is disjoint from M and if $\langle L, \pi_M(L)\rangle$ is a grid.

A line L of $\mathbb{I}_n$ is said to be of *type IV* if it is disjoint from M and if $\langle L, \pi_M(L)\rangle$ is a $W(2)$-quad.

Theorem 1.1 is a consequence of the following lemma.

Lemma 3.2 (*a*) *θ induces a bijection between the set of lines of Type I of $\mathbb{I}_n$ and the set of lines of Type I of $\mathcal{S}_1(n)$.*

(*b*) *θ induces a bijection between the set of lines of Type II of $\mathbb{I}_n$ and the set of lines of Type II of $\mathcal{S}_1(n)$.*

(*c*) *θ induces a bijection between the set of lines of Type III of $\mathbb{I}_n$ and the set of lines of Type III of $\mathcal{S}_1(n)$.*

(*d*) *θ induces a bijection between the set of lines of Type IV of $\mathbb{I}_n$ and the set of lines of Type IV of $\mathcal{S}_1(n)$.*

Proof. (a) Obviously, the map $\{x, y, z\} \mapsto \{(x, x), (y, y), (z, z)\}$ defines a bijection between the set of lines of Type I of $\mathbb{I}_n$ and the set of lines of Type I of $\mathcal{S}_1(n)$.

(b) Let $L = \{x, y, z\}$ be a line of Type II of $\mathbb{I}_n$ and suppose x is the unique point of L contained in M. Let G denote the unique grid-quad of $\mathbb{I}_n$ containing L. Then $G \cap M$ is a line $\{x, y', z'\}$. Clearly, $\theta(L) = \{(x, x), (x, y'), (x, z')\}$ is a line of Type II of $\mathcal{S}_1(n)$.

Conversely, suppose that $\{(x, x), (x, y'), (x, z')\}$ is a line of Type II of $\mathcal{S}_1(n)$. Let G denote the unique grid-quad of $\mathbb{I}_n$ containing the line $\{x, y', z'\}$ and let L denote the unique line of G through x different from $\{x, y', z'\}$. Then L is the unique line of $\mathbb{I}_n$ which is mapped by θ on the line $\{(x, x), (x, y'), (x, z')\}$ of $\mathcal{S}_1(n)$.

(c) Let $\{x, y, z\}$ be a line of Type III of $\mathbb{I}_n$ and let G be the grid-quad $\langle L, \pi_M(L)\rangle$ of $\mathbb{I}_n$. Put $\theta(x) = (x_1, x_2)$, $\theta(y) = (y_1, y_2)$ and $\theta(z) = (z_1, z_2)$. Then $\pi_M(L) = \{x_1, y_1, z_1\}$, $x_2, y_2, z_2 \in \pi_M(L)$, $x_1 \neq x_2$, $y_1 \neq y_2$ and $z_1 \neq z_2$. Let O_x, O_y and O_z be the unique elements of $\mathcal{O}_1$ such that $x \in O_x \subseteq G$, $y \in O_y \subseteq G$ and $z \in O_z \subseteq G$. Then $\{O_x, O_y, O_z\}$ is a partition of G. Since $O_x \cap M = \{x_2\}$, $O_y \cap M = \{y_2\}$ and $O_z \cap M = \{z_2\}$, $\pi_M(L) = \{x_2, y_2, z_2\}$. Now, since $x_1 \neq x_2$, $y_1 \neq y_2$ and $z_1 \neq z_2$, $\theta(L)$ must be a line of Type III of $\mathcal{S}_1(n)$.

Conversely, let $\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\}$ be a line of Type III of $\mathcal{S}_1(n)$. Let x denote the unique point of $\mathbb{I}_n$ for which $\theta(x) = (x_1, x_2)$. Then x is contained in the unique grid-quad G of $\mathbb{I}_n$ containing the line $\{x_1, y_1, z_1\} = \{x_2, y_2, z_2\}$. Let L denote the unique line of G through x different from xx_1. Then L is the unique line of $\mathbb{I}_n$ which is mapped by θ on $\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\}$.

