Partial ovoids and partial spreads of classical finite polar spaces

J. De Beule*, A. Klein, K. Metsch and L. Storme†

July 4, 2008

Abstract

We survey the main results on ovoids and spreads, large maximal partial ovoids and large maximal partial spreads, and on small maximal partial ovoids and small maximal partial spreads in classical finite polar spaces. We also discuss the main results on the spectrum problem on maximal partial ovoids and maximal partial spreads in classical finite polar spaces.

Keywords: Partial ovoids, partial spreads, classical finite polar spaces

MSC: 05B25, 51E20

1 Classical finite polar spaces

The classical finite polar spaces play an important role in incidence geometry. They consist of the non-singular quadrics, the non-singular hermitian varieties, and the symplectic spaces in projective spaces of odd dimension (see [7] and [27] for an introduction to polar spaces).

The interest in these incidence structures follows first of all from the fact that they are classical geometrical objects. The study of substructures contained in these classical finite polar spaces also contributes to their geometrical importance. The substructures involved include (partial) ovoids and (partial) spreads.

*The work on this survey article initiated while the first author was visiting the Justus-Liebig-Universität Gießen, Germany. The first author thanks the Fund for Scientific Research - Flanders (Belgium) for a research grant.

†The work on this survey article initiated while the fourth author was visiting the Justus-Liebig-Universität Gießen, Germany, with an Alexander von Humboldt Fellowship. The fourth author wishes to thank the Alexander von Humboldt Foundation for granting him this Fellowship.

‡The research of the fourth author was also supported by the Project *Combined algorithmic and theoretical study of combinatorial structures* between the Fund for Scientific Research Flanders-Belgium (FWO-Flanders) and the Bulgarian Academy of Sciences.

We survey the main results on these (partial) ovoids and (partial) spreads. The focus is first of all put on the known (non-)existence results on ovoids and spreads in classical finite polar spaces. Then the attention is drawn to the main results on large partial ovoids and large partial spreads. This is then followed by focusing on small maximal partial ovoids and small maximal partial spreads.

After the discussion of these results, attention is paid to the spectrum problem on maximal partial ovoids and maximal partial spreads.

The main results are gathered in tables to make them easily accessible to the readers. We also give information on some of the most important techniques used in obtaining these results.

To present the results, we proceed as follows.

In Section 2, we first present the basic definitions and results on classical finite polar spaces. Section 3 presents an overview of the main results. To conclude, we present in Section 4 the most important proof techniques.

2 Basic results

2.1 Polar spaces

A *polar space* is a geometry that satisfies the "one or all axiom":

- Let l be a line and let P be a point not on l. Then either P is connected to exactly one point of l by a line or P is connected to all points of l by lines.

A polar space contains projective spaces as subspaces. A projective subspace of maximal dimension is called a *generator*.

A polar space is *classical* if its points and lines are the totally isotropic points and lines of a projective space with respect to some non-degenerate sesquilinear form. There exist five different types of classical finite polar spaces:

- The elliptic quadric $Q^-(2n+1,q)$, $n \geq 2$, formed by all points of $PG(2n+1,q)$ which satisfy the standard equation $x_0x_1+\cdots+x_{2n-2}x_{2n-1}+f(x_{2n},x_{2n+1}) = 0$ where f is an irreducible polynomial of degree 2 over $\mathbb{F}_q$.
- The parabolic quadric $Q(2n,q)$, $n \geq 2$, formed by all points of $PG(2n,q)$ which satisfy the standard equation $x_0x_1 + \cdots + x_{2n-2}x_{2n-1} + x_{2n}^2 = 0$.
- The hyperbolic quadric $Q^+(2n+1,q)$, $n \geq 1$, formed by all points of $PG(2n+1,q)$ which satisfy the standard equation $x_0x_1+\cdots+x_{2n}x_{2n+1} = 0$.
- The symplectic polar space $W(2n+1,q)$, $n \geq 1$, which consists of all points of $PG(2n+1,q)$ together with the totally isotropic lines with respect to the standard symplectic form $\theta(x,y) = x_0y_1-x_1y_0+\cdots+x_{2n}y_{2n+1}-x_{2n+1}y_{2n}$.
- The hermitian polar space $H(n,q^2)$, $n \geq 3$, formed by all points of $PG(n,q^2)$ which satisfy the standard equation $x_0^{q+1} + \cdots + x_n^{q+1} = 0$.

For the results to come, it is useful to know the number of points and generators of these polar spaces. We summarise these numbers in the following theorems (for a proof, see for example [25]).

Theorem 1

- $Q^-(2n+1,q)$ *has* $\frac{(q^n-1)(q^{n+1}+1)}{q-1}$ *points.*
- $Q(2n,q)$ *has* $\frac{q^{2n}-1}{q-1}$ *points.*
- $Q^+(2n+1,q)$ *has* $\frac{(q^n+1)(q^{n+1}-1)}{q-1}$ *points.*
- $W(2n+1,q)$ *has* $\frac{q^{2n+2}-1}{q-1}$ *points.*
- $H(n,q^2)$ *has* $\frac{(q^{n+1}+(-1)^n)(q^n-(-1)^n)}{q^2-1}$ *points.*

Theorem 2

- $Q^-(2n+1,q)$ *has* $(q^2+1)(q^3+1)\cdots(q^{n+1}+1)$ *generators.*
- $Q(2n,q)$ *has* $(q+1)(q^2+1)(q^3+1)\cdots(q^n+1)$ *generators.*
- $Q^+(2n+1,q)$ *has* $2(q+1)(q^2+1)\cdots(q^n+1)$ *generators.*
- $W(2n+1,q)$ *has* $(q+1)(q^2+1)\cdots(q^{n+1}+1)$ *generators.*
- $H(2n,q^2)$ *has* $(q^3+1)(q^5+1)\cdots(q^{2n+1}+1)$ *generators.*
- $H(2n+1,q^2)$ *has* $(q+1)(q^3+1)\cdots(q^{2n+1}+1)$ *generators.*

2.2 Ovoids and spreads

Definition 3

Let P be a classical finite polar space.

A partial ovoid *$\mathcal{O}$ is a set of points with the property that no generator contains more than one point of $\mathcal{O}$. A partial ovoid $\mathcal{O}$ is* maximal *if it is not a proper subset of an other partial ovoid. A partial ovoid $\mathcal{O}$ is called an* ovoid *if every generator contains exactly one point of $\mathcal{O}$.*

A partial spread *$\mathcal{S}$ is a set of pairwise disjoint generators. A partial spread $\mathcal{S}$ is* maximal *if it is not a proper subset of an other partial spread. A partial spread $\mathcal{S}$ is called a* spread *if all points of P are covered by the elements of $\mathcal{S}$.*

¿From Theorem 1 and Theorem 2 we get directly the number of points in an ovoid and the number of generators in a spread.

Theorem 4

An ovoid in $Q^-(2n-1,q)$, $Q(2n,q)$, $Q^+(2n+1,q)$ or $W(2n-1,q)$ has q^n+1 points. An ovoid of $H(2n,q^2)$ or $H(2n+1,q^2)$ has $q^{2n+1}+1$ points.

A spread of $Q^-(2n+1,q)$, $Q(2n,q)$, $Q^+(2n+1,q)$ or $W(2n-1,q)$ contains q^n+1 generators. A spread of $H(2n,q^2)$ or $H(2n+1,q^2)$ contains $q^{2n+1}+1$ generators.

Proof. One has to do the proof for all different types of polar spaces. We show it on the example of a hyperbolic quadric $Q^+(2n+1,q)$.

By Theorem 2, $Q^+(2n+1,q)$ has $2(q+1)(q^2+1)\cdots(q^n+1)$ generators. Going to the factor space of one point, one sees a hyperbolic quadric $Q^+(2n-1,q)$, i.e. every point lies in $2(q+1)(q^2+1)\cdots(q^{n-1}+1)$ generators. Thus an ovoid must have $[2(q+1)(q^2+1)\cdots(q^n+1)]/[2(q+1)(q^2+1)\cdots(q^{n-1}+1)] = q^n+1$ elements.

