On a class of hyperplanes of the symplectic and Hermitian dual polar spaces

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Abstract

Let Δ be a symplectic dual polar space $DW(2n-1, \mathbb{K})$ or a Hermitian dual polar space $DH(2n-1, \mathbb{K}, \theta)$, $n \geq 2$. We define a class of hyperplanes of Δ arising from its Grassmann-embedding and discuss several properties of these hyperplanes. The construction of these hyperplanes allows us to prove that there exists an ovoid of the Hermitian dual polar space $DH(2n-1, \mathbb{K}, \theta)$ arising from its Grassmann-embedding if and only if there exists an empty θ-Hermitian variety in $\mathrm{PG}(n-1, \mathbb{K})$. Using this result we are able to give the first examples of ovoids in thick dual polar spaces of rank at least 3 which arise from some projective embedding. These are also the first examples of ovoids in thick dual polar spaces of rank at least 3 for which the construction does not make use of transfinite recursion.

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1 Introduction

1.1 Basic definitions

Let Π be a non-degenerate thick polar space of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. The geometry Δ is called a *dual polar space* (Cameron [2]). There exists a bijective correspondence between the non-empty convex subspaces of Δ and the possibly empty singular subspaces of Π: if α is a singular subspace of Π, then the set of all maximal singular subspaces containing α is a convex subspace of Δ. If x and y are two points of Δ, then $\mathrm{d}(x, y)$ denotes the distance between x and y in the collinearity graph of Δ. The maximal distance between two points of a convex subspace A of Δ is called the *diameter* of A. The diameter of Δ is equal to n. The convex subspaces of diameter 2, 3, respectively $n-1$, are called the *quads*, *hexes*, respectively *maxes*, of Δ. The points and lines contained in a convex subspace of diameter $\delta \geq 2$ define a dual polar space of rank δ. In particular, the points and lines contained in a quad define a generalized quadrangle (Payne and Thas [17]). If $*_1, *_2, \ldots, *_k$ are points or convex subspaces of Δ, then we denote by $\langle *_1, *_2, \ldots, *_k \rangle$ the smallest convex subspace of Δ containing $*_1, *_2, \ldots, *_k$. The convex subspaces through a point x of Δ define a projective space of dimension $n-1$ which we will denote by $Res_\Delta(x)$. For every point x of Δ, let $x^\perp$ denote the set of points equal to or collinear with x. The dual polar space Δ is a near polygon (Shult and Yanushka [19]; De Bruyn [6]) which means that for every point x and every line L, there exists a unique point on L nearest to x. More generally, for every point x and every convex subspace A, there exists a unique point $\pi_A(x)$ in A nearest to x and $\mathrm{d}(x,y) = \mathrm{d}(x, \pi_A(x)) + \mathrm{d}(\pi_A(x), y)$ for every point y of A. We call $\pi_A(x)$ the *projection* of x onto A.

A *hyperplane* of a point-line geometry $\mathcal{S}$ is a proper subspace of $\mathcal{S}$ meeting each line. An *ovoid* of a point-line geometry $\mathcal{S}$ is a set of points of $\mathcal{S}$ meeting each line in a unique point. Every ovoid is a hyperplane. If Δ is a dual polar space of rank $n \geq 2$, then for every point x of Δ, the set H_x of points of Δ at distance at most $n-1$ from x is a hyperplane of Δ, called the *singular hyperplane* of Δ *with deepest point* x. If A is a convex subspace of diameter δ of Δ and H_A is a hyperplane of A, then the set of points of Δ at distance at most $n-\delta$ from H_A is a hyperplane of Δ, called the *extension* of H_A.

Now, suppose Δ is a thick dual polar space. Then every hyperplane of Δ is a maximal subspace by Shult [18, Lemma 6.1]. If H is a hyperplane of Δ and Q is a quad of Δ, then either $Q \subseteq H$ or $Q \cap H$ is a hyperplane of Q. By Payne and Thas [17, 2.3.1], one of the following cases then occurs: (i) $Q \subseteq H$, (ii) there exists a point x in Q such that $x^\perp \cap Q = H \cap Q$, (iii) $Q \cap H$ is a subquadrangle of Q, or (iv) $Q \cap H$ is an ovoid of Q. If case (i), case (ii), case (iii), respectively case (iv), occurs, then we say that Q is *deep*, *singular*, *subquadrangular*, respectively *ovoidal*, with respect to H.

A *full embedding* of a point-line geometry $\mathcal{S}$ into a projective space Σ is an injective mapping e from the point-set P of $\mathcal{S}$ to the point-set of Σ satisfying (i) $\langle e(P) \rangle = \Sigma$ and (ii) $e(L) := \{e(x) \,|\, x \in L\}$ is a line of Σ for every line L of $\mathcal{S}$. If $e: \mathcal{S} \rightarrow \Sigma$ is a full embedding, then for every hyperplane α of Σ, $H(\alpha) := e^{-1}(e(P) \cap \alpha)$ is a hyperplane of $\mathcal{S}$; we will say that the hyperplane $H(\alpha)$ *arises from the embedding* e.

1.2 Overview

Let $n \in \mathbb{N} \setminus \{0,1\}$, let $\mathbb{K}$ be a field and let ζ be a non-degenerate symplectic or Hermitian polarity of $\mathrm{PG}(2n-1, \mathbb{K})$. If ζ is a Hermitian polarity, we assume that there exists a totally isotropic subspace of maximal dimension $n-1$. Notice that such a subspace always exists in the symplectic case. In the case ζ is a Hermitian polarity of $\mathrm{PG}(2n-1,\mathbb{K})$, let θ be the associated involutory automorphism of $\mathbb{K}$ and let $\mathbb{K}_0$ be the fix-field of θ.

Let Π be the polar space of the totally isotropic subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$ (with respect to ζ) and let Δ be its associated dual polar space. In the symplectic case, we denote Π and Δ by $W(2n-1,\mathbb{K})$ and $DW(2n-1,\mathbb{K})$, respectively. In the Hermitian case, we denote Π and Δ by $H(2n-1,\mathbb{K},\theta)$ and $DH(2n-1,\mathbb{K},\theta)$, respectively.

Let π be an arbitrary $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$ and let H_π denote the set of all maximal totally isotropic subspaces meeting π.

Suppose Δ is the symplectic dual polar space $DW(2n-1,\mathbb{K})$. We will show in Section 2.1 that H_π is a hyperplane of Δ. We call any hyperplane of $DW(2n-1,\mathbb{K})$ arising from an $(n-1)$-dimensional subspace π of $\mathrm{PG}(2n-1,\mathbb{K})$ a *hyperplane of type (S)*. ("S" refers to Symplectic.) This class of hyperplanes is already implicitly described in the literature.

Let $\mathcal{G}$ be the Grassmannian of the $(n-1)$-dimensional subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$. The points of $\mathcal{G}$ are the $(n-1)$-dimensional subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$ and the lines are all the sets $\{C \mid A \subset C \subset B\}$, where A and B are subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$ satisfying $\dim(A) = n-2$, $\dim(B) = n$ and $A \subset B$. If α is an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$, then the set of all $(n-1)$-dimensional subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$ meeting α is a hyperplane G_α of the geometry $\mathcal{G}$ (see e.g. [14]). Now, the dual polar space $DW(2n-1,\mathbb{K})$ can be regarded as a subspace of $\mathcal{G}$ and the hyperplane G_α will give rise to a hyperplane of $DW(2n-1,\mathbb{K})$. This is precisely the hyperplane H_α of $DW(2n-1,\mathbb{K})$ defined above.

In the case of the Grassmannian $\mathcal{G}$, there is essentially only one type of hyperplane which can be constructed in this way. This is not the case for the symplectic dual polar space $DW(2n-1,\mathbb{K})$. The isomorphism type depends on the size of the radical of π.

In Section 2.1 we will discuss several properties of the hyperplanes of type (S). Some of these properties turn out to be important for other applications (see e.g. [8] and [9]). In Section 2.2, we will give an alternative description of these hyperplanes in terms of certain objects of the dual polar space, and in Section 2.3, we will prove that all these hyperplanes arise from the so-called Grassmann-embedding of $DW(2n-1,\mathbb{K})$.

Now, suppose Δ is the Hermitian dual polar space $DH(2n-1,\mathbb{K},\theta)$.

(i) If π is a totally isotropic subspace, then H_π is a hyperplane of Δ, namely the singular hyperplane of Δ with deepest point π.

