Intriguing sets in partial quadrangles

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Abstract. The point-line geometry known as a *partial quadrangle* (introduced by Cameron in 1975) has the property that for every point/line non-incident pair (P, ℓ), there is at most one line through P concurrent with ℓ. So in particular, the well-studied objects known as *generalised quadrangles* are each partial quadrangles. An *intriguing set* of a generalised quadrangle is a set of points which induces an equitable partition of size two of the underlying strongly regular graph. We extend the theory of intriguing sets of generalised quadrangles by Bamberg, Law and Penttila to partial quadrangles, which gives insight into the structure of hemisystems and other intriguing sets of generalised quadrangles.

1. Introduction

A set of points $\mathcal{I}$ of a generalised quadrangle is defined in [**3**] to be *intriguing* if the number of points of $\mathcal{I}$ collinear to an arbitrary point P is a constant h_1 if P lies in $\mathcal{I}$, and another constant h_2 if P resides outside of $\mathcal{I}$. For example, a line of a generalised quadrangle is such an object where h_1 is the number of points on a line, and $h_2 = 1$. Eisfeld [**16**] asks whether such sets have a natural geometric interpretation, and it is shown in [**3**] that the intriguing sets of a generalised quadrangle are precisely the m-ovoids and tight sets introduced by J. A. Thas [**22**] and S. E. Payne [**20**] respectively. If one looks to the point graph of a generalised quadrangle, one will find a strongly regular graph. The associated Bose-Mesner algebra of this graph decomposes into an orthogonal decomposition of three eigenspaces of the adjacency matrix, one of which is the one-dimensional subspace generated by the "all 1's" vector. The other two eigenspaces correspond naturally to the two types of intriguing sets; the positive eigenvalue corresponds to the tight sets, and the negative eigenvalue corresponds to the m-ovoids [**3**, Theorem 4.1]. In this paper we consider the algebraic combinatorics of a partial quadrangle.

A *partial quadrangle* was introduced by P. J. Cameron [**7**] as a geometry of points and lines such that every two points are on at most one line, there are $s+1$ points on a line, every point is on $t+1$ lines and satisfying the following two important properties:

(i) for every point P and every line ℓ not incident with P, there is at most one point on ℓ collinear with P;
(ii) there is a constant μ such that for every pair of non-collinear points (X, Y) there are precisely μ points collinear with X and Y.

With the above specifications, we say that the partial quadrangle has *parameters* (s, t, μ), or that it is a partial quadrangle $\mathsf{PQ}(s, t, \mu)$. Note that the point-graph of this object is strongly regular (see Section 2).

The only known partial quadrangles, which are not generalised quadrangles, are

- triangle-free strongly regular graphs (i.e., partial quadrangles with $s = 1$);
- one of three exceptional examples, namely they arise from linear representation of one of the Coxeter 11-cap of $\mathsf{PG}(4, 3)$, the Hill 56-cap of $\mathsf{PG}(5, 3)$ or the Hill 78-cap of $\mathsf{PG}(5, 4)$;
- or arise from removing points from a generalised quadrangle of order (s, s^2).

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We will now be more precise for this last class of partial quadrangles. Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) and let P be a point of $\mathcal{G}$. Then by removing all those points $P^\perp$ which are collinear with P results in a partial quadrangle $\mathsf{PQ}(s-1, s^2, s(s-1))$ (see [**8**, pp. 4]). We will often refer to this construction as a generalised quadrangle minus the perp of a point. Similarly, we can remove a certain type of m-ovoid from $\mathcal{G}$ to obtain a partial quadrangle [**8**, Prop. 2.2]. A *hemisystem (of points)* of $\mathcal{G}$, where s is odd, is a set of points $\mathcal{H}$ of $\mathcal{G}$ such that every line meets $\mathcal{H}$ in $(s+1)/2$ points (i.e., it is an m-ovoid with $m = (s+1)/2$). By considering the incidence structure restricted to $\mathcal{H}$, we obtain a partial quadrangle $\mathsf{PQ}((s-1)/2, s^2, (s-1)^2/2)$. In 2005, Cossidente and Penttila [**11**] constructed new hemisystems of the classical generalised quadrangle $\mathsf{Q}^-(5, q)$, and thus new partial quadrangles, and this work has recently been extended in [**1**] to every flock generalised quadrangle.

For generalised quadrangles, it has been shown that an m-ovoid and an i-tight set intersect in mi points [**3**, Theorem 4.3]. From this observation, one can prove or reprove interesting results in the forum of generalised quadrangles. For partial quadrangles, the theory still holds; there are two types of intriguing sets according to the parity of the associated eigenvalue, and there is a similar "intersection result" (see Section 2.3). In Section 3, we investigate and in some cases classify, the intriguing sets of triangle-free strongly regular graphs; the *thin* partial quadrangles. The section that follows concerns the two known families of *thick* partial quadrangles which arise from (i) deleting the perp of a point, or from (ii) deleting a hemisystem. In both cases, we look to the deleted point set, which we nominate as "infinity", and analyse the situation for when an intriguing set of the ambient generalised quadrangle gives rise to an intriguing set of the partial quadrangle obtained by removing infinity. In the case of a generalised quadrangle minus the perp of a point, we give some strong combinatorial information in Section 5 on the structure of incumbent intriguing sets, which manifests in a characterisation of the *positive* intriguing sets arising from tight sets of the ambient generalised quadrangle, and a characterisation of the *negative* intriguing sets. The intriguing sets of partial quadrangles obtained from hemisystems have less combinatorial structure, however, we are able to deduce certain relationships between intriguing sets of the ambient generalised quadrangle and the partial quadrangle (see Section 6). In Section 7, we return to isolated examples of partial quadrangles, and this time on the exceptional examples arising from caps of projective spaces via linear representation.