By Theorem 1, $Q^+(2n+1,q)$ has $\frac{(q^n+1)(q^{n+1}-1)}{q-1}$ points. Each generator is a projective space of dimension n, that contains $\frac{q^{n+1}-1}{q-1}$ points. Thus a spread must contain $[\frac{(q^n+1)(q^{n+1}-1)}{q-1}]/[\frac{q^{n+1}-1}{q-1}] = q^n+1$ elements. □

2.3 Useful observations

There exist several relations between classical finite polar spaces that can be used to translate results. In this subsection, we summarise these relations.

2.3.1 Isomorphisms and anti-isomorphisms

- $Q(4,q)$ is isomorphic to the dual of $W(3,q)$. Thus every result on spreads of $Q(4,q)$ is also a result on ovoids of $W(3,q)$ and vice versa. Similarly, every result on ovoids of $Q(4,q)$ is also a result on spreads of $W(3,q)$ and vice versa [39].
- For q even, $Q(4,q)$ and $W(3,q)$ are self-dual. In this case, every result on ovoids is also a result on spreads [39].
- For q even, $Q(2n,q)$ is isomorphic to $W(2n-1,q)$. Thus every result on $Q(2n,q)$, q even, is also a result on $W(2n-1,q)$, q even [46].
- $Q^-(5,q)$ is isomorphic to the dual of $H(3,q^2)$ [39].

2.3.2 The Klein correspondence

The generators of $Q^+(5,q)$ fall into two groups. Two planes from the same group always intersect in a point. Two planes from different groups are either disjoint or intersect in a line. The geometry with the generators of type 1 as "points", the points of $Q^+(5,q)$ as "lines" and the generators of type 2 as "planes" is a 3-dimensional projective space $PG(3,q)$ (see [26] for details). By these properties, every (partial) ovoid of $Q^+(5,q)$ corresponds to a (partial) line spread of $PG(3,q)$.

An ovoid of $Q^+(5,q)$ is an elliptic quadric $Q^-(3,q)$ if and only if the corresponding line spread of $PG(3,q)$ is regular. An ovoid of $Q^+(5,q)$ is contained in a parabolic quadric $Q(4,q)$ contained in $Q^+(5,q)$ if and only if the corresponding line spread of $PG(3,q)$ is also a spread of a symplectic geometry $W(3,q)$.

2.3.3 Triality

In $Q^+(7,q)$, the generators fall into two groups; each group has size $(q+1)(q^2+1)(q^3+1)$ which is exactly the number of points of $Q^+(7,q)$. We call two generators from different groups incident if they intersect in a point. With this definition, the three classes of objects points, generators of group 1, and generators of group 2 become interchangeable (see [25] for more details and further references).

This means that every result on (partial) ovoids of $Q^+(7,q)$ is also a result on (partial) spreads and vice versa.

2.3.4 Intersections

Let $\mathcal{S}$ be a partial spread of $Q^+(2n+1,q)$. A hyperplane of $PG(2n+1,q)$ which is not a tangent hyperplane intersects $Q^+(2n+1,q)$ in a parabolic quadric $Q(2n,q)$. The intersection of $\mathcal{S}$ with that hyperplane forms a partial spread of this $Q(2n,q)$. Similar intersection arguments prove:

Theorem 5
If $Q^+(2n+1,q)$ has a partial spread of size s, then $Q(2n,q)$ has a partial spread of size s. Especially $Q(4n+2,q)$ has a spread if $Q^+(4n+3,q)$ has a spread.

If $Q(2n,q)$ has a partial spread of size s, then $Q^-(2n-1,q)$ has a partial spread of size s. Especially $Q^-(2n-1,q)$ has a spread if $Q(2n,q)$ has a spread.

If $H(2n+1,q^2)$ has a partial spread of size s, then $H(2n,q^2)$ has a partial spread of size s.

One can also show that a partial spread of size s in $Q(2n+2,q)$ implies a partial spread of size s in $Q^+(4n+3,q)$ (see Theorem 9).

If a polar space P can be embedded in an other polar space P', we know that every partial ovoid of P also is a partial ovoid of P'. We summarise these observations in the following theorem.

Theorem 6
If $Q^-(2n-1,q)$ has a partial ovoid of size s, then $Q(2n,q)$ has a partial ovoid of size s.

If $Q(2n,q)$ has a partial ovoid of size s, then $Q^+(2n+1,q)$ has a partial ovoid of size s. Especially $Q^+(2n+1,q)$ has an ovoid if $Q(2n,q)$ has an ovoid.

If $H(2n,q^2)$ has a partial ovoid of size s, then $H(2n+1,q^2)$ has a partial ovoid of size s.

2.3.5 Quadratic extensions

Consider the polar space $Q^-(4n+1,q)$. The ambient space $PG(4n+1,q)$ can be embedded in a quadratic extension $PG(4n+1,q^2)$. The polar space $Q^-(4n+1,q)$ extends to $Q^+(4n+1,q^2)$. On this $Q^+(4n+1,q^2)$, it is possible to choose a projective $2n$-space π with $\pi \cap \pi^\sigma = \emptyset$, where π^σ is the conjugate of π with respect to the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$. The lines of $Q^-(4n+1,q)$

whose extensions intersect π and π^σ form a line spread L of $Q^-(4n+1,q)$ and the intersections of these lines with π form a hermitian variety $H(2n,q^2)$. Let G be a generator of this $H(2n,q^2)$, then the union of lines of L whose extensions intersect in G form a generator G' of $Q^-(4n+1,q)$. Thus every (partial) spread of $H(2n,q^2)$ defines a (partial) spread of $Q^-(4n+1,q)$. Let $\mathcal{O}$ be a partial ovoid of $Q^-(4n+1,q)$. The lines of L through the points of $\mathcal{O}$ intersect $H(2n,q^2)$ in the points of a partial ovoid of $H(2n,q^2)$.

A similar correspondence exists between $Q^+(4n+3,q)$ and $H(2n+1,q^2)$. We summarise these correspondences in the following theorem.

Theorem 7
If $Q^-(4n+1,q)$, $n \geq 2$, has a partial ovoid of size s, then $H(2n,q^2)$ also has a partial ovoid of size s.

If $H(2n,q^2)$ has a partial spread of size s, then $Q^-(4n+1,q)$ also has a partial spread of size s.

If $Q^+(4n+3,q)$, $n \geq 1$, has a partial ovoid of size s, then $H(2n+1,q^2)$ also has a partial ovoid of size s.

If $H(2n+1,q^2)$ has a partial spread of size s, then $Q^+(4n+3,q)$ also has a partial spread of size s.

An other application of a quadratic extension can be found in [42].

3 Overview of the results

This section contains tables that summarise the known results on partial ovoids and partial spreads.

3.1 Ovoids and spreads

Tables 1 and 2 give the main known results on the (non-)existence of ovoids and spreads.

Non-existence proofs for ovoids and spreads are normally good upper bounds on the size of a partial ovoid or partial spread; see the next subsection for an overview on large partial ovoids and large partial spreads. Some rare exceptions to this rule are non-existence proofs for ovoids that use a classification of ovoids in a lower dimension. The non-existence of ovoids of $Q(6,q)$, $q > 3$, q prime [3], and $H(5,4)$ [13] are examples for this type of proof.

The existence of ovoids and spreads is proved by the construction of examples. The most difficult and interesting of these constructions are ovoids of $Q^+(7,q)$: see Subsection 4.3 for an example.