(ii) If π is not totally isotropic, then H_π is not a hyperplane of Δ. If $e : \Delta \to \Sigma$ denotes the Grassmann-embedding of Δ, then we show in Section 3.1 that there exists a subspace γ_π of co-dimension 2 in Σ such that $e(H_\pi) = e(P) \cap \gamma_\pi$, where P denotes the point-set of Δ. If α is a hyperplane of Σ through γ_π, then $H(\alpha)$ is a hyperplane of Δ which (regarded as point-line geometry) contains H_π as a hyperplane.

Any hyperplanes of $DH(2n-1, \mathbb{K}, \theta)$ which is obtained as in (i) or (ii) is called a *hyperplane of type (H)* of $DH(2n-1, \mathbb{K}, \theta)$. ("$H$" refers to Hermitian.) Making use of these hyperplanes of type (H) of $DH(2n-1, \mathbb{K}, \theta)$, we prove the following in Section 3.2.

Theorem 1.1 (Section 3.2) *The dual polar space $DH(2n-1, \mathbb{K}, \theta)$, $n \geq 2$, has an ovoid arising from its Grassmann-embedding if and only if there exists an empty θ-Hermitian variety in* $\mathrm{PG}(n-1, \mathbb{K})$.

Now, suppose the dual polar space $DW(2n-1, \mathbb{K}_0)$ is isometrically embedded as a subspace in $DH(2n-1, \mathbb{K}, \theta)$. (Up to isomorphism, there exists a unique such embedding, see [11, Theorem 1.5].) Then every ovoid of $DH(2n-1, \mathbb{K}, \theta)$ intersects $DW(2n-1, \mathbb{K}_0)$ in an ovoid of $DW(2n-1, \mathbb{K}_0)$. By [12, Theorem 1.1], the full embedding of $DW(2n-1, \mathbb{K}_0)$ induced by the Grassmann-embedding of $DH(2n-1, \mathbb{K}, \theta)$ is isomorphic to the Grassmann-embedding of $DW(2n-1, \mathbb{K}_0)$. By Theorem 1.1, we then have

Corollary 1.2 *If there exists an empty θ-Hermitian variety in* $\mathrm{PG}(n-1, \mathbb{K})$, *$n \geq 2$, then the dual polar space $DW(2n-1, \mathbb{K}_0)$ has ovoids arising from its Grassmann-embedding.*

The ovoids alluded to in Theorem 1.1 and Corollary 1.2 are the first examples (for $n \geq 3$) of ovoids in thick dual polar spaces of rank at least 3 which arise from some projective embedding. They are also the first examples of ovoids in thick dual polar spaces of rank at least 3 for which the construction does not make use of transfinite recursion. (Using transfinite recursion it is rather easy to construct ovoids in infinite dual polar spaces, see Cameron [3].)

In Section 4, we will discuss the finite Hermitian case. We will prove that if π is an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1, q^2)$ which is not totally isotropic, then there are precisely $q+1$ hyperplanes in $DH(2n-1, q^2)$ which contain H_π as a hyperplane and that all these hyperplanes are isomorphic. Some other properties of these hyperplanes are investigated.

2 The symplectic case

2.1 Definition and properties of the hyperplanes of type (S)

Consider in $\mathrm{PG}(2n-1, \mathbb{K})$, $n \geq 2$, a symplectic polarity ζ and let $W(2n-1, \mathbb{K})$ and $\Delta = DW(2n-1, \mathbb{K})$ denote the corresponding polar space and dual polar space. Let π be an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1, \mathbb{K})$ and let H_π be the set of all maximal totally isotropic subspaces meeting π.

Lemma 2.1 *If α is a maximal totally isotropic subspace of $W(2n-1,\mathbb{K})$, then* $\dim(\pi \cap \alpha) = \dim(\pi^\zeta \cap \alpha)$.

Proof. Put $\beta = \pi \cap \alpha$ and $k = \dim(\beta)$. The space β^ζ has dimension $2n-2-k$ and contains the $(n-1)$-dimensional subspaces π^ζ and α. Hence, $\dim(\pi^\zeta \cap \alpha) \geq k = \dim(\pi \cap \alpha)$. By symmetry, also $\dim(\pi \cap \alpha) \geq \dim(\pi^\zeta \cap \alpha)$. ∎

Corollary 2.2 $H_\pi = H_{\pi^\zeta}$. ∎

Lemma 2.3 *Through every point x of* $\mathrm{PG}(2n-1,\mathbb{K})$ *not contained in $\pi \cup \pi^\zeta$, there exists a maximal totally isotropic subspace disjoint from π (and hence also from π^ζ).*

Proof. We will prove the lemma by induction on n.

Suppose first that $n = 2$. Let L be a line through x contained in the plane x^ζ and not containing the point $x^\zeta \cap \pi$. Then L satisfies the required conditions.

Suppose next that $n \geq 3$. The totally isotropic subspaces through x determine a polar space of type $W(2n-3,\mathbb{K})$ which lives in the quotient space x^ζ/x. Since $\dim(x^\zeta \cap \pi) = n-2$ (recall that $x \notin \pi^\zeta$), the subspace $\pi' = \langle x, x^\zeta \cap \pi \rangle$ of x^ζ/x has dimension $n-2$ (in x^ζ/x). By the induction hypothesis, there exists a maximal totally isotropic subspace in $W(2n-3,\mathbb{K})$ disjoint from π'. Hence, in $W(2n-1,\mathbb{K})$ there exists a maximal totally isotropic subspace through x disjoint from π. ∎

Proposition 2.4 *The set H_π is a hyperplane of $DW(2n-1,\mathbb{K})$.*

Proof. First, we show that H_π is a subspace. Let α_1 and α_2 be two maximal totally isotropic subspaces meeting π such that $\dim(\alpha_1 \cap \alpha_2) = n-2$ and let α_3 denote an arbitrary maximal totally isotropic subspace through $\alpha_1 \cap \alpha_2$. If $\alpha_1 \cap \alpha_2 \cap \pi \neq \phi$, then obviously α_3 meets π. Suppose now that $\alpha_1 \cap \alpha_2 \cap \pi = \emptyset$, $\alpha_1 \cap \pi = \{x_1\}$ and $\alpha_2 \cap \pi = \{x_2\}$. Then $(\alpha_1 \cap \alpha_2)^\zeta = \langle \alpha_1, \alpha_2 \rangle$. So, $\alpha_3 \subseteq \langle \alpha_1, \alpha_2 \rangle$ meets the line x_1x_2 and hence also π. In each of the two cases, $\alpha_3 \in H_\pi$. This proves that H_π is a subspace. By Lemma 2.3, H_π is a proper subspace.

We will now prove that H_π is a hyperplane. Let β denote an arbitrary totally isotropic subspace of dimension $n-2$ and let L_β denote the set of all maximal totally isotropic subspaces containing β. Obviously, $L_\beta \subseteq H$ if $\beta \cap \pi \neq \emptyset$. If $\beta \cap \pi = \emptyset$, then β^ζ is an n-dimensional subspace which has (at least) a point x in common with π. Obviously, $\langle \beta, x \rangle$ is a point of L_β contained in H. ∎

Definition. We say that a hyperplane H of $\Delta = DW(2n-1,\mathbb{K})$ is of *type (S)* if it is of the form H_π for a certain $(n-1)$-dimensional subspace π of $\mathrm{PG}(2n-1,\mathbb{K})$.

Consider the following three types of points in the hyperplane H_π of $\Delta = DW(2n-1,\mathbb{K})$.

- Type I: maximal totally isotropic subspaces α for which $\dim(\alpha \cap \pi) = \dim(\alpha \cap \pi^\zeta) = 0$ and $\alpha \cap \pi = \alpha \cap \pi^\zeta$.

- Type II: maximal totally isotropic subspaces α for which $\dim(\alpha \cap \pi) = \dim(\alpha \cap \pi^\zeta) = 0$ and $\alpha \cap \pi \neq \alpha \cap \pi^\zeta$.

- Type III: maximal totally isotropic subspaces α for which $\dim(\alpha \cap \pi) = \dim(\alpha \cap \pi^\zeta) \geq 1$.

For every point x of H_π, let $\Lambda(x)$ denote the set of lines through x which are contained in H_π. Then $\Lambda(x)$ can be regarded as a set of points of $Res_\Delta(x)$.

