Tight sets, weighted m-covers, weighted m-ovoids, and minihypers

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July 4, 2008

Abstract

Minihypers are substructures of projective spaces introduced to study linear codes meeting the Griesmer bound. Recently, many results in finite geometry were obtained by applying characterization results on minihypers [8, 17, 18]. In this paper, using characterization results on certain minihypers, we present new results on tight sets in classical finite polar spaces and weighted m-covers, and on weighted m-ovoids of classical finite generalized quadrangles. The link with minihypers gives us characterization results of i-tight sets in terms of generators and Baer subgeometries contained in the hermitian and symplectic polar spaces, and in terms of generators for the quadratic polar spaces. We also present extendability results on partial weighted m-ovoids and partial weighted m-covers, having small deficiency, to weighted m-covers and weighted m-ovoids of classical finite generalized quadrangles. As a particular application, we prove in an alternative way the extendability of 53-, 54-, and 55-caps of $\mathrm{PG}(5,3)$, contained in a non-singular elliptic quadric $\mathrm{Q}^-(5,3)$, to 56-caps contained in this elliptic quadric $\mathrm{Q}^-(5,3)$.

1 Introduction

Let $\mathrm{PG}(n,q)$ denote the n-dimensional projective space over $\mathbb{F}_q$, the finite field of order q. By π_r, we always denote an r-dimensional subspace of $\mathrm{PG}(n,q)$ and by $v_{r+1} := \frac{q^{r+1}-1}{q-1}$, we denote the number of points of an r-dimensional projective space.

Definition 1.1 (Hamada and Tamari [21, 22]) *An* $\{f,m;n,q\}$-minihyper *is a pair* (F,w)*, where* F *is a subset of the point set of* $\mathrm{PG}(n,q)$ *and* w *is a weight function* $w : \mathrm{PG}(n,q) \to \mathbb{N} : P \mapsto w(P)$*, satisfying*

1. $w(P) > 0 \Leftrightarrow P \in F$,
2. $\sum_{P\in F} w(P) = f$*, and*

*The research of this author is supported by a research grant of the Research council of Ghent University.

3. $\min\{\sum_{P \in H} w(P) : H \text{ is a hyperplane}\} = m$.

The weight function w determines the set F completely. When this function has range $\{0, 1\}$, then (F, w) is determined completely by the set F and the minihyper is denoted by F.

¿From this definition, it follows that an $\{f, m; n, q\}$-minihyper is a (weighted) *m-fold blocking set with respect to hyperplanes*, i.e. every hyperplane contains at least m points of this set, defined in [24]. This link with (multiple) blocking sets is very important in obtaining characterization results on minihypers.

Although minihypers were first introduced to study the problem of linear codes meeting the Griesmer bound [20, 21], characterization results on minihypers can be used to solve problems in finite geometry, see [8], [18] and [4] for applications on substructures of finite projective spaces and generalized quadrangles. We refer to [33] for a survey on the use of minihypers in the study of linear codes meeting the Griesmer bound and in the study of geometrical problems.

In this paper, we present new applications. In Section 3, we contribute to the study of i-tight sets, as defined and studied in [1, 7, 9]. In Section 5, we concentrate on the study of partial weighted m-ovoids and partial weighted m-covers in classical finite generalized quadrangles.

The classical finite polar spaces are the non-singular elliptic quadrics $\mathrm{Q}^-(2n+1, q)$, $n \geq 2$, the non-singular hyperbolic quadrics $\mathrm{Q}^+(2n+1, q)$, $n \geq 1$, the non-singular parabolic quadrics $\mathrm{Q}(2n, q)$, $n \geq 2$, the symplectic spaces $\mathrm{W}(2n+1, q)$, $n \geq 1$, and the non-singular hermitian varieties $\mathrm{H}(n, q^2)$, $n \geq 3$ [25]. For a classical finite polar space different from $\mathrm{Q}(2n, q)$, q even, $\perp$ denotes the polarity corresponding to this classical finite polar space. For the classical finite polar space $\mathrm{Q}(2n, q)$, q even, $P^\perp$ denotes the tangent hyperplane to $\mathrm{Q}(2n, q)$ in a point P of $\mathrm{Q}(2n, q)$.

To conclude this introduction, we state some definitions and results on blocking sets in $\mathrm{PG}(2, q)$.

A *blocking set* in $\mathrm{PG}(2, q)$ is a set of points in $\mathrm{PG}(2, q)$ that meets every line. A blocking set in $\mathrm{PG}(2, q)$ is called *trivial* when it contains a line. For information on blocking sets, we refer to [24]. Let $q + \epsilon_q$ denote the size of the smallest non-trivial blocking sets in $\mathrm{PG}(2, q)$. In the next table, we give exact values on ϵ_q and lower bounds on ϵ_q.

q	ϵ_q	Condition	
square	$= \sqrt{q} + 1$		[5]
odd prime	$=(q+3)/2$		[3]
$q = p^{3h}$, $p \geqslant 7$ prime, $h \geqslant 1$	$= q^{2/3} + 1$		[29, 30, 31]
$q = p^h$, p prime, $h \geq 4$	$\geq q + q/(p^e+1) - 1$	$e < h$ largest divisor of h	[13]

Table 1: Exact values and lower bounds on ϵ_q

2 Minihypers contained in quadrics

In this section, we present new characterizations of minihypers whose point sets are contained in classical finite polar spaces, see Theorems 2.8 and 2.10. These characterization results are used in the proofs of the characterization results for i-tights sets in Section 3.

Lemma 2.1 (Govaerts and Storme [16]) *Suppose that F is a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $0 \leqslant \delta \leqslant (q+1)/2$, $0 \leqslant \mu \leqslant n-1$. If H is a hyperplane containing more than δv_μ points of F, then every $(n-\mu-1)$-space in H contains at least one point of F.*

This implies that $H \cap F$ is a blocking set with respect to the $(n-\mu-1)$-spaces in H. The next theorem will be crucial in the proof of the new characterization results of Theorem 2.8 and 2.10.

Theorem 2.2 (Szőnyi and Weiner [34]) *A minimal blocking set B with respect to the k-dimensional spaces of $\mathrm{PG}(n,q)$, $n \geqslant 3$, $q = p^h$, $p > 2$ prime, $h \geq 1$, of size $|B| < q^{n-k} + q^{n-k}/2$, intersects every line in zero points or in $1 \pmod p$ points.*

The next result of Govaerts and Storme will be used to classify $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihypers on quadrics.

Lemma 2.3 (Govaerts and Storme [16]) *Let (F, w) be a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $0 \leqslant \delta \leqslant (q+1)/2, 0 \leqslant \mu \leqslant n-1$, and containing a μ-space π_μ. Then the minihyper (F', w') defined by the weight function w', where*

- *$w'(P) = w(P) - 1$, for $P \in \pi_\mu$, and*
- *$w'(P) = w(P)$, for $P \in \mathrm{PG}(n,q) \setminus \pi_\mu$,*

is a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; n, q\}$-minihyper.

Let F be an $\{xv_{r+1}, xv_r; 2r+1, q\}$-minihyper, where $x \leqslant q/2 - 1$, on $\mathrm{Q}^+(2r+1,q)$. We prove that F is the union of x pairwise disjoint subspaces $\mathrm{PG}(r,q)$. This is already known for $\mathrm{Q}^+(5,q)$, see [8]. We rely on Theorem 2.2.

Lemma 2.4 *Let π_r be an r-dimensional space containing exactly one point of F. There exists a $2r$-dimensional space $\mathrm{PG}(2r,q)$ through π_r containing more than xv_r points of F.*

Proof. Suppose that every $2r$-dimensional space $\mathrm{PG}(2r,q)$ through π_r has xv_r points of F. Count the size of the set

$$X = \{(P, H) : P \in F \backslash \pi_r, H \text{ a hyperplane through } \pi_r, P \in H\}.$$

Starting with P, we have that $|X| = (|F|-1)v_r$, since there are v_r hyperplanes through π_r and P. Starting with H, we have $|X| = v_{r+1}(xv_r - 1)$. For $|F| = xv_{r+1}$, this gives a contradiction. □

The following result is standard.

