On codewords in the dual code of classical generalized quadrangles and classical polar spaces

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Abstract

In [8], the codewords of small weight in the dual code of the code of points and lines of $\mathcal{Q}(4,q)$ are characterized. Using geometrical arguments, we characterize the codewords of small weight in the dual code of the code of points and generators of $\mathcal{Q}^+(5,q)$ and $\mathcal{H}(5,q^2)$. For the dual codes of the codes of $\mathcal{Q}^+(5,q)$, q even, and $\mathcal{Q}(4,q)$, q even, we investigate the codewords with the largest weights. We show that there exists an interval such that for every even number k in this interval, there is a codeword in the dual code of $\mathcal{Q}^+(5,q)$, q even, with weight k.

For $\mathcal{Q}(4,q)$, q even, we show that there is an empty interval in the weight distribution of the dual of the code of $\mathcal{Q}(4,q)$. To prove this, we show that a blocking set of $\mathcal{Q}(4,q)$, q even, of size q^2+1+r, where $0 < r < (q+4)/6$, contains an ovoid of $\mathcal{Q}(4,q)$, improving on [4, Theorem 9].

Finally, we present lower bounds on the weight of the codewords in the dual of the code of points and k-spaces of $\mathcal{Q}^+(2n+1,q)$, $\mathcal{Q}(2n,q)$, $\mathcal{Q}^-(2n+1,q)$, and $\mathcal{H}(n,q^2)$. For q even and k sufficiently small, we determine the maximum weight of these codes and characterize the codewords of maximum weight. We also link our results to sets of even type in these polar spaces.

Keywords: linear code, blocking set, ovoid, polar space, generalized quadrangle

1 Definitions

Let $\mathrm{PG}(d,q)$ denote the projective space of dimension d over the finite field $\mathbb{F}_q$, $q=p^h$, $h \geq 1$, p prime, and let θ_d denote the number of points in $\mathrm{PG}(d,q)$, i.e.

$$\theta_d = \frac{q^{d+1}-1}{q-1}.$$

Denote by $\mathcal{Q}^+(2n+1,q)$, $\mathcal{Q}(2n,q)$ and $\mathcal{Q}^-(2n+1,q)$, the non-singular hyperbolic quadric of $\mathrm{PG}(2n+1,q)$, resp. the non-singular parabolic quadric of $\mathrm{PG}(2n,q)$, resp. the non-singular elliptic quadric of $\mathrm{PG}(2n+1,q)$. The subspaces of maximal dimension on a quadric are called the *generators* of the quadric. The dimension g of a generator is $n-1$ for $\mathcal{Q}^-(2n+1,q)$ and $\mathcal{Q}(2n,q)$,

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and n for $\mathcal{Q}^+(2n+1,q)$. Denote by $\mathcal{H}(d,q^2)$ the Hermitian variety defined by a non-degenerate Hermitian form in the projective space $\mathrm{PG}(d,q^2)$. The generators of $\mathcal{H}(d,q^2)$ have dimension $\lfloor (d-1)/2 \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

For $\mathcal{Q}^-(2n+1,q)$, $\mathcal{Q}(2n,q)$, q odd, $\mathcal{Q}^+(2n+1,q)$ or $\mathcal{H}(d,q^2)$, let $\perp$ denote the related polarity, and let $\pi^\perp$ denote the image of a subspace π under $\perp$. A subspace entirely contained in $\mathcal{Q}^-(2n+1,q)$, $\mathcal{Q}(2n,q)$, q odd, $\mathcal{Q}^+(2n+1,q)$ or $\mathcal{H}(d,q^2)$ is called *self-polar* (with respect to $\perp$).

In the next section, we introduce the code arising from a polar space and its dual code. We use results on the code of points and k-spaces of $\mathrm{PG}(n,q)$ to obtain results on the codewords of small weight in the code of a polar space. In the following sections, we investigate the codewords in the dual code of a polar space. In [8], the codewords of small weight in the dual code of the code of points and lines of classical generalized quadrangles are characterized. In [15], the authors characterized codewords of small weight in the dual code of non-classical generalized quadrangles arising from the linear representation of geometries. In both papers, the authors use the term *LDPC*-code of a generalized quadrangle to describe the dual code arising from this generalized quadrangle.

In this paper, we continue this research by studying the codes and the dual codes (i.e. LDPC-codes) arising from classical polar spaces.

2 Codewords in the code of classical polar spaces

We define the incidence matrix $A = (a_{ij})$ of points and k-spaces of the polar space $\mathcal{P}$ in a projective space defined over a finite field of characteristic p as the matrix whose rows are indexed by the k-spaces contained in $\mathcal{P}$ and whose columns are indexed by the points of $\mathcal{P}$, and with entry

$$a_{ij} = \begin{cases} 1 & \text{if point } j \text{ belongs to } k\text{-space } i, \\ 0 & \text{otherwise.} \end{cases}$$

The p-ary linear code $\mathrm{C}_k(\mathcal{P})$, $k \leq g$, with g the dimension of the generators of $\mathcal{P}$, of points and k-spaces of $\mathcal{P}$ is the $\mathbb{F}_p$-span of the rows of the incidence matrix A. If we consider the code $\mathrm{C}_g(\mathcal{P})$, we simply denote this by $\mathrm{C}(\mathcal{P})$. The *support* of a codeword c, denoted by *supp*(c), is the set of all non-zero positions of c. The *weight* of c is the number of non-zero positions of c and is denoted by $wt(c)$. Often we identify the support of a codeword with the corresponding set of points of $\mathcal{P}$. We let c_P denote the symbol of the codeword c in the coordinate position corresponding to the point P, and let (c_1, c_2) denote the scalar product in $\mathbb{F}_p$ of two codewords c_1, c_2 of $\mathrm{C}_k(\mathcal{P})$.

The dual code $\mathrm{C}_k(\mathcal{P})^\perp$ is the set of all vectors orthogonal to all codewords of $\mathrm{C}_k(\mathcal{P})$, hence

$$\mathrm{C}_k(\mathcal{P})^\perp = \{v \in V(|\mathcal{P}|,p) || (v,c) = 0,\ \forall c \in \mathrm{C}_k(\mathcal{P})\}.$$

This means that for all $c \in \mathrm{C}_k(\mathcal{P})^\perp$ and all k-spaces K contained in $\mathcal{P}$, we have $(c,K) = 0$.

Similarly, one can define the incidence matrix $B = (b_{ij})$ of points and k-spaces of the projective space $\mathrm{PG}(n,q)$ as the matrix whose rows are indexed by the k-spaces of $\mathrm{PG}(n,q)$ and whose columns are indexed by the points of $\mathrm{PG}(n,q)$, and with entry

$$a_{ij} = \begin{cases} 1 & \text{if point } j \text{ belongs to } k\text{-space } i, \\ 0 & \text{otherwise.} \end{cases}$$

The p-ary linear code $\mathrm{C}_k(n,q)$ of points and k-spaces of $\mathrm{PG}(n,q)$, $q = p^h$, p prime, $h \geq 1$, is the $\mathbb{F}_p$-span of the rows of the incidence matrix B.

Using these definitions, it is clear that $\mathrm{C}_k(\mathcal{P})$, with $\mathcal{P}$ a polar space embedded in $\mathrm{PG}(\nu,q)$, is a subcode of $\mathrm{C}_k(\nu,q)$.

Hence, we can use the following result on the codewords of small weight in the code of points and k-spaces of $\mathrm{PG}(n,q)$ to obtain a result on $\mathrm{C}_k(\mathcal{P})$.

Result 1. *[9] The minimum weight of $\mathrm{C}_k(n,q)$ is equal to θ_k and a codeword of minimum weight corresponds to a scalar multiple of an incidence vector of a k-space. There are no codewords in $\mathrm{C}_k(n,q)$ with weight in $]\theta_k, (12\theta_k+2)/7[$ if $p = 7$ and there are no codewords in $\mathrm{C}_k(n,q)$ with weight in $]\theta_k, (12\theta_k+6)/7[$ if $p > 7$. If q is prime, there are no codewords of $\mathrm{C}_k(n,q)$ with weight in $]\theta_k, 2q^k[$.*

Corollary 2. *The minimum weight of $\mathrm{C}_k(\mathcal{P})$, with $\mathcal{P}$ a polar space embedded in $\mathrm{PG}(\nu,q)$, is equal to θ_k and a codeword of minimum weight corresponds to a scalar multiple of the incidence vector of a k-space contained in $\mathcal{P}$. There are no codewords in $\mathrm{C}_k(\mathcal{P})$ with weight in $]\theta_k, (12\theta_k+2)/7[$ if $p = 7$ and there are no codewords with weight in $]\theta_k, (12\theta_k+6)/7[$ if $p > 7$. If q is prime, there are no codewords in $]\theta_k, 2q^k[$.*

For the dual code $\mathrm{C}_k(\mathcal{P})^\perp$, however, we cannot use the results for $\mathrm{C}_k(n,q)^\perp$, since $\mathrm{C}_k(\mathcal{P})^\perp$ is not a subcode of $\mathrm{C}_k(n,q)^\perp$. In the following sections, we investigate the dual code $C(\mathcal{P})^\perp$ for $\mathcal{P} = \mathcal{Q}(4,q), \mathcal{H}(5,q^2)$, $\mathcal{Q}^+(5,q)$, and we present some results on the code of the polar spaces $\mathcal{Q}^+(2n+1,q)$, $\mathcal{Q}(2n,q)$, $\mathcal{Q}^-(2n+1,q)$, and $\mathcal{H}(2n+1,q^2)$, q even.

3 The dual code of $\mathcal{Q}(4,q)$, q even

A *generalized quadrangle* Γ is a set of points and lines such that:

(a) any two distinct points are on at most one line,

(b) every line is incident with $s+1$ points and every point is incident with $t+1$ lines,

(c) if a point p is not incident with the line L, then there is exactly one line through p intersecting L.

The generalized quadrangle Γ is said to have *order* (s,t) or s if $s = t$; the number of points of Γ is $(s+1)(st+1)$ and the number of lines is $(t+1)(st+1)$. Dualizing Γ we get a generalized quadrangle of order (t,s). For more information on generalized quadrangles, we refer to [14].

Let $\mathcal{Q}(4,q)$ be a non-singular parabolic quadric in the projective space $\mathrm{PG}(4,q)$; the set of points and the set of lines of $\mathcal{Q}(4,q)$ form a generalized quadrangle of order q and hence $\mathcal{Q}(4,q)$ has $(q+1)(q^2+1)$ points and $(q+1)(q^2+1)$ lines. The points of $\mathrm{PG}(3,q)$ and the self-polar lines of a symplectic polarity $\perp$ form the generalized quadrangle $\mathcal{W}(q)$ of order q. The following theorem describes the connection between $\mathcal{Q}(4,q)$ and $\mathcal{W}(q)$.

Theorem 3. *([14, Theorem 3.2.1])*

1. *The generalized quadrangle* $\mathcal{Q}(4,q)$ *is isomorphic to the dual of* $\mathcal{W}(q)$.
2. *The generalized quadrangles* $\mathcal{Q}(4,q)$ *and* $\mathcal{W}(q)$ *are self-dual if and only if* q *is even.*

A *blocking set* of a generalized quadrangle Γ is a set B of points such that every line of Γ contains at least one point of B. A blocking set B is called *minimal* if no proper subset of B is still a blocking set. A set $\mathcal{O}$ of points of Γ is called an *ovoid* if every line of Γ contains exactly one point of $\mathcal{O}$, hence $\mathcal{O}$ is a set of $st+1$ pairwise non-collinear points. It is well-known that $\mathcal{Q}(4,q)$, for q even, has an ovoid (see [7]).

The binary dual code $\mathrm{C}_1(\mathcal{Q}(4,q))^{\perp} = \mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, $q = 2^h$, defined by $\mathcal{Q}(4,q)$ is a code of length $(q+1)(q^2+1)$ which has the incidence matrix H of points and lines of $\mathcal{Q}(4,q)$ as parity check matrix. This implies that the vector c belongs to $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$ if and only if $cH^T = 0$.

In [8], the authors prove that the minimum weight of the p-ary code $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, $q = p^h$, p prime, is $2(q+1)$ and characterize the small weight codewords using the following geometrical property.

Let c *be a codeword of* $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$*, then* $supp(c)$ *defines a set* $\mathcal{S}$ *of points of* $\mathcal{Q}(4,q)$ *such that every line of* $\mathcal{Q}(4,q)$ *contains 0 or at least 2 points of* $\mathcal{S}$.

This condition is a necessary condition for a codeword of the p-ary code $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, $q = p^h$. For the binary code $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, $q = 2^h$, we have the following stronger condition.

$(*)$ *c is a codeword of* $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, $q = 2^h$*, if and only if every line of* $\mathcal{Q}(4,q)$ *contains an even number of points of* $\mathcal{S}$.

In this section, we investigate the large weight codewords of the dual code $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$ of $\mathcal{Q}(4,q)$, q even.

Let c be a codeword of $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, q even, then since a line has an odd number $q+1$ of points, $\mathcal{S} = supp(c)$ cannot contain a line because of $(*)$. Let $\mathcal{B}$ be the complement of $\mathcal{S}$ in $\mathcal{Q}(4,q)$, then we have the following necessary condition.

$(**)$ $\mathcal{B}$ *is a blocking set of* $\mathcal{Q}(4,q)$.

We use both conditions $(*)$ and $(**)$ to find large weight codewords of $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, q even.