Open problems

The most interesting open problems on ovoids and spreads are:

Space	Existence	References
$Q^-(2n+1,q)$	$n>1$: No	[46]
$Q(4,q)$	Yes	[28, 32, 40, 49]
$Q(6,q)$	q even: No	[46]
	$q=3^n$: Yes	[28, 45, 46]
	$q>3$, q prime: No	[38]
$Q(2n,q)$	$n\geq 4$: No	[21, 46]
$Q^+(3,q)$	Yes	
$Q^+(5,q)$	Yes	[26]
$Q^+(7,q)$	$q=3^h$: Yes	[8, 28, 50]
	$q=2^h$: Yes	[15, 28]
	$q=p^h$, $p\equiv 2 \mod 3$, p prime, h odd: Yes	[28]
	q prime: Yes	[10, 36]
$Q^+(2n+1,q)$	$q=p^h$, p prime, $p^n>\binom{2n+p}{2n+1}-\binom{2n+p-2}{2n+1}$: No	[4]
$W(3,q)$	q even: Yes	[44]
	q odd: No	[44]
$W(2n+1,q)$	$n>1$: No	[46]
$H(2n,q^2)$	$n\geq 2$: No	[46]
$H(3,q^2)$	Yes	[39, 47, 49]
$H(5,4)$	No	[13]
$H(2n+1,q^2)$	$q=p^h$, p prime, $p^{2n+1}>\binom{2n+p}{2n+1}^2-\binom{2n+p-1}{2n+1}^2$: No	[37]

Table 1: Existence and non-existence results on ovoids

Space	Existence	References
$Q^-(5,q)$	Yes	[39, 47, 49]
$Q^-(2n+1,q)$	q even: Yes	[15, 45, 46]
$Q(2n,q)$	$n\geq 2$, q even: Yes	[15, 45, 46, 49]
$Q(6,q)$	q odd, q prime: Yes	
	q odd, $q\not\equiv 1 \mod 3$: Yes	[8, 10, 28, 36, 50]
$Q(4n,q)$	q odd: No	[44, 48]
$Q^+(4n+1,q)$	No	
$Q^+(3,q)$	Yes	
$Q^+(7,q)$	q odd, q prime: Yes	
	q odd, $q\not\equiv 1 \mod 3$: Yes	[8, 10, 28, 36, 50]
$Q^+(4n+3,q)$	q even: Yes	[15, 45, 46]
$W(2n+1,q)$	Yes	
$H(2n+1,q^2)$	No	[46, 48]
$H(4,4)$	No	[5]

Table 2: Existence and non-existence results on spreads

Space	Recursion	References
$Q^-(2n+1,q)$	$x_{n,q} \leq 2 + \frac{q^n+1}{q^{n-1}+1}(x_{n-1,q}-2)$	[29]
$Q(2n,q)$	$x_{n,q} \leq 1 + q(x_{n-1,q}-1)$	[12]
$Q^+(2n+1,q)$	$x_{n,q} \leq 2 + \frac{q^n-1}{q^{n-1}-1}(x_{n-1,q}-2)$	[12]
$W(2n+1,q)$	$x_{n,q} \leq 2 + (q-1)x_{n-1,q}$	[12]
$H(2n,q^2)$	$x_{n,q^2} \leq q^2 x_{n-1,q^2} - q^2 + 1$	[11]
$H(2n+1,q^2)$	$x_{n,q^2} \leq q^2 x_{n-1,q^2} - q^2 + 1$	[11]

Table 3: Recursive bounds on the size of partial ovoids

Space	Upper bound	Lower bound	References
$Q^-(5,q)$	$q=2$: $\frac{1}{2}(q^3+q+2)=6$ $q=3$: $\frac{1}{2}(q^3+q+2)=16$ $\frac{1}{2}(q^3+q+2)$	$q=2$: 6 $q=3$: 16 q^2+1	[11, 26] [11, 16] [1, 11]
$Q(6,q)$	$q>13$, q prime: q^3-2q+1		[12]
$Q(8,q)$	q odd, q not a prime: $q^4-q\sqrt{q}$		[12]
$W(5,q)$	$1+\frac{q}{2}(\sqrt{5q^4+6q^3+7q^2+6q+1}-q^2-q-1)$		[12]
$H(4,q^2)$	$q^5-q^4+q^3+1$	q^4+1	[11, 33]

Table 4: Large partial ovoids in small dimensions

- Lower the bound on n in Corollaries 12 and 14 stating that $H(2n+1,q^2)$ and $Q^+(2n+1,q)$, n large, have no ovoid.

  Conjecture: $H(5,q^2)$ and $Q^+(9,q)$ have no ovoid.

- Does $Q(6,q)$, $q>3$ odd, q not a prime, have an ovoid?

- Construct ovoids for $Q^+(7,q)$.

3.2 Large partial ovoids and large partial spreads

After the study of ovoids and spreads, the next natural question is the size of the largest partial ovoid or spread, when ovoids or spreads do not exist. Indeed, most non-existence results are in fact upper bounds on the largest size of a partial ovoid or partial spread.

For partial ovoids, we can obtain recursive bounds on the size of partial ovoids. Table 3 summarises these recursions. In Table 3, $x_{n,q}$ denotes the upper bound on the size of a partial ovoid on the corresponding classical finite polar space in $\mathrm{PG}(2n,q)$ or $\mathrm{PG}(2n+1,q)$.

Bounds on the size of partial ovoids in high dimension are obtained by applying these recursions to the bounds in small dimensions. Therefore, Table 4 lists only the small dimensions. The table contains only the cases in which the non-existence of ovoids has been proven. For convenience, we include Table 5 which gives the bounds of Table 4 combined with the recursive bounds of Table 3.

Space	Upper bound	Ref.
$W(2n+1,q)$, $q>2$	$2\frac{(q-1)^{n-2}-1}{q-2}+(q-1)^{n-2}\cdot q^3+1-q(\sqrt{q}-1)(q-\sqrt{q}+1)$	[12]
$Q(2n+2,q)$, q even, $q>2$	$2\frac{(q-1)^{n-2}-1}{q-2}+(q-1)^{n-2}\cdot q^3+1-q(\sqrt{q}-1)(q-\sqrt{q}+1)$	[12]
$Q(2n,q)$, $n\geq 4$, q odd, q not prime	$q^n+1-q^{n-4}(q\sqrt{q}+1)$	[12]
$Q(2n,q)$, $n\geq 3$, $q>13$ prime	$q^n-2q^{n-2}+1$	[12]
$Q^-(2n+1,q)$, $n\geq 3$	$2+\frac{1}{2}\cdot\frac{q^n-1}{q+1}\cdot(q^2+q+2)$	[12]
$H(2n,q^2)$, $n\geq 3$	$q^{2n+1}+1-q^{2(n-3)}(q^6-q^5-1)-q^3\cdot\frac{q^{2(n-2)}-1}{q^2-1}$	[11]
$H(2n+1,q^2)$, $n\geq 2$	$q^{2n+1}+1$	[11]

Table 5: Large partial ovoids

Space	Upper bound	Lower bound	References
$Q(4n,q)$	q odd: $q^n+1-\delta$		[20]
$Q^+(4n+1,q)$	2	2	
$H(3,q^2)$	$q=2$: 6	$q=2$: 6	[11, 26]
	$q=3$: 16	$q=3$: 16	[11, 16]
	$\frac{1}{2}(q^3+q+2)$	q^2+1	[1, 11]
$H(5,q^2)$	q^3+1	q^3+1	[14]
$H(2n+1,q^2)$	n odd: $q^{2n+1}-q^{\frac{1}{2}(3n+4)}+q^{\frac{1}{2}(3n+3)}$		[11]
	n even: $q^{2n+1}+1+q^n(q-1)-$		
	$-q^n\sqrt{q^{n+1}(q-1)+(q-1)^2}$		[11]

Table 6: Large partial spreads

Most of these results are proven by double counting (see Subsection 4.1). The upper bounds on the size of partial spreads use as additional proof technique extension results (see Subsection 4.2) and a special geometric property of $H(2n+1,q^2)$ (see Subsection 4.6). The upper bound $q^n+1-\delta$ is related to the problem of the classification of the blocking sets in $\mathrm{PG}(2,q)$. In the table entry for $Q(4n,q)$, $n\geq 2$, q odd, there always holds that $\delta\geq\epsilon$ where $q+1+\epsilon$ is the size of the smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$. In cases that the smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$ are characterized, larger values of δ are allowed. For instance, for q an odd square, $q>16$, the results of [19] imply that $\delta\geq q^{5/8}/\sqrt{2}+1$.