Lemma 2.5 *If an s-dimensional space π_s intersects a quadric Q in at least three hyperplanes of π_s, then $\pi_s \subset Q$.*

Lemma 2.6 *Let $\widetilde{B}$ be a minimal 1-fold blocking set with respect to the r-dimensional subspaces of π_{2r} contained in a hyperplane section $\pi_{2r} \cap \mathrm{Q}^+(2r+1,q)$ of $\mathrm{Q}^+(2r+1,q)$, with $|\widetilde{B}| \leqslant q^r + q^r/2$. Then, for any positive integer t, every t linearly independent points of $\widetilde{B}$ span a $(t-1)$-dimensional subspace π_{t-1} completely contained in $\mathrm{Q}^+(2r+1,q)$.*

Proof. This is true for $t = 2$. Indeed, let $R_1, R_2 \in \widetilde{B}$ be 2 linearly independent points. By Theorem 2.2, the line $\langle R_1, R_2\rangle$ must contain at least $1 + p$ points of $\widetilde{B}$. This means that this line contains at least 3 points of $\mathrm{Q}^+(2r+1,q)$, so lies completely on $\mathrm{Q}^+(2r+1,q)$.

We will argue by induction on t, so suppose that the lemma is true for some t. Let π_{t-1} be a $(t-1)$-dimensional space on $\mathrm{Q}^+(2r+1,q)$, spanned by t linearly independent points of $\widetilde{B}$. Let R be a point of $\widetilde{B}\backslash\pi_{t-1}$. Take two sets of $t-1$ points of these t points. By induction, we know that these sets together with R are two sets of t linearly independent points of $\widetilde{B}$, so they define two $(t-1)$-dimensional spaces in $\mathrm{Q}^+(2r+1,q)$. Together with π_{t-1}, this gives three $(t-1)$-dimensional spaces on $\mathrm{Q}^+(2r+1,q)$ that span a t-dimensional space π_t. Theorem 2.5 implies that π_t is a subspace contained in $\mathrm{Q}^+(2r+1,q)$. □

Lemma 2.7 *Let $\widetilde{B}$ be a minimal 1-fold blocking set with respect to the r-dimensional subspaces of π_{2r} contained in a hyperplane section $\pi_{2r} \cap \mathrm{Q}^+(2r+1,q)$ of $\mathrm{Q}^+(2r+1,q)$, with $|\widetilde{B}| \leqslant q^r + q^r/2$. Then $\widetilde{B}$ is the point set of an r-dimensional subspace π_r of π_{2r}.*

Proof. Since $|\widetilde{B}| \geqslant v_{r+1}$, we can find at least $r+1$ linearly independent points in $\widetilde{B}$. This means by the previous lemma that $\langle\widetilde{B}\rangle = \pi_x \subset \mathrm{Q}^+(2r+1,q)$, with $x \geqslant r$. But since $\pi_x \subset \mathrm{Q}^+(2r+1,q)$, x can be at most r. We conclude that $x = r$ and that $\widetilde{B}$ is the point set of an r-dimensional subspace π_r of π_{2r}. □

Theorem 2.8 *An $\{xv_{r+1}, xv_r; 2r+1, q\}$-minihyper F contained in $\mathrm{Q}^+(2r+1,q)$, with $x \leqslant q/2 - 1$, consists of x pairwise disjoint r-dimensional spaces.*

Proof. Consider a point P' of F. There exists an r-dimensional space π_r through P' only containing that point of F. To find a $2r$-dimensional space π_{2r} through π_r that contains more than xv_r points of F, we use Lemma 2.4.

The space π_{2r} intersects F in a 1-fold blocking set B with respect to the r-dimensional spaces in π_{2r} (Lemma 2.1). Let $\widetilde{B}$ be a minimal blocking set contained in B.

We determine the maximal possible size of $\widetilde{B}$. As the blocking set $\pi_{2r} \cap F$ is the intersection of a hyperplane $H = \pi_{2r}$ with the minihyper F, results from [15, Lemma 1.1], [20, Theorem 2.3] state that this is a

$$\left\{\sum_{i=0}^{r} \epsilon_i v_{i+1}, \sum_{i=0}^{r} \epsilon_i v_i; 2r, q\right\}\text{-minihyper,}$$

with $\sum_{i=0}^{r} \epsilon_i \leqslant x$.

Every r-dimensional subspace in π_{2r} intersects such a minihyper $F \cap H$ in at least ϵ_r points [20, Theorem 2.5]. Since π_r contains only one point of $F \cap H$, ϵ_r must be equal to 1. So $|H \cap F| \leqslant v_{r+1} + (x-1)v_r \leqslant q^r + q^r/2$. By Lemma 2.6, $\widetilde{B}$ is the point set of an r-dimensional subspace.

¿From Lemma 2.3, it follows that $F \backslash \widetilde{B}$ is an $\{(x-1)v_{r+1}, (x-1)v_r; 2r+1, q\}$-minihyper F'. Repeating the previous arguments x times, implies that F consists of x pairwise disjoint r-dimensional subspaces. □

Corollary 2.9 *Let F be an $\{xv_{r+1}, xv_r; 2r+1, q\}$-minihyper on $\mathrm{Q}^+(2r+1,q)$, with $x \leq q/2 - 1$. If r is even, then necessarily $x \leqslant 2$.*

Proof. This follows from the fact that at most two r-dimensional spaces of $\mathrm{Q}^+(2r+1,q)$, r even, can be disjoint to each other. □

The following results can be obtained using similar arguments.

Theorem 2.10 (1) *An $\{xv_r, xv_{r-1}; 2r, q\}$-minihyper F contained in $\mathrm{Q}(2r,q)$, with $x \leqslant q/2-1$, consists of x pairwise disjoint $(r-1)$-dimensional spaces.*

(2) *An $\{xv_r, xv_{r-1}; 2r+1, q\}$-minihyper F contained in $\mathrm{Q}^-(2r+1,q)$, with $x \leqslant q/2-1$, consists of x pairwise disjoint $(r-1)$-dimensional spaces.*

3 Minihypers and i-tight sets

We will consider i-tight sets on quadratic polar spaces, hermitian polar spaces and symplectic polar spaces. The link with minihypers gives us some nice characterization results of i-tight sets in terms of generators and Baer subgeometries contained in these hermitian and symplectic polar spaces, and in terms of generators for the quadratic polar spaces, see Theorems 3.9, 3.12, and 3.6 respectively.

Definition 3.1 (Bamberg, Kelly, Law, and Penttila [1]) *A set $\mathcal{T}$ of points of a finite polar space of rank $r \geqslant 2$ over a finite field of order q is i*-tight *if*

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\frac{q^{r-1}-1}{q-1} + q^{r-1} & \text{if } P \in \mathcal{T}, \\ i\frac{q^{r-1}-1}{q-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

Example 3.2 A classical example of an i-tight set in a classical finite polar space $\mathcal{P}$ is a union of i pairwise disjoint generators of $\mathcal{P}$.

3.1 Tight sets and Baer subgeometries on hermitian varieties

Example 3.3 Consider the hermitian variety $\mathrm{H}(2r+1,q)$, q square. A $(\sqrt{q}+1)$-tight set can be constructed using a particular example of a Baer subgeometry contained in $\mathrm{H}(2r+1,q)$.

Let $\epsilon \in \mathbb{F}_q$, q odd, such that $\epsilon^{\sqrt{q}} = -\epsilon$, hence $\epsilon \notin \mathbb{F}_{\sqrt{q}}$, or let $\epsilon \in \mathbb{F}_{\sqrt{q}}, q$ even. Up to a projectivity, the hermitian variety $\mathrm{H}(2r+1,q)$ consists of the set of points whose coordinates satisfy the equation

$$\epsilon(X_1X_0^{\sqrt{q}} - X_0X_1^{\sqrt{q}} + X_3X_2^{\sqrt{q}} - \ldots + X_{2r+1}X_{2r}^{\sqrt{q}} - X_{2r}X_{2r+1}^{\sqrt{q}}) = 0.$$

Each hyperplane of $\mathrm{PG}(2r+1,q)$ intersects the standard Baer subgeometry $\mathrm{PG}(2r+1,\sqrt{q}) = \{(x_0,\ldots,x_{2r+1}) : x_i \in \mathbb{F}_{\sqrt{q}}\}$ in either a $\mathrm{PG}(2r,\sqrt{q})$ or a $\mathrm{PG}(2r-1,\sqrt{q})$.