Proposition 4. *If c is a codeword of* $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, $q = 2^h$*, then* $wt(c) \leq q^3+q$*, and if* $wt(c) = q^3+q$*, then* $supp(c)$ *is the complement of an ovoid.*

Proof. Let c be a codeword of $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$. By condition $(**)$, the complement of $supp(c)$ defines a blocking set B. Hence, a codeword of large weight corresponds to a blocking set of small size. The smallest size for a blocking set of $\mathcal{Q}(4,q)$ is that of an ovoid, i.e. q^2+1. Moreover, by condition $(*)$, the complement of an ovoid defines a codeword, and it has weight $(q+1)(q^2+1)-(q^2+1) = q^3+q$. □

Remark 5. *An ovoid in the projective space* $\mathrm{PG}(3,q)$*,* $q>2$*, is a set of* q^2+1 *points, no three of which are collinear. When* q *is odd, the only ovoids are the elliptic quadrics, but the classification of ovoids for even* q *is an open problem. There are only two known classes of ovoids: the elliptic quadrics (which exist for all prime powers* q*) and the Tits ovoids which exist only for* $q=2^h$*, where* $h \geq 3$ *is odd. Only for* $q \leq 32$*, ovoids have been characterized as one of these types (see [10]). By results of Thas [18] and Tits [19], every ovoid of* $\mathrm{PG}(3,q)$ *corresponds to an ovoid of* $\mathcal{W}(3,q)$ *and vica versa. Since for* q *even,* $\mathcal{W}(3,q)$ *is isomorphic to* $\mathcal{Q}(4,q)$*, the classification of ovoids in* $\mathrm{PG}(3,q)$*,* q *even, is the same as for* $\mathcal{Q}(4,q)$*,* q *even.*

In Proposition 4, we determined that the maximum weight of a codeword of $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$ is equal to q^3+q. The next goal is to find information on the spectrum of the large weight codewords of $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, q even. We will show that there are no codewords in $\mathrm{C}(\mathcal{Q}(4,q))^{\perp}$, q even, with weight in $]q^3+(5q-4)/6, q^3+q-1]$.

Since $\mathcal{Q}(4,q)$ is self-dual for even q (see Theorem 3), we can use the dual structure as well. The dual of a blocking set B of a generalized quadrangle Γ is a cover $\mathcal{C}$. A *cover* $\mathcal{C}$ is a set of lines such that every point of Γ lies on at least one line of $\mathcal{C}$. If every point of Γ lies on exactly one line of $\mathcal{C}$, then $\mathcal{C}$ is a *spread* of Γ, i.e. $\mathcal{C}$ is a partition of the point set of Γ into q^2+1 pairwise non-concurrent lines. A cover $\mathcal{C}$ is *minimal* if there is no cover properly contained in $\mathcal{C}$. The multiplicity $\mu(P)$ of a point P is the number of lines of $\mathcal{C}$ through it. The *excess* of a point P, denoted by $e(P)$, is equal to $\mu(P)-1$. A *multiple* point P of $\mathcal{C}$ is a point with $e(P)>0$. The excess of a line L is the sum of the excesses of the points on L.

A line L of Γ is called a *good line* for $\mathcal{C}$ when $L \notin \mathcal{C}$ and L does not have multiple points of $\mathcal{C}$. We have the following result for a cover of $\mathcal{Q}(4,q)$.

Lemma 6. *([4, Lemma 2]) A cover* $\mathcal{C}$ *of* $\mathcal{Q}(4,q)$ *of size* q^2+1+r*,* $0 \leq r \leq q$*, always has a good line.*

In the proof of the next lemmas, we use weighted minihypers.

Definition 7. *A* weighted $\{f,m;N,q\}$-minihyper *is a pair* (F,w)*, where* F *is a subset of the point set of* $\mathrm{PG}(N,q)$ *and* w *is a weight function* $w: \mathrm{PG}(N,q) \to \mathbb{N}: x \mapsto w(x)$ *satisfying*

1. $w(x) > 0 \Longleftrightarrow x \in F$,
2. $\sum_{x\in F} w(x) = f$,
3. $\min\{\sum_{x\in H} w(x) || H$ *is a hyperplane of* $\mathrm{PG}(N,q)\} = m$.

Remark 8. *When the weight function* w *has values in* $\{0,1\}$*, then we say that the minihyper is* non-weighted *and we refer to it by* F*.*

We have the following result on non-weighted minihypers of $\mathcal{Q}(4,q)$.

Lemma 9. *([3, Lemma 3.2]) Let* F *be a non-weighted* $\{x(q+1),x;4,q\}$*-minihyper,* $x<\frac{q}{2}$*, contained in* $\mathcal{Q}(4,q)$*. Then* F *is the union of* x *pairwise disjoint lines.*

We prove a similar result for weighted minihypers of $\mathcal{Q}(4,q)$.

Consider x lines $L_1, \ldots, L_x$, where a given line may occur more than once. The *sum* $L_1 + \cdots + L_x$ is the weighted set F of points with weight function w, satisfying $w(P) = j$ if P belongs to j lines in $L_1, \ldots, L_x$. For example, if $L_1 = \ldots = L_x$, then all points of L_1 have weight x, and all other points have weight zero.

Lemma 10. *Let (F, w) be a weighted $\{x(q+1), x; 4, q\}$-minihyper, $x < \frac{q}{2}$, contained in $\mathcal{Q}(4, q)$. Then (F, w) is a sum of x lines.*

Proof. Let $k = \min\{w(Q) || Q \in F\}$ and let P be a point with $w(P) = k$. Consider a tangent plane π to F in P, i.e. $\pi \cap F = \{P\}$. Let $S_1, S_2, \ldots, S_{q+1}$ be the $q+1$ solids through π. At least one of them, say S_1, contains more than x points of F (counted according to their weight). By [5, Lemma 2.1], $S_1 \cap F$ is a blocking set with respect to the planes of S_1. Let B be the minimal blocking set inside $S_1 \cap F$. With the same arguments as in [3, Lemma 3.2], we get that B contains a line, say L. Hence, we have a line L of $\mathcal{Q}(4, q)$ completely contained in F through a point P of minimum weight k. We construct a minihyper (F', w') in $\mathrm{PG}(4, q)$ in the following way: if $Q \in L$, then $w'(Q) = w(Q) - 1$, $w'(Q) = w(Q)$ otherwise. By [5, Lemma 2.2], (F', w') is a $\{(x-1)(q+1), x-1; 4, q\}$-minihyper. Repeating these arguments until all the points have weight zero, we get that (F, w) is a sum of x lines. $\square$

Lemma 11. *Let $\mathcal{C}$ be a cover of $\mathcal{Q}(4, q)$ of size $q^2 + 1 + r$ and let E be the set of multiple points of $\mathcal{C}$. Then (E, w), with $w(P) = e(P)$, is a $\{r(q+1), r; 4, q\}$-minihyper.*

Proof. See the proof of [4, Theorem 7, Part 1]. $\square$

Theorem 12. *Let $\mathcal{C}$ be a cover of $\mathcal{Q}(4, q)$, q even, of size $q^2 + 1 + r$, where $0 < r < \frac{q+4}{6}$. Then $\mathcal{C}$ contains a spread of $\mathcal{Q}(4, q)$.*

Proof. Lemma 6 shows that there exists a good line L for the cover $\mathcal{C}$. Let $M_1, M_2, \ldots, M_{q+1}$ be the lines of $\mathcal{C}$ intersecting L. Let $L^\perp$ be the plane defined by L and by the nucleus of $\mathcal{Q}(4, q)$, then the planes $L^\perp, \langle L, M_1 \rangle, \langle L, M_2 \rangle, \ldots, \langle L, M_{q+1} \rangle$ define a $(q+2)$-set S in the quotient geometry $\mathrm{PG}(2, q)_L$ of L, such that every line of $\mathrm{PG}(2, q)_L$ intersects S in $0, 1$ or 2 points, except for at most r lines which can contain, in total, at most $3r$ points of S (see Theorem 3 of [4]). Hence, at least $q + 2 - 3r$ elements of S are internal nuclei. Since $q + 2 - 3r > \frac{q}{2}$, every point of S is an internal nucleus (see [1]), i.e. S has only 0- and 2-secants. This implies that every hyperbolic quadric containing L contains 0 or 2 lines of $\mathcal{C}$ intersecting L.

Since $r > 0$, there exist two intersecting lines M_1 and M_2 of $\mathcal{C}$. There are q hyperbolic quadrics through $M_1 \cup M_2$. Assume that one of them contains a good line M intersecting M_2, then it has another line of $\mathcal{C}$, say M_2', intersecting M. Hence the line M_1 has at least two multiple points. By Lemmas 10 and 11, we know that the multiple points form a sum $\mathcal{L}$ of r lines.

Suppose that the cover $\mathcal{C}$ is minimal. The lines M_1 and M_2 are not contained in $\mathcal{L}$, hence the total excess of M_1 and M_2 is at most r. So at most $2(r-1)$ hyperbolic quadrics through M_1 and M_2 contain a good line. Let $\mathcal{Q}^+(3, q)$ be one of the hyperbolic quadrics of $\mathcal{Q}(4, q)$ through $M_1 \cup M_2$ not containing a good line and consider a regulus $\mathcal{R}$ of $\mathcal{Q}^+(3, q)$. A line of $\mathcal{R}$ cannot be a good line, hence, it is either a line of the cover $\mathcal{C}$ or it has at least one multiple point.

Hence, we have at least $q+1$ multiple points in $\mathcal{Q}^+(3,q)$. Since $q+1 > 2r$, at least one line of the sum $\mathcal{L}$ is contained in $\mathcal{Q}^+(3,q)$ and we know that it is not M_1 nor M_2. So we find that $q-2r+2 \leq r$, that is $r > \frac{q+2}{3}$, leading us to a contradiction. This implies that the cover $\mathcal{C}$ is not minimal.

Since for every $0 < r < \frac{q+4}{6}$, a cover $\mathcal{C}$ of size q^2+1+r is not minimal, the minimal cover contained in $\mathcal{C}$ has size q^2+1, and hence, is a spread. $\square$

Corollary 13. *Let B be a blocking set of $\mathcal{Q}(4,q)$, q even, of size q^2+1+r, where $0 < r < \frac{q+4}{6}$. Then B contains an ovoid of $\mathcal{Q}(4,q)$.*

Proof. This follows from Theorems 12 and 3(2). $\square$

Remark 14. *Theorem 8 of [4] shows that a cover of $\mathcal{Q}(4,q)$, q even, $q \geq 32$, of size q^2+1+r, $0 < r \leq \sqrt{q}$, contains a spread of $\mathcal{Q}(4,q)$. Theorem 12 improves on this theorem for all values of q, q even. In the same way, Corollary 13 improves on [4, Theorem 9].*

Theorem 15. *There are no codewords with weight in $]q^3+\frac{5q-4}{6}, q^3+q-1]$ in $\mathrm{C}(\mathcal{Q}(4,q))^\perp$, q even.*

Proof. Let c be a codeword of $\mathrm{C}(\mathcal{Q}(4,q))^\perp$ with weight in $]q^3+\frac{5q-4}{6}, q^3+q-1]$ and let B be the complement of *supp*(c) in $\mathcal{Q}(4,q)$. This implies that B is a blocking set of $\mathcal{Q}(4,q)$ of size less than $q^2+1+\frac{q+4}{6}$. Corollary 13 shows that B contains an ovoid of $\mathcal{Q}(4,q)$, say $\mathcal{O}$. Let c' be the codeword of weight q^3+q of $\mathrm{C}(\mathcal{Q}(4,q))^\perp$ defined by the complement of $\mathcal{O}$. Since $\mathrm{C}(\mathcal{Q}(4,q))^\perp$ is a linear code, $c'' = c+c'$ is a codeword of $\mathrm{C}(\mathcal{Q}(4,q))^\perp$. Moreover, it has weight at least 1 and less than $\frac{q+4}{6}$. This is a contradiction since the minimum weight of $\mathrm{C}(\mathcal{Q}(4,q))^\perp$ is $2(q+1)$ (see [8]). $\square$

Let $\mathrm{C}(\mathcal{W}(q))^\perp$ be the dual binary code arising from the symplectic polarity $\mathcal{W}(q)$, q even. We can translate the results about $\mathrm{C}(\mathcal{Q}(4,q))^\perp$ to $\mathrm{C}(\mathcal{W}(q))^\perp$ by using Theorem 3(2).

Theorem 16. *The largest weight of $\mathrm{C}(\mathcal{W}(q))^\perp$, q even, is q^3+q, which corresponds to codewords defined by the complement of an ovoid, and there are no codewords of weight in $]q^3+\frac{5q-4}{6}, q^3+q-1]$.*

We list some examples of large weight codewords.

Example 17. *Let $\mathcal{W}(q)$ be a symplectic polarity of $\mathrm{PG}(3,q)$. Now $\mathcal{W}(q)$, q even, has ovoids $\mathcal{O}$, and they all satisfy the following properties: a line L is self-polar with respect to $\mathcal{W}(q)$ if and only if L is a tangent line of $\mathcal{O}$, and a line M, intersecting $\mathcal{O}$ in two points, has $M^\perp$ skew to $\mathcal{O}$ (see [6]). Let c be the codeword defined by the complement of $\mathcal{O}$ and let c' be a codeword of minimum weight $2(q+1)$, hence c' is defined by two non-self-polar lines M and $M^\perp$ (see [8]). Then $c+c'$ is a codeword of $\mathrm{C}(\mathcal{W}(q))^\perp$ of weight q^3-q+2.*

Example 18. *Let q be even. Let $\mathrm{AG}(3,q)$ be the set of the affine points of $\mathrm{PG}(3,q)$, that is the set of points of $\mathrm{PG}(3,q) \setminus \pi$, where π is a plane: every line of $\mathrm{PG}(3,q)$ contains 0 or q points of $\mathrm{AG}(3,q)$, hence $\mathrm{AG}(3,q)$ defines a codeword of $\mathrm{C}(\mathcal{W}(q))^\perp$, q even, of weight q^3.*

Example 19. *Let q be even. Let c be a codeword of weight q^3 defined by the set $\mathrm{AG}(3,q) = \mathrm{PG}(3,q) \setminus \pi$ and let c' be a codeword of weight $2(q+1)$ defined by two lines L and $L^\perp$, such that $L \subseteq \pi$ and $L^\perp$ intersects π in one point. The codeword $c + c'$ of $\mathrm{C}(\mathcal{W}(q))^\perp$, q even, has weight $q^3 + 2$.*

Remark 20. *By Theorem 3, these examples correspond to codewords of weight q^3+2, q^3 or $q^3 - q + 2$ in $\mathrm{C}(\mathcal{Q}(4,q))^\perp$, q even.*

A set of points $\mathcal{S}$ such that every line of $\mathcal{Q}(4,q)$ contains an even number of points of $\mathcal{S}$ is often called *a set of even type*. This implies that results on the binary code $\mathrm{C}(\mathcal{Q}(4,q))^\perp$ are in fact results on sets of even type of $\mathcal{Q}(4,q)$.