The lower bounds in Table 4 and Table 6 indicate the largest known partial ovoid or partial spread in the classical finite polar space. Note the sharpness of the bound q^3+1 on the size of a partial spread in $H(5,q^2)$.

Open problems

- Little is known about the construction of large partial ovoids and large partial spreads. Find good constructions (i.e. lower bounds) and improve the upper bounds.

Space	Upper bound	Lower bound	References
$Q^-(5,q)$	$q=2^{2h+1}, h\geq 1: \frac{q^2+7q+8\sqrt{2q}+10}{2}$ $+(2q+4\sqrt{2q}+4)\log(q^4-q^2)$	$q\geq 4$: $2q+2$	[12, 16, 35]
$Q^-(2n+1,q)$		$n\geq 3$: $2q+1$	[12]
$Q(2n,q)$		$n=2$, q odd: $1.419q$	[9, 12]
		$n=3$, $q=3,5,7$: $2q$	[12]
		$n=3$, q odd, $q\geq 9$: $2q-1$	[12]
		$n=4$, $q=3$: $2q$	[12]
		$n\geq 4$, q odd, $(n,q)\neq(4,3)$: $2q+1$	[12]
	q even: $q+1$	q even: $q+1$	[9, 12]
$Q^+(5,q)$		$2q$	[12, 17]
$Q^+(2n+1,q)$		$n\geq 3, q>3$: $2q+1$	[12]
$W(2n+1,q)$	$q+1$	$q+1$	[12, 9]
$H(3,q^2)$	$q>2$ even: q^2+1	$q>2$, q even: q^2+1	[2]
		q odd: $q^2+1+\frac{4}{9}q$	[34]
$H(4,q^2)$		q^2+q+2	[35]
$H(n,q^2)$		$n\geq 5$: q^2+q+1	[11]

Table 7: Small maximal partial ovoids

Space	Upper bound	Lower bound	References
$Q^-(2n+1,q)$		$n\geq 3$: q^2+1	[12]
$Q^+(4n+3,q)$		$q\geq 7$ and $n\geq 1$: $2q+1$	[31]
$W(2n+1,q)$		$q\geq 5$ and $n\geq 2$: $2q+1$	[31]
$H(2n,q^2)$		$n\geq 3$: q^3+1	[12]
$H(2n+1,q^2)$		$q\geq 13$ and $n\geq 2$: $2q+3$	[31]

Table 8: Small maximal partial spreads

- A very interesting case is that of large partial spreads of $H(3,q^2)$. The upper bound in this case is known to be sharp for $q=2$ and $q=3$. What happens for $q\geq 4$? The proof of the upper bound $\frac{1}{2}(q^3+q+2)$ gives many properties that a potential example must have (see Subsection 4.6 and [11]).

3.3 Small maximal partial ovoids and small maximal partial spreads

Small maximal partial ovoids and small maximal partial spreads have recently drawn much attention. The proof techniques in this area are mostly variations of the triple counting technique (see Subsection 4.5), sometimes combined with a geometric property like the one described in Subsection 4.6. Table 7 and Table 8 present the known results on small maximal partial ovoids and small maximal partial spreads.

Space	Interval	References
$Q(4,q)$, $q=2^{4h}$, $h\geq 2$	$\frac{q^2+194q+10q\lfloor 48\log(q+1)\rfloor-190}{10}\leq k\leq\frac{9q^2-69q+440}{10}$	[41]
$Q(4,q)$, $q=2^{4h+1}$, $h\geq 2$	$\frac{q^2+198q+10q\lfloor 48\log(q+1)\rfloor-230}{10}\leq k\leq\frac{9q^2-68q+430}{10}$	[41]
$Q(4,q)$, $q=2^{4h+2}$, $h\geq 1$	$\frac{q^2+196q+10q\lfloor 48\log(q+1)\rfloor-210}{10}\leq k\leq\frac{9q^2-66q+410}{10}$	[41]
$Q(4,q)$, $q=2^{4h+3}$, $h\geq 1$	$\frac{q^2+192q+10q\lfloor 48\log(q+1)\rfloor-170}{10}\leq k\leq\frac{9q^2-67q+420}{10}$	[41]

Table 9: Spectrum on maximal partial spreads

Space	Interval	References
$Q^+(5,q)$, q odd, $q\geq 7$	$\frac{q^2+13}{2}\leq k\leq q^2-q+2$	[22, 23, 24]
$Q^+(5,q)$, q even, $q\geq q_0$	$\frac{5q^2+q+16}{8}\leq k\leq q^2-q+2$	[18]
$Q(4,q)$, $q=2^{4h}$, $h\geq 2$	$\frac{q^2+194q+10q\lfloor 48\log(q+1)\rfloor-190}{10}\leq k\leq\frac{9q^2-69q+440}{10}$	[41]
$Q(4,q)$, $q=2^{4h+1}$, $h\geq 2$	$\frac{q^2+198q+10q\lfloor 48\log(q+1)\rfloor-230}{10}\leq k\leq\frac{9q^2-68q+430}{10}$	[41]
$Q(4,q)$, $q=2^{4h+2}$, $h\geq 1$	$\frac{q^2+196q+10q\lfloor 48\log(q+1)\rfloor-210}{10}\leq k\leq\frac{9q^2-66q+410}{10}$	[41]
$Q(4,q)$, $q=2^{4h+3}$, $h\geq 1$	$\frac{q^2+192q+10q\lfloor 48\log(q+1)\rfloor-170}{10}\leq k\leq\frac{9q^2-67q+420}{10}$	[41]

Table 10: Spectrum on maximal partial ovoids

3.4 Spectrum results

A *spectrum result* on maximal partial ovoids or maximal partial spreads is a result stating that for a large interval for the parameter k, there exist maximal partial ovoids or maximal partial spreads of size k for every integer in that interval. In Tables 9 and 10, we present results on the spectrum of maximal partial ovoids and maximal partial spreads on $Q(4,q)$, q even. (Here, $\lfloor x\rfloor$ denotes the largest integer smaller than or equal to x.) Note that these results are also valid for $W(q)$, q even, since this classical generalised quadrangle is isomorphic to $Q(4,q)$, q even (see Subsection 2.3.1).

By the Klein correspondence (Subsection 2.3.2), partial ovoids of $Q^+(5,q)$ are linked to partial spreads of $\mathrm{PG}(3,q)$, for which there are important results of Heden [22, 23, 24].

The results of [22, 23, 24] originally were stated as a spectrum result on maximal partial spreads of $\mathrm{PG}(3,q)$. As an additional application, they were used to obtain a spectrum-like result on maximal partial spreads of the elliptic quadric $Q^-(5,q)$.

Theorem 8 (see [6])
Suppose that $\mathcal{S}$ is a maximal partial spread of $\mathrm{PG}(3,q)$ of size $q^2+1-\delta$. Then there exists a maximal partial spread of $Q^-(5,q)$ of size $q^3+1-q\delta$.

Theorem 9 (see [48])
Let $n\geq 1$. Then $Q^+(4n+3,q)$ has a maximal partial spread of size s if and only if $Q(4n+2,q)$ has a maximal partial spread of size s.

Proof. We will use the intersection technique from Subsection 2.3.4.

The generators of $Q^+(4n+3,q)$ fall into two groups and generators of the same group intersect in a space of odd dimension, while generators of different groups intersect in a space of even dimension. Thus every partial spread of $Q^+(4n+3,q)$ must consist of generators of the same class.

Now take a hyperplane π of $\mathrm{PG}(4n+3,q)$ intersecting $Q^+(4n+3,q)$ in a parabolic quadric $Q(4n+2,q)$.

Let $\mathcal{S}$ be a partial spread of generators of type 1 of $Q^+(4n+3,q)$. Then the intersections of elements of $\mathcal{S}$ with π define a partial spread $\mathcal{S}'$ of $Q(4n+2,q)$.