2. Some algebraic graph theory and intriguing sets

2.1. Intriguing sets of strongly regular graphs. A regular graph Γ, with v vertices and valency k, is *strongly regular* with parameters (v, k, λ, μ) if (i) any two adjacent vertices are both adjacent to λ common vertices; (ii) any two non-adjacent vertices are both adjacent to μ common vertices. If A is the adjacency matrix of the strongly regular graph Γ, then A has three eigenvalues and satisfies the equation $A^2 = kI + \lambda A + \mu(J - I - A)$ where I is the identity matrix and J is the all-ones matrix. The all-ones vector $\mathbb{1}$ is an eigenvector of A with eigenvalue k. The remaining two eigenvalues e^+ and e^- satisfy the quadratic equation $x^2 = k + \lambda x + \mu(-1 - x)$. Hence $\mu - k = e^+e^-$ and $\lambda - \mu = e^+ + e^-$. (Since A has 0 trace, we deliberately write e^+ and e^- since one eigenvalue must be non-negative and the other is negative).

As mentioned in the introduction, a strongly regular graph comes equipped with its Bose-Mesner algebra, the 3-dimensional matrix algebra generated by A, I and J. Now the Bose-Mesner algebra of a strongly regular graph is a commutative algebra of real symmetric matrices, and so it has an orthogonal decomposition into idempotents. Moreover, there exist so-called *minimal idempotents* E_0, E_1, E_2 such that the product of any two is zero, and such that they add up to the identity matrix. To obtain these matrices, one can take the Gram matrices of the orthogonal projections to the three eigenspaces of A. So for a strongly regular graph with eigenvalues k (the valency), e^+ and e^-, we can take the following minimal idempotents (n.b., n

is the size of A):

$$E_0 = \frac{1}{n}J,$$
$$E_1 = \frac{1}{e^+ - e^-}\left(A - e^- I - \frac{k-e^-}{n}J\right),$$
$$E_2 = \frac{1}{e^- - e^+}\left(A - e^+ I - \frac{k-e^+}{n}J\right).$$

All of the above content is standard in the theory of association schemes and can be found in a textbook such as [**17**].

We say that a proper subset of vertices $\mathcal{I}$ of a strongly regular graph Γ is an *intriguing set* with parameters (h_1, h_2) if there are two constants h_1 and h_2 such that the number of elements of $\mathcal{I}$ adjacent to any vertex of $\mathcal{I}$ is h_1, and the number of elements of $\mathcal{I}$ adjacent to any vertex of $\Gamma \setminus \mathcal{I}$ is h_2. So necessarily, the subgraph induced by $\mathcal{I}$ is regular of valency h_1. We will call h_1 and h_2 the *intersection numbers* of $\mathcal{I}$, and note that we have made a slight difference here in comparison to the definition in [**3**]; our parameter h_1 will always be one less than the analogue in [**3**] due to "adjacency" being an anti-reflexive relation. It turns out (see Lemma 2.1) that $h_1 - h_2$ is an eigenvalue of the adjacency matrix, hence $h_1 \neq h_2$, and so we define $\mathcal{I}$ to be a *positive* or *negative* intriguing set according to whether $h_1 - h_2$ is equal to e^+ or e^-.

From the algebraic graph theoretic point of view, an intriguing set of a strongly regular graph is a set of vertices whose characteristic vector is annihilated by one of the minimal idempotents E_1 or E_2. This simple observation allows us to design algorithms to search for intriguing sets. A characteristic vector of a set of points has values 0 or 1, so an intriguing set corresponds to a set of rows of a minimal idempotent which add to the zero vector. One can reduce the problem by taking the row echelon reduced form of the given minimal idempotent or by using subgroups of the induced permutation group on the points to obtain collapsed matrices with constant row sums.

The following results follow in the same way as in [**2**] (see also [**16**]). We use the notation $\mathbb{1}_{\mathcal{I}}$ for the characteristic vector of $\mathcal{I}$.

Lemma 2.1. *Let $\mathcal{I}$ be an intriguing set of a strongly regular graph Γ, and let the intersection numbers of $\mathcal{I}$ be h_1 and h_2. Let v and k be the number of vertices and the valency of Γ respectively, and let A be the adjacency matrix of Γ. Then:*

(i) $(h_1 - h_2 - k)\mathbb{1}_{\mathcal{I}} + h_2\mathbb{1}$ *is an eigenvector of A with eigenvalue $h_1 - h_2$;*
(ii) $|\mathcal{I}| = h_2 v/(k - h_1 + h_2)$.

Proof. The proof of (i) is just a straight-forward calculation, so we provide the proof for part (ii). Let A be the adjacency matrix of Γ. Since A is a real symmetric matrix, the eigenvector $(h_1 - h_2 - k)\mathbb{1}_{\mathcal{I}} + h_2\mathbb{1}$ is orthogonal to the all-ones vector $\mathbb{1}$ with eigenvalue k. So $((h_1 - h_2 - k)\mathbb{1}_{\mathcal{I}} + h_2\mathbb{1}) \cdot \mathbb{1} = 0$ and hence:

$$-(h_1 - h_2 - k)\mathbb{1}_{\mathcal{I}} \cdot \mathbb{1} = h_2\mathbb{1} \cdot \mathbb{1}$$

from which the conclusion follows. □

Lemma 2.2. *Let $\mathcal{I}^+$ and $\mathcal{I}^-$ be positive and negative intriguing sets respectively of a strongly regular graph Γ and let v be the total number of vertices. Then*

$$|\mathcal{I}^+ \cap \mathcal{I}^-| = |\mathcal{I}^+||\mathcal{I}^-|/v.$$

Proof. Just as in [**2**, Theorem 4], we use the fact that the eigenvectors corresponding to $\mathcal{I}^+$ and $\mathcal{I}^-$ (see Lemma 2.1) are orthogonal from which the result easily follows. □

One can obtain new intriguing sets by taking unions of disjoint intriguing sets of the same type. Moreover, the complement of an intriguing set is also intriguing, and of the same type. These observations will be important in the study of intriguing sets of strongly regular graphs. The proof of the next lemma is easy and is left to the reader.