Corollary 21. *The largest set of even type of $\mathcal{Q}(4,q)$, q even, corresponds to the complement of an ovoid. There are no sets of even type of $\mathcal{Q}(4,q)$, q even, with size in the interval $]q^3 + \frac{5q-4}{6}, q^3 + q - 1]$.*

4 Codewords of small weight in the dual code of $\mathcal{Q}^+(5,q)$

A codeword c in the code $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$ satisfies $(c,\pi) = 0$, for all planes $\pi \subseteq \mathcal{Q}^+(5,q)$. Hence, c defines a set S of points of $\mathcal{Q}^+(5,q)$ such that every plane of $\mathcal{Q}^+(5,q)$ contains zero or at least 2 points of S. Moreover, the sum of symbols c_P of the points P in this plane equals zero. Using the Klein correspondence, this set S of points of $\mathcal{Q}^+(5,q)$ corresponds to a set $\mathcal{S}$ of lines in $\mathrm{PG}(3,q)$ such that:

(1) Every plane of $\mathrm{PG}(3,q)$ contains 0 or at least 2 lines of $\mathcal{S}$.

(2) Every point of $\mathrm{PG}(3,q)$ lies on 0 or at least 2 lines of $\mathcal{S}$.

(3) The sum of symbols of lines of $\mathcal{S}$ going through a fixed point of $\mathrm{PG}(3,q)$ equals zero.

(4) The sum of symbols of lines of $\mathcal{S}$ lying in a fixed plane of $\mathrm{PG}(3,q)$, equals zero.

Example 22. *Let $\mathcal{S}$ be the set of $2q+2$ lines of a hyperbolic quadric $\mathcal{Q}^+(3,q)$ in $\mathrm{PG}(3,q)$, $q = p^h$, p prime, $h \geq 1$, where all $q+1$ lines of one regulus get symbol $\alpha \in \mathbb{F}_p$, and the $q+1$ lines of the opposite regulus get symbol $-\alpha$. It is easy to check that this set $\mathcal{S}$ satisfies the conditions (1)-(4). Under the Klein correspondence, the set $\mathcal{S}$ corresponds to a set S of points of $\mathcal{Q}^+(5,q)$, consisting of two conics, lying in two skew polar planes of $\mathcal{Q}^+(5,q)$.*

In Theorem 31, we will prove that the codewords of minimum weight in $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$ correspond to this example.

Example 23. *Let $\mathcal{S}$ be the set of $4q$ lines through two fixed points P and R of $\mathrm{PG}(3,q)$, $q = p^h$, p prime, $h \geq 1$, lying in two fixed planes π_1 and π_2 through PR, different from the line PR, where all lines of $\mathcal{S}$ through P in π_1 and all lines of $\mathcal{S}$ through R in π_2 get symbol α, and all other lines of $\mathcal{S}$ get symbol $-\alpha$. Under the Klein correspondence, the set $\mathcal{S}$ corresponds to a set of $4q$ points in $\mathcal{Q}^+(5,q)$, lying on four lines through a fixed point Q, where these four lines define a quadrangle on the base $\mathcal{Q}^+(3,q)$ of the cone $T_Q(\mathcal{Q}^+(5,q)) \cap \mathcal{Q}^+(5,q)$, where $T_Q(\mathcal{Q}^+(5,q)$ denotes the tangent hyperplane through Q.*

In Theorem 32, we will prove that the codewords of weight at most $4q$ are the codewords in Examples 22 and 23.

Lemma 24. *Let $\mathcal{S}$ be a set of lines in $\mathrm{PG}(3,q)$, satisfying conditions (1)-(2), with $|\mathcal{S}| \leq 4(q+1)$. A point of $\mathrm{PG}(3,q)$ lies on at most four or on at least $q-1$ lines of $\mathcal{S}$, when $q \geq 16$.*

Proof. Let R be a point of $\mathrm{PG}(3,q)$ and suppose that there are x lines of $\mathcal{S}$ through R. Every plane through one of these x lines of $\mathcal{S}$ through R has to contain a second line of $\mathcal{S}$ because of condition (1). Since we assume that there are exactly x lines of $\mathcal{S}$ through R, these extra lines do not pass through R, hence, we count any of those extra lines once. Only the planes spanned by two of the x lines of $\mathcal{S}$ through R do not necessarily need to contain an extra line. This gives in total at least

$$x(q+1) - x(x-1)$$

extra lines, which can be at most $4(q+1)-x$. If $q \geq 16$, then this implies $x \leq 4$ or $x \geq q-1$. $\square$

Lemma 25. *Let $\mathcal{S}$ be a set of lines in $\mathrm{PG}(3,q)$, $q \geq 19$, satisfying conditions (1)-(2), with $|\mathcal{S}| \leq 4(q+1)$. A point R of $\mathrm{PG}(3,q)$ lying on at least 5 lines of $\mathcal{S}$ lies on at least $5q/3 - 38$ lines of $\mathcal{S}$.*

Moreover, R lies in a plane π_1 with at least $2q/3 - 25$ lines of $\mathcal{S}$ through R in π_1, and in a plane $\pi_2 \neq \pi_1$ with at least $3q/7 - 18$ lines of $\mathcal{S}$ through R in π_2.

Proof. Let R be a point of $\mathrm{PG}(3,q)$ on at least 5 lines of $\mathcal{S}$. Lemma 24 implies that there are at least $q-1$ lines of $\mathcal{S}$ through R, say $L_1, L_2, \ldots, L_s$, $s \geq q-1$, hence there are t points R_i, $i = 1, \ldots, t$, $t \geq q(q-1)$, on L_i, $i = 1, \ldots, s$, that have to lie on a second line of $\mathcal{S}$. There are at most $4q+4-q+1 = 3q+5$ lines in $\mathcal{S} \setminus \{L_1, \ldots, L_s\}$. Hence, there is a line with at least

$$\frac{q(q-1)}{3q+5} \geq \frac{q-3}{3}$$

points of $\{R_i || i = 1, \ldots, t\}$. This implies that there is a plane π through R containing at least $(q-3)/3$ lines of $\mathcal{S}$ through R.

Suppose that π contains $x \geq (q-3)/3$ lines $L_1, \ldots, L_x$ of $\mathcal{S}$ through R. Every plane through L_i, $i = 1, \ldots, x$, has to contain a second line of $\mathcal{S}$. Suppose there are y lines through R, not in π. Then the number of extra lines of $\mathcal{S}$ needed is $xq - yx$, which has to be at most $4q+4-x-y$. This implies that

$$\frac{(q+1)(x-4)}{x-1} \leq y.$$

Since x is at least $(q-3)/3$, and $(q+1)(x-4)/(x-1)$ increases as x increases, y is at least $(q+1)(q-15)/(q-6) \geq q-13$ if $q \geq 19$.

Hence, we find at least $q-13$ lines of $\mathcal{S}$ through R, not in π.

This implies that there are at least $q - 13 + (q-3)/3$ lines of $\mathcal{S}$ through R. Repeating the previous argument, we get that there is a plane π' through R containing at least

$$\frac{q(4q/3-14)}{4q+4-4q/3+14} \geq \frac{q}{2} - 10$$

lines of $\mathcal{S}$, and, again repeating the same calculations, that R lies in total on at least $q/2 - 10 + q - 13$ lines of $\mathcal{S}$.

Repeating again, yields that R lies on a plane π'' with at least $3q/5 - 20$ lines of $\mathcal{S}$ through R, and one last time, yields that R lies on a plane π_1 with at least $2q/3 - 25$ lines of $\mathcal{S}$ through R.

The same arguments show that there are at least $q - 13$ lines of $\mathcal{S}$, not in the plane π_1, and that there is a plane π_2 with at least

$$\frac{(q-13)q}{4q+4-q+13-2q/3+25} \geq 3q/7 - 18$$

lines of $\mathcal{S}$ through R in π_2. □

Corollary 26. *Let $\mathcal{S}$ be a set of lines in* $\mathrm{PG}(3,q)$*, $q \geq 19$, satisfying conditions (1)-(2), with $|\mathcal{S}| \leq 4(q+1)$. If there is a plane with at least 5 lines of $\mathcal{S}$ lying in this plane, then this plane contains at least $5q/3 - 38$ lines of $\mathcal{S}$.*

Proof. Since the conditions (1)-(2) are self-dual, the dual of Lemma 25 holds. □

Corollary 27. *Let $\mathcal{S}$ be a set of lines in* $\mathrm{PG}(3,q)$*, $q \geq 51$, satisfying conditions (1)-(2), with $|\mathcal{S}| \leq 4(q+1)$. If there is a point R lying on at least 5 lines of $\mathcal{S}$, then there are two planes through R containing at least $5q/3 - 38$ lines of $\mathcal{S}$.*

Proof. Since $q \geq 51$, the 2 planes π_1 and π_2 through R, found in Lemma 25, each contain more than 4 lines. Using Corollary 26, this implies that these two planes π_1 and π_2 each contain at least $5q/3 - 38$ lines of $\mathcal{S}$. □

Corollary 28. *Let $\mathcal{S}$ be a set of lines in* $\mathrm{PG}(3,q)$*, $q \geq 51$, satisfying conditions (1)-(2), with $|\mathcal{S}| \leq 4(q+1)$. If there is a plane π with at least 5 lines of $\mathcal{S}$, then there are two points in π on at least $5q/3 - 38$ lines of $\mathcal{S}$.*

Proof. This is the dual of Corollary 27. □

Lemma 29. *Let $\mathcal{S}$ be a set of lines in* $\mathrm{PG}(3,q)$*, $q > 124$, satisfying conditions (1)-(2), with $|\mathcal{S}| \leq 4(q+1)$. If there is a point R lying on at least 5 lines of $\mathcal{S}$, then $\mathcal{S}$ consists of the $4q$ lines through R and a fixed point S in two planes through RS, different from the line RS.*

Proof. Since R lies on at least 5 lines of $\mathcal{S}$, Corollary 27 implies that R lies on two planes with at least $5q/3 - 38$ lines of $\mathcal{S}$.

Suppose that there are 3 planes with at least this number of lines of $\mathcal{S}$. Then there are at least $3(5q/3 - 38) - 3 > 4q + 4$ lines in $\mathcal{S}$ since $q > 124$; a contradiction. This implies that there are exactly two planes, π_1 and π_2 containing more than 5 lines of $\mathcal{S}$, and dually, that there are exactly 2 points R and R' on at least 5 lines of $\mathcal{S}$. It follows from Corollary 28 that R and R' are contained in $\pi_1 \cap \pi_2$.

Let M_i be the lines of $\mathcal{S}$ that are contained in $\pi_1 \cup \pi_2$. Then the number of lines M_i is at least $2(5q/3 - 38) - 1$. Denote the intersection points of the lines M_i with $\pi_1 \cap \pi_2$ by R, S_1, S_2, ...

Suppose that there is a point S_j of $\pi_1 \cap \pi_2$ such that all lines of $\mathcal{S}$ through it are contained in exactly one of the planes π_1 and π_2, say π_1. Let M be a line of $\mathcal{S}$ through S_j, $M \neq \pi_1 \cap \pi_2$. The q planes through M, different from π_1, all contain a second line of $\mathcal{S}$, not in $\pi_1 \cup \pi_2$. All points of those q lines have to lie on another line of $\mathcal{S}$, not in $\pi_1 \cup \pi_2$, which is not yet chosen. This implies that there are $2q-1$ lines of $\mathcal{S}$, not in $\pi_1 \cup \pi_2$; a contradiction, since $|\mathcal{S}| \leq 4q+4$, the number of lines of $\mathcal{S}$ in $\pi_1 \cup \pi_2$ is at least $10q/3 - 77$, and $q > 124$. Moreover, the same arguments prove that each point S_j lies on a line of $\mathcal{S}$ in π_2.

Suppose that there are x points R, S_i, $i = 1, \ldots, x-1$, with $x \geq 3$. One of the planes, say π_1, has at most $2q+3$ lines of $\mathcal{S}$, since otherwise $|\mathcal{S}| > 4q+4$. Hence, there are at least $xq - 2q - 3$ lines through R and S_i, $i = 1, \ldots, x-1$, in π_1 not in $\mathcal{S}$, so there is a point R or S_k lying on at least $(xq - 2q - 3)/x$ lines $N_1, \ldots, N_t$ in π_1 not of $\mathcal{S}$. As proven before, R or S_k lies on a line M' of $\mathcal{S}$ in π_2. Since all planes through M' and a line of $\{N_1, \ldots, N_t\}$ have to contain a second line, not in π_1 and π_2, there are at least $(xq - 2q - 3)/x$ lines of $\mathcal{S}$ not in π_1 and π_2. Choosing one of these lines gives $q-1$ points that have to lie on lines of $\mathcal{S}$ that are not yet counted. This implies that there are at least $q - 1 + (xq - 2q - 3)/x$ lines of $\mathcal{S}$, not in $\pi_1 \cup \pi_2$. If $q > 124$ and $x \geq 3$, this is a contradiction.