On the other hand, let $\mathcal{S}'$ be a partial spread of $Q(4n+2,q)$. Every element G of $\mathcal{S}'$ lies in two generators of $Q^+(4n+3,q)$; one of each type. Thus there exists a unique set $\mathcal{S}$ of generators of type 1 in $Q^+(4n+3,q)$ with the property that the intersection of the elements of $\mathcal{S}$ with π defines $\mathcal{S}'$. Furthermore, since $\mathcal{S}'$ is a partial spread of $Q(4n+2,q)$, two different elements of $\mathcal{S}$ can share at most a point. But since the intersection of two generators of type 1 must be of odd dimension, this means that each two generators in $\mathcal{S}$ must be skew, i.e. $\mathcal{S}$ is a partial spread of $Q^+(4n+3,q)$.

This establishes a one-to-one correspondence between the partial spreads of $Q^+(4n+3,q)$ which only contain generators of type 1 and the partial spreads of $Q(4n+2,q)$, which proves the theorem.

The same proof shows the equivalence between the maximality of the partial spread of $Q^+(4n+3,q)$ and the induced partial spread of $Q(4n+2,q)$. □

4 Proof techniques

We now present a number of the most important techniques involved in obtaining the results of the tables.

4.1 Double counting

Double counting is one of the most basic proof techniques in combinatorics. One of the simplest applications of double counting in this topic on partial spreads and partial ovoids is the non-existence of ovoids in $Q^-(2n+1,q)$, $W(2n+1,q)$, and $H(2n,q^2)$ for $n>1$, see [46]. We demonstrate the technique for $Q^-(2n+1,q)$. For a classical finite polar space P, let $O(P)$ be the size of the largest partial ovoids of P.

Theorem 10 (see [29])
For $n \geq 3$, we have

$$O(Q^-(2n+1,q)) \leq \frac{q^n+1}{q^{n-1}+1}(O(Q^-(2n-1,q))-2)+2. \tag{1}$$

Proof. Let $\mathcal{O}$ be a partial ovoid of $Q^-(2n+1,q)$, with $|\mathcal{O}| = O(Q^-(2n+1,q))$. Let θ be the polarity of $PG(2n+1,q)$ corresponding to $Q^-(2n+1,q)$. Let $x,y \in \mathcal{O}$, with $x \neq y$. Define $\pi = (xy)^\theta$. Then π is a $(2n-1)$-dimensional

subspace and $\pi \cap Q^-(2n+1,q)$ is an elliptic quadric $Q^-(2n-1,q) = Q$. Since $x, y \in \pi^\theta$, we conclude that $\pi \cap \mathcal{O} = \emptyset$.

Now we count the number N of pairs (u, v), with $u \in Q$, $v \in \mathcal{O} \backslash \{x, y\}$, such that uv is a line of $Q^-(2n+1,q)$. For each possible v, the point u must lie in the non-singular quadric $(xyv)^\theta \cap Q^-(2n+1,q)$. This quadric contains $\frac{q^{2n-2}-1}{q-1}$ points, i.e. $N = (O(Q^-(2n+1,q)) - 2)\frac{q^{2n-2}-1}{q-1}$.

On the other hand, for each $u \in Q$, the points of $\mathcal{O} \cap u^\theta$ define a partial ovoid in u^θ/u. Since $x, y \in \mathcal{O} \cap u^\theta$, for each possible $u \in Q$, there exist at most $O(Q^-(2n-1,q)) - 2$ corresponding points v of $\mathcal{O}$. Therefore we have $N \leq \frac{(q^{n-1}-1)(q^n+1)}{q-1}(O(Q^-(2n-1,q)) - 2)$.

All together we get:

$$(O(Q^-(2n+1,q)) - 2)\frac{q^{2n-2}-1}{q-1} \leq \frac{(q^{n-1}-1)(q^n+1)}{q-1}(O(Q^-(2n-1,q)) - 2).$$

Simplifying, we get (1). □

Perhaps the most surprising application of double counting is the case of $H(2n+1,q^2)$.

Lemma 11 (see [29])
Let $\mathcal{O}$ be an ovoid of $H(2n+1,q^2)$. Let π be a plane which intersects $H(2n+1,q^2)$ in a polar space of type $H(2,q^2)$. Let $m = |\pi \cap \mathcal{O}| \geq 2$.

Then there exists an ovoid $\mathcal{O}'$ of $H(2n-1,q^2)$ and a plane π', which intersects $H(2n-1,q^2)$ in a polar space of type $H(2,q^2)$, and with $|\pi' \cap \mathcal{O}'| > m$.

An inductive application of this lemma gives the following corollary.

Corollary 12 (see [29])
The hermitian polar space $H(2n+1,q^2)$, $n > q^3$, has no ovoid.

This bound is weaker than the bound found by algebraic methods (see Subsection 4.4), but it gives additional information about the possible ovoids in small dimension. In [13], J. De Beule and K. Metsch use Lemma 11 together with the classification of ovoids of $H(3,4)$ to prove that $H(5,4)$ has no ovoid.

We will not prove Lemma 11 in this article. Instead, we will present a similar argument for hyperbolic quadrics.

Lemma 13
Assume that the hyperbolic quadric $Q^+(2n+1,q)$, $n > 2$, has an ovoid $\mathcal{O}$. Let π be a 3-dimensional space that contains $m \geq 1$ points of $\mathcal{O}$ and which intersects $Q^+(2n+1,q)$ in an elliptic quadric $Q^-(3,q)$.

Then the number a of points of $\mathcal{O}$ that generate together with π a 4-space that intersects $Q^+(2n+1,q)$ in a cone over the elliptic quadric $Q^-(3,q)$ is

$$a = q^{n-1} - q^{n-2} - q + 1 + m(q^{n-2} - 1).$$

Proof. The proof is inspired by Thas [48].

Let b be the number of points of $\mathcal{O}$ that generate together with π a 4-space that intersects $Q^+(2n+1,q)$ in a parabolic quadric $Q(4,q)$. Then $a+b = q^n+1-m$.

Now we count the number N of pairs (P,O) with the following properties: P lies in $Q^+(2n+1,q) \cap \pi^\perp$, O is a point of $\mathcal{O}$ not in π, and PO is a line of $Q^+(2n+1,q)$.

Now $\pi^\perp$ intersects $Q^+(2n+1,q)$ in an elliptic quadric $Q^-(2n-3,q)$. Each point of that $Q^-(2n-3,q)$ is connected to $q^{n-1}+1$ points of the ovoid $\mathcal{O}$, thus

$$N = \frac{(q^{n-2}-1)(q^{n-1}+1)}{q-1}(q^{n-1}+1-m).$$

For each of the a points O that generate together with π a 4-space that contains a cone over the elliptic quadric $Q^-(3,q)$, we find that $\langle \pi, O\rangle^\perp$ intersects $Q^+(2n+1,q)$ in a cone over an elliptic quadric $Q^-(2n-5,q)$, not containing any points of $\mathcal{O}$, for otherwise these points of $\mathcal{O}$ would be collinear with the points of $\mathcal{O}$ in π. Thus each of these points is counted in $\frac{(q^{n-3}-1)(q^{n-2}+1)}{q-1}q+1$ pairs.

Similarly, for each of the b other points of $\mathcal{O}\setminus\pi$, we find that $\langle \pi, O\rangle^\perp$ intersects $Q^+(2n+1,q)$ in a parabolic quadric $Q(2n-4,q)$, i.e. these points are counted in $\frac{q^{2n-4}-1}{q-1}$ pairs. Thus

$$N = a(\frac{(q^{n-3}-1)(q^{n-2}+1)}{q-1}q+1) + b\frac{q^{2n-4}-1}{q-1}.$$

Now we have two linear equations in a and b. Solving the system proves the lemma. □

As a corollary, we obtain the following result.

Corollary 14
$Q^+(2n+1,q)$, $n > q^2$, has no ovoid.