Proposition 2.5
- *If α is a point of Type I, then $\Lambda(x)$ is a hyperplane of $Res_\Delta(x)$.*

- *If α is a point of Type II, then $\Lambda(x)$ is the union of two distinct hyperplanes of $Res_\Delta(x)$.*

- *If α is a point of Type III, then $\Lambda(x)$ coincides with the whole point set of $Res_\Delta(x)$.*

Proof. Let α be a point of Type I and let x denote the unique point contained in $\alpha \cap \pi = \alpha \cap \pi^\zeta$. Let β be an $(n-2)$-dimensional subspace of α. If β contains the point x, then the line of $DW(2n-1, \mathbb{K})$ corresponding with β obviously is contained in H_π. If β does not contain the point x, then $\beta^\zeta \cap \pi = \{x\}$, and it follows that α is the unique point of the line of $DW(2n-1, \mathbb{K})$ corresponding with β which is contained in H_π. [If $\beta^\zeta \cap \pi$ would be a line L, then L must be a totally isotropic line through x and $\langle \beta, L \rangle$ would be a totally isotropic subspace of dimension n, which is impossible.] Hence, there exists a unique max $A(\alpha)$ through α such that the lines of $DW(2n-1, \mathbb{K})$ through α which are contained in H_π are precisely the lines of $A(\alpha)$ through α.

Let α be a point of Type II and let x_1 and x_2 be the points contained in $\alpha \cap \pi$ and $\alpha \cap \pi^\zeta$, respectively. Let β be an $(n-2)$-dimensional subspace of α. If β contains at least one of the points x_1 and x_2, then by Lemma 2.1, every maximal totally isotropic subspace through β meets π, proving that the line of $DW(2n-1, \mathbb{K})$ corresponding with β is contained in H_π. Suppose now that $\beta \cap \{x_1, x_2\} = \emptyset$. If $\alpha' \neq \alpha$ is a maximal totally isotropic subspace through β meeting π in a point $x \neq x_1$, then $\beta = x^\perp \cap \alpha$ contains the point x_2, a contradiction. So, if $\beta \cap \{x_1, x_2\} = \emptyset$, then α is the unique point of the line of $DW(2n-1, \mathbb{K})$ corresponding with β which is contained in H_π. It follows that there are two distinct maxes $A_1(\alpha)$ and $A_2(\alpha)$ through α such that the lines through α contained in H_π are precisely the lines through α which are contained in $A_1(\alpha) \cup A_2(\alpha)$.

If α is a point of Type III, then every $(n-2)$-dimensional subspace of α contains a point of π. It follows that every line through α is contained in H_π. ■

Proposition 2.6 *Let M be a max of $DW(2n-1, \mathbb{K})$ and let x be the point of* $\mathrm{PG}(2n-1, \mathbb{K})$ *corresponding with M. Then M is contained in H_π if and only if $x \in \pi \cup \pi^\zeta$.*

Proof. If $x \in \pi \cup \pi^\zeta$, then every maximal totally isotropic subspace through x meets π and hence $M \subseteq H_\pi$. If $x \notin \pi \cup \pi^\zeta$, then M is not contained in H_π by Lemma 2.3. ■

The following proposition is obvious.

Proposition 2.7 *If π is a maximal totally isotropic subspace, then H_π is the singular subspace with deepest point π.* ■

Proposition 2.8 *Let $n \geq 3$ and suppose that the subspace π is singular. Let x be a point of π such that $\pi \subseteq x^\zeta$. The maximal totally isotropic subspaces through x define a convex subspace $A \cong DW(2n-3, \mathbb{K})$ of $DW(2n-1, \mathbb{K})$. Let G_π denote the hyperplane of type (S) of A consisting of all maximal totally isotropic subspaces containing a line of π through x. Then the hyperplane H_π of $DW(2n-1, \mathbb{K})$ is the extension of the hyperplane G_π of A.*

Proof. Let α denote an arbitrary point of $DW(2n-1, \mathbb{K})$. If α is a point of A, then $\alpha \in H_\pi$ since α contains the point x of π.

Suppose now that α does not contain the point x. Let α' denote the unique maximal totally isotropic subspace through x meeting α in a space β of dimension $n-2$. Then α' is the projection of α onto A.

Suppose $\alpha \in H_\pi$. If u is a point of α contained in π, then the line xu is contained in α', proving that $\alpha' \in G_\pi$. Conversely, suppose that $\alpha' \in G_\pi$. If L is a line of π through x contained in α', then L meets the hyperplane β of α'. Hence, $\alpha \cap \pi \neq \emptyset$ and $\alpha \in H_\pi$.

So, a point of $DW(2n-1, \mathbb{K})$ not contained in A belongs to H_π if and only if its projection on A belongs to G_π. This proves that H_π is the extension of G_π. ■

Proposition 2.9 *Suppose H_π is a hyperplane of type (S) of $DW(2n-1, \mathbb{K})$ and let A be a convex subspace of $DW(2n-1, \mathbb{K})$ of diameter at least 2. Then either $A \subseteq H_\pi$ or $A \cap H_\pi$ is a hyperplane of type (S) of A.*

Proof. Let α be the totally isotropic subspace corresponding with A. If α meets π, then $A \subseteq H_\pi$. So, we will suppose that α is disjoint from π. Put $\dim(\alpha) = n-1-i$ with $i \geq 2$. The totally isotropic subspaces through α define a polar space $W(2i-1, \mathbb{K})$ which lives in the quotient space α^ζ/α. The space α^ζ is $(n-1+i)$-dimensional and hence $\alpha^\zeta \cap \pi$ has dimension at least $i-1$. Let π' be the subspace generated by α and $\alpha^\zeta \cap \pi$. The dimension of the quotient space α^ζ/α is $2i-1$ and the dimension of π' in this quotient space is at least $i-1$. If this dimension is at least i, then every maximal totally isotropic subspace through α meets $\alpha^\zeta \cap \pi$ and hence $A \subseteq H_\pi$. If the dimension is precisely $i-1$, then the hyperplane $H \cap A$ of A has type (S). ■

Proposition 2.10 *Every hyperplane H_π of type (S) of $DW(3, \mathbb{K})$ is either a singular hyperplane or a grid.*

Proof. In this case π is a line of $\mathrm{PG}(3, \mathbb{K})$. If π is totally isotropic, then H_π is a singular hyperplane.

If π is not totally isotropic, then the points of H_π are precisely the lines meeting π and π^ζ. The lines of H_π are the points of $\pi \cup \pi^\zeta$, see Proposition 2.6. It is now easily seen that H_π defines a grid. ■

Propositions 2.9 and 2.10 have the following corollary:

Corollary 2.11 *A hyperplane of type (S) does not admit ovoidal quads.* ■

Proposition 2.12 *Every hyperplane H_π of type (S) of $DW(5, \mathbb{K})$ is either a singular hyperplane or the extension of a grid.*

Proof. In this case π is a plane which is always singular. So, H_π is isomorphic to the extension of a hyperplane of type (S) in $DW(3, \mathbb{K})$. This proves the lemma. [In fact, the following holds: if π is totally isotropic, then H_π is a singular hyperplane; if π contains a unique singular point, then H_π is the extension of a grid.] ■

2.2 Alternative description of the hyperplanes

In this section, we will give an alternative description of the hyperplanes of type (S). We will restrict ourselves to those hyperplanes H_π, where π is nonsingular. (This is not so restrictive in view of Proposition 2.8.) The fact that π is nonsingular implies that $\dim(\pi)$ is odd.

Consider the dual polar space $DW(4n-1, \mathbb{K})$ with $n \geq 2$. Let π be a nonsingular subspace of dimension $2n-1$. Let $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$, and let π_i, $i \in \{1, 2\}$ be a nonsingular subspace of π of dimension $2n_i - 1$ such that $\pi_2 = \pi_1^\zeta \cap \pi$. Then π_1 and π_2 are disjoint and $\langle \pi_1, \pi_2 \rangle = \pi$. Let Ω_i, $i \in \{1, 2\}$, denote the set of maxes of $DW(4n-1, \mathbb{K})$ corresponding with the points of π_i. Then every max of Ω_1 intersects every max of Ω_2 in a convex subspace of diameter $2n-2$. Let X denote the set of points which are contained in a max of Ω_1 and a max of Ω_2.

Proposition 2.13 *H_π consists of those points of $DW(4n-1, \mathbb{K})$ at distance at most 1 from X.*

Proof. Notice that every point of π is contained in a line which meets π_1 and π_2.

Now, let α denote an arbitrary point of H_π, i.e. α is a totally isotropic subspace and there exists a point $x \in \alpha \cap \pi$. Let L denote a line through x meeting π_1 and π_2. There exists a maximal totally isotropic subspace α' through L meeting α in at least an $(2n-2)$-dimensional subspace. Obviously, $\alpha' \in X$ and $\mathrm{d}(\alpha, \alpha') \leq 1$.

Now, let α denote an arbitrary point of $DW(4n-1, \mathbb{K})$ at distance at most 1 from a point α' of X. The totally isotropic subspace α' contains a point $x_1 \in \pi_1$

and a point $x_2 \in \pi_2$. Since $\dim(\alpha \cap \alpha') \geq 2n-2$, the line x_1x_2 meets $\alpha \cap \alpha'$ and hence also α. This proves that $\alpha \in H_\pi$. ■

Example. Consider the dual polar space $DW(7, \mathbb{K})$. Suppose Ω_1 and Ω_2 are two sets of mutually disjoint hexes satisfying the following properties.