For a hyperplane π with equation $a_0X_0+\cdots+a_{2r+1}X_{2r+1} = 0$, its conjugate hyperplane $\pi^{\sqrt{q}}$ with respect to the standard Baer subgeometry $\mathrm{PG}(2r+1,\sqrt{q})$ has equation $a_0^{\sqrt{q}}X_0 + \cdots + a_{2r+1}^{\sqrt{q}}X_{2r+1} = 0$. Now $\pi = \pi^{\sqrt{q}}$ if and only if for some scalar $t \in \mathbb{F}_q^*$, $\forall i$, $ta_i \in \mathbb{F}_{\sqrt{q}}$. Let $P = (x_0,\ldots,x_{2r+1}) \in \pi \cap \mathrm{PG}(2r+1,\sqrt{q})$, then P lies also in $\pi^{\sqrt{q}}$. So

$$\begin{aligned} \pi \cap \mathrm{PG}(2r+1,\sqrt{q}) &= \pi^{\sqrt{q}} \cap \mathrm{PG}(2r+1,\sqrt{q}) && (1)\\ &= \pi \cap \pi^{\sqrt{q}} \cap \mathrm{PG}(2r+1,\sqrt{q}), && (2)\end{aligned}$$

but if $\pi \neq \pi^{\sqrt{q}}$, then $\pi \cap \mathrm{PG}(2r+1,\sqrt{q}) = \mathrm{PG}(2r-1,\sqrt{q})$. If $\pi = \pi^{\sqrt{q}}$, then $\pi \cap \mathrm{PG}(2r+1,\sqrt{q}) = \mathrm{PG}(2r,\sqrt{q})$, since the intersection is invariant under the conjugation $x \mapsto x^{\sqrt{q}} : (\pi \cap \pi^{\sqrt{q}})^{\sqrt{q}} = \pi^{\sqrt{q}} \cap \pi$.

Denote the polarity associated with the hermitian variety by $\perp$. Consider a point $P \in \mathrm{H}(2r+1,q)$, let $P = (x_0,x_1,\ldots,x_{2r+1})$. The tangent hyperplane $\pi = P^\perp$ to $\mathrm{H}(2r+1,q)$ at the point P satisfies the equation

$$\epsilon(X_1x_0^{\sqrt{q}} - X_0x_1^{\sqrt{q}} + \cdots + X_{2r+1}x_{2r}^{\sqrt{q}} - X_{2r}x_{2r+1}^{\sqrt{q}}) = 0,$$

its conjugate, $\pi^{\sqrt{q}}$ satisfies the equation

$$-\epsilon(X_1x_0 - X_0x_1 + \cdots + X_{2r+1}x_{2r} - X_{2r}x_{2r+1}) = 0.$$

They are equal if and only if $x_i = tx_i^{\sqrt{q}}, t \in \mathbb{F}_q^*, i = 0,1,\ldots,2r+1$, so if $P \in \mathrm{PG}(2r+1,\sqrt{q})$. Hence,

$$P^\perp \cap \mathrm{PG}(2r+1,\sqrt{q}) = \begin{cases} \mathrm{PG}(2r,\sqrt{q}) & \text{if } P \in \mathrm{PG}(2r+1,\sqrt{q}),\\ \mathrm{PG}(2r-1,\sqrt{q}) & \text{if } P \notin \mathrm{PG}(2r+1,\sqrt{q}).\end{cases}$$

These intersections are of sizes equal to the intersection numbers of the definition of an i-tight set with $i = \sqrt{q}+1$. So we conclude that this Baer subgeometry $\mathrm{PG}(2r+1,\sqrt{q})$ is a $(\sqrt{q}+1)$-tight set in $\mathrm{H}(2r+1,q)$.

The preceding example was also stated in [1, Section 5.2]. Their approach was as follows: they considered the embedding of $\mathrm{W}(2r+1,\sqrt{q})$ in $\mathrm{H}(2r+1,q)$ and proved that this defines a $(\sqrt{q}+1)$-tight set in $\mathrm{H}(2r+1,q)$. We now prove the converse. The next theorem characterizes a Baer subgeometry $\mathrm{PG}(2r+1,\sqrt{q})$ contained in the hermitian variety $\mathrm{H}(2r+1,q)$ defining a $(\sqrt{q}+1)$-tight set as a symplectic polar space contained in the hermitian variety.

Theorem 3.4 *Suppose that a subgeometry* $\mathrm{PG}(2r+1,\sqrt{q}) \subset \mathrm{H}(2r+1,q)$ *defines a* $(\sqrt{q}+1)$*-tight set. Then the hermitian polarity of* $\mathrm{H}(2r+1,q)$ *induces a symplectic polarity in this Baer subgeometry.*

Proof. Since this Baer subgeometry $\mathrm{PG}(2r+1,\sqrt{q})$ defines a $(\sqrt{q}+1)$-tight set $\mathcal{T}$, we have the following intersection numbers:

$$|P^\perp \cap \mathcal{T}| = \begin{cases} (\sqrt{q}+1)\frac{q^r-1}{q-1} + q^r = \frac{\sqrt{q}^{2r+1}-1}{\sqrt{q}-1} & \text{if } P \in \mathcal{T}, \\ (\sqrt{q}+1)\frac{q^r-1}{q-1} = \frac{\sqrt{q}^{2r}-1}{\sqrt{q}-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

Let $\mathcal{H}$ be the set of hyperplanes of $\mathrm{PG}(2r+1,\sqrt{q})$. Define η: $\mathrm{PG}(2r+1,\sqrt{q}) \to \mathcal{H} : P \mapsto P^\perp \cap \mathrm{PG}(2r+1,\sqrt{q})$, with $\perp$ the hermitian polarity. Note that $P^\perp \cap \mathrm{PG}(2r+1,\sqrt{q})$ indeed is a hyperplane of $\mathrm{PG}(2r+1,\sqrt{q})$ since $|P^\perp \cap \mathcal{T}| = (\sqrt{q}^{2r+1}-1)/(\sqrt{q}-1)$.

Then η is a bijection from the point set of $\mathrm{PG}(2r+1,\sqrt{q})$ to the set of hyperplanes of $\mathrm{PG}(2r+1,\sqrt{q})$ since the hyperplanes $P^\perp \cap$ $\mathrm{PG}(2r+1,\sqrt{q})$ are extendable to hyperplanes of $\mathrm{PG}(2r+1,q)$, and distinct points of $\mathrm{H}(2r+1,q)$ have distinct tangent hyperplanes.

Now η is involutory starting from $\perp$. If P, P_1, P_2 are collinear in $\mathrm{PG}(2r+1,\sqrt{q})$, then $P^\perp \cap P_1^\perp \cap P_2^\perp$ is a $(2r-1)$-dimensional subspace of $\mathrm{PG}(2r+1,q)$. In fact, it is a $(2r-1)$-dimensional subspace of $\mathrm{PG}(2r+1,\sqrt{q})$ since $P^{\perp\sqrt{q}} = P^\perp, P_1^{\perp\sqrt{q}} = P_1^\perp, P_2^{\perp\sqrt{q}} = P_2^\perp$. So

$$P^{\perp\sqrt{q}} \cap P_1^{\perp\sqrt{q}} = P^\perp \cap P_1^\perp = (P^\perp \cap P_1^\perp)^{\sqrt{q}}.$$

So η is a polarity of $\mathrm{PG}(2r+1,\sqrt{q})$; since $P \in P^\eta$ for all points P of $\mathrm{PG}(2r+1,\sqrt{q})$, η is necessarily symplectic. □

3.2 Characterization results on tight sets

We now turn to the characterization problem of i-tight sets in the classical finite polar spaces. These i-tight sets are linked to minihypers. The following lemma is implicitly proven in [1, Theorem 13]. We however state and prove it again here to make the article self-contained.

Lemma 3.5 *An* i*-tight set, with* $i > 1$*, on* $\mathrm{W}(2r+1,q), \mathrm{Q}^+(2r+1,q)$*, or* $\mathrm{H}(2r+1,q)$ *generates the whole space.*

Proof. Let $\mathcal{T}$ be this i-tight set. Then

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\frac{q^r-1}{q-1} + q^r & \text{if } P \in \mathcal{T}, \\ i\frac{q^r-1}{q-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

So $\mathcal{T}$ is not contained in a tangent hyperplane if $i > 1$. This finishes the proof for $\mathrm{W}(2r+1,q)$.