LEMMA 2.3. *Suppose we have a strongly regular graph Γ and let $\mathcal{A}$ and $\mathcal{B}$ be two intriguing sets of the same type, that is, they give rise to eigenvectors with the same eigenvalue. Then:*

(a) *If $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{B} \setminus \mathcal{A}$ is an intriguing set of the same type as $\mathcal{B}$ and $\mathcal{A}$;*
(b) *If $\mathcal{A}$ and $\mathcal{B}$ are disjoint, then $\mathcal{A} \cup \mathcal{B}$ is an intriguing set of the same type as $\mathcal{A}$ and $\mathcal{B}$;*
(c) *The complement $\mathcal{A}'$ of $\mathcal{A}$ in Γ is an intriguing set of the same type as $\mathcal{A}$.*

Below we give a simple example of how to determine (by hand) the intriguing sets of the Petersen graph.

2.2. Example: Intriguing sets of the Petersen graph. The two minimal idempotents we will consider of the Petersen graph are:

$$E_1 = \frac{1}{6}\begin{bmatrix} 3 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 3 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 3 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 3 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 3 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 3 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 3 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & 3 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 3 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 3 \end{bmatrix},$$

$$E_2 = \frac{1}{15}\begin{bmatrix} 6 & -4 & 1 & 1 & -4 & -4 & 1 & 1 & 1 & 1 \\ -4 & 6 & -4 & 1 & 1 & 1 & 1 & -4 & 1 & 1 \\ 1 & -4 & 6 & -4 & 1 & 1 & 1 & 1 & 1 & -4 \\ 1 & 1 & -4 & 6 & -4 & 1 & -4 & 1 & 1 & 1 \\ -4 & 1 & 1 & -4 & 6 & 1 & 1 & 1 & -4 & 1 \\ -4 & 1 & 1 & 1 & 1 & 6 & -4 & 1 & 1 & -4 \\ 1 & 1 & 1 & -4 & 1 & -4 & 6 & -4 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 & 1 & -4 & 6 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 & 1 & 1 & -4 & 6 & -4 \\ 1 & 1 & -4 & 1 & 1 & -4 & 1 & 1 & -4 & 6 \end{bmatrix}.$$

To obtain the intriguing sets, we first look for rows of E_1 which add to the zero vector. We will identify the vertices of the Petersen graph, and hence the rows of E_1, with $\{1, 2, \ldots, 10\}$. Since the Petersen graph is vertex transitive, we may suppose without loss of generality that 1 is in our putative intriguing set $\mathcal{I}$. It turns out that the stabiliser of 1 in the automorphism group of the Petersen graph has as orbits $\{2, 5, 6\}$ and $\{3, 4, 7, 8, 9, 10\}$. We can see this by looking at the values in the first column of E_1. So we may suppose without loss of generality that $3 \in \mathcal{I}$. So far, our two rows of $\mathcal{I}$ add to $\frac{1}{6}(2, 2, 2, 0, 0, 0, -2, -2, -2, 0)$. It turns out that $\mathcal{I}$ can only be one of $\{1, 3, 4, 6, 8, 9\}$, $\{1, 3, 5, 7, 8, 10\}$ or $\{1, 3, 7, 9\}$; and we can exclude the first two since the collection of intriguing sets is closed under complements and hence we can regard only those of size at most 5. The set $\{1, 3, 7, 9\}$ corresponds to a 4-coclique of the Petersen graph. In fact, there are in total five 4-cocliques of the Petersen graph.

For the second minimal idempotent E_2, we similarly assume that 1 and 2 are contained in our putative intriguing set $\mathcal{I}$. The sum of the first two rows of E_2 is $\frac{1}{15}(2, 2, -3, 2, -3, -3, 2, -3, 2, 2)$, and in order to cancel this vector, we must complete $\mathcal{I}$ to one of $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 6, 10\}$, $\{1, 2, 5, 8, 9\}$, or $\{1, 2, 6, 7, 8\}$. It then follows that the twelve pentagons (5-cycles) are intriguing sets of the Petersen graph.

2.3. Intriguing sets of partial quadrangles, the basics. Let $\mathcal{P}$ be a point-line incidence structure whose point graph is strongly regular. Then a set of points of $\mathcal{P}$ is an *intriguing set* if it corresponds to an intriguing set of the point graph. We will use the symbol $\perp$ to denote the collinearity relation on points, so $P^\perp$ will denote the set of all points collinear to P. However, we will also extend the graph theoretic notion of adjacency to geometries by writing $P^\sim$ to mean the set of all points collinear **but not equal to** P; that is, the *neighbours* of P. (Thus our point graphs have no loops, and our adjacency matrices will have 0's on the diagonal).