- Every line L meeting two distinct hexes of Ω_i, $i \in \{1,2\}$, meets every hex of Ω_i. Moreover, the hexes of Ω_i cover all the points of L.
- Every hex of Ω_1 intersects every hex of Ω_2 in a quad.

Let X denote the set of points which are contained in a hex of Ω_1 and a hex of Ω_2 and let H be the set of points at distance at most 1 from X. Then H is a hyperplane of type (S) of $DW(7, \mathbb{K})$ arising from a nonsingular 3-dimensional subspace.

2.3 The hyperplanes of type (S) arise from embedding

Put $I = \{1, 2, \ldots, 2n\}$ with $n \geq 2$. Suppose X is an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1, \mathbb{K})$ generated by the points $(x_{i,1}, \ldots, x_{i,2n})$, $1 \leq i \leq n$, of $\mathrm{PG}(2n-1, \mathbb{K})$. For every $J = \{i_1, i_2, \ldots, i_n\} \in \binom{I}{n}$ with $i_1 < i_2 < \cdots < i_n$, we define

$$X_J := \begin{vmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_n} \\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{2,i_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,i_1} & x_{n,i_2} & \cdots & x_{n,i_n} \end{vmatrix}.$$

The elements X_J, $J \in \binom{I}{n}$, are the coordinates of a point $f(X)$ of $\mathrm{PG}(\binom{2n}{n}-1, \mathbb{K})$ and this point does not depend on the particular set of n points which we have chosen as generating set for X. The image $\{f(X) \mid \dim(X) = n-1\}$ of f is a so-called Grassmann-variety $\mathcal{G}_{2n-1,n-1,\mathbb{K}}$ of $\mathrm{PG}(\binom{2n}{n}-1, \mathbb{K})$ which we will shortly denote by $\mathcal{G}$. If α and β are subspaces of $\mathrm{PG}(2n-1, \mathbb{K})$ satisfying $\dim(\alpha) = n-2$ and $\dim(\beta) = n$, then $\{f(X) \mid \dim(X) = n-1, \alpha \subset X \subset \beta\}$ is a line of $\mathrm{PG}(\binom{2n}{n}-1, \mathbb{K})$. For more background information on the topic of Grassmann-varieties, we refer to Hirschfeld and Thas [15, Chapter 24].

Let X and Y be two $(n-1)$-dimensional subspaces of $\mathrm{PG}(2n-1, \mathbb{K})$. Suppose that X is generated by the points $(x_{i,1}, \ldots, x_{i,2n})$, $1 \leq i \leq n$, and that Y is generated by the points $(y_{i,1}, \ldots, y_{i,2n})$, $1 \leq i \leq n$. Then $X \cap Y \neq \emptyset$ if and only if

$$\begin{vmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,2n} \\ y_{1,1} & y_{1,2} & \cdots & y_{1,2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,2n} \end{vmatrix} = 0,$$

i.e., if and only if

$$\sum_{J\in\binom{I}{n}} (-1)^{\sigma(J)} X_J Y_{I\setminus J} = 0, \tag{1}$$

where $\sigma(J) = (1+\cdots+n) + \Sigma_{j\in J} j$. (Expand according to the first n rows.) The following lemma is an immediate corollary of formula (1).

Lemma 2.14 *Let π be a given $(n-1)$-dimensional subspace of* $\mathrm{PG}(2n-1,\mathbb{K})$ *and let V_π denote the set of all $(n-1)$-dimensional subspaces of* $\mathrm{PG}(2n-1,\mathbb{K})$ *meeting π. Then there exists a hyperplane A_π of* $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ *satisfying the following property: if π' is an $(n-1)$-dimensional subspace of* $\mathrm{PG}(2n-1,\mathbb{K})$*, then $\pi' \in V_\pi$ if and only if $f(\pi') \in A_\pi$.* ∎

Now, consider a symplectic polarity ζ in $\mathrm{PG}(2n-1,\mathbb{K})$ and let $W(2n-1,\mathbb{K})$ and $DW(2n-1,\mathbb{K})$ denote the associated polar and dual polar spaces. A point α of $DW(2n-1,\mathbb{K})$ is an $(n-1)$-dimensional totally isotropic subspace. So, $f(\alpha)$ is a point of $\mathcal{G} \subseteq \mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$. A line β of $DW(2n-1,\mathbb{K})$ is a totally isotropic subspace of dimension $n-2$ and the points of β (in $DW(2n-1,\mathbb{K})$) are all the $(n-1)$-dimensional subspaces through β contained in β^ζ. It follows that f defines a full embedding e_{gr} of $DW(2n-1,\mathbb{K})$ in a certain subspace $\mathrm{PG}(N-1,\mathbb{K})$ of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$. The value of N is equal to $\binom{2n}{n}-\binom{2n}{n-2}$, see e.g. Burau [1, 82.7] or De Bruyn [7]. We call e_{gr} the *Grassmann-embedding* of $DW(2n-1,\mathbb{K})$.

Proposition 2.15 *Let π be an $(n-1)$-dimensional subspace of* $\mathrm{PG}(2n-1,\mathbb{K})$ *and let H_π denote the associated hyperplane of $DW(2n-1,\mathbb{K})$. Then H_π arises from the Grassmann-embedding of $DW(2n-1,\mathbb{K})$.*

Proof. Let A_π denote a hyperplane of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ satisfying the following: an $(n-1)$-dimensional subspace π' of $\mathrm{PG}(2n-1,\mathbb{K})$ meets π if and only if $f(\pi') \in A_\pi$. Suppose that A_π contains $\mathrm{PG}(N-1,\mathbb{K})$. Then every maximal totally isotropic subspace would meet π, which is impossible, see Lemma 2.3. Hence A_π intersects $\mathrm{PG}(N-1,\mathbb{K})$ in a hyperplane B_π of $\mathrm{PG}(N-1,\mathbb{K})$. Obviously, the hyperplane H_π of $DW(2n-1,\mathbb{K})$ arises from the hyperplane B_π of $\mathrm{PG}(N-1,\mathbb{K})$. ∎

3 The Hermitian case

3.1 A hyperplane of a hyperplane

Let $n \geq 2$, let $\mathbb{K}_0$ be a field, let $\mathbb{K}$ be a quadratic Galois-extension of $\mathbb{K}_0$ and let θ be the unique non-trivial element in $Gal(\mathbb{K}/\mathbb{K}_0)$. Consider in $\mathrm{PG}(2n-1,\mathbb{K})$ a nondegenerate θ-Hermitian variety $H(2n-1,\mathbb{K},\theta)$ of maximal Witt-index n and let ζ denote the Hermitian polarity of $\mathrm{PG}(2n-1,\mathbb{K})$ associated with $H(2n-1,\mathbb{K}',\theta)$. Let $\Delta := DH(2n-1,\mathbb{K},\theta)$ denote the dual polar space corresponding with $H(2n-1,\mathbb{K},\theta)$. Put $I = \{1,2,\ldots,2n\}$.

Suppose X is an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$ generated by the points $(x_{i,1},\ldots,x_{i,2n})$, $1 \leq i \leq n$. For every $J = \{i_1, i_2, \ldots, i_n\} \in \binom{I}{n}$ with $i_1 < i_2 < \cdots < i_n$, we define

$$X_J = \begin{vmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_n} \\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{2,i_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,i_1} & x_{n,i_2} & \cdots & x_{n,i_n} \end{vmatrix}.$$

The elements X_J, $J \in \binom{I}{n}$, are the coordinates of a point $f(x)$ of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ and this point does not depend on the particular set of n points with we have chosen as generating set for X. Now, let $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K}_0)$ denote the subgeometry of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ consisting of all the points of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ whose coordinates can be chosen in the subfield $\mathbb{K}_0 \subseteq \mathbb{K}$. The following proposition follows from Cooperstein [5] and De Bruyn [10].

Proposition 3.1 ([5], [10]) *Let $\overline{f}$ be the restriction of f to the set of points of $\Delta = DH(2n-1,\mathbb{K},\theta)$. Then there exists a projectivity ϕ of* $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ *such that $e := \phi \circ \overline{f}$ defines a full embedding of Δ into* $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K}_0)$.

Definition. The full embedding alluded to in Proposition 3.1 is called the *Grassmann-embedding* of $DH(2n-1,\mathbb{K},\theta)$.