For $\mathrm{Q}^+(2r+1,q)$ and $\mathrm{H}(2r+1,q)$, a non-degenerate hyperplane section is a $(\frac{q^r-1}{q-1})$-ovoid [1]. An m-ovoid and an i-tight set intersect in mi points [1]. So here in $i(\frac{q^r-1}{q-1})$ points. So $\mathcal{T}$ is not contained in a non-degenerate hyperplane. □

We obtain that an i-tight set $\mathcal{T}$ on one of the classical finite polar spaces $\mathrm{W}(2r+1,q), \mathrm{Q}^+(2r+1,q), \mathrm{H}(2r+1,q)$ is a set of $i(q^{r+1}-1)/(q-1)$ points intersecting every hyperplane in at least $i(q^r-1)/(q-1)$ points. This means that $\mathcal{T}$ is an $\{i(q^{r+1}-1)/(q-1), i(q^r-1)/(q-1); 2r+1, q\}$-minihyper (Definition 1.1).

We now use known characterization results on minihypers to get new information on i-tight sets in the classical finite polar spaces $\mathrm{W}(2r+1,q), \mathrm{Q}^+(2r+1,q)$, and $\mathrm{H}(2r+1,q)$. For the first characterization result, we rely on Theorem 2.8 and Corollary 2.9.

Theorem 3.6 *An i-tight set on $\mathrm{Q}^+(2r+1,q)$, with $2 < i \leq q/2-1$, can only exist for r odd. When r is odd, then such an i-tight set is the union of i pairwise disjoint generators of $\mathrm{Q}^+(2r+1,q)$.*

For every $r \geq 1$, a 1-tight or 2-tight set on $\mathrm{Q}^+(2r+1,q)$ consists of one generator or of two disjoint generators.

The following general characterization result on minihypers is known.

Theorem 3.7 (Govaerts and Storme [15]) *A $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper F, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2}+1, 2\mu+1 \leqslant n$, is a union of pairwise disjoint μ-spaces and Baer subgeometries $\mathrm{PG}(2\mu+1, \sqrt{q})$.*

In the preceding results of Theorems 2.8 and 2.10 on quadrics, we could exclude the Baer subgeometries, since there are no Baer subgeometries $\mathrm{PG}(d, \sqrt{q})$ contained in a non-singular quadric in $\mathrm{PG}(d,q)$. But what can we say about these Baer subgeometries contained in the hermitian variety? We will now study the correspondence between these Baer subgeometries and i-tight sets on the hermitian variety $\mathrm{H}(2r+1,q)$.

Lemma 3.8 *Let $P \in \mathrm{H}(2r+1,q)$, let $P^\perp$ share a $PG(2r, \sqrt{q})$ with $\mathrm{H}(2r+1,q)$, then $P \in \mathrm{PG}(2r, \sqrt{q})$.*

Proof. Assume that $P \notin \mathrm{PG}(2r, \sqrt{q})$.

Then P lies on the extension of a unique line of $\mathrm{PG}(2r, \sqrt{q})$ (the line $PP^{\sqrt{q}}$) and P projects $\mathrm{PG}(2r, \sqrt{q})$ onto a cone with vertex R and base $\mathrm{PG}(2r-2, \sqrt{q})$.

Now this $\mathrm{PG}(2r, \sqrt{q})$ lies on $\langle P, \mathrm{H}(2r-1,q)\rangle$. Since the projection $\langle R, \mathrm{PG}(2r-2, \sqrt{q})\rangle$ lies completely on $\mathrm{H}(2r+1,q)$, it lies in the tangent hyperplane $R^\perp$ w.r.t. $\mathrm{H}(2r-1,q)$. But $R^\perp$ w.r.t. $\mathrm{H}(2r-1,q)$ has dimension $2r-2$, and $\langle R, \mathrm{PG}(2r-2, \sqrt{q})\rangle$ generates a $(2r-1)$-space, so we get a contradiction.

Hence, $P \in \mathrm{PG}(2r, \sqrt{q})$. $\square$

Theorem 3.9 *Let $\mathcal{T}$ be an i-tight set in $\mathrm{H}(2r+1,q)$, with $q > 16$ and $i < q^{5/8}/\sqrt{2}+1$, then $\mathcal{T}$ is a union of pairwise disjoint Baer subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$ and generators $\mathrm{PG}(r,q)$, where the hermitian polarity $\perp$ induces a symplectic polarity in every Baer subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$ contained in $\mathcal{T}$.*

Proof. This i-tight set defines an $\{i(q^{r+1}-1)/(q-1), i(q^r-1)/(q-1); 2r+1, q\}$-minihyper contained in $\mathrm{H}(2r+1,q)$.

By Theorem 3.7, this minihyper is a union of pairwise disjoint r-dimensional spaces and Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$. It is possible to take away an r-dimensional space $\mathrm{PG}(r,q)$ from $\mathcal{T}$ and reduce $\mathcal{T}$ to an $(i-1)$-tight set (Lemma 2.3).

So from now on, we assume that $\mathcal{T}$ is a union of δ pairwise disjoint Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$. This implies that $i=\delta(\sqrt{q}+1)$. Denote the Baer subgeometries in $\mathcal{T}$ by $\pi_i, i=1,2,\ldots,\delta$.

Consider a point P of $\mathcal{T}$. Then

$$\begin{aligned} |P^\perp\cap\mathcal{T}| &= \delta(\sqrt{q}+1)\left(\frac{q^r-1}{q-1}\right)+q^r & (3)\\ &= |\mathrm{PG}(2r,\sqrt{q})|+(\delta-1)|\mathrm{PG}(2r-1,\sqrt{q})|. & (4)\end{aligned}$$

So $P^\perp$ must intersect the Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$, contained in $\mathcal{T}$, once in $\mathrm{PG}(2r,\sqrt{q})$ and $\delta-1$ times in a $\mathrm{PG}(2r-1,\sqrt{q})$. By the preceding lemma, $P\in\mathrm{PG}(2r,\sqrt{q})$.

The preceding arguments, including the proof of the preceding theorem, now imply that the hermitian polarity $\perp$ induces a symplectic polarity in every Baer subgeometry π_i contained in $\mathcal{T}$. □

3.3 i-Tight sets on $\mathrm{W}(2r+1,q)$

Let $\mathcal{T}$ be an i-tight set on $\mathrm{W}(2r+1,q)$, q square, $i<\frac{q^{5/8}}{\sqrt{2}}+1$. Then $\mathcal{T}$ is a union of pairwise disjoint $\mathrm{PG}(2r+1,\sqrt{q})$ and $\mathrm{PG}(r,q)$.

Lemma 3.10 *Let $\mathcal{T}$ be an i-tight set on $\mathrm{W}(2r+1,q)$, q square, $i<\frac{q^{5/8}}{\sqrt{2}}+1$. If $\mathcal{T}$ contains a subspace $\mathrm{PG}(r,q)=U$, then $U^\perp$ is also contained in $\mathcal{T}$.*

Proof. For $P\in\mathcal{T}$, $|P^\perp\cap\mathcal{T}|=i(\frac{q^r-1}{q-1})+q^r$. We know that $\mathcal{T}$ defines an $\{i(q^{r+1}-1)/(q-1), i(q^r-1)/(q-1); 2r+1,q\}$-minihyper, which is a union of pairwise disjoint subspaces $\mathrm{PG}(r,q)$ and Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$ (Theorem 3.7).

Assume that $\mathcal{T}$ consists of δ distinct $\mathrm{PG}(2r+1,\sqrt{q})$ and $i-\delta(\sqrt{q}+1)$ distinct $\mathrm{PG}(r,q)$. Then

$$\begin{aligned} |P^\perp\cap\mathcal{T}| &= |\mathrm{PG}(2r,\sqrt{q})|+(\delta-1)|\mathrm{PG}(2r-1,\sqrt{q})|+(i-\delta(\sqrt{q}+1))|\mathrm{PG}(r-1,q)|\\ &= \delta|\mathrm{PG}(2r-1,\sqrt{q})|+|\mathrm{PG}(r,q)|+(i-\delta(\sqrt{q}+1)-1)|\mathrm{PG}(r-1,q)|.\end{aligned}$$

So $P^\perp\cap\mathcal{T}$ either contains

1. one $\mathrm{PG}(2r,\sqrt{q})$, $\delta-1$ distinct $\mathrm{PG}(2r-1,\sqrt{q})$, and $i-\delta(\sqrt{q}+1)$ distinct $\mathrm{PG}(r-1,q)$ of $\mathcal{T}$ or,

2. δ distinct $\mathrm{PG}(2r-1,\sqrt{q})$, one $\mathrm{PG}(r,q)$, and $i-\delta(\sqrt{q}+1)-1$ distinct $\mathrm{PG}(r-1,q)$ of $\mathcal{T}$.

