Partial covers of $\mathrm{PG}(n, q)$

S. Dodunekov L. Storme G. Van de Voorde *

August 27, 2008

Abstract

In this paper, we investigate some properties of partial covers of $\mathrm{PG}(n, q)$. We show that a set of $q + a$ hyperplanes, $q \geq 81$, $a < (q - 1)/3$, or $q > 13$ and $a \leq (q - 10)/4$, that does not cover $\mathrm{PG}(n, q)$, does not cover at least $q^{n-1} - aq^{n-2}$ points, and that this bound is sharp. In the planar case, we show that if there are at most $q + a$ non-covered points, $q \geq 81$, $a < (q - 1)/3$, the non-covered points are collinear. In this case, the bound on a is sharp. Moreover, for $\mathrm{PG}(n, q)$, we show that for $q \geq 81$ and $a < (q - 1)/3$, or $q > 13$ and $a \leq (q - 10)/4$, if the number of non-covered points is at most q^{n-1}, then all non-covered points are contained in one hyperplane.

1 Introduction

Let $\mathrm{PG}(n, q)$ denote the n-dimensional projective space over the finite field $\mathbb{F}_q$ with q elements, where $q = p^h$, p prime, $h \geq 1$. We denote the number of points in $\mathrm{PG}(n, q)$ by θ_n, i.e., $\theta_n = \frac{q^{n+1}-1}{q-1}$.

Let $\mathcal{C}$ be a family of $q + a$ hyperplanes of $\mathrm{PG}(n, q)$. Denote by $\mathcal{C}(P)$ the set of hyperplanes of $\mathcal{C}$ containing P. A $(q + a)$-cover $\mathcal{C}$ of $\mathrm{PG}(n, q)$ is a family $\mathcal{C}$ of $q + a$ different hyperplanes in $\mathrm{PG}(n, q)$ such that $|\mathcal{C}(P)| \geq 1, \forall P \in \mathrm{PG}(n, q)$. A *partial* $(q + a)$-cover $\mathcal{S}$ is a set of $q + a$ hyperplanes such that there is at least one point Q in $\mathrm{PG}(n, q)$ such that $|\mathcal{S}(Q)| = 0$. A point H for which $|\mathcal{S}(H)| = 0$, is called a *hole*.

A *blocking set* of $\mathrm{PG}(n, q)$ is a set B of points such that each hyperplane of $\mathrm{PG}(n, q)$ contains at least one point of B. A blocking set B is called *trivial* if it contains a line of $\mathrm{PG}(n, q)$. If a hyperplane contains exactly one point of a blocking set B in $\mathrm{PG}(n, q)$, it is called a *tangent hyperplane* to B, and a point P of B is called *essential* when it belongs to a tangent hyperplane to B. A blocking set B is called *minimal* when no proper subset of B is also a blocking set, i.e., when each point of B is essential. A blocking set of $\mathrm{PG}(n, q)$ is called *small* if it contains less than $3(q + 1)/2$ points.

It is clear that a cover of $\mathrm{PG}(n, q)$ is a dual blocking set. Dualizing the above definitions yields that a cover $\mathcal{C}$ is called *trivial* if it contains all hyperplanes through a certain $(n - 2)$-space, *minimal* if no proper subset of $\mathcal{C}$ is a

*This author is supported by the Fund for Scientific Research Flanders (FWO - Vlaanderen).

cover, and *small* if it contains less than $3(q+1)/2$ hyperplanes. A hyperplane π is *essential* to a cover $\mathcal{C}$ if there is a point $P \in \pi$ such that $\mathcal{C}(P) = \{\pi\}$

The following reducibility result, proven for $n = 2$ in [6, Remark 3.3], will be used in this article.

Result 1. *[5, Corollary 1] A blocking set B of size smaller than $2q$ in $\mathrm{PG}(n,q)$ is uniquely reducible to a minimal blocking set.*

We extend the following result of Blokhuis and Brouwer to general dimension in Theorem 10.