Let $\mathcal{G}$ be a generalised quadrangle of order (s, t). The point graph of $\mathcal{G}$ is strongly regular with parameters:

$$v = (s+1)(st+1), \quad k = s(t+1), \quad \lambda = s - 1, \quad \mu = t + 1,$$

and hence has three eigenvalues, one of which is the valency k. The other two eigenvalues, commonly known as the *principal eigenvalues*, are $s-1$ and $-t-1$. The eigenvalues of the point graph of a partial quadrangle with parameters (s,t,μ) are accordingly:

the valency	$s(t+1)$
positive	$e^+ := (-\mu - 1 + s + \sqrt{(\mu-1-s)^2+4st})/2$
negative	$e^- := (-\mu - 1 + s - \sqrt{(\mu-1-s)^2+4st})/2.$

From the above definition, a nonempty subset of points $\mathcal{I}$ of a partial quadrangle $\mathsf{PQ}(s,t,\mu)$ is intriguing if there are two constants h_1 and h_2 such that

$$|P^{\sim} \cap \mathcal{I}| = \begin{cases} h_1 & \text{if } P \in \mathcal{I}, \\ h_2 & \text{otherwise} \end{cases}$$

where P runs over the points of the partial quadrangle. In other words, if A is the adjacency matrix of the point graph, then $\mathcal{I}$ is intriguing if and only if its characteristic function $\mathbb{1}_{\mathcal{I}}$ satisfies the following relation:

$$A\mathbb{1}_{\mathcal{I}} = (h_1 - h_2)\mathbb{1}_{\mathcal{I}} + h_2\mathbb{1}$$

where $\mathbb{1}$ is the "all 1's" map. Recall from Lemma 2.1 that $h_1 - h_2$ is an eigenvalue of A. So a *positive* intriguing set has $h_1 - h_2 = e^+$ and a *negative* intriguing set has $h_1 - h_2 = e^-$. The number of points of the partial quadrangle is

$$\frac{s(t+1)(\mu+st)}{\mu} + 1.$$

3. Intriguing sets of the known thin partial quadrangles

A *thin* partial quadrangle is simply a triangle-free strongly regular graph. There are only seven known such graphs (see [**10**, Chapter 8]) and we explore and classify below the intriguing sets of these geometries, for which many of the well-known interesting regular subgraphs of these graphs predominate. Firstly, it is not difficult to see that the pentagon contains no intriguing sets. The Petersen graph was dealt with in Section 2, and so it remains to consider the Clebsch, Hoffman-Singleton, Gewirtz, M_{22} and Higman-Sims graphs.

The Clebsch graph on 16 vertices. In the Clebsch graph on 16 vertices, the only negative intriguing sets are the ten subgraphs isomorphic to $4K_2$, each stabilised by a group of order 192; which are maximal subgroups of the full group $2^4 : S_5$. As for positive intriguing sets, the only examples are the forty C_4's, a disjoint pair of C_4's, and complements of these. The Clebsch graph is small enough that we can give a simple computer-free proof for the negative intriguing sets. Here is a commonly used construction of the Clebsch graph. We have a special vertex ∞, a set of five vertices $V_1 = \{1,2,3,4,5\}$ and the subsets of V_1 of size two, which we denote $V_2 = \{12,13,14,15,23,24,25,34,35,45\}$. The vertex ∞ is adjacent to all the members of V_1, the set V_1 is a coclique and V_2 forms a Petersen graph whereby two elements are adjacent if they are disjoint. A vertex i in V_1 is adjacent to those vertices in V_2 whose label contains i as one of its coordinates (e.g., 3 is adjacent to 23).

Let $\mathcal{I}$ be a negative intriguing set of the Clebsch graph with parameters (h_1,h_2). Since the Clebsch graph is vertex transitive, we may suppose that $\infty \in \mathcal{I}$. Moreover, the stabiliser of ∞ has V_1 and V_2 as two of its orbits, so we may also suppose without loss of generality that $1 \in \mathcal{I}$. Since V_1 is a coclique, there are no further elements of V_1 inside $\mathcal{I}$, and we know now that $h_1 = 1$. In fact, $\mathcal{I}$ must be a union of edges and have size 8, as $h_2 = 4$. No element of V_2 adjacent to 1 can be in $\mathcal{I}$, so we can consider 12 and 15 as external elements. For there to be 4 elements in $\mathcal{I}$ adjacent to 12 (resp. 15), we must have that 34 and 45 are in $\mathcal{I}$. The only edges of V_2 with no vertex adjacent to 1 are $\{34,25\}$, $\{45,23\}$ and $\{35,24\}$. By considering 15, we see that all of these edges must also be inside $\mathcal{I}$ and so it follows that $\mathcal{I} = \{\infty,1,34,25,45,23,35,24\}$. Hence the only negative intriguing sets are the ten subgraphs isomorphic to $4K_2$. Alternatively, we can look to the minimal idempotent E which annihilates $\mathbb{1}_{\mathcal{I}}$:

$$-\frac{1}{8}\begin{bmatrix}
-5 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1\\
1 & -5 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1\\
1 & 1 & -5 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1\\
1 & 1 & 1 & -5 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1\\
1 & 1 & 1 & -1 & -5 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1\\
-1 & 1 & 1 & 1 & 1 & -5 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1\\
1 & 1 & -1 & 1 & 1 & 1 & -5 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1\\
1 & -1 & 1 & 1 & 1 & 1 & 1 & -5 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1\\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -5 & 1 & 1 & 1 & 1 & 1 & -1 & 1\\
-1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -5 & 1 & 1 & 1 & 1 & 1 & -1\\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -5 & 1 & 1 & -1 & 1 & 1\\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -5 & -1 & 1 & 1 & 1\\
-1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -5 & 1 & 1 & 1\\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -5 & 1 & 1\\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -5 & 1\\
1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -5
\end{bmatrix}.$$

The points ∞ and 1 in the above argument correspond to the first and sixth rows above, which add to $-\frac{1}{8}(-6, 2, 2, 2, 2, -6, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0)$. The only way to cancel this vector out by adding other rows of E, is to use all remaining rows or just the remainder from the first eight rows.