Assume that $P^\perp \cap \mathcal{T}$ contains a subgeometry $\mathrm{PG}(2r, \sqrt{q})$, then P is the only element of $\mathcal{T}$ containing this $\mathrm{PG}(2r, \sqrt{q})$ in its polar hyperplane $P^\perp$ since $\langle \mathrm{PG}(2r, \sqrt{q})\rangle_{\mathbb{F}_q} = \mathrm{PG}(2r, q)$. This hyperplane must be $P^\perp$. So at most $\delta|\mathrm{PG}(2r+1, \sqrt{q})|$ points P of $\mathcal{T}$ share a subgeometry $\mathrm{PG}(2r, \sqrt{q})$ with $\mathcal{T}$ in their polar hyperplane $P^\perp$.

For a subspace $\mathrm{PG}(r, q) = U$ in $\mathcal{T}$, $U \neq U^\perp$, we can remove U from $\mathcal{T}$ to obtain an $(i-1)$-tight set. Now $\dim U^\perp = r$, so at most $(i - \delta(\sqrt{q}+1))|\mathrm{PG}(r, q)|$ points of $\mathcal{T}$ share a $\mathrm{PG}(r, q)$ with $\mathcal{T}$ in their polar hyperplane $P^\perp$.

So at most $\delta(\sqrt{q}+1)|\mathrm{PG}(r,q)| + (i - \delta(\sqrt{q}+1))|\mathrm{PG}(r,q)| \leqslant i|\mathrm{PG}(r,q)|$ points of $\mathcal{T}$ share a subgeometry $\mathrm{PG}(2r, \sqrt{q})$ or a subspace $\mathrm{PG}(r, q)$ with $\mathcal{T}$. Since every point of $\mathcal{T}$ contains a subgeometry $\mathrm{PG}(2r, \sqrt{q})$ or a subspace $\mathrm{PG}(r, q)$ in the intersection of its polar hyperplane $P^\perp$ with $\mathcal{T}$, we obtain equality in both cases.

So $|\mathrm{PG}(r, q)|$ points of $\mathcal{T}$ lie in $U^\perp$, for U a subspace $\mathrm{PG}(r, q)$ in $\mathcal{T}$. If $U^\perp \neq U'$ for all r-spaces U' in $\mathcal{T}$, then all other r-spaces U' of $\mathcal{T}$ share at most an $(r-1)$-dimensional space with $U^\perp$. This is also true for U itself. Then for at least $|U| - (i - \delta(\sqrt{q}+1))|\mathrm{PG}(r-1, q)|$ points P of $\mathcal{T}$, P lies in $U^\perp$, and P lies in a subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$ of $\mathcal{T}$. This number is at least $|\mathrm{PG}(r,q)| - i|\mathrm{PG}(r-1,q)| + \delta(\sqrt{q}+1)|\mathrm{PG}(r-1,q)| \geqslant q^r/2$.

We know that $\dim U^\perp = r$, so $U^\perp$ intersects every subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$ in $\mathcal{T}$ in at most a subgeometry $\mathrm{PG}(r, \sqrt{q})$ containing at most $(\sqrt{q})^r$ points of this subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$. But $\mathcal{T}$ must then have at least $(\sqrt{q})^r/2$ distinct $\mathrm{PG}(2r+1, \sqrt{q})$. Now $r \geqslant 1$, so this would imply that $i/(\sqrt{q}+1) \geqslant \sqrt{q}/2$. This is false since $\mathcal{T}$ contains $\delta \leq i/(\sqrt{q}+1)$ distinct Baer subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$. Here, $i/(\sqrt{q}+1) < (q^{5/8}/\sqrt{2}+1)/(\sqrt{q}+1) < \sqrt{q}/2$, so we have a contradiction. We can conclude that $U^\perp$ also lies in $\mathcal{T}$. □

Lemma 3.11 *Let $\mathcal{T}$ be an i-tight set on $\mathrm{W}(2r+1, q)$, q square, $i < \frac{q^{5/8}}{\sqrt{2}} + 1$. If $\mathcal{T}$ contains subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$, then they are all pairwise disjoint, and moreover they are either invariant under the symplectic polarity or they come in pairs $\{\mathrm{PG}(2r+1, \sqrt{q})_1, \mathrm{PG}(2r+1, \sqrt{q})_2\}$, where $\mathrm{PG}(2r+1, \sqrt{q})_1 \cap \mathrm{PG}(2r+1, \sqrt{q})_2 = \emptyset$ and where $P^\perp \cap \mathrm{PG}(2r+1, \sqrt{q})_2 = \mathrm{PG}(2r, \sqrt{q})$ for all $P \in \mathrm{PG}(2r+1, \sqrt{q})_1$.*

Proof. By using the arguments of the preceding theorem, if $\mathcal{T}$ contains subspaces $U \equiv \mathrm{PG}(r, q)$, then either $U = U^\perp$, or $U \neq U^\perp$, and then $U, U^\perp$ both lie in $\mathcal{T}$. In the first case, U can be deleted from $\mathcal{T}$ to obtain an $(i-1)$-tight set, and in the second case, U and $U^\perp$ can be deleted from $\mathcal{T}$ to obtain an $(i-2)$-tight set. So, from now on, we assume that $\mathcal{T}$ consists of a union of pairwise disjoint subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$.

Assume that $\mathcal{T}$ consists of δ distinct $\mathrm{PG}(2r+1, \sqrt{q})_i$, $i = 1, \ldots, \delta$. For every point $P \in \mathrm{PG}(2r+1, \sqrt{q})_i$, $P^\perp$ intersects one $\mathrm{PG}(2r+1, \sqrt{q})_j$, $j \in \{1, \ldots, \delta\}$, in a subgeometry $\mathrm{PG}(2r, \sqrt{q})$ and intersects all other subgeometries $\mathrm{PG}(2r+1, \sqrt{q})_j$, $j \in \{1, \ldots, \delta\}$, in a subgeometry $\mathrm{PG}(2r-1, \sqrt{q})$.

Consider all hyperplanes $\mathrm{PG}(2r, \sqrt{q})$ of $\mathrm{PG}(2r+1, \sqrt{q})_1$. They in fact form a dual subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$. Each hyperplane $\mathrm{PG}(2r, \sqrt{q})$ defines a unique $\mathrm{PG}(2r, q) = P^\perp$. So the points P of $\mathcal{T}$ for which $P^\perp$ contains a hyperplane $\mathrm{PG}(2r, \sqrt{q})$ of $\mathrm{PG}(2r+1, \sqrt{q})_1$ form themselves a subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$. This subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$

is contained in $\mathcal{T}$, so it is either $\mathrm{PG}(2r+1,\sqrt{q})_1$ itself or it is another subgeometry $\mathrm{PG}(2r+1,\sqrt{q})_2$.

Assume that it is another subgeometry $\mathrm{PG}(2r+1,\sqrt{q})_2$. There are $|\mathrm{PG}(2r,\sqrt{q})|$ hyperplanes of $\mathrm{PG}(2r+1,\sqrt{q})_1$ through a point R in $\mathrm{PG}(2r+1,\sqrt{q})_1$, so $R^\perp$ contains $|\mathrm{PG}(2r,\sqrt{q})|$ points of $\mathrm{PG}(2r+1,\sqrt{q})_2$. So we get the pairing $\{\mathrm{PG}(2r+1,\sqrt{q})_1, \mathrm{PG}(2r+1,\sqrt{q})_2\}$. □

Theorem 3.12 *Let $\mathcal{T}$ be an i-tight set of $\mathrm{W}(2r+1,q)$, q square, $i < \frac{q^{5/8}}{\sqrt{2}}+1$. Then $\mathcal{T}$ is a union of pairwise disjoint r-dimensional spaces $\mathrm{PG}(r,q)$ and Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$. Moreover, these r-dimensional spaces $\mathrm{PG}(r,q)$ and Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$ can be described in the following more detailed way: $\mathcal{T}$ is a union of generators of $\mathrm{W}(2r+1,q)$, pairs of r-dimensional spaces $\{U,U^\perp\}$, with $U\cap U^\perp=\emptyset$, subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$ invariant under the symplectic polarity, and of pairs $\{\mathrm{PG}(2r+1,\sqrt{q})_1, \mathrm{PG}(2r+1,\sqrt{q})_2\}$, where $P^\perp\cap \mathrm{PG}(2r+1,\sqrt{q})_2 = \mathrm{PG}(2r,\sqrt{q})$ for all $P\in \mathrm{PG}(2r+1,\sqrt{q})_1$.*

Proof. This characterization result follows from the preceding lemmas. □

Remark 3.13 (1) *The preceding theorem shows that a possible construction for i-tight sets in $\mathrm{W}(2r+1,q)$ is to consider two disjoint Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$, that are each others image under the symplectic polarity.*

It is still an open problem whether such an example exists. An exhaustive search for such a 6-tight set in $\mathrm{W}(3,4)$ gave no such example [14, 28]. *One of the referees gave us the following proof for the non-existence of such a 6-tight set in $\mathrm{W}(3,4)$. We wish to thank this referee for giving us this proof.*

Theorem 3.14 *The symplectic polar space $\mathrm{W}(3,4)$ does not have a 6-tight set which is the union of two disjoint Baer subgeometries $\mathrm{PG}(3,2)$ which are each others image under the symplectic polarity.*

Proof. The isometry group $\mathrm{PSp}(4,4)$ of $\mathrm{W}(3,4)$ has three orbits on Baer subgeometries $\mathrm{PG}(3,2)$:

1. those which are invariant under the symplectic polarity (there are 1360 of them);
2. those which share 11 lines with their perp, 9 of which are totally isotropic (there are 27200 of them);
3. those which share 7 lines with their perp, all totally isotropic (there are 20400 of them).