Hence, there are only 2 points R, S_1. Suppose that there is a line L' of $\mathcal{S}$, not through R or S_1.

All planes through L' have to contain a second line of $\mathcal{S}$, this implies that there are at least $q-1$ lines of $\mathcal{S}$, not in $\pi_1 \cup \pi_2$. Let L'' be one of those lines. All points of L'', except for $L' \cap L''$, $L'' \cap \pi_1$ and $L'' \cap \pi_2$ have to lie on a second line of $\mathcal{S}$, which is not yet counted. This implies that there are at least $q - 1 + q - 4 = 2q - 5$ lines of $\mathcal{S}$, not in $\pi_1 \cup \pi_2$. Since $q > 124$, this is a contradiction.

This yields that all lines of $\mathcal{S}$ go through R and S_1, and denote from now on $S_1 = R'$. Suppose that there is a line M_1 of $\mathcal{S}$ not lying in $\pi_1 \cup \pi_2$, and suppose w.l.o.g. that it contains R. Every point of $M_1 \setminus \{R\}$ lies on a second line of $\mathcal{S}$. This line can only go through S_1. But then the plane $\langle M_1, S_1 \rangle$ contains at least $q+1$ lines of $\mathcal{S}$, a contradiction.

This implies that all lines of $\mathcal{S}$ are contained in the two planes π_1 and π_2. It is easy to see that $\mathcal{S}$ consists of all lines through R and R' in $\pi_1 \cup \pi_2$, except for $\pi_1 \cap \pi_2$. The line $\pi_1 \cap \pi_2$ cannot be in $\mathcal{S}$, since in that case, any plane, different from π_1 and π_2, contains only the line $\pi_1 \cap \pi_2$ of $\mathcal{S}$. $\square$

Lemma 30. *Let $\mathcal{S}$ be a set of lines in* $\mathrm{PG}(3, q)$*, $q > 88$, satisfying conditions (1)-(2), with $|\mathcal{S}| \leq 4(q+1)$. If there are no points lying on at least 5 lines of $\mathcal{S}$, then $\mathcal{S}$ contains more than $q - 6$ lines of each regulus of a hyperbolic quadric $\mathcal{Q}^+(3, q)$.*

Proof. Let L be a line of $\mathcal{S}$, let $R_1, \ldots, R_{q+1}$ be the points of L. The point R_i lies on a second line of $\mathcal{S}$, say L_i. If there are more than four lines of $\mathcal{S}$ in one plane, Corollary 28 shows that there is a point on more than 4 lines.

Suppose that two of the lines L_i, say L_1 and L_2, have a point in common. Then there can be at most one of the other lines L_i, say L_3, that is contained in $\langle L_1, L_2 \rangle$, since otherwise, there would be more than four lines of $\mathcal{S}$ in this plane, a contradiction. From this, and the fact that three lines in a plane can meet in at most three points, we get that there are at most $3(q+1)$ points of the lines L_i's, not in L, that are on another L_j. This leaves at least $q^2 - 3(q+1)$ points P_i on the lines L_i that have to lie on a second line of $\mathcal{S}$. There are at most $4q + 4 - (q+2) = 3q + 2$ lines of $\mathcal{S}$ left. Hence, there is a line L' containing at least $\frac{q^2-3(q+1)}{3q+2} \geq (q-4)/3$ of the points P_i. Note that the line L' is skew to L, otherwise there would be a plane with at least $(q-4)/3+1$ lines of $\mathcal{S}$ in it, a contradiction.

Let $L_1, \ldots, L_s$ be the $s \geq (q-4)/3$ lines of $\mathcal{S}$ intersecting both L and L'. On $L_i, i = 1, \ldots, s$, there are at least $(q-1)s$ points Q_k that have to lie on a second line of $\mathcal{S}$; at most $2(q+1-s)$ of them lie on one of the lines $L_{s+1}, \ldots, L_{q+1}$. This gives at least $(q-1)s - 2(q+1-s) = (s-2)(q+1) \geq (q+1)((q-10)/3)$ points that have to lie on one of the $4q+4-(q+3) = 3q+1$ lines of $\mathcal{S} \setminus (\{L, L'\} \cup \{L_i | i = 1 \ldots, q+1\})$. Hence, there is a line L'' containing at least

$$\frac{(q+1)((q-10)/3)}{3q+1} \geq (q-10)/9$$

points of $\{Q_k, k = 1, \ldots, t \geq (q-1)s\}$.

So we find three skew lines L, L', L'', defining a hyperbolic quadric $\mathcal{Q} = \mathcal{Q}^+(3, q)$ with in one regulus at least $(q-10)/9$ lines of $\mathcal{S}$. Suppose there are x lines of $\mathcal{S}$ in this regulus of $\mathcal{Q}$ and t lines of $\mathcal{S}$ in the opposite regulus of $\mathcal{Q}$. Let $x \geq t$. This implies that $(q+1-x)t + (q+1-t)x$ points of $\mathcal{Q}$ have to lie on a second line of $\mathcal{S}$. This number is at most $2(4q+4-x-t)$, since a line of $\mathcal{S}$, not in $\mathcal{Q}$ can intersect $\mathcal{Q}$ in at most 2 points. From

$$(q+1-x)t + (q+1-t)x \leq 2(4q+4-x-t), \tag{1}$$

and $x, t \leq q+1$, we get that

$$x - t \leq 8,$$

and since $x \geq (q-10)/9$, that $t \geq (q-10)/9 - 8$. Set $x = t + i$, $i = 0, \ldots, 8$. From inequality (1), it follows that

$$(q+1-t-i)t + (q+1-t)(t+i) \leq 8q + 8 - 2t - 2i - 2t,$$

hence that

$$-2t^2 + 2t(q+3-i) + (i-8)(q+1) + 2i \leq 0. \tag{2}$$

Let $t = (q-10)/9 - 8$, then inequality (2) becomes an inequality, with the left hand side reaching a maximum for $i = 8$. Filling in $i = 8$ yields a contradiction for $q > 88$. Let $t = q-6$, then inequality (2) becomes an inequality, with the left hand side reaching a maximum for $i = 0$. Filling in $i = 0$ yields a contradiction for $q > 13$.

Hence, $\mathcal{Q}$ is a hyperbolic quadric with more than $q-6$ lines of $\mathcal{S}$ in each regulus. $\square$

Theorem 31. *Let c be a codeword of weight at most $2q+2$ of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$, $q > 88$, then supp(c) corresponds to a hyperbolic quadric $\mathcal{Q}^+(3,q)$ in $\mathrm{PG}(3,q)$, with all lines in one regulus symbol α, and all lines in the opposite regulus symbol $-\alpha$.*

Proof. As seen in the introduction to this section, a codeword c of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$ corresponds to a set of lines $\mathcal{S}$ in $\mathrm{PG}(3,q)$, satisfying conditions $(1)-(4)$. Looking at all planes of $\mathrm{PG}(3,q)$ through a line of $\mathcal{S}$ shows that $d(\mathrm{C}(\mathcal{Q}^+(5,q))^\perp) \geq q+2$.

It follows from Lemma 29 that $\mathcal{S}$ is a set of lines such that there is no point of $\mathrm{PG}(3,q)$ lying on more than 4 lines of $\mathcal{S}$. Lemma 30 shows that at least $2q-12$ lines of $\mathcal{S}$ are contained in a hyperbolic quadric $\mathcal{Q}$ of $\mathrm{PG}(3,q)$; at least $q-6$ lines L_i, $i=1,\ldots,s$, $s \geq q-6$, of $\mathcal{S}$ in a regulus $\mathcal{R}$ and at least $q-6$ lines M_j, $j=1,\ldots,t$, $t \geq q-6$, in the opposite regulus $\mathcal{R}'$ of $\mathcal{Q}$.

Since $2q+2-2(q-6)=14$, there are at most 14 lines of $\mathcal{S}$ not contained in $\mathcal{Q}$, which gives at most 28 points Q_i on $\mathcal{Q}$ that lie on lines of $\mathcal{S}$, not contained in $\mathcal{Q}$. Suppose that some line of $\mathcal{Q}$ does not belong to $\mathcal{S}$. There is a line, say L_1, of $\{L_i||i=1,\ldots,s\}$ containing none of the points Q_i, since $q-6>28$. Suppose that L_1 has symbol α in the corresponding codeword c of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$. Then the lines of $\{M_i||i=1,\ldots,t\}$, which all intersect L_1, all have symbol $-\alpha$. Since $q-6>28$, there is a line, say M_x, of $\{M_i||i=1,\ldots,t\}$ containing none of the points Q_i. Then the lines of $\{L_i||i=1,\ldots,s\}$, which all intersect M_x, all have symbol α.

Give all lines of the regulus $\mathcal{R}$ of $\mathcal{Q}$ containing L_i the symbol α, and give all lines of the opposite regulus $\mathcal{R}'$ the symbol $-\alpha$. As seen in Example 22, this set of lines corresponds to a codeword c' of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$.

If $c \neq c'$, then $c-c'$ is a non-zero codeword of the linear code $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$ with weight at most $wt(c)+wt(c')-2wt(c\cap c') \leq 2q+2+2q+2-2(2q-12)=28$.

Since $28<q+2$, this is a contradiction. Hence, $c=c'$ and the theorem is proven. $\square$

Theorem 32. *Let c be a codeword of weight at most $4q+4$ of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$, $q>124$, then supp(c) corresponds via the Klein correspondence to one of the following configurations of lines in $\mathrm{PG}(3,q)$:*

1. *a hyperbolic quadric $\mathcal{Q}^+(3,q)$ in $\mathrm{PG}(3,q)$, with all lines in one regulus symbol α, and all lines in the opposite regulus symbol $-\alpha$,*

2. *a linear combination of two codewords of type (1),*

3. *$4q$ lines through two fixed points R and S in two planes π_1 and π_2 through RS, the line RS not included, where the lines through R in π_1 and the lines through S in π_2 have symbol β, and the other lines have symbol $-\beta$.*

Proof. As seen in the introduction to this section, a codeword c of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$ corresponds to a set $\mathcal{S}$ of lines in $\mathrm{PG}(3,q)$, satisfying conditions $(1)-(4)$. If there is a point in $\mathrm{PG}(3,q)$ lying on at least 5 lines of $\mathcal{S}$, Lemma 29 shows that $\mathcal{S}$ is the set of the $4q$ lines through two fixed points R and S in two planes π_1 and π_2 through RS, the line RS not included. If a line of $\mathcal{S}$ through R in π_1 has symbol β, it is easy to see that all lines of $\mathcal{S}$ through R in π_2, and all lines of $\mathcal{S}$ through S in π_1, have symbol $-\beta$. The same argument implies that the lines of $\mathcal{S}$ through S in π_2 have symbol β. This proves part (3) of the statement.

If all points of $\mathrm{PG}(3,q)$ lie on at most 4 lines of $\mathcal{S}$, Lemma 30 shows that there is a hyperbolic quadric $\mathcal{Q}$ with at least $q-6$ lines L_i, $i=1,\ldots,s$, $s \geq q-6$, of $\mathcal{S}$ in a regulus $\mathcal{R}$ and at least $q-6$ lines M_j, $j=1,\ldots,t$, $t \geq q-6$, of $\mathcal{S}$ in the opposite regulus $\mathcal{R}'$ of $\mathcal{Q}$. There are at most $4q+4-(2q-12)=2q+16$ lines of $\mathcal{S}$ not on $\mathcal{Q}$. This gives in total at most $4q+32$ points Q_i on $\mathcal{Q}$ lying on lines of $\mathcal{S}$, not contained in $\mathcal{Q}$.

Suppose that each of the $q-6$ lines of $\mathcal{R}\cap\mathcal{S}$ contains at least 7 points Q_i, then the number of points Q_i would be at least $7(q-6)>4q+32$, a contradiction if $q > 24$. Hence, there is a line, say L_1, with at most 6 such points Q_i. Suppose that L_1 has symbol α, then there are at least $q-6-6$ lines M_i with symbol $-\alpha$. There is one of these $t \geq q-12$ lines $M_1, \ldots, M_t$ containing at most 6 points Q_i. Suppose that all lines $M_1, \ldots, M_t$ contain at least 7 points Q_i, then the number of points Q_i is at least $7(q-12)$, which is larger than $4q+32$ if $q > 38$, a contradiction. This implies that at least $q-12$ lines L_i have symbol α.

Give all lines of $\mathcal{R}$ the symbol α and give all lines of $\mathcal{R}'$ the symbol $-\alpha$. As seen in Example 22, this set of lines corresponds to a codeword c' of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$.

If $c \neq c'$, then $c - c'$ is a non-zero codeword of the linear code $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$ with weight at most $wt(c)+2q+2-2(2q-24) < wt(c)-4 \leq 4q$, if $q > 27$. Hence, the codeword $c-c'$ is a codeword of weight smaller than $4q$. By induction on the weight of the codewords, it is a hyperbolic quadric with weight $2q+2$, so c is a linear combination of two codewords of type (1). $\square$

5 Codewords of large weight in the dual code of $\mathcal{Q}^+(5,q)$, q even

As seen in the previous section, a codeword c in the binary code $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$, $q = 2^h$, defines a set S of points of $\mathcal{Q}^+(5,q)$ such that every plane of $\mathcal{Q}^+(5,q)$ contains an even number of points of S. Hence, the complement of such a codeword corresponds to a set B of points such that every plane of $\mathcal{Q}^+(5,q)$ contains at least one point of B. Such a set is called a *blocking set* of $\mathcal{Q}^+(5,q)$. If every plane contains exactly one point of B, this set is called an *ovoid* of $\mathcal{Q}^+(5,q)$. These ovoids exist (see [7]) and have size q^2+1. This implies that the codewords of maximal weight in $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$, q even, correspond to the complement of an ovoid, hence have size $|\mathcal{Q}^+(5,q)| - (q^2+1) = (1+q^2)(q^2+q)$.