Definition. For every hyperplane H of Δ and for every point x of H, let $\Lambda_H(x)$ denote the set of lines through x contained in H. Then $\Lambda_H(x)$ is a set of points of $Res_\Delta(x)$. For a proof of the following proposition, see Cardinali and De Bruyn [4, Corollary 1.5] or Pasini [16, Theorem 9.3].

Proposition 3.2 *Let H be a hyperplane of $DH(2n-1,\mathbb{K},\theta)$ arising from the Grassmann-embedding of $DH(2n-1,\mathbb{K},\theta)$. Then for every point x of $DH(2n-1,\mathbb{K},\theta)$, $\Lambda_H(x)$ is a possibly degenerate θ-Hermitian variety of $Res_\Delta(x) \cong$* $\mathrm{PG}(n-1,\mathbb{K})$.

Important remark. In Proposition 3.2, the complete point-set of $Res_\Delta(x)$ must be regarded as a degenerate θ-Hermitian variety.

Let π be an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$ and let H_π be the set of all maximal totally isotropic subspaces meeting π. In the following lemma, we collect some properties of the set H_π. The proofs of these properties are completely similar to the ones given in the symplectic case.

Lemma 3.3 (*i*) *If α is a maximal totally isotropic subspace of $H(2n-1,\mathbb{K},\theta)$, then* $\dim(\pi \cap \alpha) = \dim(\pi^\zeta \cap \alpha)$. *Hence, $H_\pi = H_{\pi^\zeta}$.*

(*ii*) *For every point x of $H(2n-1,\mathbb{K},\theta) \setminus (\pi \cup \pi^\zeta)$, there exists a maximal totally isotropic subspace disjoint from π (and hence also from π^ζ).*

(*iii*) *H_π is a subspace of $DH(2n-1,\mathbb{K},\theta)$.*

Proposition 3.4 *(i) If π is a totally isotropic $(n-1)$-dimensional subspace of* $\mathrm{PG}(2n-1,\mathbb{K})$*, then H_π is a hyperplane of $DH(2n-1,\mathbb{K},\theta)$, namely the singular hyperplane of $DH(2n-1,\mathbb{K},\theta)$ with deepest point π.*

(ii) If π is not totally isotropic, then H_π is not a hyperplane of $DH(2n-1,\mathbb{K},\theta)$.

Proof. Part (i) is trivial. So, suppose π is an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$ which is not totally isotropic. Then there exists a point $x \in \pi \setminus H(2n-1,\mathbb{K},\theta)$. Now, x^ζ is a $(2n-2)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$ containing the $(n-1)$-dimensional subspace π^ζ. By Lemma 3.3 (ii), there exists a maximal totally isotropic subspace α disjoint from $\pi \cup \pi^\zeta$. Then $\alpha \cap x^\zeta$ is an $(n-2)$-dimensional subspace of x^ζ disjoint from $\pi \cup \pi^\zeta$. So, $\alpha \cap x^\zeta$ corresponds with a line of Δ.

We show that $\langle \alpha \cap x^\zeta, x\rangle$ is the unique $(n-1)$-dimensional subspace through $\alpha \cap x^\zeta$ which is contained in $(\alpha \cap x^\zeta)^\zeta$ and which meets π. If this would not be the case, then $(\alpha \cap x^\zeta)^\zeta \cap \pi$ contains a line L through x. Then L^ζ contains $(\alpha \cap x^\zeta)$ and also π^ζ. Hence, $x^\zeta = \langle \pi^\zeta, \alpha \cap x^\zeta\rangle \subseteq L^\zeta$, a contradiction.

Since $x \notin H(2n-1,\mathbb{K},\theta)$, the line corresponding with $\alpha \cap x^\zeta$ is disjoint from H_π. This proves that H_π is not a hyperplane. ■

Let V_π denote the set of all $(n-1)$-dimensional subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$ meeting π. Then there exists a hyperplane A_π of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ satisfying the following: if π' is an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$, then $\pi' \in V_\pi$ if and only if $f(\pi') \in A_\pi$ (Similar proof as Lemma 2.14). So, with ϕ as in Proposition 3.1, we have that

$$e(H_\pi) = \phi(A_\pi) \cap e(P),$$

where P is the point-set of Δ. Now, put $\phi(A_\pi) = \beta_\pi$ and suppose β_π is described by the equation $\sum_{J \in \binom{I}{n}} a_J X_J = 0$. Then we denote by β'_π the hyperplane of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K})$ described by the equation $\sum_{J\in\binom{I}{n}} a_J^\theta X_J$. Then β_π and β'_π intersect $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K}_0)$ is a subspace γ_π of co-dimension 1 or 2, and

$$e(H_\pi) = \beta_\pi \cap \beta'_\pi \cap e(P) = \gamma_\pi \cap e(P).$$

If $\beta_\pi = \beta'_\pi$, then γ_π is a hyperplane of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K}_0)$ and H_π is a hyperplane of Δ. In that case π is totally isotropic. If $\beta_\pi \neq \beta'_\pi$, then γ_π has co-dimension 2 and H_π is not a hyperplane. In that case π is not totally isotropic. The following proposition is now obvious.

Proposition 3.5 *Every hyperplane of* $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K}_0)$ *through γ_π gives rise to a hyperplane of Δ which either is equal to H_π (if π is totally isotropic) or contains H_π as a hyperplane (if π is not totally isotropic).*

Definition. Any hyperplane which can be obtained as described in Proposition 3.5 is called a *hyperplane of type (H)*. Every singular hyperplane is a hyperplane of type (H).

Proposition 3.6 *Suppose π is not totally isotropic. Let H be a hyperplane of Δ which contains H_π as a hyperplane and let α be a point of H not contained in H_π. Then the following properties hold:*

(*i*) *The set $\Lambda_H(\alpha)$ of lines through α which are contained in H are precisely the lines through α meeting H_π.*

(*ii*) *The set $\Lambda_H(\alpha)$, regarded as set of points of $Res_\Delta(\alpha)$, is a possibly degenerate θ-Hermitian variety of $Res_\Delta(\alpha)$, which is isomorphic to the θ-Hermitian variety $H(2n-1,\mathbb{K},\theta)\cap\pi$ of π.*

Proof. Part (i) is trivial. We will regard α as a totally isotropic $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$ which is disjoint from π and hence also from π^ζ. For every subspace δ of π, let δ^μ be the subspace $\delta^\zeta\cap\alpha$. Then μ defines an isomorphism between the projective spaces π and $Res_\Delta(\alpha)$. If p is a point of $\pi\cap H(2n-1,\mathbb{K},\theta)$, then the line p^μ through α is completely contained in H since p^μ contains two points of H (namely α and $\langle p, p^\zeta\cap\alpha\rangle$). If p is a point of $\pi\setminus H(2n-1,\mathbb{K},\theta)$, then $(p^\zeta\cap\alpha)^\zeta$ contains α and $\langle p^\zeta\cap\alpha, p\rangle$ and hence coincides with $\langle\alpha,p\rangle$. Since $\langle\alpha,p\rangle$ intersects π in the point $p\notin H(2n-1,\mathbb{K},\theta)$, the line p^μ contains a unique point of H, namely α (see part (i)). The proposition follows. ■

Proposition 3.7 *Suppose π is not totally isotropic. Let H be a hyperplane of Δ arising from a hyperplane of* $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K}_0)$ *through γ_π and let α be a point of H. Then the following holds.*

(*i*) *If $\alpha\notin H_\pi$, then $\Lambda_H(\alpha)$ is a θ-Hermitian variety of $Res_\Delta(\alpha)$ which is isomorphic to the θ-Hermitian variety $H(2n-1,\mathbb{K},\theta)\cap\pi$ of π.*

(*ii*) *If $\alpha\in H_\pi$ and the generator α contains a line of π, then $\Lambda_H(\alpha)$ consists of the whole point-set of $Res_\Delta(\alpha)$.*

(*iii*) *If $\alpha\in H_\pi$ and the generator α intersects π in a point not belonging to π^ζ, then $\Lambda_H(\alpha)$ is a degenerate θ-Hermitian variety (a cone) with top an $(n-3)$-dimensional subspace of $Res_\Delta(\alpha)$ and with base a Baer subline of a line of $Res_\Delta(\alpha)$.*

Proof. Claim (i) is precisely Proposition 3.6. Claim (ii) is trivial. So, suppose that the generator α intersects π in a unique point $y_1\notin\pi^\zeta$. Then by Lemma 3.3, the generator α also intersects π^ζ in a unique point, say y_2. Let F_i, $i\in\{1,2\}$ denote the max through α corresponding with y_i.