Proof. Suppose that $Q^+(2n+1,q)$ has an ovoid. Then we can find a 3-dimensional space π intersecting $Q^+(2n+1,q)$ in an elliptic quadric $Q^-(3,q)$ that contains at least 3 points of the ovoid. By Lemma 13, there exists a cone over that elliptic quadric that contains more than 3 points of $\mathcal{O}$. Projecting this cone, we find an ovoid of $Q^-(2n-1,q)$ with a 3-dimensional space π' intersecting $Q^-(2n+1,q)$ in an elliptic quadric $Q^-(3,q)$ that contains at least 4 points of this ovoid.

Repeating the argument $n-2$ times, we find an ovoid of $Q^+(5,q)$ which has $n+1$ points inside a 3-space intersecting $Q^+(5,q)$ in an elliptic quadric $Q^-(3,q)$. Since $Q^-(3,q)$ has only q^2+1 points, we find $n \leq q^2$. □

4.2 Extension results

Extension results have the following form.

Suppose that a partial ovoid or partial spread of a classical finite polar space P has almost as many points or generators as an ovoid or spread of P, then it can be extended to an ovoid or spread of P.

Depending on whether an ovoid or a spread exists or not, we either obtain a spectrum result or an upper bound on the size of a partial ovoid or spread. Let us have a look at an example.

Theorem 15 (see [30])
Suppose that S is a maximal partial spread of $Q(4,q)$. Then either S is a spread or $|S| \leq q^2 - q + 1$.

Proof. Put $\delta := q^2 + 1 - |S|$. We assume that $0 < \delta < q$ and derive a contradiction. Let H be the set of $\delta(q+1)$ points of $Q(4,q)$ not covered by the elements of S. We call these points *holes*. Since S is maximal, H contains no line.

Embed $Q(4,q)$ in the natural way in $PG(4,q)$. Every hyperplane of $PG(4,q)$ meets $Q(4,q)$ in 1 (mod q) points. As $|S| = q^2 + 1 - \delta$, it follows that every hyperplane meets H in δ (mod q) points. As $\delta < q$, this implies that every hyperplane contains at least δ holes.

Consider a hole P. The tangent hyperplane $P^\perp$ on P meets $Q(4,q)$ in a cone with vertex P over a parabolic quadric $Q(2,q)$. Every line of S meets $P^\perp$ in a unique point. As $P^\perp$ contains $q^2 + q + 1$ points of $Q(4,q)$, then $P^\perp$ contains $q + \delta$ holes. If π is a plane of $P^\perp$, then each of the q hyperplanes on π other than $P^\perp$ contains at least δ holes. As the number of holes is $(q+1)\delta$ and as $P^\perp$ contains more than δ holes, it follows that π must contain a hole. Hence, $P^\perp \cap H$ meets every plane of $P^\perp$, i.e. the $q+\delta$ holes in $P^\perp$ form a blocking set w.r.t. the planes in $P^\perp$.

Blocking sets which lie in a quadric were studied in [30], where it was shown that in the case above, the blocking set must contain a line. This is however a contradiction to the maximality of S. □

Since for q odd, it is known that $Q(4,q)$ has no spread (Table 2), this proves that a partial spread of $Q(4,q)$, q odd, has at most q^2-q+1 elements. This was first proven in [43]. For q even, $Q(4,q)$ has a spread (Table 2) and the above result gives us information on the upper part of the spectrum (which was first proven in [6]). The above proof is the first that works for all q. It can also be generalised to $Q^-(5,q)$.

Theorem 16 (see [30])
Suppose that $\mathcal{S}$ is a maximal partial spread of $Q^-(5,q)$. Then either $\mathcal{S}$ is a spread or $|\mathcal{S}| \leq q^3 - q + 1$.

4.3 Ovoids of $Q^+(7,q)$

The hyperbolic quadric $Q^+(7,q)$ is the only polar space for which ovoids are known to exist for some q but no construction is known for all q. Currently, no ovoids of $Q^+(7,q)$ are known for $q \equiv 1 \pmod 6$ and q not a prime. For all other q, ovoids are known.

The known constructions of ovoids are via coordinates and large automorphism groups. As an example of the methods used in this case, we sketch the construction of ovoids of $Q^+(7,q)$, $q = p^h$, $p \equiv 2 \pmod 3$, h odd, found by Kantor [28].

In $\mathbb{F}_q$, we let $\bar{\alpha} = \alpha^q$, $\mathrm{Tr}(\alpha) = \alpha + \bar{\alpha}$ and $N(\alpha) = \alpha \cdot \bar{\alpha}$.

Let

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and let

$$\begin{aligned} V &= \{M \mid \mathrm{trace}(M) = 0, J^{-1}MJ = \bar{M}^{tr}\} \\ &= \left\{ \begin{pmatrix} \alpha & \beta & c \\ \gamma & a & \bar{\beta} \\ b & \bar{\gamma} & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta, \gamma \in \mathbb{F}_{q^2}, a, b, c \in \mathbb{F}_q, a + Tr(\alpha) = 0 \right\}. \end{aligned}$$

$$Q = \left\{ \begin{pmatrix} \alpha & \beta & c \\ \gamma & a & \bar{\beta} \\ b & \bar{\gamma} & \bar{\alpha} \end{pmatrix} \mid \alpha^2 + \alpha\bar{\alpha} + \bar{\alpha}^2 + Tr(\beta\gamma) + bc = 0 \right\}$$

defines a hyperbolic quadric $Q^+(7,q)$ on V. Let

$$\mathcal{O} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \bar{\rho} & \bar{\rho}\sigma & N(\rho) \\ \bar{\sigma} & N(\sigma) & \rho\bar{\sigma} \\ 1 & \sigma & \rho \end{pmatrix} \middle| Tr(\rho) + N(\sigma) = 0 \right\}.$$

Note that $\mathrm{PG}(3,q)$ acts 2-transitively on $\mathcal{O}$, so to prove that $\mathcal{O}$ is an ovoid, it is sufficient to see that

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

are not collinear on Q.

4.4 Algebraic techniques

For each classical finite polar space $\mathcal{P}$, we define the incidence matrix $I(\mathcal{P}) = (i_{P,G})$, where P runs over all points of $\mathcal{P}$ and G runs over all generators of the polar space $\mathcal{P}$, by $i_{P,G} = 1$ if $P \in G$ and $i_{P,G} = 0$ if $P \notin G$.

Now let $\mathcal{O}$ be a partial ovoid of $\mathcal{P}$. For each $P \in \mathcal{O}$, choose a generator G_P through P and let $\mathcal{O}' = \{G_P \mid P \in \mathcal{O}\}$.

The incidence matrix $I(\mathcal{P})$ contains the submatrix $(i_{P,G})_{P\in\mathcal{O},G\in\mathcal{O}'}$, which is the identity matrix. Therefore, $I(\mathcal{P})$ has at least the rank $|\mathcal{O}|$.

Thus we get an upper bound on the size of the partial ovoids in $\mathcal{P}$ if we compute the rank of $I(\mathcal{P})$. We are still free to choose the characteristic of the underlying field. The appropriate rank to compute is the p-rank, where p is the characteristic of the field we used in the definition of $\mathcal{P}$.

The determination of the p-ranks is rather complicate, so we state only the results. For the proofs, see [4, 37].

Result 17
Let $q = p^e$, p prime, $e \geq 1$, then

$$\operatorname{rank}_p(I(\mathrm{PG}(n,q))) = \binom{p+n-1}{n}^e + 1.$$

If Q is the incidence structure of all points of a non-singular quadric of $\mathrm{PG}(n,q)$ and all hyperplanes of $\mathrm{PG}(n,q)$, then

$$\operatorname{rank}_p(I(Q)) = \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^e + 1.$$

If H is the incidence structure of all points of a non-singular hermitian variety in $\mathrm{PG}(n,q^2)$ and all hyperplanes of $\mathrm{PG}(n,q^2)$, then

$$\operatorname{rank}_p(I(H)) = \left[\binom{2n+p}{2n+1}^2 - \binom{2n+p-1}{2n+1}^2\right]^e + 1.$$

If n and q are even, then for any quadric $Q^*(n,q)$,

$$\operatorname{rank}_p(I(Q^*(n,q))) = n^e + 1.$$

Again, the exact values for the rank of these incidence matrices imply upper bounds on the sizes of partial ovoids of the considered classical finite polar spaces $\mathcal{P}$. In fact, for large dimensions n, depending on the characteristic p of the underlying field over which $\mathcal{P}$ is defined, these dimensions are smaller than the size of ovoids of $\mathcal{P}$; hence, they exclude the possibility of ovoids of $\mathcal{P}$. For the precise bounds, we refer to Table 1.