If $wt(c) = (1+q^2)(q^2+q) - r$, then $\mathcal{Q}^+(5,q) \setminus supp(c)$ is a set S of points, which is a blocking set of size q^2+1+r meeting every plane of $\mathcal{Q}^+(5,q)$ in an odd number of points since q is even. Using the Klein correspondence, this set S of points of $\mathcal{Q}^+(5,q)$ corresponds to a set $\mathcal{S}$ of lines in $\mathrm{PG}(3,q)$ such that:

(1) every plane of $\mathrm{PG}(3,q)$ contains an odd number of lines of $\mathcal{S}$.

(2) every point of $\mathrm{PG}(3,q)$ lies on an odd number of lines of $\mathcal{S}$.

A *cover* $\mathcal{C}$ of $\mathrm{PG}(3,q)$ is a set $\mathcal{L}$ of lines such that every point lies on at least one line of $\mathcal{L}$. For a study of covers of $\mathrm{PG}(3,q)$, we refer to [2].

Lemma 33. *A codeword of* $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$*, q even, has even weight.*

Proof. Since c is a codeword of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$, condition (2) shows that the complement of *supp*(c) corresponds to a set $\mathcal{S}$ of lines of $\mathrm{PG}(3,q)$, such that every point lies on an odd number of lines of $\mathcal{S}$. If $wt(c) = (1+q^2)(q^2+q) - r$, then $|\mathcal{S}| = q^2+1+r$, and $\mathcal{S}$ defines a cover of size q^2+1+r. A double counting of the number of pairs ($P \in \mathrm{PG}(3,q)$, line L of $\mathcal{S}$ through P) yields that $r(q+1)$ is the sum of the excesses of the multiple points. Since every point of $\mathrm{PG}(3,q)$ lies on an odd number of lines of $\mathcal{S}$, every point has even excess, so in total, the sum of all excesses is even. Since q is even, this implies that r is even, hence that $wt(c)$ is even. $\square$

Theorem 34. *There are codewords in* $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$*, q even, of weight* $(1+q^2)(q^2+q)-2i$*, where* $i=0,1,\dots,q/2$*.*

Proof. We give an explicit construction of these codewords. Let T be a regular spread of $\mathrm{PG}(3,q)$, let L be a line of T and let $\mathcal{R}_1,\dots,\mathcal{R}_q$ be q reguli of T through L which pairwise only share L. Replace $2i$ of the reguli $\mathcal{R}_1,\dots,\mathcal{R}_q$ by their opposite reguli. Put the line L back. Let $\mathcal{S}$ be the set of $(q-2i)q+1+2i(q+1)=q^2+1+2i$ lines obtained in this way.

Condition (1) holds since every plane through L contains exactly $2i+1$ lines of $\mathcal{S}$, and a plane, not through L, contains exactly one line of $\mathcal{S}$. Condition (2) holds since a point not on L lies on exactly one line of $\mathcal{S}$, while a point of L lies on $2i+1$ lines of $\mathcal{S}$.

This implies that via the Klein correspondence the complement of $\mathcal{S}$ is a codeword of $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$ of weight $(1+q^2)(q^2+q)-2i$. □

Remark 35. *It is interesting to notice the difference between the possible large weight codewords in* $\mathrm{C}(\mathcal{Q}(4,q))^\perp$*, q even, and* $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$*, q even. In* $\mathrm{C}(\mathcal{Q}(4,q))^\perp$*, q even, Theorem 15 shows there is an empty interval in the weight enumerator, whereas Theorem 34 constructs codewords in* $\mathrm{C}(\mathcal{Q}^+(5,q))^\perp$*, q even, for every even value in* $[(1+q^2)(q^2+q)-q,(1+q^2)(q^2+q)]$*.*

6 Codewords of small weight in the dual code of $\mathcal{H}(5,q^2)$

A codeword c in the p-ary code $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$, $q=p^h$, p prime, $h\geq 1$, satisfies $(c,\pi)=0$ for all planes π contained in $\mathcal{H}(5,q^2)$. Hence, c defines a set S of points of $\mathcal{H}(5,q^2)$ such that every plane of $\mathcal{H}(5,q^2)$ contains zero or at least 2 points of S. Moreover, the sum of the symbols c_P of the points P in one plane equals zero.

Example 36. *Let Γ be a Hermitian curve $\mathcal{H}(2,q^2)\subseteq\mathcal{H}(5,q^2)$, lying in the plane $\pi\not\subseteq\mathcal{H}(5,q^2)$, let Γ' be the Hermitian curve $\mathcal{H}(5,q^2)\cap\pi^\perp$. Let μ be a plane of $\mathcal{H}(5,q^2)$ through a point $Q\in\Gamma$. Since $\pi^\perp\subset Q^\perp$, $\mu=\mu^\perp\subset Q^\perp$, and $\mu\subseteq\mathcal{H}(5,q^2)$, the planes $\pi^\perp$ and μ intersect in at least one point of $\mathcal{H}(5,q^2)$.*

Hence, $S=\Gamma\cup\Gamma'$ is a set of $2(q^3+1)$ points such that every plane of $\mathcal{H}(5,q^2)$ contains zero or at least two points of S. Giving all points of Γ symbol α and all points of Γ' symbol $-\alpha$, yields a codeword of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ of weight $2(q^3+1)$.

Example 37. *Let π be a plane of $\mathrm{PG}(5,q^2)$ intersecting $\mathcal{H}(5,q^2)$ in a cone Γ with vertex P and base a Baer subline. Let Γ' be the intersection of $\pi^\perp$ with $\mathcal{H}(5,q^2)$. It is easy to show that $S=(\Gamma\cup\Gamma')\setminus\{P\}$ is a set such that every plane of $\mathcal{H}(5,q^2)$ contains zero or at least 2 points of S. Giving all points of $\Gamma\setminus\{P\}$ symbol α, all points of $\Gamma'\setminus\{P\}$ symbol $-\alpha$, and the point P symbol zero, yields a codeword of weight $2(q^3+q^2)$ in $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$.*

We will now characterize the two smallest weight codewords of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ to be the codewords of Example 36 and Example 37 (see Theorem 43).

Lemma 38. *Let c be a codeword of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ with weight at most $2(q^3+q^2)$, let π be a plane intersecting $\mathcal{H}(5,q^2)$ in a Hermitian curve $\Gamma\cong\mathcal{H}(2,q^2)$, containing x points of supp(c), and $\Gamma'\cong\mathcal{H}(2,q^2)=\pi^\perp\cap\mathcal{H}(5,q^2)$, containing t points of supp(c). Then $\max(x,t)\leq(13q+19)/2$ or $\min(x,t)\geq(2q^3-5q-5)/2$.*

Proof. Let c be a codeword of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ with weight at most $2(q^3+q^2)$, let π be a plane intersecting $\mathcal{H}(5,q^2)$ in a Hermitian curve $\Gamma \cong \mathcal{H}(2,q^2)$, containing x points of $supp(c)$, and $\Gamma' \cong \mathcal{H}(2,q^2) = \pi^\perp \cap \mathcal{H}(5,q^2)$, containing t points of $supp(c)$. Consider a line L through a point of $\Gamma \cap supp(c)$ and a point of $\Gamma' \setminus supp(c)$. Then there are $x(q^3+1-t)$ such lines, where each such line lies in $q+1$ planes of $\mathcal{H}(5,q^2)$, which yields $x(q^3+1-t)(q+1)$ planes of $\mathcal{H}(5,q^2)$ passing through these lines. All these planes need an extra point of $supp(c) \setminus (\pi_1 \cup \pi_2)$.

A point R of $\mathcal{H}(5,q^2) \setminus (\pi_1 \cup \pi_2)$ is collinear with at most $q+1$ points of Γ. Hence, it lies on at most $q+1$ lines of $\mathcal{H}(5,q^2)$ containing a point of Γ and Γ'. Each such line lies in $q+1$ planes of $\mathcal{H}(5,q^2)$, so every such point R blocks at most $(q+1)^2$ planes through a point of Γ and a point of Γ'. So we need at least $x(q^3+1-t)(q+1)/(q+1)^2$ points in $supp(c) \setminus (\pi_1 \cup \pi_2)$. Doing the symmetrical calculation starting from the lines L through a point of $\Gamma \setminus supp(c)$ and a point of $\Gamma' \cap supp(c)$, and taking the average of both calculations and adding the $x+t$ points of $(\Gamma \cup \Gamma') \cap supp(c)$, yields that there are in total at least

$$x(q^3+1-t)/(2q+2) + t(q^3+1-x)/(2q+2) + x + t \leq 2(q^3+q^2) \qquad (3)$$

points in $supp(c)$.

Since t is at most q^3+1, and omitting the term $x+t$, this gives that

$$(x+t)(q^3+1) - 2x(q^3+1) \leq 4(q+1)(q^3+q^2). \qquad (4)$$

Suppose that $\min(x,t) = x$, otherwise switch t and x. If we substitute $t = x+i$, then inequality (4) yields that $i < 4q+9$.

Rewriting inequality (3) shows that

$$2x^2 - 2x(q^3+2q+3-i) - i(q^3+2q+3) + 4q^2(q+1)^2 \geq 0.$$

This implies, together with $0 \leq i \leq 4q+8$, that $x \leq (5q+3)/2$, or $x \geq (2q^3-5q-5)/2$. □

Lemma 39. *Let c be a codeword of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ with weight at most $2(q^3+q^2)$, let π be a plane intersecting $\mathcal{H}(5,q^2)$ in a cone Γ with vertex P and base a Baer subline, containing x points of $supp(c)$, and $\Gamma' = \pi^\perp \cap \mathcal{H}(5,q^2)$ a cone with vertex P and base a Baer subline in $\pi^\perp$, containing t points of $supp(c)$. Then $\max(x,t) \leq (17q^2-270q+10)/2$ or $\min(x,t) \geq q^3-6q^2$.*

Proof. We consider the lines L of $\mathcal{H}(5,q^2)$ through a point of $(\Gamma \cap supp(c)) \setminus \{P\}$ and a point of $\Gamma' \setminus (\{P\} \cup supp(c))$. There are at least $(x-1)(q^3+q^2-1-t)$ such lines, and all such lines L lie in $q+1$ planes of $\mathcal{H}(5,q^2)$. One of these planes is $\langle L, P\rangle$, but we do not consider this plane.

We have at least $(x-1)(q^3+q^2-1-t)q$ such planes, only intersecting π and $\pi^\perp$ in one point, different from P. Each of those planes needs at least one extra point of $supp(c)$, which lies in $supp(c) \setminus (\pi \cup \pi^\perp)$. A point of $supp(c) \setminus (\pi \cup \pi^\perp)$ is collinear with at most q^2 points of $\Gamma \setminus \{P\}$, so it lies on at most q^2 lines L, which lie in q of those planes. Hence, we need at least $(x-1)(q^3+q^2-1-t)q/q^3$ points in $supp(c) \setminus (\pi \cup \pi^\perp)$. Doing again the symmetrical calculation and taking the average yields that there are at least

$$(x-1)(q^3+q^2-1-t)/(2q^2) + (t-1)(q^3+q^2-1-x)/(2q^2) + x + t \leq 2(q^3+q^2) \quad (5)$$

points in $supp(c)$.

Since t is at most q^3+q^2+1, and omitting the term $x+t$, this gives that

$$(x+t-2)(q^3+q^2-1)-2x(q^3+q^2+1)\leq 4q^2(q^3+q^2). \tag{6}$$

Suppose that $\min(x,t)=x$, otherwise switch t and x. If we substitute $t=x+i$, then inequality (6) yields that $i<4q^2+5$.

Rewriting inequality (5) shows that

$$2x^2-2x(q^3+3q^2-i)-(1+i-iq^3-3iq^2)+4q^2(q^3+q^2)\geq 0.$$

This implies, together with $0\leq i\leq 4q^2+5$, that $x\leq 9(q^2-30q)/2$ or $x\geq q^3-6q^2$ if $q>327$. □

Lemma 40. *Let c be a codeword of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ of weight at most $2(q^3+q^2)$. A plane π of $\mathcal{H}(5,q^2)$ contains at most $2q^2+2q+2$ points of supp(c).*

Proof. Let π contain x points of $supp(c)$. Let P be a point of $\pi\cap supp(c)$. If we project from P onto its quotient geometry $\mathcal{H}(3,q^2)$, then the points of $supp(c)\cap P^\perp$ are projected onto a blocking set of $\mathcal{H}(3,q^2)$ with respect to lines, and the points of π are projected onto a line L of $\mathcal{H}(3,q^2)$. There are $(1+q)(q^3+1)-1-(q^2+1)q=q^4$ lines of $\mathcal{H}(3,q^2)$ skew to L. A point $R\notin L$ lies on q of those lines, so we need at least q^3 extra points to block the lines of $\mathcal{H}(3,q^2)$ skew to L.

Hence, $P^\perp$ contains at least q^3 points of $supp(c)$ outside of π.