So, in the second and third case, there is a line of $\mathrm{PG}(3,4)$ containing 3 points of the first Baer subgeometry and 3 points of the second Baer subgeometry; these two sets of size 3 necessarily intersect in at least one point. Hence, there cannot be a 6-tight set in $\mathrm{W}(3,4)$ obtained by two disjoint Baer subgeometries which are paired by the symplectic polarity. □

4 Weighted m-covers and weighted m-ovoids

The topic of Section 5 will be applications of minihypers to the study of weighted m-covers and weighted m-ovoids. We first state the required definitions.

If P is a point of $\mathrm{PG}(n,q)$, then $\mathsf{star}(P)$ denotes the set of lines of $\mathrm{PG}(n,q)$ through P. Let $\mathrm{Q}(4,q)$ denote the non-singular parabolic quadric in $\mathrm{PG}(4,q)$, $\mathrm{Q}^-(5,q)$ the non-singular elliptic quadric in $\mathrm{PG}(5,q)$, $\mathrm{W}(3,q)$ the 3-dimensional symplectic space over $\mathbb{F}_q$, and $\mathrm{H}(3,q^2)$ the non-singular hermitian variety in $\mathrm{PG}(3,q^2)$.

Definition 4.1 *A* weighted set B *of a projective space or of a classical finite polar space is a set of points, to which a weight function* w *is associated, which satisfies the following properties:*

1. $w(P) \geq 0$ *for all points* P *of the projective space or classical finite polar space,*
2. $w(P) > 0$ *if and only if* $P \in B$.

The intersection size *of a weighted set* B *with an other set* B' *is the sum* $\sum_{P \in B \cap B'} w(P)$.

Definition 4.2 *Let* Q *be either* $Q(4,q)$ *or* $\mathrm{Q}^-(5,q)$.

A weighted m-ovoid $\mathcal{O}$ *on* Q *is a weighted set of points on* Q *such that each line of* Q *contains exactly* m *points of* $\mathcal{O}$.

A partial weighted m-ovoid $\mathcal{O}$ *on* Q *is a weighted set of points on* Q *such that each line of* Q *contains at most* m *points of* $\mathcal{O}$.

If $\mathcal{O}$ is a weighted m-ovoid on $\mathrm{Q}(4,q)$ (respectively $\mathrm{Q}^-(5,q)$), then $|\mathcal{O}| = m(q^2+1)$ (respectively $m(q^3+1)$). An example of a weighted m-ovoid, when ovoids exist, is obtained by simply taking a sum of m ovoids.

In the case of $m = (q+1)/2$, q odd, we prefer the notion of a *weighted hemisystem.*

Construction 4.3 Consider a conic $\mathcal{C}$ in $Q(4,q)$, q odd, such that the perp $\mathcal{C}^\perp$ is an external line L of $Q(4,q)$. Take $m \leqslant (q+1)/2$ points $P_1,\ldots,P_m$ of L such that $P_i^\perp \cap Q(4,q) = Q_i^-(3,q), i = 1,\ldots,m$. We have that $\mathcal{C} \subset Q_i^-(3,q)$, $\forall i$. Since $Q^-(3,q)$ is an ovoid of $Q(4,q)$, every line of $Q(4,q)$ has one point in common with each $Q_i^-(3,q)$. Hence, $\cup_{i=1}^m Q_i^-(3,q)$ is a weighted m-ovoid of $Q(4,q)$.

This is an example of a weighted m-ovoid, where the $q+1$ points of $\mathcal{C}$ have weight m and all the other points of the weighted m-ovoid have weight 1. In the case $m = (q+1)/2$, q odd, we have constructed a weighted hemisystem.

The dual of an m-ovoid on $\mathrm{Q}(4,q)$ is a weighted m-cover of $\mathrm{W}(3,q)$, so every point of $\mathrm{W}(3,q)$ is covered m times. The dual of a $\mathrm{Q}^-(3,q)$ on $Q(4,q)$ is a regular spread of $\mathrm{W}(3,q)$. In the dual of Construction 4.3, the lines coming from the points of $\mathcal{C}$ will have weight m and the other lines of the weighted m-cover will have weight 1.

Remark 4.4 *In* $\mathrm{PG}(3,q)$*, there exist* 2*-covers which cannot be partitioned into two disjoint spreads of* $\mathrm{PG}(3,q)$*. The example for* q *odd is due to Ebert* [12]*, and the example for* *q even is due to Drudge* [11]. *Both examples consist of lines of a symplectic space* $\mathrm{W}(3, q)$*, so are in fact* 2*-covers of* $\mathrm{W}(3, q)$*.*

Since $\mathrm{Q}(4, q)$ (respectively $\mathrm{Q}^-(5, q)$) is the point-line dual of $\mathrm{W}(3, q)$ (respectively $\mathrm{H}(3, q^2)$), see e.g. [26, §3.2], it makes sense to introduce the dual notion.

Definition 4.5 *Let* P *be either* $\mathrm{W}(3, q)$ *or* $\mathrm{H}(3, q^2)$*.*

A partial dual weighted m-ovoid $\mathcal{O}^*$ *is a set of lines in* P *such that each point of* P *is incident with at most* m *lines of* $\mathcal{O}^*$*. We will also use the name* partial weighted m-cover *for a partial dual weighted* m*-ovoid.*

Definition 4.6 *The* deficiency δ *of a partial (dual) weighted* m*-ovoid of* Q *is the number of (lines) points that it lacks to be an (dual)* m*-ovoid.*

A partial weighted m*-ovoid (or cover) of* Q *is called* maximal *when it is not contained in a larger partial weighted* m*-ovoid (or cover) of* Q*.*

5 The link with minihypers

Theorem 5.1 *Suppose that* $\mathcal{O}^*$ *is a partial weighted* m*-cover of* $\mathrm{W}(3, q)$*, having deficiency* δ*. Define as follows a weight function* w*:*

$$w : \mathrm{PG}(3, q) \to \mathbb{N} : P \mapsto m - |\mathsf{star}(P) \cap \mathcal{O}^*|.$$

If F *is the set of points of* $\mathrm{PG}(3, q)$ *with positive weight, then* (F, w) *is a* $\{\delta(q + 1), \delta; 3, q\}$*-minihyper.*

Proof. The weight of $\mathrm{PG}(3, q)$ equals

$$\begin{aligned} w(\mathrm{PG}(3, q)) = \sum_{P \in \mathrm{PG}(3,q)} w(P) &= m(q^3 + q^2 + q + 1) - |\mathcal{O}^*|(q + 1) \\ &= \delta(q + 1), \end{aligned}$$

since $|\mathcal{O}^*| = m(q^2 + 1) - \delta$.