We claim that the lines through α contained in H_π are precisely the lines through α contained in $F_1\cup F_2$. Obviously, every line through α contained in $F_1\cup F_2$ is also contained in H_π. Suppose β is an $(n-2)$-dimensional subspace of α not containing y_1, y_2 and α' is a maximal totally isotropic subspace through β meeting π in a point $y_1'\neq y_1$. Then since $y_2\in {y_1'}^\zeta$ and $\beta\subseteq {y_1'}^\zeta$, $\alpha\subseteq {y_1'}^\zeta$. This is however impossible since $\langle y_1',\alpha\rangle$ is not totally isotropic. So, every line through α which is not contained in $F_1\cup F_2$ is also not contained in H_π.

Now, let $\mathcal{W}$ denote the set of hyperplanes of Δ arising from a hyperplane of $\mathrm{PG}(\binom{2n}{n}-1,\mathbb{K}_0)$ through γ_π. Then for every $H\in\mathcal{W}$, $\Lambda_H(\alpha)$ is a possibly degenerate θ-Hermitian variety of $Res_\Delta(\alpha)$ containing every line through α which is contained in $F_1 \cup F_2$. So, either $\Lambda_H(\alpha)$ is the whole set of points of $Res_\Delta(\alpha)$ or $\Lambda_H(\alpha)$ is as described in (iii) above. Since the hyperplanes of $\mathcal{W}$ partition the set of points of $P \setminus H_\pi$, every line through α not contained in $F_1 \cup F_2$ is contained in a unique hyperplane of $\mathcal{W}$. This implies that every $\Lambda_H(\alpha)$, $H \in \mathcal{W}$, is as described in (iii) above. ∎

3.2 Ovoids arising from the Grassmann-embedding of Δ

Up to present, no ovoid is known to exist in a finite thick dual polar space of rank at least 3. The same conclusion does not hold for infinite thick dual polar spaces due to constructions using transfinite recursion, see Cameron [3]. An unanswered question up to now was whether there exist ovoids in possibly infinite thick dual polar spaces of rank at least 3 which arise from projective embeddings. In this subsection, we will construct the first examples of such ovoids. In fact we will give necessary and sufficient conditions for the existence of ovoids of the Hermitian dual polar space $DH(2n-1, \mathbb{K}, \theta)$ which arise from its Grassmann-embedding. Recall also that if $DH(2n-1, \mathbb{K}, \theta)$ has ovoids arising from its Grassmann-embedding, then also the dual polar space $DW(2n-1, \mathbb{K}_0)$ has ovoids arising from its Grassmann-embedding since $DW(2n-1, \mathbb{K}_0)$ can be embedded as a subspace in $DH(2n-1, \mathbb{K}, \theta)$ such that the projective embedding of $DW(2n-1, \mathbb{K}_0)$ induced by the Grassmann-embedding of $DH(2n-1, \mathbb{K}, \theta)$ is isomorphic to the Grassmann-embedding of $DW(2n-1, \mathbb{K}_0)$.

Theorem 3.8 *The dual polar space $DH(2n-1, \mathbb{K}, \theta)$, $n \geq 2$, has ovoids arising from its Grassmann-embedding if and only if* $\mathrm{PG}(n-1, \mathbb{K})$ *has an empty θ-Hermitian variety, in which case there even exists a partition of ovoids arising from the Grassmann-embedding.*

Proof. Suppose H is an ovoid arising from the Grassmann-embedding of $DH(2n-1, \mathbb{K}, \theta)$ and let x be a point of H. Then $\Lambda_H(x) = \emptyset$ and hence $Res_\Delta(x) \cong \mathrm{PG}(n-1, \mathbb{K})$ admits an empty θ-Hermitian variety by Proposition 3.2.

Conversely, suppose $\mathrm{PG}(n-1, \mathbb{K})$ admits an empty θ-Hermitian variety. Let $\sum_{i=0}^{n-1} a_{ij} X_i X_j^\theta$ ($a_{ij}^\theta = a_{ji}$) denote such an empty Hermitian variety (with respect to a given reference system). Consider then the following Hermitian variety in $\mathrm{PG}(2n-1, \mathbb{K})$ (again with respect to a certain reference system):

$$\sum_{i=0}^{n-1} a_{ij} X_i X_j^\theta + (X_0 X_n^\theta + X_n X_0^\theta) + (X_1 X_{n+1}^\theta + X_{n+1} X_1^\theta)$$

$$+ \cdots + (X_{n-1} X_{2n-1}^\theta + X_{2n-1} X_{n-1}^\theta) = 0.$$

This Hermitian variety is non-singular and its maximal singular subspaces have maximal possible dimension $n-1$ (e.g. $X_0 = X_1 = \cdots = X_{n-1} = 0$). Now, let π be the subspace $X_n = X_{n+1} = \cdots = X_{2n-1} = 0$ of $\mathrm{PG}(2n-1, \mathbb{K})$. Then $H_\pi = \emptyset$. So, if $e : \Delta \to \Sigma$ denotes the Grassmann-embedding of Δ, then there exists a subspace γ_π of co-dimension 2 in Σ having empty intersection with $e(P)$, where P denotes the point-set of $DH(2n-1,\mathbb{K},\theta)$. (Recall $e(P)\cap\gamma_\pi = e(H_\pi)$.) Now, let $\mathcal{W}$ denote the set of hyperplanes of $DH(2n-1,\mathbb{K},\theta)$ arising from a hyperplane of Σ through γ_π. If a hyperplane $W\in\mathcal{W}$ contains a line L, then $e(L)$ meets γ_π, a contradiction. Hence, all elements of $\mathcal{W}$ are ovoids and $\mathcal{W}$ defines a partition of $DH(2n-1,\mathbb{K},\theta)$ into ovoids. ∎

Examples. (1) Let $\mathbb{K}_0=\mathbb{R}$, $\mathbb{K}=\mathbb{C}$ and let θ be the complex conjugation $\bar{\cdot}$. The Hermitian variety $X_0\overline{X_0}+X_1\overline{X_1}+\cdots+X_{n-1}\overline{X_{n-1}}$ of $\mathrm{PG}(n-1,\mathbb{C})$ is empty. Hence, $DH(2n-1,\mathbb{C},\bar{\cdot})$ and $DW(2n-1,\mathbb{R})$ admit partitions in ovoids with each ovoid arising from the Grassmann-embedding of the dual polar space.

(2) Let $\mathbb{K}_0=\mathbb{Q}$, $\mathbb{K}=\mathbb{Q}(\sqrt{2})$ and let θ be the automorphism $q_1+\sqrt{2}q_2\mapsto q_1-\sqrt{2}q_2$ ($q_1,q_2\in\mathbb{Q}$) of $Q(\sqrt{2})$. The Hermitian variety $X_0^{\theta+1}+X_1^{\theta+1}+\cdots+X_{n-1}^{\theta+1}=0$ of $\mathrm{PG}(n-1,\mathbb{Q}(\sqrt{2}))$ is empty. Hence, $DH(2n-1,\mathbb{Q}(\sqrt{2}),\theta)$ and $DW(2n-1,\mathbb{Q})$ admit partitions in ovoids.

4 Discussion of the finite Hermitian case

Let $H(2n-1,q^2)$, $n\geq 2$, be a nonsingular Hermitian variety in $\mathrm{PG}(2n-1,q^2)$ and let $\Delta = DH(2n-1,q^2)$ be the associated dual polar space. Let P denote the point-set of Δ (i.e. the set of generators of $H(2n-1,q^2)$). Notice that every quad of Δ is isomorphic to $DH(3,q^2)\cong Q^-(5,q)$ and that every hyperplane of $Q^-(5,q)$ is either a $Q(4,q)$-subquadrangle or a singular hyperplane (see Payne and Thas [17]). Let $e:\Delta\to\Sigma=\mathrm{PG}(\binom{2n}{n}-1,q)$ be the Grassmann-embedding of Δ.

Let π be an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,q^2)$ which is not a generator of $H(2n-1,q^2)$ and let H_π be the set of generators of $H(2n-1,q^2)$ meeting π. Then by the above, we know that there exists a subspace γ_π of co-dimension 2 in Σ such that $H_\pi = e^{-1}(\gamma_\pi\cap e(P))$. Let $\widetilde{\beta_\pi}$ and $\widetilde{\beta'_\pi}$ be two hyperplanes of Σ such that $\gamma_\pi=\widetilde{\beta_\pi}\cap\widetilde{\beta'_\pi}$.