We count the number of pairs (P,R), $P\in\pi\cap supp(c)$, $R\in supp(c)\setminus\pi$, with PR a line of $\mathcal{H}(5,q^2)$. There are at least xq^3 such pairs, moreover, a point $R\in supp(c)\setminus\pi$ is collinear with q^2+1 points of π, hence, there are at most $2(q^3+q^2)(q^2+1)$ such pairs. This implies that

$$xq^3\leq 2(q^3+q^2)(q^2+1),$$

from which it follows that $x\leq 2q^2+2q+2$. □

Remark 41. *The minimum weight of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ is at least q^3+2.*

Proof. Let P be a point of $\pi\cap supp(c)$. If we project from P onto its quotient geometry $\mathcal{H}(3,q^2)$, then the points of $supp(c)\cap P^\perp$ are projected onto a blocking set of $\mathcal{H}(3,q^2)$ with respect to lines. Such a blocking set of $\mathcal{H}(3,q^2)$ has at least q^3+1 points. This implies that the minimum weight of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ is at least q^3+2. □

Lemma 42. *Let c be a codeword of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$, $q>893$, of weight at most $2(q^3+q^2)$, then there is a plane $\pi\not\subseteq\mathcal{H}(5,q^2)$ containing more than $(17q^2-270q+10)/2$ points of supp(c).*

Proof. Let c be a codeword of $\mathrm{C}(\mathcal{H}(5,q^2))^\perp$ of weight at most $2(q^3+q^2)$ and let P_i, $i=1,\ldots,3$, be points of $supp(c)$. As seen in Remark 41, the set $P_i^\perp\cap supp(c)$ contains at least q^3+2 points. Using this, together with the fact that $|(P_1^\perp\cup P_2^\perp\cup P_3^\perp)\cap supp(c)|$ is at most $2(q^3+q^2)$, and

$$|(P_1^\perp\cup P_2^\perp\cup P_3^\perp)\cap supp(c)|=$$

$$|P_1^\perp\cap supp(c)|+|P_2^\perp\cap supp(c)|+|P_3^\perp\cap supp(c)|-|P_1^\perp\cap P_2^\perp\cap supp(c)|$$

$-|P_1^\perp \cap P_3^\perp \cap supp(c)| - |P_2^\perp \cap P_3^\perp \cap supp(c)| + |P_1^\perp \cap P_2^\perp \cap P_3^\perp \cap supp(c)|$

yields that $\max(|P_1^\perp \cap P_2^\perp \cap supp(c)|, |P_1^\perp \cap P_3^\perp \cap supp(c)|, |P_2^\perp \cap P_3^\perp \cap supp(c)|)$ is at least $(q^3 - 2q^2 + 6)/3$.

Hence, for any three points Q_i, $i = 1, \ldots, 3$, in $supp(c)$, $|Q_i^\perp \cap Q_j^\perp \cap supp(c)| \geq (q^3 - 2q^2 + 6)/3$ for some $i \neq j \in \{1, \ldots, 3\}$. Denote the number of distinct 3-dimensional spaces $Q_i^\perp \cap Q_j^\perp$ such that $|Q_i^\perp \cap Q_j^\perp \cap supp(c)|$ is at least $(q^3 - 2q^2 + 6)/3$ by x, and suppose that two such distinct 3-spaces share less than $q^3/105$ common points with $supp(c)$. Then

$$\sum_{i=0}^{x-1} (q^3 - 2q^2 + 6)/3 - i(q^3/105) \leq 2(q^3 + q^2). \qquad (7)$$

It is easy to see that there are at least seven 3-spaces $Q_i^\perp \cap Q_j^\perp \cap supp(c)$ such that $|Q_i^\perp \cap Q_j^\perp \cap supp(c)|$ is at least $(q^3 - 2q^2 + 6)/3$. Filling in $x = 7$ in inequality (7) yields a contradiction since $q \geq 263$. This implies that there is a plane containing at least $q^3/105 > (17q^2 - 270q + 10)/2$ points of $supp(c)$ if $q > 893$. $\square$

Theorem 43. *Let c be a codeword of $\mathrm{C}(\mathcal{H}(5, q^2))^\perp$, $q > 893$, of weight at most $2(q^3 + q^2)$. Then one of the following cases holds:*

- *$wt(c) = 2(q^3 + 1)$ and $supp(c)$ corresponds to the union of 2 Hermitian curves Γ and Γ' of $\mathcal{H}(5, q^2)$, in polar planes π and $\pi^\perp$, where $\pi \not\subseteq \mathcal{H}(5, q^2)$. In the codeword c, all points of Γ have symbol α and all points of Γ' have symbol $-\alpha$.*

- *$wt(c) = 2(q^3 + q^2)$ and $supp(c)$ corresponds to the union of two cones Δ and Δ' of $\mathcal{H}(5, q^2)$, both with vertex P and base a Baer subline, where $\Delta \subset \pi$ for some plane $\pi \not\subseteq \mathcal{H}(5, q^2)$ and $\Delta' \subset \pi^\perp$. In the codeword c, all points of Δ have symbol α and all points of Δ' have symbol $-\alpha$, except for the point P which has symbol zero.*

Proof. According to Lemma 42, we find a plane π with more than $(17q^2 - 270q + 10)/2$ points of $supp(c)$, and Lemma 40 shows that $\pi^\perp \neq \pi$. Lemmas 38 and 39 show that there are at least $q^3 - 6q^2$ points P_i of $supp(c)$ contained in this plane π, and at least $q^3 - 6q^2$ points Q_j of $supp(c)$ contained in $\pi^\perp$. Hence, there are at most $14q^2$ points of $supp(c)$ left.

Let P_1 be a non-vertex point in $\pi \cap supp(c)$, with symbol α. Then all points Q_j, except for at most $14q^2$, have symbol $-\alpha$. But this implies that all points P_i, except for at most $14q^2$, have symbol α.

Let c' be the codeword of $\mathrm{C}(\mathcal{H}(5, q^2))^\perp$ such that all points of $\mathcal{H}(5, q^2) \cap \pi$ have symbol α, and all points of $\mathcal{H}(5, q^2) \cap \pi^\perp$ symbol $-\alpha$. If $\mathcal{H}(5, q^2) \cap \pi$ is a cone with vertex P, give the point P symbol zero in c'.

Then $c - c'$ is a codeword of the linear code $\mathrm{C}(\mathcal{H}(5, q^2))^\perp$, it has weight at most $2(q^3 + q^2) + 2(q^3 + q^2) - 2(2q^3 - 12q^2) = 28q^2$. Since the minimum weight of $\mathrm{C}(\mathcal{H}(5, q^2))^\perp$ is at least $q^3 + 2$, $c = c'$, and the theorem is proven. $\square$

7 Examples of small weight codewords in the dual code of $\mathcal{Q}^-(5,q)$ and $\mathcal{H}(4,q^2)$

In this section, we give a geometric description of some small weight codewords in the codes $\mathrm{C}_1(\mathcal{Q}^-(5,q))^\perp = \mathrm{C}(\mathcal{Q}^-(5,q))^\perp$ and $\mathrm{C}_1(\mathcal{H}(4,q^2))^\perp = \mathrm{C}(\mathcal{H}(4,q^2))^\perp$.

Let c be a codeword of the p-ary code $\mathrm{C}(\mathcal{Q}^-(5,q))^\perp$, $q = p^h$, p prime, $h \geq 1$, and let S be *supp*(c). Then S defines a set of points of $\mathcal{Q}^-(5,q)$ such that every line of $\mathcal{Q}^-(5,q)$ contains 0 or at least 2 points of S, and such that the sum of the symbols of the points on a line equals zero.

The minimum weight of $\mathrm{C}(\mathcal{Q}^-(5,q))^\perp$ is not known. Let us first derive a trivial lower bound on this minimum weight. Let P be a point of S. Since $P^\perp$ is a cone with vertex P and base $\mathcal{Q}^-(3,q)$, it is clear that $|P^\perp \cap S| \geq q^2+2$. Hence, the minimum weight of $\mathrm{C}(\mathcal{Q}^-(5,q))^\perp$ is at least q^2+2.

However, this trivial lower bound is weak: the *bit-oriented bound*, the *parity-oriented bound*, and the *tree bound*, developed by Tanner ([16], [17], [20]), yield that the minimum weight of $\mathrm{C}(\mathcal{Q}^-(5,q))^\perp$ is at least q^3+q+2. In the following proposition, we construct a codeword of weight less than twice this lower bound.

Proposition 44. *Let d be the minimum weight of the p-ary code* $\mathrm{C}(\mathcal{Q}^-(5,q))^\perp$*, $q = p^h$, p prime, $h \geq 1$, then*

$$q^3+q+2 \leq d \leq 2(q^3-q^2+q).$$

Proof. The left hand side of this inequality follows from [16], [17], [20].

Let P_1 and P_2 be non-collinear points of $\mathcal{Q}^-(5,q)$, with $P_1^\perp \cap P_2^\perp$ an elliptic quadric $\mathcal{Q} = \mathcal{Q}^-(3,q)$. Let c be the vector where all points of the cone $P_1\mathcal{Q} \setminus \mathcal{Q}$ have symbol α, all points of $P_2\mathcal{Q} \setminus \mathcal{Q}$ have symbol $-\alpha$, and all other points have symbol zero.

Let P be a point of *supp*(c); assume w.l.o.g. that P is collinear with P_1 and let L be a line of $\mathcal{Q}^-(5,q)$ through P. There are two possibilities. Either L is a line of $P_1\mathcal{Q}$, and then the sum of the symbols of the points on L is $q \cdot \alpha = 0 \pmod p$, or L is a line not in $P_1\mathcal{Q}$. There are q^2+1 lines through P, and every one of the q^2+1 lines through P_2 has exactly one point collinear with P since $\mathcal{Q}-(5,q)$ is a generalized quadrangle. Hence, if L is not a line of $P_1\mathcal{Q}$, it intersects exactly one of the lines of $P_2\mathcal{Q}$ in a point with symbol $-\alpha$. This implies that for every line through P, the sum of the symbols of the points on this line equals zero, hence c is a codeword, and it has weight $2(q^2+1)(q-1)+2$. □

For the code $\mathrm{C}(\mathcal{H}(4,q^2))^\perp$, we can proceed in the same way. Again, by [16], [17], and [20], we find a lower bound on the minimum weight $q^5-q^4+q^3+q^2+2$, strongly improving the trivial lower bound q^3+2.

In the following proposition, we construct a small weight codeword in the code $\mathrm{C}(\mathcal{H}(4,q^2))^\perp$.

Proposition 45. *Let d be the minimum weight of the p-ary code* $\mathrm{C}(\mathcal{H}(4,q^2))^\perp$*, $q = p^h$, p prime, $h \geq 1$, then*

$$q^5-q^4+q^3+q^2+2 \leq d \leq 2(q^5-q^3+q^2).$$

Proof. The left hand side of this inequality follows from [16], [17], [20].

Let P_1 and P_2 be non-collinear points of $\mathcal{H}(4,q^2)$, with $P_1^\perp \cap P_2^\perp$ a Hermitian curve $\mathcal{H} = \mathcal{H}(2,q^2)$. Let c be the vector where all points of the cone $P_1\mathcal{H} \setminus \mathcal{H}$ have symbol α, all points of $P_2\mathcal{H} \setminus \mathcal{H}$ have symbol $-\alpha$, and all other points of $\mathcal{H}(4,q^2)$ have symbol zero. Using the same proof as for Proposition 44, c is a codeword, and it has weight $2(q^2-1)(q^3+1)+2 = 2(q^5-q^3+q^2)$. □

8 Codewords of small and large weight in the dual code of $\mathcal{Q}^+(2n+1,q)$

Let $\mathcal{Q}^+(2n+1,q)$ be a hyperbolic quadric in $\mathrm{PG}(2n+1,q)$. The number of its points is $(q^{2n+1}-1)/(q-1)+q^n$. As usual, let $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$ denote the p-ary dual code of $\mathcal{Q}^+(2n+1,q)$ defined by the incidence matrix of the points and the k-dimensional subspaces of $\mathcal{Q}^+(2n+1,q)$, $q=p^h$, p prime, $h \geq 1$. Let c be a codeword of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$, let S be the set of points defined by *supp*(c), and let B be the complement of S in $\mathcal{Q}^+(2n+1,q)$. We have the following geometrical condition:

If c is a codeword of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$ then every k-subspace of $\mathcal{Q}^+(2n+1,q)$ contains zero or at least 2 points of S.

If q is even, we can deduce more:

$(*)$ *c is a codeword of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$ if and only if every k-subspace of $\mathcal{Q}^+(2n+1,q)$ contains an even number of points of S.*
$(**)$ *B is a blocking set with respect to the k–spaces of $\mathcal{Q}^+(2n+1,q)$.*

We have the following result on the minimum weight of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$, $1 \leq k \leq n$.

Proposition 46. *Let d be the minimum weight for the code $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$, then*

$$d \geq 1 + \frac{q^n-1}{q^k-1}(q^{n-1}+1).$$

Proof. Let c be a codeword of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$ and let S be the set of points defined by *supp*(c). If P is a point of S, then, by $(*)$, every k–space of $\mathcal{Q}^+(2n+1,q)$ through P must contain at least another point of S. The number of k–spaces of $\mathcal{Q}^+(2n+1,q)$ through P is the number of $(k-1)$–spaces of $\mathcal{Q}^+(2n-1,q)$, and this number equals (see [7, Chapter 22])

$$M := \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{i+1}-1} \cdot \prod_{i=n-k}^{n-1} (q^i+1)$$

and the number of k–spaces of $\mathcal{Q}^+(2n+1,q)$ through two collinear points of $\mathcal{Q}^+(2n+1,q)$ is

$$N := \prod_{i=0}^{k-2} \frac{q^{n-i-1}-1}{q^{i+1}-1} \cdot \prod_{i=n-k}^{n-2} (q^i+1),$$

hence

$$|S| \geq 1 + \frac{M}{N} = 1 + \frac{q^n-1}{q^k-1}(q^{n-1}+1).$$

□

In this way, we have derived a lower bound on the minimum weight of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$. In the next proposition and its corollary, we derive an upper bound on the minimum weight of $\mathrm{C}_n(\mathcal{Q}^+(2n+1,q))^\perp$ by constructing a codeword of small weight.