(d) Let $L = \{x, y, z\}$ be a line of Type IV of $\mathbb{I}_n$. Put $\theta(x) = (x_1, x_2)$, $\theta(y) = (y_1, y_2)$ and $\theta(z) = (z_1, z_2)$. Then $\pi_M(L) = \{x_1, y_1, z_1\}$. Recall that $Q := \langle L, \pi_M(L)\rangle$ is a $W(2)$-quad. Let G_x denote the unique grid-quad of $\mathbb{I}_n$ containing the line $L_x = xx_1$ and let A denote the hex $\langle G_x, Q\rangle$. Since A contains a grid-quad, $A \cong \mathbb{I}_3$. So, the unique grid-quads G_y and G_z through respectively $L_y = yy_1$ and $L_z = zz_1$ are also contained in A. Now, let Q' denote the unique $W(2)$-quad of $A \cong \mathbb{I}_3$ through L_z different from Q. Then the reflection (in A) of G_x about Q' is a grid-quad through L_y which necessarily coincides with G_y. So, the lines $G_x \cap M$, $G_y \cap M$ and $Q' \cap M$ are contained in a grid-quad. It follows that the lines $G_x \cap M$, $G_y \cap M$ and $G_z \cap M$ are not contained in a grid-quad. Hence, the points $x_1, x_2, y_1, y_2, z_1, z_2$ are contained in the $W(2)$-quad $A \cap M$, but not in a grid-quad. Now, let O_x, O_y and O_z denote the unique elements of $\mathcal{O}_1$ such that $x \in O_x \subseteq G_x$, $y \in O_y \subseteq G_y$ and $z \in O_z \subseteq G_z$. Let O'_x denote the ovoid $\pi_{G_y}(O_x)$ of G_y (cf. Lemma 2.2). Since $(O_x, O_y) \in R$, $|O'_x \cap O_y| = 1$. Hence, $O'_x \cap O_y = \{y_1\}$. This implies that $x_2 \not\sim y_2$. In a similar way one shows that $y_2 \not\sim z_2$ and $x_2 \not\sim z_2$. It is now clear that $\theta(L)$ is a line of Type IV of $\mathcal{S}_1(n)$.

Conversely, suppose that $\{(x_1,x_2),(y_1,y_2),(z_1,z_2)\}$ is a line of Type IV of $\mathcal{S}_1(n)$. Let Q denote the unique $W(2)$-quad containing x_1, x_2, y_1, y_2, z_1 and z_2. Let x, y and z denote the unique points of $\mathbb{I}_n$ for which $\theta(x)=(x_1,x_2)$, $\theta(y)=(y_1,y_2)$ and $\theta(z)=(z_1,z_2)$. Let G_x (G_y, respectively G_z) denote the unique grid-quad of $\mathbb{I}_n$ containing x_1x_2 (y_1y_2, respectively z_1z_2). Then $x\in G_x$, $y\in G_y$ and $z\in G_z$. Let y' denote the unique point of G_y collinear with x (cf. Lemma 2.2) and let L be the line xy'. Since $\pi_M(y')\in y_1y_2$ and $\pi_M(y')\sim\pi_M(x)=x_1$, we have $\pi_M(y')=y_1$. So, $\theta(y')=(y_1,y_2')$ where y_2' is some point of $y_1y_2\setminus\{y_1\}$. By (a), (b) and (c), we know that L is a line of Type IV of $\mathbb{I}_n$ and by the first paragraph of (d), we know that $x_2\not\sim y_2'$. Hence, $y_2'=y_2$ and $y'=y$. It is also clear that the third point of the line xy must be mapped to the point (z_1,z_2). So, $L=\{x,y,z\}$. By the above discussion, L is the unique line of $\mathbb{I}_n$ which is mapped by θ on $\{(x_1,x_2),(y_1,y_2),(z_1,z_2)\}$. ■

4 Proof of Theorem 1.2

Let P denote the (common) point-set of $\mathcal{S}_1(n)$ and $\mathcal{S}_2(n)$. For every point x of $DQ(2n-2,2)$, we define $\theta[(x,x)]=(x,x)$. For every $(x,y)\in P$ with $x\neq y$, we define $\theta[(x,y)]=(z,y)$, where z denotes the third point on the line xy. Obviously, $\theta^2=Id_P$. So, θ is a permutation of the set P. We show that θ defines an isomorphism from $\mathcal{S}_1(n)$ to $\mathcal{S}_2(n)$.