The Hoffman-Singleton graph on 50 vertices. The intriguing sets of the Hoffman-Singleton graph (on 50 vertices) correspond naturally to the maximal subgroups of its automorphism group $\mathsf{PSU}(3,5).2$. For the negative intriguing sets, we have one-hundred 15-cocliques (stabilised by A_7), the 252 subgraphs isomorphic to $5C_5$ (each stabilised by a $5_+^{1+2} : 8 : 2$) and pairs of disjoint 15-cocliques (each stabilised by an M_{10}). The positive intriguing sets are also very interesting: the 525 Petersen subgraphs (stabiliser: $2S_5.2$), a pair of disjoint Petersen subgraphs (stabiliser: D_{20}) and three disjoint Petersen subgraphs (stabiliser: $\mathsf{GL}(2,3) : 2$). The remaining intriguing sets are complements of those above, and by computer, these are fully classified.

The Gewirtz graph on 56 vertices. By computer, the only negative intriguing sets of the Gewirtz graph (on 56 vertices) are the forty-two 16-cocliques, the 105 subgraphs isomorphic to $6C_4$, the 480 Coxeter subgraphs (the graph on the antiflags of the Fano plane), the 112 Sylvester subgraphs (the complement is $10K_2$) and complements of these. The only positive intriguing sets are isomorphic to the six regular subgraphs on 14 vertices shown below, and those of greater size obtained from a union of disjoint subgraphs or the complement of such a subgraph. The six different types of positive intriguing sets of size 14 form single orbits under the automorphism group of the Gewirtz graph.

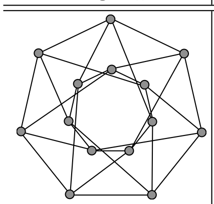
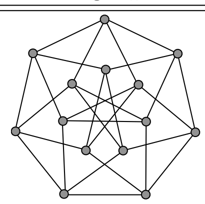
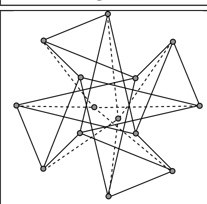
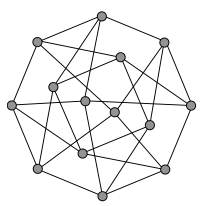
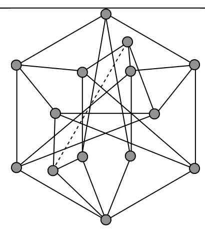
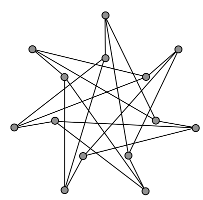
Subgraph	Aut. group	Subgraph	Aut. group	Subgraph	Aut. group
	D_{28}		D_{14}		$C_2 \times S_4$
	D_8		D_{12}		$\mathsf{PSL}(3,2) : C_2$

TABLE 1. The positive intriguing sets of size 14 in the Gewirtz graph. The first is a circulant and the last is the co-Heawood graph.

The Higman-Sims M_{22}-graph on 77 vertices. Two of the natural subgraphs of the M_{22}-graph are the 21-cocliques and the odd graphs O_4. These, and their complements, are the only negative intriguing sets of the M_{22}-graph. A full classification of the positive intriguing sets of the M_{22}-graph was not possible by computer. However, we do have complete information of the positive intriguing sets which admit a nontrivial automorphism group. There are two interesting positive intriguing sets which generate all the known examples. The first is a particular regular subgraph on 11 vertices (see the figure below) and the second is the incidence graph of the complement of a biplane on 11 points (i.e., 22 vertices). There exist disjoint triples of subgraphs of the first kind, and there exist disjoint pairs consisting of one of each type of subgraph.

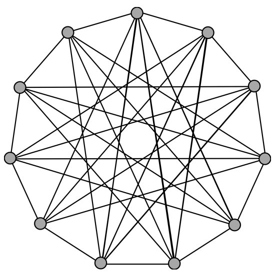


Figure 1. Circulant on 11 vertices.

The Higman-Sims graph on 100 vertices. The only negative intriguing sets of the Higman-Sims graph are the 704 Hoffman-Singleton subgraphs. The known positive intriguing sets are as follows: (i) a tetravalent circulant on 10-vertices (see the figure below), (ii) the graph which Brouwer [**5**] calls $BD(K_5)$ (which is $K_{5,5}$ minus a matching), (iii) bipartite on 20 vertices, (iv) point-plane non-incidence graph of $\mathsf{PG}(3,2)$ (30 vertices), (v) 2-coclique extension of the Petersen graph (20 vertices), (vi) a regular subgraph on 40 vertices which Brouwer [**5**] denotes "a pair of splits from the same family", a union of up to three disjoint subgraphs of type (i), and a union of up to three disjoint subgraphs of type (ii). The positive intriguing sets admitting a group that does not have order a power of 2, have been classified.

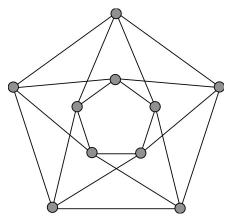


Figure 2. Tetravalent circulant on 10 vertices.