Result 2. *[2] Let B be a blocking set in $\mathrm{PG}(2,q)$. If $|B| = 2q - s$, then there are at least $s + 1$ tangent lines through each essential point of B.*

Finally, for q a prime, we use the following result, proven by Blokhuis [1] for $n = 2$, which shows that a small blocking set in $\mathrm{PG}(n,q)$, q prime, is trivial.

Result 3. *[4] Let B be a non-trivial blocking set in $\mathrm{PG}(n,p)$, where p is an odd prime. Then*

$$|B| \geq 3(p+1)/2.$$

2 Partial covers of $\mathrm{PG}(2,q)$

In this section, we investigate planar partial $(q+a)$-covers. We show that, under the condition that $q > 13$ and $a \leq (q-10)/4$ or $q \geq 81$ and $a < (q-1)/3$, the number of holes is at least $q - a$ (Theorem 8), and if the number of holes is not too large, all holes are collinear (Theorems 4 and 6).

In Corollary 9, we show that if q is prime, a partial $(q+a)$-cover contains q lines through a fixed point.

Theorem 4. *Let $\mathcal{S}$ be a partial $(q+a)$-cover of $\mathrm{PG}(2,q)$, with $0 \leq a \leq (q-10)/4$, $q > 13$, with at most $q+a$ holes. Then there are at most q holes and these holes are collinear.*

Proof. Denote the set of holes by $\mathcal{H}$, denote $|\mathcal{H}|$ by x, and assume that $x \leq q+a$. Suppose that there are three non-collinear points in $\mathcal{H}$, otherwise the theorem is proven. The set $\mathcal{H}$ can be covered by at most $(x+1)/2$ lines, denote the set of these lines by $\mathcal{L}$. The set $\mathcal{S} \cup \mathcal{L}$ is a cover in $\mathrm{PG}(2,q)$. Since the size of $\mathcal{S} \cup \mathcal{L}$ is at most $q+a+(q+a+1)/2 \leq 2q$, there is a unique minimal cover $\mathcal{C}$ contained in $\mathcal{S} \cup \mathcal{L}$ (Result 1).

Let ℓ_y be a y-secant to $\mathcal{H}$ with $y \leq (q-3a-1)/2$. Interchanging ℓ_y by y other lines gives, together with the lines of $\mathcal{S}$, another cover $\mathcal{C}'$, with $|\mathcal{C} \cup \mathcal{C}'| \leq q + a + (q+a+1)/2 + (q-3a-1)/2 \leq 2q$. Hence, by the unique reducibility property, there is a unique minimal cover contained in $\mathcal{C} \cup \mathcal{C}'$, hence in $\mathcal{C} \cap \mathcal{C}'$. This minimal cover does not contain ℓ_y. This implies that to cover the holes, only lines with more than $(q-3a-1)/2$ holes are essential, we will call them *long* secants.

If there is exactly one *long* secant, then the theorem is proven. Remark that if there is only one long secant and there are q holes, then $a = 0$, since otherwise, there would be more than $q+1$ lines through a point.

Suppose that there are exactly z long secants essential to cover the holes. These z long secants, together with the $q+a$ lines of $\mathcal{S}$, form a cover $\mathcal{C}''$. Then there is a line L with less than $(q+a+1+\binom{z}{2})/z$ holes. Suppose to the contrary that any line contains at least $(q+a+1+\binom{z}{2})/z$ holes, then there are at least $z(q+a+1+\binom{z}{2})/z-\binom{z}{2} = q+a+1 > q+a$ holes, a contradiction. We construct a new cover by replacing this line L with less than $(q+a+1+\binom{z}{2})/z$ random lines, one through each hole on L. In total, with the z long secants and the lines of $\mathcal{S}$, this set of lines constitutes a cover $\mathcal{C}'''$ of size at most $q+a+z+(q+a+1+\binom{z}{2})/z$. If