Let $L=\{x,y,z\}$ be an arbitrary line of $DQ(2n-2,2)$. Then θ maps the line $\{(x,x),(y,y),(z,z)\}$ of $\mathcal{S}_1(n)$ to the line $\{(x,x),(y,y),(z,z)\}$ of $\mathcal{S}_2(n)$, the line $\{(x,x),(x,y),(x,z)\}$ of $\mathcal{S}_1(n)$ to the line $\{(x,x),(z,y),(y,z)\}$ of $\mathcal{S}_2(n)$ and the line $\{(x,y),(y,z),(z,x)\}$ of $\mathcal{S}_1(n)$ to the line $\{(z,y),(x,z),(y,x)\}$ of $\mathcal{S}_2(n)$. Clearly, every line $\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$ of $\mathcal{S}_2(n)$ where $\{x_1,x_2,x_3\}=\{y_1,y_2,y_3\}$ is a line of $DQ(2n-2,2)$ can be obtained in this way.

Now, let $\{(x,x'),(y,y'),(z,z')\}$ be an arbitrary line of Type IV of $\mathcal{S}_1(n)$. Let x'' (y'', respectively z'') denote the unique third point of the line xx' (yy', respectively zz'). We show that x'' is collinear with y'. Since y is the unique point of $\{x,y,z\}$ collinear with y', the points x and y' are not collinear. Now, also x' and y' are not collinear. It follows that x'' and y' are collinear. In a completely similar way one shows that $x''\sim z'$, $y''\sim x'$, $y''\sim z'$, $z''\sim x'$ and $z''\sim y'$. This implies that $\{x'',y'',z''\}$ and $\{x',y',z'\}$ are orthogonal hyperbolic lines of $DQ(2n-2,2)$. So, θ maps lines of Type IV of $\mathcal{S}_1(n)$ to lines of $\mathcal{S}_2(n)$.

Conversely, let $\{(x'', x'), (y'', y'), (z'', z')\}$ be a line of $\mathcal{S}_2(n)$, where $\{x'', y'', z''\}$ and $\{x', y', z'\}$ are two orthogonal hyperbolic lines of $DQ(2n-2, 2)$. Let x (y, respectively z) denote the unique third point of the line $x'x''$ ($y'y''$, respectively $z'z''$). The point x is not collinear with y' (since $y' \sim x''$) and y'' (since $y'' \sim x'$) and hence is collinear with y. In a similar way, one shows that $x \sim z$ and $y \sim z$. So, $\{x, y, z\}$ is a line of $DQ(2n-2, 2)$. Since x', y', z' are mutually noncollinear points of $DQ(2n-2, 2)$, the points x, y, z, x', y' and z' cannot be contained in a grid. It follows that $\{(x, x'), (y, y'), (z, z')\}$ is a line of $\mathcal{S}_1(n)$ which is mapped by θ to the line $\{(x'', x'), (y'', y'), (z'', z')\}$ of $\mathcal{S}_2(n)$. This finishes the proof that θ defines an isomorphism from $\mathcal{S}_1(n)$ to $\mathcal{S}_2(n)$.