It is remarkable that even the p-rank of the incidence matrix of $\mathrm{PG}(n,q)$ already excludes ovoids in large dimensions.

4.5 Counting of triples

In 1982, Glynn introduced a new counting technique to prove a lower bound on the size of maximal partial spreads of $\mathrm{PG}(3,q)$ [17]. By the Klein correspondence, this is equal to a result on small maximal partial ovoids of $Q^+(5,q)$. The proof of Glynn becomes simpler inside $Q^+(5,q)$; indeed it has a simple generalisation to (hyperbolic) quadrics of any dimension.

Theorem 18 (see [12])
A maximal partial ovoid of $Q^+(5,q)$ has at least $2q$ points. A maximal partial ovoid of $Q^+(2n+1,q)$, $n \geq 3$, has at least $2q+1$ points.

Proof. Let $\mathcal{O}$ be a maximal partial ovoid of $Q^+(2n+1,q)$. Let $w = |\mathcal{O}|$ and denote by n_i the number of points of $Q^+(2n+1,q)\backslash\mathcal{O}$ that are joined to exactly i points of $\mathcal{O}$ by lines of $Q^+(2n+1,q)$. Then we have:

$$\sum_i n_i = |Q^+(2n+1,q)| - w,$$
$$\sum_i n_i i = wq|Q^+(2n-1,q)|,$$
$$\sum_i n_i i(i-1) = w(w-1)|Q^+(2n-1,q)|,$$
$$\sum_i n_i i(i-1)(i-2) = w(w-1)(w-2)|Q(2n-2,q)|.$$

The first equation just states that every point of $Q^+(2n+1,q)\setminus\mathcal{O}$ is counted once. The second equation is obtained by double counting pairs (u,v), with $u \in Q^+(2n+1,q)\backslash\mathcal{O}$ and v in $\mathcal{O}$, such that uv is a line of $Q^+(2n+1,q)$. The third equation is obtained by counting triples (u,v_1,v_2), with $u \in Q^+(2n+1,q)\backslash\mathcal{O}$ and $v_1 \neq v_2$ in $\mathcal{O}$, such that uv_1 and uv_2 are lines of $Q^+(2n+1,q)$. The last equation is obtained by counting quadruples (u,v_1,v_2,v_3), with $u \in Q^+(2n+1,q)\backslash\mathcal{O}$ and v_1,v_2,v_3 distinct points of $\mathcal{O}$, such that uv_1, uv_2 and uv_3 are lines of $Q^+(2n+1,q)$. Note that three points of $\mathcal{O}$ always span a plane that meets $Q^+(2n+1,q)$ in a conic $Q(2,q)$, so that their polar space intersects $Q^+(2n+1,q)$ in a parabolic quadric $Q(2n-2,q)$.

As the partial ovoid $\mathcal{O}$ is maximal, we have $n_0 = 0$. Hence,

$$\begin{aligned} 0 &\leq \sum_i n_i(i-1)(i-3)(i-4) \\ &= \sum_i n_i i(i-1)(i-2) - 5\sum_i n_i i(i-1) + 12\sum_i n_i i - 12\sum_i n_i. \end{aligned}$$

Solving this inequality, we obtain $w > 2q-1$ for $n=2$ and $w > 2q$ for $n \geq 3$.
□

4.6 A geometric property of $H(2n+1,q^2)$

This section is devoted to results that use a special property of hermitian spaces, first proven in [48].

Result 19
Let π_1, π_2 and π be mutually skew generators of $H(2n+1,q^2)$. Then the points of π that lie on a line of $H(2n+1,q^2)$, meeting π_1 and π_2, form a hermitian variety $H(n,q^2)$ in π.

One of the best applications of this property is an upper bound on the size of maximal partial spreads of $H(3,q^2)$.

Theorem 20 (see [11])
A partial spread of $H(3,q^2)$ has at most $\frac{1}{2}(q^3+q+2)$ elements.

Proof. Suppose that $\mathcal{S}$ is a partial spread of $H(3,q^2)$ and that $|\mathcal{S}| = q^3+1-\delta$. Then the number of points of $H(3,q^2)$ not covered by lines of $\mathcal{S}$ is $h = \delta(q^2+1)$. We call these points *holes*.

Consider triples (S_1,S_2,P), where S_1 and S_2 are different elements of $\mathcal{S}$ and where P is a hole. We shall estimate how many of these triples have the property that the unique line of $PG(3,q^2)$ on P that meets S_1 and S_2 belongs to $H(3,q^2)$.

To do so, we consider a hole P. Then P lies on $q+1$ lines of $H(3,q^2)$. If x_i, $i=1,\ldots,q+1$, is the number of points on the i-th line on P covered by an element of $\mathcal{S}$, then we have $\sum x_i = |\mathcal{S}|$ and hence

$$\sum x_i(x_i-1) \geq (q+1)\frac{|\mathcal{S}|}{q+1}\left(\frac{|\mathcal{S}|}{q+1}-1\right) .$$

Since the number of holes is $\delta(q^2+1)$, we find a lower bound on the number of triples.

Now choose a pair (S_1,S_2) of distinct spread elements. There are q^2+1 lines of $H(3,q^2)$ that meet S_1 and S_2. These lines cover $(q^2+1)(q^2-1)$ points of $H(3,q^2)$ not on S_1 and S_2. By Result 19, every line of $\mathcal{S}\backslash\{S_1,S_2\}$ contains $q+1$ of these points. Thus there are $(q^2+1)(q^2-1)-(|\mathcal{S}|-2)(q+1)$ holes.

Together with the lower bound, this gives

$$|\mathcal{S}|(|\mathcal{S}|-1)\left[(q^2+1)(q^2-1)-(|\mathcal{S}|-2)(q+1)\right] \geq (q^3+1-|\mathcal{S}|)(q^2+1)|\mathcal{S}|\left(\frac{|\mathcal{S}|}{q+1}-1\right).$$

After simplification, we obtain $|\mathcal{S}| \leq \frac{1}{2}(q^3+q+2)$. □

What makes this bound so remarkable is that for $q=2$ and $q=3$, the bound is sharp [15, 16]. But for $q \geq 4$, we do not know whether this bound is sharp [11].

An other theorem that makes use of the geometric property (Result 19) is the following result on small maximal partial spreads.

Theorem 21 (see [31])
For $n \geq 2$ and $q \geq 13$, every maximal partial spread of $H(2n+1,q^2)$ has at least $2q+3$ generators.

In [31], similar geometric properties were proven for $Q^+(4n+3,q)$ and $W(2n+1,q)$, leading to lower bounds on small maximal partial spreads in these polar spaces (see Table 8).

References

[1] A. Aguglia, A. Cossidente, and G.L. Ebert. Complete spans on hermitian varieties. *Des. Codes Cryptogr.*, 29:7–15, 2003.

[2] A. Aguglia, G.L. Ebert, and D. Luyckx. On partial ovoids of Hermitian surfaces. *Bull. Belgian Math. Soc., Simon Stevin*, 12(5):641–650, 2005.

[3] S. Ball, P. Govaerts, and L. Storme. On ovoids of parabolic quadrics. *Des. Codes Cryptogr.*, 38:131–145, 2006.

[4] A. Blokhuis and G.E. Moorhouse. Some p-ranks related to orthogonal spaces. *J. Algebraic Combin.*, 4:295–316, 1995.

[5] A.E. Brouwer. (Private communication).