$$q + a + z + (q + a + 1 + \binom{z}{2})/z \leq 2q, \qquad (1)$$

the unique reducibility property (Result 1) shows that there is a minimal cover contained in $\mathcal{C}'' \cap \mathcal{C}'''$, which does not contain the line L. This implies that the line L was not essential to the cover $\mathcal{C}''$, a contradiction. It is easy to see that for $z \geq 2$ and $z < 9$, inequality (1) holds for $a \leq (q-10)/4$ and $q > 13$. Hence, there are at least 9 long secants essential to the minimal cover $\mathcal{C}''$. On each of these secants, there are at least $(q-3a-1)/2$ holes, hence we have at least $9(q-3a-1)/2 - 9 \cdot 8/2$ holes. But

$$9(q-3a-1)/2 - 36 > q + a$$

if $a < (7q-63)/25$. Since $a \leq (q-10)/4$, and $(q-10)/4 < (7q-63)/25$, the theorem follows. □

T. Szőnyi and Zs. Weiner proved the following theorem (see [8]).

Result 5. *[8, Theorem 6.3] Let B be a point set in $\mathrm{PG}(2,q)$, $q \geq 81$, of size less than $3(q+1)/2$. Denote the number of 0-secants of B by δ, and assume that δ is at most $q\sqrt{q}/3$ when $|B| < q+\sqrt{q}$ and at most $\frac{q^2}{3(|B|-q)}$ otherwise. Then B can be obtained from a blocking set by deleting at most $2\delta/q$ points of this blocking set.*

Using this result, for $q \geq 81$ and $a < (q-1)/3$, we can improve on Theorem 4.

Theorem 6. *Let $\mathcal{S}$ be a partial $(q+a)$-cover of $\mathrm{PG}(2,q)$, with at most $q+a$ holes, with $0 \leq a < (q-1)/3$, $q \geq 81$. Then there are at most q holes and these holes are collinear.*

Proof. If the number of holes is smaller than q, the theorem is proven by Result 5 since $2\delta/q < 2q/q = 2$. Suppose that there are at least q and at most $q+a$ holes, then Result 5 shows that the holes lie on at most 2 lines. Repeating the arguments of Theorem 4 yields a contradiction if $a < (q-1)/3$, except for the case that the holes are collinear (and hence, the number of holes is at most q). This proves Theorem 6. □

Remark 7. *The bound $a < (q-1)/3$ is sharp. Let $a = (q-1)/3$ and let $\mathcal{S}$ be a set of $q-1$ lines L_i through a point P, and $a+1$ other lines through a fixed point, lying on one of the lines L_i. Then there are $2(q-a-1) < q+a$ holes, lying on two lines.*

Theorem 8. *Let $\mathcal{S}$ be a partial $(q+a)$-cover of* $\mathrm{PG}(2,q)$*, then there are at least $q-a$ holes, with $a<(q-1)/3$, $q\geq 81$ or $a\leq (q-10)/4$, $13<q$.*

Proof. Theorems 4 and 6 show that either there are more than $q+a$ holes, and the theorem holds, or that there are at most q holes, all on the same line, say M. Hence, $\mathcal{S}\cup\{M\}$ is a cover $\mathcal{C}$ of size $q+a+1$, which can be reduced to a minimal cover $\mathcal{C}'$ of size $q+a'+1$. Dualizing $\mathcal{C}'$ gives a minimal blocking set B of $q+a'+1$ points. Then Result 2 shows that every point of B lies on at least $q-a'$ tangent lines to B. This implies that any point of the dual of $\mathcal{C}$ lies on at least $(q-a')-(a-a')=q-a$ tangent lines. Dualizing again yields that every line of $\mathcal{C}$ contains at least $q-a$ points only lying on this line of $\mathcal{C}$. Now removing the line M shows that there are at least $q-a$ holes. $\square$