5 Proof of Theorem 1.4

Lemma 5.1 *The points of $\mathcal{S}_2(n)$ at distance 1 from the point (x, x) are precisely the points (y, y) where $y \in \Gamma_1(x)$ and the points (y, z) where $\{x, y, z\}$ a line of $DQ(2n-2, 2)$ through x.*

Proof. Let (y, z) be a point of $\mathcal{S}_2(n)$ at distance 1 from (x, x). Then $y \in x^\perp \setminus \{x\}$ and $z \in \{x, y\}^\perp \setminus \{x\}$. If $\{x, y, z'\}$ denotes the line of $DQ(2n-2, 2)$ containing x and y, then $z \in \{y, z'\}$. This proves the lemma. ∎

Lemma 5.2 *Let $\{x, y, z\}$ be a line of $DQ(2n-2, 2)$. The points of $\mathcal{S}_2(n)$ at distance 1 from the point (x, y) are precisely the points (z, z), (y, x), (y, z), (z, x) and the points (u, v) where $u \in \Gamma_1(y) \cap \Gamma_2(x)$ and $v \in \Gamma_1(u) \cap \Gamma_1(x) \setminus \{y\}$.*

Proof. Let (u, v) be a point of $\mathcal{S}_2(n)$ at distance 1 from the point (x, y). Then $u \in y^\perp \setminus \{x\}$ and $v \in \{u, x\}^\perp \setminus \{y\}$. If $u \in \{x, y, z\}$, then $u \in \{y, z\}$ and $v \in \{u, x\}^\perp \setminus \{y\} = \{x, z\}$. This gives rise to the points (z, z), (y, x), (y, z) and (z, x). If $u \notin \{x, y, z\}$, then $u \in \Gamma_1(y) \cap \Gamma_2(x)$ and v is one of the two points contained in $\Gamma_1(u) \cap \Gamma_1(x) \setminus \{y\}$. ∎

Lemma 5.3 *Let $\{x, y, z\}$ be a line of $DQ(2n-2, 2)$. Then the points (x, x) and (x, y) of $\mathcal{S}_2(n)$ lie at distance 2 from each other and have precisely two common neighbours, namely the points (z, z) and (y, z).*

Proof. Clearly, the points (x, x) and (x, y) lie at distance at least 2 from each other. Suppose (u, v) is a common neighbour of (x, x) and (x, y). Then $u \in \{x, y\}^\perp = \{x, y, z\}$ and $u \neq x$. So, $u \in \{y, z\}$. Since $v \in \{x, u\}^\perp =$

$\{x, y, z\}$ and $v \notin \{x, y\}$, $v = z$. It follows that the points (x, x) and (x, y) have precisely two common neighbours, namely the points (y, z) and (z, z). ■

Lemma 5.4 *Let $\{x, y, z\}$ be a line of $DQ(2n-2, 2)$. Then the points (x, y) and (x, z) of $\mathcal{S}_2(n)$ lie at distance 2 from each other and have precisely two common neighbours, namely the points (y, x) and (z, x).*

Proof. Clearly, the points (x, y) and (x, z) lie at distance at least 2 from each other. Suppose (u, v) is a common neighbour of (x, y) and (x, z). Then $u \in \{y, z\}^{\perp\perp} = \{x, y, z\}$ and $u \neq x$. So, $u \in \{y, z\}$. Since $v \in \{x, u\}^{\perp} = \{x, y, z\}$ and $v \notin \{y, z\}$, $v = x$. It follows that the points (x, y) and (x, z) have precisely two common neighbours, namely (y, x) and (z, x). ■

Lemma 5.5 *Let x, y and z be points of $DQ(2n-2, 2)$ such that $d(x, y) = d(x, z) = 1$ and $d(y, z) = 2$. Put $\{y, z\}^{\perp} = \{x, u_1, u_2\}$ and $\{y, z\}^{\perp\perp} = \{y, z, v\}$. Then the points (x, y) and (x, z) of $\mathcal{S}_2(n)$ have precisely two common neighbours, namely the points (u_1, v) and (u_2, v).*