4. Intriguing sets of generalised quadrangles and their interaction with embedded partial quadrangles

Before we embark on an investigation into intriguing sets of partial quadrangles which arise from point sets of generalised quadrangles, it will be necessary to revise before-hand some of what we know about intriguing sets of generalised quadrangles. As was mentioned in the introduction, an intriguing set of a generalised quadrangle is either an m-ovoid or an i-tight set. An m-ovoid is a set of points such that every line meets it in m points, and it is a *negative intriguing set* of the generalised quadrangle; that is, the difference $h_1 - h_2$ of its intersection numbers $h_1 = m(t+1) - t - 1$ and $h_2 = m(t+1)$ is negative (where $t+1$ is the number of lines on a point). An i-tight set $\mathcal{T}$ is a set of points of a generalised quadrangle $\mathcal{P}$ (of order (s,t)) such that the average number of points of $\mathcal{T}$ collinear with a given point of $\mathcal{P}$ equals the maximum possible value, namely $i+s$. A set of points is *tight* if it is i-tight for some $i \geqslant 1$. The two intersection numbers here are $h_1 = i + s - 1$ and $h_2 = i$, and so their difference $h_1 - h_2$ is positive.

As the name suggests, m-ovoids are generalisations of *ovoids*. An ovoid of a generalised quadrangle is a set of points which partitions the lines, that is, a 1-ovoid. The simplest tight sets are the 1-tight sets, which one can prove are the lines of a generalised quadrangle [**20**]. Hence the point set covered by a partial spread (a set of disjoint lines) is a ubiquitous example of a tight set of points. For more information on intriguing sets of generalised quadrangles, we refer the reader to [**3**].

The partial quadrangles that we study in the following two sections are subsets of points of generalised quadrangles, and hence, we will make use of the following notion of "intriguing at infinity".

Definition 4.1 (Intriguing at infinity). Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) and let ∞ be a set of points of $\mathcal{G}$ such that $\mathcal{G} \setminus \infty$ is a partial quadrangle. Then a set of points $\mathcal{I}$ of $\mathcal{G}$ is said to be *intriguing at infinity* (with respect to ∞) if there are two constants a_1 and a_2 such that

$$|Y^\perp \cap \mathcal{I} \cap \infty| = \begin{cases} a_1 & Y \in \mathcal{I} \setminus \infty \\ a_2 & Y \notin \mathcal{I} \cup \infty. \end{cases}$$

Theorem 4.2. *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) and let ∞ be a set of points of $\mathcal{G}$ such that $\mathcal{G} \setminus \infty$ is a partial quadrangle. Let $\mathcal{I}$ be an intriguing set of $\mathcal{G}$ with parameters (h_1, h_2). Then $\mathcal{I} \setminus \infty$ is an intriguing set of the partial quadrangle $\mathcal{G} \setminus \infty$ if and only if $\mathcal{I}$ is intriguing at infinity.*

Proof. Let A be the adjacency matrix of the point graph of $\mathcal{G}$ and let B be the adjacency matrix for the point graph of $\mathcal{G} \setminus \infty$. Let S be the matrix whose rows are indexed by the points of $\mathcal{G}$, and whose columns are indexed by the points of $\mathcal{G} \setminus \infty$, such that the (i, j)-th entry of S is equal to 1 if the i-th point of $\mathcal{G}$ is equal to the j-th point of $\mathcal{G} \setminus \infty$, and 0 otherwise. Then

$$S^T AS = B \text{ and } S^T S = I.$$

When we write $\mathbb{1}_{\mathcal{H}}^{\mathsf{PQ}}$ we mean the function $\mathbb{1}_{\mathcal{H}}$ restricted to the partial quadrangle. By supposition, we have that

$$A\mathbb{1}_{\mathcal{I}} = (h_1 - h_2)\mathbb{1}_{\mathcal{I}} + h_2\mathbb{1}.$$

Denote by ∞' the complement of ∞. Note that $\mathcal{I}$ is intriguing at infinity if and only if there exist non-negative integers a_1 and a_2 such that

$$S^T A\mathbb{1}_{\mathcal{I}\cap\infty} = (a_1 - a_2)\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} + a_2\mathbb{1}_{\infty'}^{\mathsf{PQ}}.$$

On the other hand, $\mathcal{I} \setminus \infty$ is intriguing in the partial quadrangle if and only if there exist non-negative integers h_1' and h_2' such that

$$B\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} = (h_1' - h_2')\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} + h_2'\mathbb{1}_{\infty'}^{\mathsf{PQ}}.$$

Now $A\mathbb{1}_{\mathcal{I}\cap\infty} = A(\mathbb{1}_{\mathcal{I}} - \mathbb{1}_{\mathcal{I}\cap\infty'}) = (h_1 - h_2)\mathbb{1}_{\mathcal{I}} + h_2\mathbb{1} - A\mathbb{1}_{\mathcal{I}\setminus\infty}$ and so $\mathcal{I}$ is intriguing at infinity if and only if there exist non-negative integers a_1 and a_2 such that

$$(a_1 - a_2)\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} + a_2\mathbb{1}_{\infty'}^{\mathsf{PQ}} = (h_1 - h_2)\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} + h_2\mathbb{1}_{\infty'}^{\mathsf{PQ}} - S^T A\mathbb{1}_{\mathcal{I}\setminus\infty}.$$

When we rearrange this equation, we obtain

$$S^T A\mathbb{1}_{\mathcal{I}\setminus\infty} = ((h_1 - a_1) - (h_2 - a_2))\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} + (h_2 - a_2)\mathbb{1}_{\infty'}^{\mathsf{PQ}}$$

which is equivalent to $B\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} = ((h_1 - a_1) - (h_2 - a_2))\mathbb{1}_{\mathcal{I}\setminus\infty}^{\mathsf{PQ}} + (h_2 - a_2)\mathbb{1}_{\infty'}^{\mathsf{PQ}}$. Therefore $\mathcal{I} \setminus \infty$ is an intriguing set of the partial quadrangle $\mathcal{G} \setminus \infty$ if and only if $\mathcal{I}$ is intriguing at infinity. □