Corollary 9. *A partial $(p+a)$-cover $\mathcal{S}$ of* $\mathrm{PG}(2,p)$*, p prime, with $a<(p-1)/3$, $p\geq 81$ or $a\leq (p-10)/4$, $13<p$, with at most $p+a$ holes, consists of p lines through the same point R and a random lines $l_1,\dots,l_a$, not through R.*

Proof. It follows from Theorems 4, 6 and 8 that the holes are contained in one line, say M. Then the lines of $\mathcal{S}$, together with M, constitute a cover $\mathcal{C}$ of size $q+a+1<3(p+1)/2$. Result 3, together with Result 1, shows that the unique minimal cover contained in $\mathcal{C}$ is the set of all lines through a point R. It is clear that the line M is one of the lines through R. The other a lines are random, but do not contain R. $\square$

3 Partial covers of $\mathrm{PG}(n,q)$

In this section, we extend results of Section 2 to general dimension. Theorem 16 extends Theorem 8. Corollary 17 shows that if q is prime, a partial $(q+a)$-cover, with at most q^{n-1} holes, $a<(q-1)/3$, $q\geq 81$ or $a\leq (q-10)/4$, $13<q$, contains q hyperplanes through a fixed $(n-2)$-space. In Theorem 15 we show that, if there are at most q^{n-1} holes, the holes are contained in a hyperplane.

Before proving these theorems, we need the extension of Result 2 to general dimension.

Theorem 10. *The number of tangent hyperplanes through an essential point of a blocking set B of size $q+a+1$, $|B|\leq 2q$, in* $\mathrm{PG}(n,q)$ *is at least $q^{n-1}-aq^{n-2}$.*

Proof. The arguments of this proof are based on the proof of Proposition 2.5 in [7].

For $n=2$, Result 2 proves this theorem. Assume by induction that the theorem holds for all dimensions $i\leq n-1$. Let B be a blocking set in $\pi=\mathrm{PG}(n,q)$. Since $|B|\leq 2q$, there is an $(n-2)$-space L in π that is skew to B. Let H be a hyperplane through L. Embed π in $\mathrm{PG}(2n-2,q)$. Let P be a $\mathrm{PG}(n-3,q)$, skew to π, in $\mathrm{PG}(2n-2,q)$. Then $\langle B,P\rangle$, the cone with vertex P and base B, is a blocking set with respect to the $(n-1)$-spaces of $\mathrm{PG}(2n-2,q)$. Let $H^*\neq H$ be a hyperplane through L only sharing one point Q with B. Since $|B|$ is at most $2q$, there are at least 2 tangent hyperplanes through L, hence H^* can be chosen different from H.

Let $\mathcal{S}$ be a regular $(n-2)$-spread through L and $\langle Q,P\rangle$ in W, the $(2n-3)$-dimensional space spanned by L and $\langle Q,P\rangle$. Using the André-Bruck-Bose construction (see [3]), this yields a projective plane $\mathrm{PG}(2,q^{n-1})=\Pi^W$. The arguments of [7, Proposition 2.5] show that H defines a line ℓ in Π^W, only having essential points of the blocking set $\bar{B}$ of size $1+(q+a)q^{n-2} = q^{n-1}+aq^{n-2}+1$, where $\bar{B}$ is the blocking set in $\mathrm{PG}(2, q^{n-1})$, corresponding to $\langle B, P\rangle$. This number of points comes from $\langle Q, P\rangle$ at infinity, which is one point of the blocking set, and the $q+a$ affine points R_i of B, all on a cone $\langle R_i, P\rangle$ with q^{n-2} affine points. Result 2 shows that any essential point lies of $\bar{B}$ on at least $q^{n-1}-aq^{n-2}$ tangent lines to the blocking set $\bar{B}$ in Π^W. We will show that the number of tangent lines through an essential point of the blocking set $\bar{B}$ in Π^W is a lower bound on the number of tangent hyperplanes through an essential point of B in $\mathrm{PG}(n, q)$.