Proof. Clearly, the points (x, y) and (x, z) of $\mathcal{S}_2(n)$ lie at distance at least 2 from each other. Suppose (u', v') is a common neighbour of (x, y) and (x, z). Then $u' \in \{y, z\}^{\perp} = \{x, u_1, u_2\}$ and $u' \neq x$. So, $u' \in \{u_1, u_2\}$. Since $v' \in \{x, u'\}^{\perp} = \{y, z, v\}$ and $v' \notin \{y, z\}$, $v' = v$. It follows that the points (x, y) and (x, z) have two common neighbours, namely (u_1, v) and (u_2, v). ■

Lemma 5.6 *For every point x of $DQ(2n-2, 2)$, let $P_1(x) = \{(x, y) \,|\, y \in x^{\perp}\}$ and $P_2(x) = \{(y, x) \,|\, y \in x^{\perp}\}$. Then $P_1(x)$ and $P_2(x)$ are projective sets of $\mathcal{S}_2(n) \cong \mathbb{I}_n$. For every point (x, y) of $\mathcal{S}_2(n)$, $P_1(x)$ and $P_2(y)$ are the two projective sets of $\mathcal{S}_2(n)$ containing (x, y).*

Proof. Let (x, y) be an arbitrary point of $\mathcal{S}_2(n)$. We have $|P_1(x)| = |P_2(y)| = 2^n - 1$. By Lemmas 5.3, 5.4 and 5.5, if u and v are two distinct points of $P_1(x)$, then $\mathrm{d}(u, v) = 2$ and $\langle u, v\rangle$ is a grid-quad. By symmetry, the same conclusion also holds for two distinct points u and v of $P_2(y)$. Since there are precisely $2^{n-1} - 1$ grid-quads through every point of $\mathbb{I}_n$, $P_1(x)$ and $P_2(y)$ can be constructed in the following way: let G_j, $j \in \{1, \ldots, 2^{n-1} - 1\}$, denote all the $2^{n-1} - 1$ grid-quads of $\mathcal{S}_2(n)$ through (x, y), let $O_1^{(1)}$ and $O_1^{(2)}$ denote the two ovoids of G_1 containing (x, y) and let $O_j^{(i)}$, $i \in \{1, 2\}$ and $j \in \{2, \ldots, 2^{n-1} - 1\}$, denote the set of points of G_j at distance 2 from every point of $O_1^{(i)} \setminus \{(x, y)\}$. Then $\{P_1(x), P_2(y)\} = \{\bigcup_{j=1}^{2^{n-1}-1} O_j^{(i)} \,|\, i \in \{1, 2\}\}$.

Now, let P_1 and P_2 denote the two projective sets of $\mathcal{S}_2(n)$ through the point (x,y). Then $|P_1| = |P_2| = 2^n - 1$ and if u and v are two distinct points of P_i, $i \in \{1,2\}$, then $\mathrm{d}(u,v) = 2$ and $\langle u, v \rangle$ is a grid-quad. Similarly, as above, one then shows that $\{P_1, P_2\} = \{\bigcup_{j=1}^{2^{n-1}-1} O_j^{(i)} \,|\, i \in \{1,2\}\}$. Hence, we have $\{P_1(x), P_2(y)\} = \{P_1, P_2\}$. This proves the lemma. ■

The following proposition is precisely Theorem 1.4.

Proposition 5.7 *The point-line geometry $\mathcal{S}_3(n)$ is isomorphic to $DQ(2n,2)$.*

Proof. Consider the natural embedding of $\mathbb{I}_n$ into $DQ(2n,2)$. The dual polar space $DQ(2n,2)$ can be reconstructed in the following way from the near $2n$-gon $\mathbb{I}_n$: the points of $DQ(2n,2)$ not contained in $\mathbb{I}_n$ are in bijective correspondence with the projective sets of $\mathbb{I}_n$, the lines of $DQ(2n,2)$ not contained in $\mathbb{I}_n$ are in bijective correspondence with the sets $\{x, P_1, P_2\}$ where x is a point of $\mathbb{I}_n$ and where P_1 and P_2 are the two projective sets of $\mathbb{I}_n$ containing x. The proposition now follows from Theorem 1.2 and Lemma 5.6. ■

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