A plane π of $\mathrm{PG}(3, q)$ intersects $\mathrm{W}(3, q)$ in a pencil of lines, i.e., in the set of lines in π that pass through a given point of π. Let α denote the number of lines of $\mathcal{O}^*$ contained in π. Clearly, $\alpha \leq m$. So,

$$\begin{aligned} w(\pi) = \sum_{P \in \pi} w(P) &= m(q^2 + q + 1) - \alpha(q + 1) - (|\mathcal{O}^*| - \alpha) \\ &= \delta + q(m - \alpha) \ \geq \ \delta. \end{aligned}$$

Theorem 2.2 of [20] shows that (F, w) is a $\{\delta(q + 1), \delta; 3, q\}$-minihyper. In [20], the theorem is proven for minihypers without weights, but the proof also holds when weights are allowed. □

Corollary 5.2 *If $\mathcal{O}^*$ is a maximal partial weighted m-cover of* $\mathrm{W}(3,q)$ *with deficiency $\delta < \epsilon_q$, then δ is even.*

Proof. If $\delta < \epsilon_q$, then any $\{\delta(q+1), \delta; 3, q\}$-minihyper (F, w) can be written as a sum of lines, see [16]. Apply this result to the minihyper (F, w) associated to $\mathcal{O}^*$ (Theorem 5.1).

Suppose that L is a line of this sum. Since $\mathcal{O}^*$ is maximal, L is not a line of $\mathrm{W}(3,q)$, so $L^\perp \neq L$. Let $L = \{R_0, R_1, \ldots, R_q\}$ and $L^\perp = \{S_0, S_1, \ldots, S_q\}$. The lines of $\mathrm{W}(3,q)$ intersecting L, intersect $L^\perp$, and vice versa. If $w(R_0) + \cdots + w(R_q)$ is the total weight of the points of L, then exactly $m(q+1) - (w(R_0) + \cdots + w(R_q))$ lines of $\mathcal{O}^*$ intersect L, so exactly $m(q+1) - (w(R_0) + \cdots + w(R_q))$ lines of $\mathcal{O}^*$ intersect $L^\perp$. If $s(q+1) \leq w(R_0) + \cdots + w(R_q) < (s+1)(q+1)$, then L occurs exactly s times in the sum (F, w). So L and $L^\perp$ appear in the sum (F, w) with the same weight, so we get a pairing of the lines contained in (F, w). Hence, δ is even. □

Corollary 5.3 *If $\mathcal{O}$ is a maximal partial weighted m-ovoid of* $\mathrm{Q}(4,q)$ *with deficiency $\delta < \epsilon_q$, then δ is even.*

Proof. This follows from the duality between $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$. □

Theorem 5.4 *Suppose that $\mathcal{O}^*$ is a weighted partial m-cover of* $\mathrm{H}(3,q^2)$*, having deficiency δ. Define as follows a weight function w:*

$$w : \mathrm{PG}(3,q^2) \to \mathbb{N} : P \mapsto \begin{cases} 0 & \textit{when } P \notin \mathrm{H}(3,q^2), \\ m - |\mathsf{star}(P) \cap \mathcal{O}^*| & \textit{when } P \in \mathrm{H}(3,q^2). \end{cases}$$

If F is the set of points of $\mathrm{PG}(3,q^2)$ *with positive weight, then (F, w) is a $\{\delta(q^2+1), \delta; 3, q^2\}$-minihyper.*

Proof. The proof is similar to the proof of Theorem 5.1. □

Corollary 5.5 *If $\mathcal{O}^*$ is a weighted partial m-cover of* $\mathrm{H}(3,q^2)$ *with deficiency $\delta < \epsilon_{q^2} = q+1$, then $\mathcal{O}^*$ can be extended to a weighted m-cover of* $\mathrm{H}(3,q^2)$.

Proof. If $\delta < \epsilon_{q^2} = q+1$, then any $\{\delta(q^2+1), \delta; 3, q^2\}$-minihyper (F, w) can be written as a sum of lines, see [16]. Applying this result to the minihyper obtained as in Theorem 5.4, it follows that the set $\mathcal{O}^*$ can be extended to a weighted m-cover of $\mathrm{H}(3,q^2)$ by adding the lines that constitute the sum (F, w). □

We now apply the duality between $\mathrm{H}(3,q^2)$ and $\mathrm{Q}^-(5,q)$.

Corollary 5.6 *If $\mathcal{O}$ is a weighted partial m-ovoid of* $\mathrm{Q}^-(5,q)$ *with deficiency $\delta < \epsilon_{q^2} = q+1$, then $\mathcal{O}$ can be extended to a weighted m-ovoid of* $\mathrm{Q}^-(5,q)$.

Suppose now that $(\mathcal{H}, w)$ is a weighted hemisystem of $\mathrm{Q}^-(5,q)$, q odd. So $(\mathcal{H}, w)$ has $\sum_{x\in\mathcal{H}} w(x) = (q^3+1)(q+1)/2$ points. Associate the following linear code C to this hemisystem $(\mathcal{H}, w) = \{g_1, \ldots, g_n\}$, with $n = (q^3+1)(q+1)/2$.

Consider $G = (g_1 \cdots g_n)$ as the generator matrix of C. This defines a code C of length $n = (q^3+1)(q+1)/2$ and dimension $k = 6$. Consider the message $(u_1, \ldots, u_6)$. This message defines the codeword $x = (u_1, \ldots, u_6)G = ((u_1, \ldots, u_6)g_1, \ldots, (u_1, \ldots, u_6)g_n)$.

Consider the hyperplane $\pi_4 : u_1X_1 + \cdots + u_6X_6 = 0$ of $\mathrm{PG}(5,q)$, then $(u_1, \ldots, u_6)g_i = 0 \Leftrightarrow g_i \in \pi_4$. So the weight of x is the number of points of the hemisystem that do not lie in this hyperplane π_4. Since the minimal distance d of C is equal to the minimal weight of the non-zero codewords, we look for all different kinds of hyperplanes π_4 and how many points of $\mathcal{H}$ they contain.

a. $\pi_4 \cap \mathrm{Q}^-(5,q) = \mathrm{Q}(4,q)$. Since $\mathcal{H}$ induces a weighted $(q+1)/2$-ovoid on $Q(4,q)$,

$$|\mathcal{H} \cap Q(4,q)| = (q^2+1)\frac{q+1}{2}.$$

So this gives a codeword of weight

$$\frac{(q^3+1)(q+1)}{2} - \frac{(q^2+1)(q+1)}{2} = \frac{q+1}{2}(q^3-q^2).$$

b. $\pi_4 \cap \mathrm{Q}^-(5,q) = R\mathrm{Q}^-(3,q)$, with $R \notin \mathcal{H}$. This tangent cone contains q^2+1 lines which each contain $(q+1)/2$ points of the hemisystem. This gives a codeword of the same weight as above.

c. $\pi_4 \cap \mathrm{Q}^-(5,q) = R\mathrm{Q}^-(3,q)$, with $R \in \mathcal{H}$ and with $w(R) = a$. Then this tangent cone contains

$$a + (q^2+1)(\frac{q+1}{2} - a) = (q^2+1)\frac{q+1}{2} - aq^2$$

points, so this gives a codeword of weight $(q+1)(q^3-q^2)/2 + aq^2$.

So $d = (q+1)(q^3-q^2)/2$, and C is a $[(q^3+1)(q+1)/2, 6, (q+1)(q^3-q^2)/2]$-code.

Now we know the parameters n, k, d of this linear code C, we compare these parameters with the Griesmer bound [19, 32]:

$$\begin{aligned} n = \frac{(q^3+1)(q+1)}{2} &\geqslant \frac{(q+1)(q^3-q^2)}{2} + \frac{(q+1)(q^2-q)}{2} + \frac{(q+1)(q-1)}{2} \\ &\quad + \left\lceil \frac{q^2-1}{2q} \right\rceil + \left\lceil \frac{q^2-1}{2q^2} \right\rceil + \left\lceil \frac{q^2-1}{2q^3} \right\rceil \\ &\geqslant \frac{(q+1)(q^3-1)}{2} + \frac{q+1}{2} + 2 \\ &\geqslant \frac{(q+1)(q^3+1)}{2} - \frac{q-3}{2}. \end{aligned}$$

So the length of C has a difference of $(q-3)/2$ with relation to the Griesmer bound $g_q(k,d)$. In the case of $q=3$, we reach the Griesmer bound. For the next part, we consider $q=3$. We also rely on the following theorem. Let $g_q(k,d)$ be the Griesmer bound for linear $[n,k,d]$-codes over $\mathbb{F}_q$.