A tangent line through an affine essential point R corresponds to an $(n-1)$-space $\langle R, \Omega\rangle$, with Ω a spread element of $\mathcal{S}$. The space $\langle R, \Omega\rangle$ is not necessarily a tangent hyperplane to B in $\mathrm{PG}(n, q)$. Note that $\Omega \neq \langle Q, P\rangle$, since both are spread elements and cannot coincide since $\langle Q, P\rangle$ is an element of the blocking set, hence $\langle R, Q, P\rangle$ cannot be a tangent space.

The projection of $\langle R, \Omega\rangle$ from P onto $\mathrm{PG}(n, q)$ is an $(n-1)$-dimensional space through R in $\mathrm{PG}(n, q)$ which is skew to Q since $\Omega \cap \langle Q, P\rangle = \emptyset$, and which only has R in common with B since $\langle \Omega, R\rangle \cap \langle B, P\rangle = \{R\}$. Hence, this projection is a tangent $(n-1)$-space through R to B in $\mathrm{PG}(n, q)$. So we have shown that any tangent line in R to $\bar{B}$ gives rise to a tangent hyperplane to B. If any tangent line to $\bar{B}$ in R gives rise to a different tangent hyperplane to B, the theorem is proven.

Let η be a tangent hyperplane to B in R which is the projection of two tangent lines $\langle \Omega, R\rangle$ and $\langle \Omega', R\rangle$. The dimension of $\langle \eta, P\rangle$ is $2n-3$, and $\dim(\langle \eta, P\rangle \cap W) = 2n-4$. A hyperplane of $\mathrm{PG}(2n-3, q)$ contains exactly one element of a regular $(n-2)$-spread. Since it contains Ω and Ω', $\Omega = \Omega'$. So η is the projection of at most one such $(n-1)$-space. □

Lemma 11. *Let $\mathcal{S}$ be a partial $(q+a)$-cover of $\mathrm{PG}(n, q)$, $a < q$. If all holes are contained in a hyperplane π of $\mathrm{PG}(n, q)$, then there are at least $q^{n-1} - aq^{n-2}$ holes.*

Proof. The hyperplanes of $\mathcal{S}$, together with the hyperplane π that contains all holes, form a cover of size $q+a+1$, in which π is an essential hyperplane. Dualizing gives a blocking set B of size $q+a+1$, where the dual of π is an essential point. Theorem 10 shows that the dual of π lies on at least $q^{n-1}-aq^{n-2}$ tangent hyperplanes to B. Dualizing again shows that π contains at least $q^{n-1}-aq^{n-2}$ points that are only covered by π. Removing π shows that there are at least $q^{n-1}-aq^{n-2}$ holes. □

Remark 12. *The bound in Lemma 11 is sharp. Let $\mathcal{S}$ be the set of q hyperplanes through a fixed $(n-2)$-space π_{n-2}. Let H be the hyperplane through π_{n-2}, which is not chosen. Take a hyperplanes for which the $(n-2)$-dimensional intersections with H, go through a common $(n-3)$-space of π_{n-2}, then there are exactly $q^{n-1}-aq^{n-2}$ holes.*

Lemma 13. *Let $\mathcal{S}$ be a partial $(q+a)$-cover of $\mathrm{PG}(n, q)$, $n \geq 3$, $a < (q-1)/3$ with $q \geq 81$ or $a \leq (q-10)/4$ with $13 < q$, with at most q^{n-1} holes. A line that contains 2 holes, contains at least $a+3$ holes.*

Proof. Let L be a line with x holes, $x < q - a$. If there are at most $q + a$ holes in a plane through L, Theorems 4, 6 and 8 show that in this plane, there are at least $q - a$ holes, which are all collinear.