Lemma 4.1 *Let α be a hyperplane of Σ through γ_π, let $H(\alpha)$ be a hyperplane of Δ arising from α and let Q be a quad of Δ. Then one of the following holds:*

(1) *$Q\cap H(\alpha)=Q$ and $Q\cap H_\pi=Q$;*

(2) *$Q\cap H(\alpha)=Q$ and $Q\cap H_\pi$ is a $Q(4,q)$-subquadrangle of Q;*

(3) *$Q\cap H(\alpha)=Q$ and $Q\cap H_\pi=x^\perp\cap Q$ for a point x of Q;*

(4) *$Q\cap H(\alpha)$ is a $Q(4,q)$-subquadrangle of Q and $Q\cap H_\pi=Q\cap H(\alpha)$;*

(5) *$Q\cap H(\alpha)$ is a $Q(4,q)$-subquadrangle of Q and $Q\cap H_\pi$ is a $(q+1)\times(q+1)$-grid of $Q\cap H(\alpha)$;*

(6) *$Q\cap H(\alpha)$ is a $Q(4,q)$-subquadrangle of Q and $Q\cap H_\pi$ is a classical ovoid of $Q\cap H(\alpha)$;*

(7) *$Q\cap H(\alpha)$ is a $Q(4,q)$-subquadrangle of Q and $Q\cap H_\pi=x^\perp\cap(Q\cap H(\alpha))$ for a point x of $Q\cap H(\alpha)$;*

(8) *$Q\cap H(\alpha)=x^\perp\cap Q$ for a point x of Q and $Q\cap H_\pi=Q\cap H(\alpha)$;*

(9) *$Q \cap H(\alpha) = x^\perp \cap Q$ for a point x of Q and $Q \cap H_\pi$ is a line contained in $Q \cap H(\alpha)$;*

(10) *$Q \cap H(\alpha) = x^\perp \cap Q$ and $Q \cap H_\pi \subseteq Q \cap H(\alpha)$ is a classical ovoid of a $Q(4,q)$-quad of Q;*

(11) *$Q \cap H(\alpha) = x^\perp \cap Q$ and $Q \cap H_\pi \subseteq Q \cap H(\alpha)$ is the union of $q+1$ lines through x contained in a $Q(4,q)$-subquadrangle through x.*

Proof. Obviously, $H_\pi = H(\widetilde{\beta_\pi}) \cap H(\widetilde{\beta'_\pi}) = H(\widetilde{\beta_\pi}) \cap H(\alpha) = H(\widetilde{\beta'_\pi}) \cap H(\alpha)$. Hence, $H_\pi \cap Q = (H(\alpha) \cap Q) \cap (H(\widetilde{\beta_\pi}) \cap Q)$. Now, $H(\alpha) \cap Q$ $(H(\widetilde{\beta_\pi}) \cap Q)$ is either Q, a $Q(4,q)$-subquadrangle of Q or a singular hyperplane of Q. Combining all possibilities for $H(\alpha) \cap Q$ and $H(\widetilde{\beta_\pi}) \cap Q$, one readily finds the 11 possibilities mentioned in the lemma. ■

Now, consider the following graph Γ on the vertex set $P \setminus H_\pi$. Two vertices y_1 and y_2 are adjacent whenever one of the following conditions is satisfied:

(i) $\mathrm{d}(y_1, y_2) = 1$ and the line y_1y_2 meets H_π;

(ii) $\mathrm{d}(y_1, y_2) = 2$, $\langle y_1, y_2 \rangle \cap H_\pi$ is the union of $q+1$ lines through a point z which is collinear with y_1 and y_2;

(iii) $\mathrm{d}(y_1, y_2) = 2$, $\langle y_1, y_2 \rangle \cap H_\pi$ is a line L and $\pi_L(y_1) = \pi_L(y_2)$.

For every point x of $P \setminus H_\pi$, let

- A_x be the hyperplane of Δ arising from the hyperplane $\langle e(x), \gamma_\pi \rangle$ of Σ;
- C_x be the component of Γ containing x;
- $B_x = C_x \cup H_\pi$.

Lemma 4.2 *For every point x of $P \setminus H_\pi$, $A_x \subseteq B_x$.*

Proof. For every $i \in \{0, \dots, n\}$, consider the following property (P_i):

(P_i): If $y_1, y_2 \in A_x \setminus H_\pi$ such that $\mathrm{d}(y_1, y_2) = i$ and $y_1 \in C_x$, then also $y_2 \in C_x$.

We will prove property (P_i) by induction on i. The lemma then immediately follows from the fact that $x \in A_x \cap C_x$ and $H_\pi \subseteq B_x$. Property (P_0) trivially holds.

(1) Suppose $i = 1$ and let y_1 and y_2 be two points of $A_x \setminus H_\pi$ at distance 1 from each other such that $y_1 \in C_x$. Since H_π is a hyperplane of A_x, the line y_1y_2 meets H_π. Hence, y_1 and y_2 are adjacent vertices of Γ. Since $y_1 \in C_x$, also $y_2 \in C_x$.

(2) Suppose $i = 2$ and let y_1 and y_2 be two points of $A_x \setminus H_\pi$ at distance 2 from each other such that $y_1 \in C_x$. Now, we will apply Lemma 4.1 with $\alpha = \langle e(x), \gamma_\pi \rangle$ (so, $H(\alpha) = A_x$) and $Q = \langle y_1, y_2 \rangle$. Either case (2), (3), (5), (6), (7), (9), (10) or (11) of the lemma occurs.

In cases (2), (3), (5), (6), (7) and (10), $(Q \cap H(\alpha)) \setminus (Q \cap H_\pi)$ is connected. Since $y_1, y_2 \in (Q \cap H(\alpha)) \setminus (Q \cap H_\pi)$ and $y_1 \in C_x$, it follows that $y_2 \in C_x$ by successive application of Step (1).

Suppose case (9) occurs. Then $Q \cap H_\pi$ is a line L and $Q \cap H(\alpha)$ is a singular hyperplane of Q whose deepest point coincides with $\pi_L(y_1) = \pi_L(y_2)$ since $y_1, y_2 \in (Q \cap H(\alpha)) \setminus (Q \cap H_\pi)$. Hence, y_1 and y_2 are adjacent vertices of Γ. Since $y_1 \in C_x$, also $y_2 \in C_x$.

If case (11) occurs, then again $y_1, y_2 \in (Q \cap H(\alpha)) \setminus (Q \cap H_\pi)$ are adjacent vertices in Γ. Since $y_1 \in C_x$, also $y_2 \in C_x$.

(3) Suppose $i \geq 3$ and let y_1 and y_2 be two points of $A_x \setminus H_\pi$ at distance i from each other. Let Λ_i, $i \in \{1, 2\}$, denote the set of lines through y_i meeting H_π. Then Λ_i is a possibly degenerate Hermitian variety of $Res_\Delta(y_i)$. Let Λ_i', $i \in \{1, 2\}$, denote the set of lines of Λ_i which are contained in $\Delta' := \langle y_1, y_2 \rangle$. If Λ_2' is a hyperplane of $Res_{\Delta'}(y_2)$, then let F denote a max of $\langle y_1, y_2 \rangle$ through y_2 not containing all lines of Λ_2'. Otherwise, let F denote an arbitrary max of $\langle y_1, y_2 \rangle$ through y_2. Since $\mathrm{d}(y_1, y_2) \geq 3$, there exists more than 1 line in Λ_1'. Let L_1 be a line of Λ_1' which is different from the unique line through y_1 meeting F. Let z denote the unique point of L_1 at distance $\mathrm{d}(y_1, y_2) - 1$ from y_2. Then $\langle z, y_2 \rangle \neq F$ and there exists a line $L_2 \in \Lambda_2'$ not contained in $\langle z, y_2 \rangle$. Now, every point of L_1 has distance $i - 1$ from L_2. Since $|L_1|, |L_2| \geq 3$, there exist points $y_1' \in L_1 \setminus H_\pi$, $y_2' \in L_2 \setminus H_\pi$ at distance $i - 1$ from each other. By Step (1), $y_1' \in C_x$. By the induction hypothesis, $y_2' \in C_x$ and from Step (1), it follows again that $y_2 \in C_x$. This proves that property (P_i) holds. ■

Lemma 4.3 *If H is a hyperplane of Δ containing H_π as a hyperplane, then $B_x \subseteq H$ for every point x of $H \setminus H_\pi$.*

Proof. It suffices to show that $C_x \subseteq H$. Since $x \in H$, we must show the following: if y_1 and y_2 are two adjacent vertices of Γ such that $y_1 \in H$, then also $y_2 \in H$. We distinguish three cases.

(1) $\mathrm{d}(y_1, y_2) = 1$ and the line y_1y_2 meets H_π. Then $y_2 \in H$ since $y_1 \in H$ and $H_\pi \subseteq H$.