Theorem 5.7 [10] *Suppose that C is a $[t+g_q(k,d),k,d]$-code and $d \leqslant sq^{k-1}$. Then any generator matrix of C contains no more than $s+t$ equivalent columns.*

Since for $q=3$, the Griesmer bound is reached and also $d \leqslant q^5$, we have $t=0$ and $s=1$. This means that the generator matrix of the code has no equivalent columns. So every point of the hemisystem $\mathcal{H}$ has weight 1. For $q=3$, the hemisystem $\mathcal{H}$ is a set of $(q^3+1)(q+1)/2$ different points such that each line of $\mathrm{Q}^-(5,3)$ contains exactly $(q+1)/2=2$ points of $\mathcal{H}$. So we have that the hemisystem is in fact also a cap on $\mathrm{Q}^-(5,3)$. The largest caps on $\mathrm{Q}^-(5,3)$ have size 56. This means that we have an extendability result on partial caps on $\mathrm{Q}^-(5,3)$.

Since $\epsilon_9 = 4$, our results show that every $(56-3=53)$-cap on $\mathrm{Q}^-(5,3)$ is extendable to a maximal 56-cap on $\mathrm{Q}^-(5,3)$.

Theorem 5.8 *Every* 53-*,* 54-*, or* 55-*cap on* $\mathrm{Q}^-(5,3)$ *is extendable to a maximal* 56-*cap on* $\mathrm{Q}^-(5,3)$.

The preceding observation gives us an alternative proof for part of the results of [2, 23] where the problem of the complete caps in $\mathrm{PG}(5,3)$ was studied in detail.

Until now we have studied weighted partial m-ovoids on $\mathrm{Q}(4,q)$, but we also have a look at the weighted partial m-covers on $\mathrm{Q}(4,q)$. So $\mathcal{H}$ is a weighted set of $m(q^2+1)-\delta$ lines so that each point of $\mathrm{Q}(4,q)$ lies on at most m lines.

Theorem 5.9 *Let $\mathcal{H}$ be a weighted partial m-cover of deficiency $\delta < q$ on $\mathrm{Q}(4,q)$. Define a weight function w in the following way:*

$$w : \mathrm{PG}(4,q) \to \mathbb{N} : P \mapsto \begin{cases} 0 & \textit{when } P \notin \mathrm{Q}(4,q), \\ m - |\mathsf{star}(P) \cap \mathcal{H}| & \textit{when } P \in \mathrm{Q}(4,q). \end{cases}$$

If F is the set of points of $\mathrm{PG}(4,q)$ with positive weight, then (F,w) is a $\{\delta(q+1),\delta;4,q\}$-minihyper.

Proof. Similar to the proof of Theorem 5.1. □

Corollary 5.10 *If $\mathcal{H}$ is a weighted partial m-cover of $\mathrm{Q}(4,q)$ with deficiency $\delta < \epsilon_q$, where $q+\epsilon_q$ is the size of the smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$, then $\mathcal{H}$ can be extended to a weighted m-cover of $\mathrm{Q}(4,q)$.*

Proof. Similar to the proof of Corollary 5.2. □

Using the duality between $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$, also the following corollary holds.

Corollary 5.11 *If $\mathcal{O}$ is a weighted partial m-ovoid of* $\mathrm{W}(3,q)$ *with deficiency $\delta < \epsilon_q$, where $q+\epsilon_q$ is the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$*, then $\mathcal{O}$ can be extended to a weighted m-ovoid of* $\mathrm{W}(3,q)$.

Acknowledgement

Part of the research for this paper was done while the second author was visiting SZTAKI, the Computer and Automation Research Institute of the Hungarian Academy of Sciences. He gratefully acknowledges the hospitality and financial support extended to him.

The authors wish to thank the referees for their many suggestions and remarks for improving the original version of our manuscript. We especially thank one of the referees for giving us the proof for Theorem 3.14. We also thank John Bamberg for his suggestions for improving our original manuscript.

References

[1] J. Bamberg, S. Kelly, M. Law, and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory, Ser. A*, 114: 1293-1314, 2007.

[2] J. Barát, Y. Edel, R. Hill, and L. Storme. On complete caps in the projective geometries over $\mathbb{F}_3$. II: New improvements. *J. Combin. Math. and Combin. Computing,* 49:9–31, 2004.

[3] A. Blokhuis. Note on the size of a blocking set in $PG(2,p)$. *Combinatorica,* 14:111–114, 1994.

[4] M.R. Brown, J. De Beule, and L. Storme. Maximal partial spreads of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$. *European J. Combin.*, 24(1):73–84, 2003.

[5] A.A. Bruen. Baer subplanes and blocking sets. *Bull. Amer. Math. Soc.,* 76:342–344, 1970.

[6] P.J. Cameron and R.A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.*, 46:91–102, 1982.

[7] I. Cardinali and B. De Bruyn. Spin-embeddings, two-intersection sets and two-weight codes. *Ars Combin.*, to appear.

[8] J. De Beule, A. Hallez, and L. Storme. On the non-existence of Cameron-Liebler line classes in $PG(3,q)$. *J. Combin. Des.*, 16:342–349, 2008.

[9] B. De Bruyn. A characterization of m-ovoids and i-tight sets of polar spaces. *Adv. Geom.*, to appear.

[10] S. Dodunekov and J. Simonis. Optimal linear codes. Unpublished notes.

[11] K. Drudge. Proper 2-covers of $\mathrm{PG}(3, q)$, q even. *Geom. Dedicata*, 80:59–64, 2000.

[12] G.L. Ebert. The completion problem for partial packings. *Geom. Dedicata*, 18:261–267, 1985.

[13] S. Ferret, L. Storme, P. Sziklai, and Zs. Weiner. A $t \pmod p$ result on multiple $(n-k)$-blocking sets in $\mathrm{PG}(n, q)$. *Innov. Incidence Geom.*, to appear.

[14] GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra: http://www-gap.mcs.st-and.ac.uk/

[15] P. Govaerts and L. Storme. On a particular class of minihypers and its applications. II: Improvements for q square. *J. Combin. Theory, Ser. A*, 97:369–393, 2002.

[16] P. Govaerts and L. Storme. On a particular class of minihypers and its applications I: The result for general q. *Des. Codes Cryptogr.*, 28:51–63, 2003.

[17] P. Govaerts and L. Storme. On Cameron-Liebler line classes. *Adv. Geom.*, 4:279–286, 2004.

[18] P. Govaerts, L. Storme, and H. Van Maldeghem. On a particular class of minihypers and its applications. III: Applications. *European J. Combin.*, 23:659–672, 2002.

[19] J.H. Griesmer. A bound for error-correcting codes. *IBM J. Res. Develop.*, 4:532–542, 1960.

[20] N. Hamada. A characterization of some $[n, k, d; q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry. *Discrete Math.*, 116:229–268, 1993.

[21] N. Hamada and T. Helleseth. Codes and minihypers. *Proceedings of the Third European Workshop on Optimal Codes and Related Topics, OC'2001, June 10-16, 2001, Sunny Beach, Bulgaria*, pages 79–84, 2001.

[22] N. Hamada and F. Tamari. On a geometrical method of construction of maximal t-linearly independent sets. *J. Combin. Theory, Ser. A*, 25:14–28, 1978.

[23] R. Hill, I. Landjev, C. Jones, L. Storme, and J. Barát. On complete caps in the projective geometries over $\mathbb{F}_3$. Proceedings of the *Second Pythagorean Conference*, (Samos, Greece, May 30-June 5, 1999), *J. Geom.*, 67:127–144, 2000.

[24] J.W.P. Hirschfeld. *Projective Geometries Over Finite Fields. Second Edition.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.

[25] J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[26] S.E. Payne and J.A. Thas. *Finite Generalized Quadrangles.* Pitman, London, 1984.

[27] T. Penttila. Cameron-Liebler line classes in PG$(3, q)$. *Geom. Dedicata,* 37(3):245–252, 1991.

[28] pg - Projective Geometries: http://cage.ugent.be/∼jdebeule/pg.html

[29] O. Polverino. Small blocking sets and complete k-arcs in $PG(2, p^3)$. *Discrete Math.*, 208/209:469–476, 1999.

[30] O. Polverino. Small blocking sets in $PG(2, p^3)$. *Des. Codes Cryptogr.*, 20:319–324, 2000.

[31] O. Polverino and L. Storme. Small minimal blocking sets in $PG(2, p^3)$. *European J. Combin.*, 23:83–92, 2002.

[32] G. Solomon and J.J. Stiffler. Algebraically punctured cyclic codes. *Inform. and Control*, 8:170–179, 1965.

[33] L. Storme. Linear codes meeting the Griesmer bound, minihypers, and geometric applications. *Le Matematiche*, LIX:367–392, 2004.

[34] T. Szőnyi and Zs. Weiner. Small blocking sets in higher dimensions. *J. Combin. Theory, Ser. A*, 95:88–101, 2001.

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