This implies that any of the planes through L contains at least $q + a + 1$ holes. This implies that there are at least

$$\theta_{n-2}(q + a + 1 - x) + x$$

holes in $\mathrm{PG}(n, q)$, which has to be at most q^{n-1}. If $x = a + 2$, $\theta_{n-2}(q + a + 1 - a - 2) + a + 2 > q^{n-1}$, a contradiction. Hence, x is at least $a + 3$. □

Lemma 14. *Let $\mathcal{S}$ be a partial $(q + a)$-cover of* $\mathrm{PG}(n, q)$*, $n \geq 3$, $a < (q - 1)/3$ with $q \geq 81$ or $a \leq (q - 10)/4$ with $13 < q$, with at most q^{n-1} holes. Then every hole lies on more than $q^{n-2}/2$ lines with at least $q - a$ holes.*

Proof. Let R be a hole. There is a line L through R containing only covered points and R, otherwise there would be at least $\theta_{n-1} + 1$ holes. A plane through L contains either at most $q - 1$ holes on a line through R, different from L, or it contains at least $q + a$ holes different from R.

Suppose that there are A planes through L with at most $q - 1$ holes different from R. Then the number of holes is at least

$$A(q - a - 1) + (\theta_{n-2} - A)(q + a) + 1,$$

which has to be at most q^{n-1}. Suppose that $A = q^{n-2}/2$; we obtain a contradiction. Hence, there are more than $q^{n-2}/2$ planes with at most q holes. Theorems 4, 6 and 8 state that in each of these planes, there is a line through R containing at least $q - a - 1$ other holes, and all holes in such a plane lie on this line. □

Theorem 15. *Let $\mathcal{S}$ be a partial $(q + a)$-cover with at most q^{n-1} holes, $a < (q - 1)/3$ with $q \geq 81$ or $a \leq (q - 10)/4$ with $13 < q$. Then the holes are contained in one hyperplane of* $\mathrm{PG}(n, q)$.

Proof. For $n = 2$, this is proven in Theorem 4. Suppose by induction that this theorem holds for any dimension $i \leq n - 1$.

First, we show that there is a hyperplane π of $\mathrm{PG}(n, q)$ with at most q^{n-2} holes. Let R be a hole. There is a line L through R containing only covered points and R. Suppose that all planes through L contain more than q holes, then there would be at least $\theta_{n-2}q + 1$ holes, a contradiction. Suppose that there is an x-dimensional space π_x with at most q^{x-1} holes. Then there is an $(x + 1)$-dimensional space containing π_x with at most q^x holes. Otherwise, the number of holes would be at least $\theta_{n-x-1}(q^x + 1 - q^{x-1}) + q^{x-1}$, a contradiction if $x \leq n - 1$. Hence, by induction, there is a hyperplane π of $\mathrm{PG}(n, q)$ with at most q^{n-2} holes.

Using the induction hypothesis, all holes in π are contained in an $(n - 2)$-dimensional space π_{n-2} of π. Moreover, Lemma 11 shows that the number of holes in π_{n-2} is at least $q^{n-2} - aq^{n-3}$.

There are at least $\theta_{n-2}(q - a - 1) + 1$ holes in $\mathrm{PG}(n, q)$ since every plane through L contains at least $q - a - 1$ extra holes. Hence, there is certainly a hole R' that is not contained in π_{n-2}.

Now we distinguish between two cases.

Case 1: All lines through R' with at least $q-a$ holes are lines which intersect π_{n-2}. Lemma 14 shows that there are at least $q^{n-2}/2$ such lines. Since a line through two holes contains at least $a+3$ holes (see Lemma 13), counting the holes in $\langle R', \pi_{n-2}\rangle$ yields that this number is at least

$$q^{n-2}(q-a-1)/2 + (q^{n-2} - aq^{n-3} - q^{n-2}/2)(a+2) + 1.$$

If all holes are contained in $\langle R', \pi_{n-2}\rangle$, the theorem is proven. Suppose now that not all holes are contained in the hyperplane $\langle R', \pi_{n-2}\rangle$. Let R'' be a hole not in $\langle R', \pi_{n-2}\rangle$. Connecting R'' with all the holes in $\langle R', \pi_{n-2}\rangle$ yields at least $(a+2)(q^{n-2}(q-a-1)/2+(q^{n-2}-aq^{n-3}-q^{n-2}/2)(a+2)+1)+1$ holes, which is more than q^{n-1}, a contradiction.