(2) $\mathrm{d}(y_1, y_2) = 2$, $\langle y_1, y_2 \rangle \cap H_\pi$ is the union of $q+1$ lines through a point z which is collinear with y_1 and y_2. Since $\langle y_1, y_2 \rangle \cap H_\pi$ is a hyperplane of $\langle y_1, y_2 \rangle \cap H$ and $y_1 \in H$, $\langle y_1, y_2 \rangle \cap H$ is the singular hyperplane of $\langle y_1, y_2 \rangle$ with deepest point z. It follows that $y_2 \in H$.

(3) $\mathrm{d}(y_1, y_2) = 2$, $\langle y_1, y_2 \rangle \cap H_\pi$ is a line L and $\pi_L(y_1) = \pi_L(y_2)$. Since $L = \langle y_1, y_2 \rangle \cap H_\pi$ is a hyperplane of $H \cap \langle y_1, y_2 \rangle$, $H \cap \langle y_1, y_2 \rangle$ is a singular hyperplane with deepest point on L. Since $y_1 \in H$, the deepest point coincides with $\pi_L(y_1)$. Hence, $y_2 \in H$ since $\pi_L(y_2) = \pi_L(y_1)$. ■

Proposition 4.4 *There are $q + 1$ hyperplanes which have H_π as a hyperplane. These are the hyperplanes A_x, $x \in P \setminus H_\pi$.*

Proof. Let H be a hyperplane which has H_π as a hyperplane and let x be a point of $H \setminus H_\pi$. By Lemmas 4.2 and 4.3, $A_x \subseteq H$ and hence $A_x = H$ since A_x is a maximal subspace. ∎

Proposition 4.5 *The $q+1$ hyperplanes containing H_π as a hyperplane are all isomorphic.*

Proof. Suppose the radical of π has dimension $k \in \{-1, 0, \ldots, n-2\}$. Without loss of generality, we may suppose that $H(2n-1, q^2)$ has equation

$$(X_0 X_n^q + X_n X_0^q) + (X_1 X_{n+1}^q + X_{n+1} X_0^q) + \cdots + (X_k X_{n+k}^q + X_{n+k} X_k^q)$$

$$+ X_{k+1}^{q+1} + X_{k+2}^{q+1} + \cdots + X_{n-1}^{q+1} + X_{n+k+1}^{q+1} + X_{n+k+2}^{q+1} + \cdots + X_{2n-1}^{q+1} = 0,$$

and π has equation

$$\pi \leftrightarrow X_0 = X_1 = \cdots = X_{n-1} = 0.$$

Let ϵ be an element of $\mathbb{F}_{q^2}$ satisfying $\epsilon^{q+1} = -1$ and let ϵ' be an element of $\mathbb{F}_{q^2} \setminus \{0\}$ satisfying $\epsilon'^q = -\epsilon'$. Let L denote the following line of Δ:

$$\begin{cases} X_0 = \epsilon' \cdot X_n, X_1 = \epsilon' \cdot X_{n+1}, \ldots, X_k = \epsilon' \cdot X_{n+k}, \\ X_{k+1} = \epsilon \cdot X_{n+k+1}, \ldots, X_{n-2} = \epsilon \cdot X_{2n-2}, \\ X_{n-1} = X_{2n-1} = 0. \end{cases}$$

The $q+1$ points on this line are given by the equations

$$\begin{cases} X_0 = \epsilon' \cdot X_n, X_1 = \epsilon' \cdot X_{n+1}, \ldots, X_k = \epsilon' \cdot X_{n+k}, \\ X_{k+1} = \epsilon \cdot X_{n+k+1}, \ldots, X_{n-2} = \epsilon \cdot X_{2n-2}, \\ X_{n-1} = \bar{\epsilon} \cdot X_{2n-1}, \end{cases}$$

where $\bar{\epsilon}$ is one of the $q+1$ elements of $\mathbb{F}_{q^2}$ satisfying $\bar{\epsilon}^{q+1} = -1$. Obviously, none of the above points belongs to H_π. So, L is disjoint from H_π. It follows that the hyperplanes A_x, $x \in L$, are all the hyperplanes containing H_π as a hyperplane.

For every $\delta \in \mathbb{F}_{q^2}$ satisfying $\delta^{q+1} = 1$, the automorphism $(X_0, X_1, \ldots, X_{2n-1}) \mapsto (X_0, X_1, \ldots, X_{2n-2}, \delta \cdot X_{2n-1})$ of $\mathrm{PG}(2n-1, q^2)$ fixes $H(2n-1, q^2)$ set-wise and hence determines an automorphism θ_δ of $DH(2n-1, q^2)$ fixing H_π and L set-wise. Obviously, the group $G = \{\theta_\delta \,|\, \delta^{q+1} = 1\}$ acts regularly on the line L and hence also on the set of $q+1$ hyperplanes containing H_π as a hyperplane. ∎

Proposition 4.6 *Let $n \geq 3$ and suppose that the hyperplane π is singular. Let x be a point of π such that $\pi \subseteq x^\zeta$. The maximal totally isotropic subspaces through x define a convex subspace $A \cong DH(2n-3, q^2)$ of $DH(2n-1, q^2)$. Let G_π denote the set of all maximal totally isotropic subspaces containing a line of π through x. Let $H_1, H_2, \ldots, H_{q+1}$ denote the $q+1$ hyperplanes of $DH(2n-1, q^2)$ containing H_π as a hyperplane and let $G_1, G_2, \ldots, G_{q+1}$ denote the $q+1$ hyperplanes of A containing G_π as a hyperplane. Let $\overline{G_i}$, $i \in \{1, \ldots, q+1\}$, denote the hyperplane of $DH(2n-1, q^2)$ obtained by extending G_i. Then $\{H_1, H_2, \ldots, H_{q+1}\} = \{\overline{G_1}, \overline{G_2}, \ldots, \overline{G_{q+1}}\}$.*

Proof. It suffices to prove that each $\overline{G_i}$, $i \in \{1, \ldots, q+1\}$, contains H_π as a hyperplane. Recall that H_π is a subspace of $DH(2n-1, q^2)$.

(1) We show that $H_\pi \subseteq \overline{G_i}$. Let α denote an arbitrary maximal totally isotropic subspace meeting π. If α contains x, then $\alpha \in A \subseteq \overline{G_i}$. If α does not contain x, then the unique maximal totally isotropic subspace through x meeting α in an $(n-2)$-dimensional subspace contains the subspace $\langle x, \alpha \cap \pi \rangle$ and hence is contained in G_i. It follows that $\alpha \in \overline{G_i}$.

(2) We show that every line L contained in $\overline{G_i}$ contains a point of H_π. We distinguish three cases:

(2a) L is contained in A. Then every point of L belongs to H_π.

(2b) L meets A in a unique point. This point belongs to H_π.

(2c) L is disjoint from A. Then $\pi_A(L)$ is a line of A contained in G_i. Since G_π is a hyperplane of G_i, $\pi_A(L)$ contains a point u of G_π. The unique point of L collinear with u meets π and hence is contained in H_π. ∎

Proposition 4.7 *Let H be a hyperplane of $DH(2n-1, q^2)$ having H_π as a hyperplane. Let A be a convex subspace of $DH(2n-1, q^2)$ of diameter at least 2. Then either $A \subseteq H$ or $A \cap H$ is a hyperplane of type (H) of A.*

Proof. We suppose that A is not completely contained in H. Then $A \cap H$ is a hyperplane of A. Let α be a totally isotropic subspace corresponding with A. Since A is not contained in H, α is disjoint from π. Put $\dim(\alpha) = n-1-i$ with $i \geq 2$. The totally isotropic subspaces through α define a polar space $H(2i-1, q^2)$ which lives in the quotient space α^ζ/α. The space α^ζ is $(n-1+i)$-dimensional and hence $\alpha^\zeta \cap \pi$ has dimension at least $i-1$. Let π' be the subspace generated by α and $\alpha^\zeta \cap \pi$. The dimension of the quotient space α^ζ/α is $2i-1$ and the dimension of π' is this quotient space is at least $i-1$. Since A is not contained in H, this dimension is precisely $i-1$. Let X denote the set of maximal totally isotropic subspaces through α meeting $\alpha^\zeta \cap \pi$. (So, $X = A \cap H_\pi$.) Since every line of H meets H_π in either the whole line or a unique point, every line of the hyperplane $H \cap A$ of A meets $X = A \cap H_\pi$ is either the whole line or a unique point. It follows that the hyperplane $A \cap H$ of A is a hyperplane of type (H). [Notice that it might be possible that $H \cap A = X$. Then $H \cap A$ is a singular hyperplane of A (cf. Proposition 3.4).] ∎

Remark. In the case $n = 3$ and π is a 2-dimensional subspace of $\mathrm{PG}(5, q^2)$ intersecting $H(5, q^2)$ in a unital, the $q+1$ hyperplanes containing H_π as a hyperplane have already been described in De Bruyn and Pralle [13].

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