5. Partial quadrangles obtained by removing a point from a generalised quadrangle

Let $\mathcal{G}$ be a generalised quadrangle of order (s,t) and let P be a point of $\mathcal{G}$. Then the derived geometry with

POINTS	the points of $\mathcal{G}$ not collinear to P
LINES	the lines of $\mathcal{G}$ not incident with P.

is a $(0,1)$-*geometry*, that is, for every point P and line ℓ which are not incident in this geometry, there is at most one point on ℓ collinear with P. The point graph of this geometry will be strongly regular if and only if there is a constant c such that for any two noncollinear points X and Y of $\mathcal{G}$, not in $P^\perp$, there are c points of $\mathcal{G}$ which are collinear with all three points X, Y and P. This property occurs when and only when the parameter t is equal to s^2 (see [**4**] or [**7**]), in which case $c = s+1$, and then we obtain a partial quadrangle with parameters $(s-1, s^2, s(s-1))$. In the following lemma, we summarise the algebraic data needed to work with these kinds of partial quadrangles.

LEMMA 5.1. *Let $\mathcal{G}$ be a generalised quadrangle of order (s,s^2), let P be a point of $\mathcal{G}$, and let $\mathcal{I}$ be an intriguing set of the partial quadrangle $\mathcal{G}\setminus P^\perp$ with intersection numbers (h_1', h_2'). Then we have the following information:*

Case	*Associated eigenvalue*	*Size*
Negative intriguing set	$-s^2+s-1$	$h_2's$
Positive intriguing set	$s-1$	$h_2's^2/(s-1)$
Point set	$(s-1)(s^2+1)$	s^4

TABLE 2. Eigenvalues and sizes of intriguing sets of $\mathcal{G}\setminus P^\perp$.

THEOREM 5.2. *Let $\mathcal{G}$ be a generalised quadrangle of order (s,s^2) and let $\infty = P^\perp$ where P is a point of $\mathcal{G}$. Let $\mathcal{I}$ be an intriguing set of $\mathcal{G}$ with parameters (h_1,h_2) and which is intriguing at infinity with parameters (a_1,a_2). Then $\mathcal{I}\setminus\infty$ is an intriguing set with the same parity as $\mathcal{I}$ and we have the following possibilities for (a_1,a_2):*

Parity	*Case*	a_1	a_2	$\|\mathcal{I}\cap P^\perp\|$
m-ovoid	$P\notin\mathcal{I}$	$m(s+1)-s$	$m(s+1)$	$m(s^2+1)$
	$P\in\mathcal{I}$	$m(s+1)-2s$	$m(s+1)-s$	$m(s^2+1)-s^2$
i-tight set	$P\notin\mathcal{I}$	i/s	i/s	i
	$P\in\mathcal{I}$	$(i-1)/s+1$	$(i-1)/s+1$	$i+s$

TABLE 3. Possibilities for (a_1,a_2).

PROOF. Recall that the negative and positive eigenvalues for $\mathcal{G}$ are $-s^2-1$ and $s-1$, whilst they are $-s^2+s-1$ and $s-1$ for $\mathcal{G}\setminus P^\perp$. However, we must have that h_1-h_2 and $(h_1-a_1)-(h_2-a_2)$ are eigenvalues for the respective geometries. In Table 4, we outline the possibilities for these values depending on the four possible cases. We use the notation "$-\to+$" (for example) to denote the case that $\mathcal{I}$ is negative intriguing and $\mathcal{I}\setminus\infty$ is positive intriguing.

	h_1-h_2	$(h_1-a_1)-(h_2-a_2)$	a_1-a_2
$-\to-$	$-s^2-1$	$-s^2+s-1$	$-s$
$-\to+$	$-s^2-1$	$s-1$	$-s^2-s$
$+\to-$	$s-1$	$-s^2+s-1$	s^2
$+\to+$	$s-1$	$s-1$	0

TABLE 4. Eigenvalues for the four possible cases.

Since $a_1, a_2 \leqslant s^2+1$, we can rule out immediately the second case above. Moreover, since $|Y^\perp \cap P^\perp \cap Z^\perp| = s+1$ for any three pairwise non-collinear points Y, P, Z, and since there exists a point $Y \in \mathcal{I} \setminus \infty$ and a point $Z \in (\mathcal{G} \setminus \infty) \setminus \mathcal{I}$, it follows that $a_1 - a_2 \leqslant s^2 - s$. So the third case in the above list is ruled out too. Hence parity is preserved. Now we see what happens at infinity. Recall from Lemma 2.1 that if $\mathcal{I}$ has associated eigenvalue e and $\mathcal{I} \setminus P^\perp$ has associated eigenvalue e' (in the partial quadrangle), then

$$|\mathcal{I}| = \frac{h_2}{s(s^2+1)-e}(s+1)(s^3+1) \quad \text{and} \quad |\mathcal{I} \setminus P^\perp| = \frac{h_2 - a_2}{(s-1)(s^2+1)-e'}s^4.$$

Since $\mathcal{I}$ is intriguing we have

$$|P^\sim \cap \mathcal{I}| = \begin{cases} h_2 + e & P \in \mathcal{I} \\ h_2 & P \notin \mathcal{I}. \end{cases}$$