Proposition 47. *Let π be a non-self-polar n-dimensional space of $\mathcal{Q}^+(2n+1,q)$, $q=p^h$, p prime, $h \geq 1$. Let T be the intersection of π and $\pi^\perp$. Let c be the vector where all points of $(\pi \cap \mathcal{Q}^+(2n+1,q)) \setminus T$ have symbol α, and all points of $(\pi^\perp \cap \mathcal{Q}^+(2n+1,q)) \setminus T$ have symbol $-\alpha$. Then c is a codeword of $\mathrm{C}_n(\mathcal{Q}^+(2n+1,q))^\perp$.*

Proof. Let P be a point of *supp*(c), suppose w.l.o.g. that it has symbol α. Let μ be a generator of $\mathcal{Q}^+(2n+1,q)$ through P. Suppose first that μ does not contain a point of $\pi \cap \pi^\perp$. Then π contains 1 (mod p) points with symbol $-\alpha$. The n-spaces $\mu=\mu^\perp$ and $\pi^\perp$ are contained in the $2n$-dimensional space $P^\perp$, so they intersect in a subspace of $\mathcal{Q}^+(2n+1,q)$ of dimension at least zero. Hence, $\mu \cap pi^\perp$ contains 1 (mod p) points with symbol $-\alpha$. Suppose that μ contains points of $\pi \cap \pi^\perp$. Then π contains 0 (mod p) points with symbol α. The n-spaces $\mu=\mu^\perp$ and $\pi^\perp$ are contained in the $2n$-dimensional space $P^\perp$, so they intersect in a subspace of $\mathcal{Q}^+(2n+1,q)$ of dimension at least zero. Since μ contains points of $\pi \cap \pi^\perp$, $\mu \cap pi^\perp$ contains 0 (mod p) points with symbol $-\alpha$. This implies that for every generator through P, the sum of the symbols of the points in this generator equals zero, hence c is a codeword of $\mathrm{C}_n(\mathcal{Q}^+(2n+1,q))^\perp$. □

Corollary 48. *Let d be the minimum weight of the p-ary code $\mathrm{C}_n(\mathcal{Q}^+(2n+1,q))^\perp$, $q=p^h$, p prime, $h\geq 1$, then $d \leq 2(q^n-1)/(q-1)$ if n is even, and $d \leq 2(q^n-1)/(q-1)-2q^{(n-1)/2}$ if n is odd.*

Proof. Let n be even. The codeword, constructed in Proposition 47, starting with an n-space intersecting $\mathcal{Q}^+(2n+1,q)$ in a parabolic quadric $\mathcal{Q}(n,q)$, has weight $2(q^n-1)/(q-1)$. Let n be odd. If an n-space π intersects $\mathcal{Q}^+(2n+1,q)$ in an elliptic quadric $\mathcal{Q}^-(n,q)$, then $\pi^\perp$ intersects $\mathcal{Q}^+(2n+1,q)$ also in an elliptic quadric. Hence, the codeword constructed in Proposition 47 has weight $2(q^n-1)/(q-1)-2q^{(n-1)/2}$. □

Remark 49. *The same construction holds for codewords of small weight in $\mathrm{C}_n(\mathcal{H}(2n+1,q^2))^\perp$. Here, we obtain that the minimum weight of $\mathrm{C}_n(\mathcal{H}(2n+1,q^2))^\perp$ is at most $2(q^{n+1}+1)(q^n-1)/(q^2-1)$ if n is even, and at most $2(q^{n+1}-1)(q^n+1)/(q^2-1)$ if n is odd. Example 36 was constructed in this way.*

From now on, we will investigate codewords of large weight in $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$, q even, $k \leq n$. We use conditions $(*)$ and $(**)$ to find these large weight codewords of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$, $k \leq n$. Large weight codewords correspond to small blocking sets with respect to the k-spaces of $\mathcal{Q}^+(2n+1,q)$, hence we start by considering minimal blocking sets of $\mathcal{Q}^+(2n+1,q)$ with respect to k–subspaces.

The case $k=n$. An *ovoid* of $\mathcal{Q}^+(2n+1,q)$ is a set $\mathcal{O}$ of points such that every n-space, contained in $\mathcal{Q}^+(2n+1,q)$, contains exactly one point of $\mathcal{O}$. Obviously, ovoids of $\mathcal{Q}^+(2n+1,q)$ are the smallest possible blocking sets with respect to n-spaces. The problem of the existence of ovoids in $\mathcal{Q}^+(2n+1,q)$,

$n > 2$, $q > 3$ is still an open problem (for more details see [7]), hence we will not treat this case.

The case $k = 1$**.** The problem of determining the smallest blocking sets with respect to the lines of $Q^+(2n+1, q)$ is completely solved in the following result by Metsch.

Result 50. *[11] The two smallest blocking sets with respect to the lines of* $\mathcal{Q}^+(2n+1, q)$*,* $n \geq 2$*, are of type* $P^\perp \setminus \{P\}$*. If* $P \notin \mathcal{Q}^+(2n+1, q)$*, then* $P^\perp \setminus \{P\} = \mathcal{Q}(2n, q)$*; if* $P \in \mathcal{Q}^+(2n+1, q)$*, then* $P^\perp \setminus \{P\}$ *is a cone over a hyperbolic quadric* $\mathcal{Q}^+(2n-1, q)$ *without the vertex* P*.*

Proposition 51. *The maximum weight of* $\mathrm{C}_1(\mathcal{Q}^+(2n+1, q))^\perp$*,* q *even,* $n \geq 2$*, is* $(q^n + 1)q^n$*, the second largest weight is* q^{2n} *and the codewords of weight* $(q^n + 1)q^n$ *and* q^{2n} *are defined by the complement of a hyperplane with respect to* $\mathcal{Q}^+(2n+1, q)$*.*

Proof. Let c be a codeword of $\mathrm{C}_1(\mathcal{Q}^+(2n+1, q))^\perp$ with $wt(c) \geq q^{2n}$. The complement of $supp(c)$ defines a blocking set B with respect to lines of $\mathcal{Q}^+(2n+1, q)$ of size at most $\theta_{2n-1} + q^n$. Result 50 shows that B contains either a parabolic quadric $\mathcal{Q}(2n, q)$ or a cone over a hyperbolic quadric $\mathcal{Q}^+(2n-1, q)$ minus its vertex.

Suppose first that the blocking set B contains a parabolic quadric $\mathcal{Q} = \mathcal{Q}(2n, q)$. A line L of $\mathcal{Q}^+(2n+1, q)$ is either contained in $\mathcal{Q}$ or it intersects $\mathcal{Q}$ in one point. Hence, by conditions $(*)$ and $(**)$, the complement of $\mathcal{Q}$ defines a codeword c' of $\mathrm{C}_1(\mathcal{Q}^+(2n+1, q))^\perp$. But then $wt(c+c') = wt(c)+wt(c')-2wt(c\cap c') \leq q^n$, which is smaller than the minimum weight of $\mathrm{C}_1(\mathcal{Q}^+(2n+1, q))^\perp$ (see Proposition 46). Hence, in this case, $c = c'$, $wt(c) = q^{2n} + q^n$, and $supp(c)$ is the complement of a parabolic quadric $\mathcal{Q}(2n, q)$.

Suppose that the blocking set B contains a set $P^\perp \setminus \{P\}$, with $P \in \mathcal{Q}^+(2n+1, q)$, then the complement of c has size $\theta_{2n-1} + q^n - 1$ or $\theta_{2n-1} + q^n$. The set $P^\perp$ is such that a line is either contained in $P^\perp$ or contains one point of $P^\perp$. Hence, by conditions $(*)$ and $(**)$, the complement of $P^\perp$ defines a codeword c' of $\mathrm{C}_1(\mathcal{Q}^+(2n+1, q))^\perp$. But then $wt(c+c') \leq 1$, which implies that $c = c'$, $wt(c) = q^{2n}$, and c corresponds to the complement of a set $P^\perp$ with $P \in \mathcal{Q}^+(2n+1, q)$. ☐

Corollary 52. *The largest sets of even type in* $\mathcal{Q}^+(2n+1, q)$*,* q *even, have size* $q^{2n} + q^n$*, and they correspond to the complement of a parabolic quadric* $\mathcal{Q}(2n, q)$ *of* $\mathcal{Q}^+(2n+1, q)$*, there are no sets of even type in* $\mathcal{Q}^+(2n+1, q)$*,* q *even, with weight in* $]q^{2n}, q^{2n} + q^n[$*, and a set of even type in* $\mathcal{Q}^+(2n+1, q)$*,* q *even, of size* q^{2n} *corresponds to the complement of a cone over a hyperbolic quadric* $\mathcal{Q}^+(2n-1, q)$*.*

The case $2 \leq k \leq n-1$**.** Let $S_k\mathcal{Q}$ be a cone over a non–singular quadric $\mathcal{Q}$ with a k–dimensional vertex. A *truncated* cone $S_k\mathcal{Q}$ is the set of points $S_k\mathcal{Q}\setminus S_k$.

Result 53. *([12]) Let* B *be a blocking set of the quadric* $\mathcal{Q}^+(2n+1, q)$ *with respect to* k*–subspaces,* $2 \leq k \leq n-1$*. Then* $|B| \geq \frac{q^{n-k+1}-1}{q-1}(q^n + q^{k-2})$*. If* $|B| < \frac{q^{n-k+1}-1}{q-1}(q^n+q^{k-2}+1)$*, then* B *contains the truncated cone* $S_{k-3}\mathcal{Q}^-(2n+3-2k, q)$*.*

Hence we can prove the following theorem.

Theorem 54. *The maximum weight of* $\mathrm{C}_2(\mathcal{Q}^+(2n+1,q))^\perp$*, q even, is* $(q^n+1)(q^n+q^{n-1})$ *and it corresponds to the complement of an elliptic quadric* $\mathcal{Q}^-(2n-1,q)$ *in* $\mathcal{Q}^+(2n+1,q)$*.*

Proof. By Result 53, the smallest minimal blocking set with respect to the planes of $Q^+(2n+1,q)$ is an elliptic quadric $\mathcal{Q}^-(2n-1,q)$. Every plane of $\mathcal{Q}^+(2n+1,q)$ intersects $\mathcal{Q}^-(2n-1,q)$ in an odd number of points, hence by conditions $(*)$ and $(**)$, the complement of such an elliptic quadric $\mathcal{Q}^-(2n-1,q)$ defines a codeword of the largest weight. $\square$

Theorem 55. *The maximum weight of* $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$*, q even, with* $3 \leq k \leq n-1$*, is* $q^n(q^n+q^{n-1}+\cdots+q^{n-k+1})+q^n+q^{n-1}$ *and it corresponds to the complement of a cone* $S_{k-3}\mathcal{Q}^-(2n+3-2k,q)$ *in* $\mathcal{Q}^+(2n+1,q)$*.*

Proof. Denote the size of the complement of a cone $S_{k-3}\mathcal{Q}^-(2n+3-2k,q)$ by s. Let c be a codeword of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$ with $wt(c) \geq s$. The complement of c corresponds to a blocking set of size at most $|S_{k-3}\mathcal{Q}^-(2n+3-2k,q)|$. Hence, by Result 53, the complement of c consists of the points of a truncated cone $\mathcal{C} = S_{k-3}\mathcal{Q}^-(2n+3-2k,q)$ and some other points, which we will denote by T. Since $wt(c) \geq s$, $|T| \leq |S_{k-3}| = \theta_{k-3}$. Let c' be the codeword corresponding to the complement of the cone $\mathcal{C}'$, obtained by adding the vertex to the truncated cone $\mathcal{C}$. Since $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$ is a linear code, the vector $c+c'$ is in $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$. Moreover, it has weight

$$wt(c+c') = wt(c) + wt(c') - 2wt(c \cap c') =$$

$$|\mathcal{Q}^+(2n+1,q)| - |\mathcal{C} \cup T| + |\mathcal{Q}^+(2n+1,q)| - |\mathcal{C}'| - 2|\mathcal{Q}^+(2n+1,q)| + 2|\mathcal{C} \cup S_{k-3} \cup T|$$

$$\leq 2|S_{k-3}|.$$

But $2|S_{k-3}|$ is smaller than $1 + \frac{q^n-1}{q^k-1}(q^{n-1}+1)$ which is the minimum weight by Proposition 46. This implies that $c = c'$. Hence, the maximum weight of $\mathrm{C}_k(\mathcal{Q}^+(2n+1,q))^\perp$ is $q^n(q^n+q^{n-1}+\cdots+q^{n-k+1})+q^n+q^{n-1}$ and corresponds to the complement of a cone $S_{k-3}\mathcal{Q}^-(2n+3-2k,q)$ in $\mathcal{Q}^+(2n+1,q)$. $\square$

9 The dual code of $\mathcal{Q}(2n,q)$, q even

Let $\mathcal{Q}(2n,q)$ be a parabolic quadric in $\mathrm{PG}(2n,q)$. The number of points of $\mathcal{Q}(2n,q)$ is $(q^{2n}-1)/(q-1)$. As usual, let $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$ denote the p-ary dual code of $\mathcal{Q}(2n,q)$ defined by the incidence matrix of the points and the k-dimensional subspaces of $\mathcal{Q}(2n,q)$. Let c be a codeword of $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$, let S be the set of points defined by $supp(c)$, and let B be the complement of S in $\mathcal{Q}(2n,q)$. We have the following geometrical condition:

If c is a codeword of $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$*, then every k-subspace of* $\mathcal{Q}(2n,q)$ *contains zero or at least 2 points of S.*

If q is even, we can deduce more:

$(*)$ *c is a codeword of* $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$ *if and only if every k-subspace of* $\mathcal{Q}(2n,q)$ *contains an even number of points of S.*

(∗∗) *B is a blocking set with respect to the k–spaces of $\mathcal{Q}(2n,q)$.*

We have the following result on the minimum weight of $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$, $1 \le k \le n-1$.