[6] M.R. Brown, J. De Beule, and L. Storme. Maximal partial spreads of $T_2(O)$ and $T_3(O)$. *European J. Combin.*, 24:73–84, 2003.

[7] P.J. Cameron. *Projective and polar spaces*, volume 13 of *QMW Maths Notes*. Queen Mary and Westfield College School of Mathematical Sciences, London, 1999.

[8] P.J. Cameron, J.A. Thas, and S.E. Payne. Polarities of generalized hexagons and perfect codes. *Geom. Dedicata*, 5:525–528, 1976.

[9] M. Cimráková, S. De Winter, V. Fack, and L. Storme. On the smallest maximal partial ovoids and spreads of the generalised quadrangles $W(q)$ and $Q(4, q)$. *European J. Combin.*, 28(7):1934–1942, 2007.

[10] J.H. Conway, P.B. Kleidman, and R.A. Wilson. New families of ovoids in O_8^+. *Geom. Dedicata*, 26:157–170, 1988.

[11] J. De Beule, A. Klein, K. Metsch, and L. Storme. Partial ovoids and partial spreads in hermitian polar spaces. *Des. Codes Cryptogr.*, 2007. `http://www.springerlink.com/content/x40132rl64802720/`.

[12] J. De Beule, A. Klein, K. Metsch, and L. Storme. Partial ovoids and partial spreads in symplectic and orthogonal polar spaces. *European J. Combin.*, to appear.

[13] J. De Beule and K. Metsch. The hermitian variety $H(5, 4)$ has no ovoid. *Bull. Belgian Math. Soc. Simon Stevin*, 12:727–733, 2005.

[14] J. De Beule and K. Metsch. The maximum size of a partial spread in $H(5, q^2)$ is $q^3 + 1$. *J. Combin. Theory, Ser. A*, 114:761–768, 2007.

[15] R.H. Dye. Partitions and their stabilizers for line complexes and quadrics. *Annali di Mat.*, 114:173–194, 1977.

[16] G.L. Ebert and J.W.P. Hirschfeld. Complete systems of lines on a hermitian surface over a finite field. *Des. Codes Cryptogr.*, 17:253–268, 1999.

[17] D.G. Glynn. A lower bound for maximal partial spreads in $PG(3, q)$. *Ars Combin.*, 13:39–40, 1982.

[18] P. Govaerts, O. Heden, and L. Storme. Maximal partial spreads and the modular n-queen problem. IV. (Unpublished manuscript).

[19] P. Govaerts and L. Storme. On a particular class of minihypers and its applications. II: Improvements for q square. *J. Combin. Theory, Ser. A*, 97:369–393, 2002.

[20] P. Govaerts, L. Storme, and H. Van Maldeghem. On a particular class of minihypers and its applications. III: Applications. *European J. Combin.*, 23:659–672, 2002.

[21] A. Gunawardena and G.E. Moorhouse. The non-existence of ovoids in $O_9(q)$. *European J. Combin.*, 18:171–173, 1997.

[22] O. Heden. Partial spreads and the modular n-queen problem. *Discrete Math.*, 120:75–91, 1993.

[23] O. Heden. Partial spreads and the modular n-queen problem. II. *Discrete Math.*, 142:97–106, 1995.

[24] O. Heden. Partial spreads and the modular n-queen problem. III. *Discrete Math.*, 243:135–150, 2002.

[25] J.W.P. Hirschfeld. *Projective Geometries over Finite Fields*. Oxford University Press, Oxford, 1979.

[26] J.W.P. Hirschfeld. *Finite Projective Spaces of Three Dimensions*. Oxford University Press, Oxford, 1985.

[27] J.W.P. Hirschfeld and J.A. Thas. *General Galois Geometries*. Oxford Mathematical Monographs. Oxford University Press, 1991. New York.

[28] W.M. Kantor. Ovoids and translation planes. *Canad. J. Math.*, 34(5):153–157, 1982.

[29] A. Klein. Partial ovoids in classical finite polar spaces. *Des. Codes Cryptogr.*, 31:221–226, 2004.

[30] A. Klein and K. Metsch. New results on covers and partial spreads of polar spaces. *Innov. Incidence Geom.*, 1:19–34, 2005.

[31] A. Klein, K. Metsch, and L. Storme. Partial ovoids and partial spreads of classical finite polar spaces. *Adv. Geom.*, submitted.

[32] H. Lüneburg. *Die Suzukigruppen und ihre Geometrien*, volume 10 of *Lecture Notes in Mathematics*. Springer, Berlin, 1965.

[33] F. Mazzocca, O. Polverino, and L. Storme. Blocking sets in $PG(r, q^n)$. *Des. Codes Cryptogr.*, 44:97–113, 2007.

[34] K. Metsch. Small maximal partial ovoids of $H(3, q^2)$. *Innov. Incidence Geom.*, 3:1–12, 2006.

[35] K. Metsch and L. Storme. Maximal partial ovoids and maximal partial spreads in hermitian generalized quadrangles. *J. Combin. Des.*, 16:101–116, 2008.

[36] G.E. Moorhouse. Ovoids from the E_8 root lattice. *Geom. Dedicata*, 46:287–297, 1993.

[37] G.E. Moorhouse. Some p-ranks related to hermitian varieties. *J. Statist. Plann. Inference*, 56:229–241, 1996.

[38] C.M. O'Keefe and J.A. Thas. Ovoids of the quadric $O(2n, q)$. *European J. Combin.*, 16:87–92, 1995.

[39] S.E. Payne and J.A. Thas. *Finite Generalized Quadrangles*, volume 110 of *Research Notes in Math.* Pitman, Boston-London-Melbourne, 1984.

[40] T. Penttila and B. Williams. Ovoids in parabolic spaces. *Geom. Dedicata*, 82:1–19, 2000.

[41] C. Roessing and L. Storme. A spectrum result on maximal partial ovoids of the generalized quadrangle $Q(4, q)$, q even. *European J. Combin.*, submitted.

[42] A. Siciliano. Locally hermitian partial ovoids of unitary polar spaces and and partial ovoids of orthogonal polar spaces. *Innov. Incidence Geom.*, to appear.

[43] G. Tallini. Blocking sets with respect to planes in $PG(3, q)$ and maximal spreads of a non-singular quadric in $PG(4, q)$. In *Proceedings of the First International Conference on Blocking Sets, Giessen, 1989*, Mitteilungen des mathematischen Seminars Giessen, pages 141–147, 1991.

[44] J.A. Thas. Ovoidal translation planes. *Arch. Math.*, 23:110–112, 1972.

[45] J.A. Thas. Polar spaces, generalized hexagons and perfect codes. *J. Combin. Theory, Ser. A*, 29:87–93, 1980.

[46] J.A. Thas. Ovoids and spreads of finite classical polar spaces. *Geom. Dedicata*, 10:135–144, 1981.

[47] J.A. Thas. Semi-partial geometries and spreads of classical finite polar spaces. *J. Combin. Theory, Ser. A*, 35:58–66, 1983.

[48] J.A. Thas. Old and new results on spreads and ovoids of finite classical polar spaces. In A. Barlotti et al, editor, *Combinatorics '90*, pages 529–544, 1992.

[49] J.A. Thas and S.E. Payne. Spreads and ovoids in finite generalized quadrangles. *Geom. Dedicata*, 54:227–253, 1994.

[50] J. Tits. Les groupes simples de Suzuki et de Ree. *Sém. Bourbaki*, 210, 1960.

J. De Beule, Ghent University, Dept. of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, 9000 Ghent, Belgium
jdebeule@cage.ugent.be, http://www.debeule.eu

A. Klein, Ghent University, Dept. of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, 9000 Ghent, Belgium
klein@cage.ugent.be

K. Metsch, Justus-Liebig-Universität, Mathematisches Institut, Arndtstrasse 2, D-35392 Giessen, Germany
Klaus.Metsch@math.uni-giessen.de

L. Storme, Ghent University, Dept. of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, 9000 Ghent, Belgium
ls@cage.ugent.be, http://cage.ugent.be/∼ls