Case 2: There is a line through R' with at least $q-a$ holes skew to π_{n-2}. This yields at least

$$(q-a)(q^{n-2} - aq^{n-3})(a+1) + q^{n-2} - aq^{n-3} + q - a > q^{n-1}$$

holes, a contradiction.

□

Theorem 16. *Let $\mathcal{S}$ be a partial $(q+a)$-cover of* $\mathrm{PG}(n,q)$*, $a < (q-1)/3$ with $q \geq 81$ or $a \leq (q-10)/4$ with $13 < q$. Then there are at least $q^{n-1} - aq^{n-2}$ holes.*

Proof. This follows immediately from Theorem 15 and Lemma 11. □

Corollary 17. *A partial $(p+a)$-cover $\mathcal{S}$ of* $\mathrm{PG}(n,p)$*, p prime, $a < (p-1)/3$ with $p \geq 81$ or $a \leq (p-10)/4$ with $13 < p$, with at most p^{n-1} holes, consists of p hyperplanes through a common $(n-2)$-space π and a random hyperplanes, not through π.*

Proof. It follows from Theorem 15 that the holes are contained in one hyperplane, say μ. Then the hyperplanes of $\mathcal{S}$, together with μ, constitute a cover $\mathcal{C}$ of size $p+a+1 < 3(p+1)/2$. Result 3, together with Result 1, shows that the unique minimal cover contained in $\mathcal{C}$ is the set of all hyperplanes through an $(n-2)$-space π. Since this set covers $\mathrm{PG}(n,p)$ entirely, the hyperplane μ is one of the hyperplanes through π. The other a hyperplanes are random, but do not contain π. □

Acknowledgement: This research was supported by the Project *Combined algorithmic and theoretical study of combinatorial structures* between the Fund for Scientific Research Flanders-Belgium (FWO-Flanders) and the Bulgarian Academy of Sciences. The third author wants to thank the people at the Bulgarian Academy of Sciences in Veliko Tarnovo and Sofia for their hospitality during her research visit in the framework of this project.

References

[1] A. Blokhuis, On the size of a blocking set in $\mathrm{PG}(2,p)$. *Combinatorica* **14** (1994), no. 1, 111–114.

[2] A. Blokhuis and A.E. Brouwer, Blocking sets in Desarguesian projective planes. *Bull. London Math. Soc.* **18** (1986), 132–134.

[3] R.H. Bruck and R.C. Bose, The construction of translation planes from projective spaces. *J. Algebra* **1** (1964), 85–102.

[4] A.A. Bruen, Blocking sets and skew subspaces of projective space. *Canad. J. Math.* **32** (1980), 628–630.

[5] M. Lavrauw, L. Storme and G. Van de Voorde, On the code generated by the incidence matrix of points and k-spaces in $\mathrm{PG}(n, q)$ and its dual. *Finite Fields Appl.*, to appear.

[6] T. Szőnyi, Blocking sets in Desarguesian affine and projective planes. *Finite Fields Appl.* **3** (1997), 187–202.

[7] T. Szőnyi and Zs. Weiner, Small blocking sets in higher dimensions. *J. Combin. Theory, Ser. A* **95** (2001), 88–101.

[8] T. Szőnyi and Zs. Weiner, On stability theorems in finite geometry (Preprint).

Address of the authors:

Stefan Dodunekov:
IMI / MFI
8, Acad. G. Bonchev
1113 Sofia (Bulgaria)
stedo@moi.math.bas.bg

Leo Storme, Geertrui Van de Voorde:
Department of pure mathematics and computer algebra
Ghent University
Krijgslaan 281-S22
9000 Ghent (Belgium)
{ls,gvdvoorde}@cage.ugent.be
http://cage.ugent.be/ ∼ {ls,gvdvoorde}