Proposition 56. *Let d be the minimum weight of the code $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$, $k \le n-1$, then $d \ge 1 + \frac{q^{n-1}-1}{q^k-1}(q^{n-1}+1)$.*

Proof. Let c be a codeword of $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$ and let S be the set of points of $\mathcal{Q}(2n,q)$ defined by *supp*(c). If P is a point of S, then, by (∗), every singular k–space through P must contain at least another point of S. The number of the singular k–subspaces through P is (see [7, Chapter 22]): $M := \prod_{i=0}^{k-1} \frac{q^{n-1-i}-1}{q^{i+1}-1} \cdot \prod_{i=n-k}^{n-1}(q^i+1)$ and the number of the singular k–subspaces through two points is $N := \prod_{i=0}^{k-2} \frac{q^{n-2-i}-1}{q^{i+1}-1} \cdot \prod_{i=n-k}^{n-2}(q^i+1)$, hence $|S| \ge 1 + \frac{M}{N} = 1 + \frac{q^{n-1}-1}{q^k-1}(q^{n-1}+1)$. □

We are interested in finding the large weight codewords of $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$, q even, and we use conditions (∗) and (∗∗).

If we consider $\mathcal{Q}(2n,q)$ to be embedded in $\mathcal{Q}^+(2n+1,q)$, then every blocking set of $\mathcal{Q}(2n,q)$ with respect to subspaces of dimension k of $\mathcal{Q}(2n,q)$ is a blocking set of $\mathcal{Q}^+(2n+1,q)$ with respect to subspaces of dimension $k+1$ of $\mathcal{Q}^+(2n+1,q)$. So, for $1 \le k \le n-2$, a blocking set B of smallest size consists of the non–singular points of a quadric of type $S_{k-2}\mathcal{Q}^-(2n+1-2k,q)$ (Result 53; for more details, see [12]). From this consideration and by similar arguments of the previous section, we get the following result:

Theorem 57. *The maximum weight for $\mathrm{C}_k(\mathcal{Q}(2n,q))^\perp$, q even, $1 \le k \le n-2$ is $q^n(q^{n-1}+q^{n-2}+\cdots+q^{n-k})+q^{n-1}$ and it corresponds to the complement of a cone $S_{k-2}\mathcal{Q}^-(2n+1-2k,q)$ in $\mathcal{Q}(2n,q)$.*

Corollary 58. *The largest sets of even type in $\mathcal{Q}(2n,q)$, q even, have size $q^{2n-1}+q^{n-1}$, and correspond to the complement of an elliptic quadric $\mathcal{Q}(2n-1,q)$.*

10 The dual code of $\mathcal{Q}^-(2n+1,q)$, q even

Let $\mathcal{Q}^-(2n+1,q)$ be an elliptic quadric in $\mathrm{PG}(2n+1,q)$. The number of points of $\mathcal{Q}^-(2n+1,q)$ is $(q^{2n+1}-1)/(q-1)-q^n$. As usual, let $\mathrm{C}_k(\mathcal{Q}^-(2n+1,q))^\perp$, $k \le n-1$, denote the dual code of the p-ary code of $\mathcal{Q}^-(2n+1,q)$ defined by the points and by the k–dimensional subspaces of $\mathcal{Q}^-(2n+1,q)$.

Let c be a codeword of $\mathrm{C}_k(\mathcal{Q}^-(2n+1,q))^\perp$, let S be the set of points defined by *supp*(c) and let B be the complement of S in $\mathcal{Q}^-(2n+1,q)$. As seen before, we can use condition (∗) to obtain a lower bound on the minimum weight of the code $\mathrm{C}_k(\mathcal{Q}^-(2n+1,q))^\perp$.

Proposition 59. *Let d be the minimum weight of the code* $\mathrm{C}_k(\mathcal{Q}^-(2n+1,q))^\perp$, $1 \le k \le n-1$, *then* $d \ge 1 + \frac{q^{n-1}-1}{q^k-1}(q^n+1)$.

Proof. The number of the singular k–subspaces through a point P of $\mathcal{Q}^-(2n+1,q)$ is (see [7, Chapter 22]) $M := \prod_{i=0}^{k-1} \frac{q^{n-1-i}-1}{q^{i+1}-1} \cdot \prod_{i=n-k+1}^{n} (q^i+1)$ and the number of the singular k–subspaces through two points is $N := \prod_{i=0}^{k-2} \frac{q^{n-2-i}-1}{q^{i+1}-1} \cdot \prod_{i=n-k+1}^{n-1} (q^i+1)$, hence $|S| \ge 1 + \frac{M}{N} = 1 + \frac{q^{n-1}-1}{q^k-1}(q^n+1)$. □

Now consider $\mathcal{Q}^-(2n+1,q)$ embedded in $\mathcal{Q}^+(2n+3,q)$, then every blocking set of $\mathcal{Q}^-(2n+1,q)$ with respect to subspaces of dimension k of $\mathcal{Q}^-(2n+1,q)$ is a blocking set of $\mathcal{Q}^+(2n+3,q)$ with respect to subspaces of dimension $k+2$ of $\mathcal{Q}^+(2n+1,q)$. So, for $1 \le k \le n-3$, a blocking set B of smallest size consists of the non–singular points of a quadric of type $S_{k-1}\mathcal{Q}^-(2n+1-2k,q)$ ([12]).

Hence, for the codewords of large weight we have:

Theorem 60. *The maximum weight for* $\mathrm{C}_k(\mathcal{Q}^-(2n+1,q))^\perp$, *$q$ even,* $1 \le k \le n-3$, *is* $q^{2n-k+1}\theta_{k-1}$ *and it corresponds to the complement of a cone* $S_{k-1}\mathcal{Q}^-(2n+1-2k,q)$ *in* $\mathcal{Q}^-(2n+1,q)$.

Corollary 61. *The largest sets of even type in* $\mathcal{Q}^-(2n+1,q)$, *q even, have size* q^{2n} *and correspond to the complement of a cone over an elliptic quadric* $\mathcal{Q}^-(2n-1,q)$.

11 The dual code of $\mathcal{H}(n,q^2)$, q even

Let $\mathcal{H}(n,q^2)$ be the Hermitian variety in the projective space $\mathrm{PG}(n,q^2)$. This variety contains subspaces of dimension at most $\lfloor \frac{n-1}{2} \rfloor$ and the number of its points is $(q^n-(-1)^n)(q^{n+1}-(-1)^{n+1})/(q^2-1)$. Let $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$, $q=p^h$, p prime, $h \ge 1$, be the p-ary dual code of $\mathcal{H}(n,q^2)$ defined by the points and by the k–dimensional subspaces of $\mathcal{H}(n,q^2)$. Let S be the set of points corresponding to *supp*(c), and let B be the complement of S.

We have again the following geometrical condition:

If c is a codeword of $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ *then every k-subspace of* $\mathcal{H}(n,q^2)$ *contains zero or at least 2 points of S.*

And if q is even, we can deduce more:

$(*)$ *c is a codeword of* $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ *if and only if every k-subspace of* $\mathcal{H}(n,q^2)$ *contains an even number of points of S.*
$(**)$ *B is a blocking set with respect to the k–spaces of* $\mathcal{H}(n,q^2)$.

Proposition 62. *Let d be the minimum weight of the code* $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$, *then*

$$d \ge 1 + \frac{(q^{n-1}-(-1)^{n-1})(q^{n-2}-(-1)^{n-2})}{q^{2k}-1}.$$

Proof. Let c be a codeword of $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ and let S be the set of points defined by $supp(c)$. If P is a point of S, then every singular k–space through P must contain at least another point of S. The number of the k–subspaces of $\mathcal{H}(n,q^2)$ through P is (see [7, Chapter 23])

$$M := \prod_{i=n-2k}^{n-1}(q^i-(-1)^i)/\prod_{j=1}^{k}(q^{2j}-1)$$

and the number of the k–subspaces of $\mathcal{H}(n,q^2)$ through two collinear points is

$$N := \prod_{i=n-2k}^{n-3}(q^i-(-1)^i)/\prod_{j=1}^{k-1}(q^{2j}-1),$$

hence $|S| \geq 1+\frac{M}{N} = 1+\frac{(q^{n-1}-(-1)^{n-1})(q^{n-2}-(-1)^{n-2})}{q^{2k}-1}$. □

Let $S_i\mathcal{H}(n,q^2)$ be a cone with i–dimensional vertex S_i over a Hermitian variety $\mathcal{H}(n,q^2)$. In [13], Metsch proves several results about the blocking sets of Hermitian varieties and we summarize them in the following result.

Result 63. *[13] Let B be a minimal blocking set of a Hermitian variety $\mathcal{H}(n,q^2)$ with respect to the k–subspaces and let $k \leq \frac{n-3}{2}$. If $|B| \leq q^{2(n-k-7)}|\mathcal{H}(7,q^2)| + q^3$, then B is of type $S_i\mathcal{H}(n-k-1-i,q^2)\setminus S_i$. Thus, if n is even, then the smallest blocking set is $S_{k-1}\mathcal{H}(n-2k,q^2)\setminus S_{k-1}$; when n is odd, the smallest blocking set is $S_{k-2}\mathcal{H}(n-2k+1,q^2)\setminus S_{k-2}$.*

This result enables us to find the large weight codewords of the code $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$, $1 \leq k \leq \frac{n-3}{2}$, q even.

Theorem 64. *If n is even, the maximum weight of $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$, q even, $1 \leq k < (n-3)/2$ is $q^{2n-2k+1}\frac{q^{2k}-1}{q^2-1}$ and it corresponds to the complement of a cone $S_{k-1}\mathcal{H}(n-2k,q^2)$. If n is odd, then the maximum weight of $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ is $q^{2n-2k+1}\frac{q^{2k}-1}{q^2-1}+q^{n-1}$ and it corresponds to the complement of a cone $S_{k-2}\mathcal{H}(n-2k+1,q^2)$.*

Proof. Let c be a codeword of $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ with $wt(c)$ at least the size of the complement of a cone $S_{k-1}\mathcal{H}(n-2k,q^2)$ if n is even, and $S_{k-2}\mathcal{H}(n-2k+1,q^2)$ if n is odd. The complement of c corresponds to a blocking set of size at most $|S_{k-1}\mathcal{H}(n-2k,q^2)|$ if n is even, and $|S_{k-2}\mathcal{H}(n-2k+1,q^2)|$ if n is odd. Hence, by Result 63, the complement of c consists of the points of a truncated cone and some other points. One can check that $S_{k-1}\mathcal{H}(n-2k,q^2)\setminus S_{k-1}$ is the only truncated cone with size less than $|S_{k-1}\mathcal{H}(n-2k,q^2)|$ if n is even, and $S_{k-2}\mathcal{H}(n-2k+1,q^2)$ is the only truncated cone with size less than $|S_{k-2}\mathcal{H}(n-2k+1,q^2)|$ if n is odd. Hence, the complement of c consists of a truncated cone $\mathcal{C}_1 = S_{k-1}\mathcal{H}(n-2k,q^2)$ if n is even, or $\mathcal{C}_2 = S_{k-2}\mathcal{H}(n-2k+1,q^2)$ if n is odd, and some other points, which we will denote by T. Since $wt(c)$ is at least the size of the complement of a cone $S_{k-1}\mathcal{H}(n-2k,q^2)$ if n is even, and $S_{k-2}\mathcal{H}(n-2k+1,q^2)$ if n is odd, $|T| \leq |S_{k-1}| = \theta_{k-1}$ if n is even, and $|T| \leq |S_{k-2}| = \theta_{k-2}$ if n is odd. Let c' be the codeword corresponding to the complement of the cone $\mathcal{C}'$, obtained by adding the vertex to the truncated cone

$\mathcal{C}_1$ if n is even, and $\mathcal{C}_2$ if n is odd. Since $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ is a linear code, the vector $c+c'$ is in $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$. Moreover, it has weight

$$wt(c+c') = wt(c) + wt(c') - 2wt(c \cap c') =$$

$$|\mathcal{H}(n,q^2)| - |\mathcal{C} \cup T| + |\mathcal{H}(n,q^2)| - |\mathcal{C}'| - 2(|\mathcal{H}(n,q^2)| - |\mathcal{C} \cup S_{k-i} \cup T|)$$
$$\leq 2|S_{k-i}|,$$

where $i=1$ if n is even, and $i=2$ if n is odd and where C is the truncated cone C_1 is n is even and C is the truncated cone C_2 is n is odd. But $2|S_{k-i}|$ is smaller than $1+\frac{(q^{n-1}-(-1)^{n-1})(q^{n-2}-(-1)^{n-2})}{q^{2k}-1}$ if $k<(n-3)/2$, which is the minimum weight by Proposition 62. This implies that $c=c'$. Hence, the maximum weight of $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ for n even is $q^{2n-2k+1}\frac{q^{2k}-1}{q^2-1}$ and it corresponds to the complement of a cone $S_{k-1}\mathcal{H}(n-2k,q^2)$. For n odd, the maximum weight of $\mathrm{C}_k(\mathcal{H}(n,q^2))^\perp$ is $q^{2n-2k+1}\frac{q^{2k}-1}{q^2-1}+q^{n-1}$ and it corresponds to the complement of a cone $S_{k-2}\mathcal{H}(n-2k+1,q^2)$. □

Corollary 65. *The largest set of even type of $\mathcal{H}(n,q^2)$, q even, have size q^{2n-1} and corresponds to the complement of a cone over a Hermitian variety $\mathcal{H}(n-2,q^2)$ if n is even, and to the complement of a Hermitian variety $\mathcal{H}(n-1,q^2)$ if n is odd.*

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