Negative case: In the first case of Table 4, s^2+1 divides h_2, and

$$|\mathcal{I} \cap P^\perp| = |\mathcal{I}| - |\mathcal{I} \setminus P^\perp| = \frac{h_2(s^3+1)}{s^2+1} - (h_2 - a_2)s = a_2 s - \frac{h_2(s-1)}{s^2+1}.$$

As we know that $\mathcal{I}$ is an m-ovoid (for some m), $h_2 = m(s^2+1)$ and we have the following values:

Case	a_1	a_2	$\lvert\mathcal{I} \cap P^\perp\rvert$
$P \notin \mathcal{I}$	$m(s+1)-s$	$m(s+1)$	$m(s^2+1)$
$P \in \mathcal{I}$	$m(s+1)-2s$	$m(s+1)-s$	$m(s^2+1)-s^2$

Positive case: In the last case of Table 4, we have

$$|\mathcal{I} \cap P^\perp| = |\mathcal{I}| - |\mathcal{I} \setminus P^\perp| = h_2(s+1) - \frac{h_2 - a_2}{s-1}s^2 = \frac{a_2 s^2 - h_2}{s-1}.$$

As we know that $\mathcal{I}$ is an i-tight set (for some i), $h_2 = i$ and we have the following values:

Case	a_1	a_2	$\lvert\mathcal{I} \cap P^\perp\rvert$
$P \notin \mathcal{I}$	i/s	i/s	i
$P \in \mathcal{I}$	$(i-1)/s+1$	$(i-1)/s+1$	$i+s$

□

5.1. Positive intriguing sets. We now characterise the positive intriguing sets of a partial quadrangle obtained from removing the perp of a point of a generalised quadrangle $\mathcal{G}$, which are induced from intriguing sets of $\mathcal{G}$.

THEOREM 5.3 (Positive Intriguing $\longleftrightarrow$ Lines at Infinity). *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) and let $\infty = P^\perp$ where P is a point of $\mathcal{G}$. Let $\mathcal{I}$ be a positive intriguing set of $\mathcal{G}$, and let $y = (i-1)/s + 1$ if $P \in \mathcal{I}$, or let $y = i/s$ if $P \notin \mathcal{I}$. Then $\mathcal{I} \setminus \infty$ is an intriguing set of $\mathcal{G} \setminus \infty$ if and only if $\mathcal{I} \cap \infty$ consists of y lines through P.*

PROOF. First suppose that $\mathcal{I}$ is a positive intriguing set of $\mathcal{G}$. Then $|\mathcal{I}| = (s+1)i$, for some i. Assume that $\mathcal{I}$ intersects ∞ in y lines through P. Then we have

$$y = \begin{cases} i/s & P \notin \mathcal{I} \\ (i-1)/s + 1 & P \in \mathcal{I} \end{cases}$$

and it follows that $\mathcal{I}$ is intriguing at infinity with parameter y, and hence by Lemma 4.2, $\mathcal{I} \setminus \infty$ is an intriguing set of $\mathcal{G} \setminus \infty$.

Conversely, let $\mathcal{I}$ be a positive intriguing set of $\mathcal{G}$ and suppose that $\mathcal{I}$ is intriguing at infinity with parameters (a_1, a_2). By Lemma 4.2, we have that $a_1 = a_2 = y$ with

$$y = \begin{cases} i/s & P \notin \mathcal{I} \\ (i-1)/s + 1 & P \in \mathcal{I}. \end{cases}$$

By counting pairs $(Y,(Z,Z'))$ with $Y \in \mathcal{I} \setminus \infty$ and $Z, Z' \in \mathcal{I} \cap \infty$ with $Y \sim Z$, $Y \sim Z'$ and $Z \not\sim Z'$, we have

$$s^4 y(y-1)/2 = s^2 x$$

where x denotes the number of pairs (Z,Z') as above. Hence $x = s^2 y(y-1)/2$. From Lemma 4.2, we also know that the equation $|\mathcal{I} \cap \infty \setminus \{P\}| = ys$ is independent of whether $P \in \mathcal{I}$ or $P \notin \mathcal{I}$. Finally, it is easy to see that a set of ys points in $P^\perp \setminus \{P\}$ has the minimum number $y(y-1)/2s^2$ of non-collinear pairs (or the maximum number $ys(s-1)/2$ of collinear pairs) if and only if it consists of y lines through P. □

Remark 5.4 ($\mathsf{Q}(4,q) \longrightarrow$ Positive Intriguing). Let $\mathcal{G}$ be the generalised quadrangle $\mathsf{Q}^-(5,q)$ and let P be a point of $\mathcal{G}$. Consider a $\mathsf{Q}(4,q)$ embedded in $\mathsf{Q}^-(5,q)$. Then $\mathsf{Q}(4,q) \setminus P^\perp$ is a positive intriguing set of $\mathcal{G} \setminus P^\perp$ if and only if $\mathsf{Q}(4,q) \cap P^\perp$ is a tangent hyperplane to $\mathsf{Q}(4,q)$.

5.2. Negative intriguing sets. Segre [**21**] proved that if an m-ovoid of $\mathsf{Q}^-(5,q)$ exists, then $m = (q+1)/2$; that is, it is a hemisystem. Thas [**22**] extended this result to all generalised quadrangles of order (s,s^2), s odd. We have an alternative proof of Thas for when the m-ovoid has a particular property. Suppose $\mathcal{P}$ is a partial quadrangle obtained from removing the perp of a point from $\mathcal{G}$. If an m-ovoid meets $\mathcal{P}$ in a negative intriguing set, then it follows from the theorem below that $m = (s+1)/2$.

Theorem 5.5 (Negative Intriguing $\longrightarrow$ Hemisystems). *Let $\mathcal{G}$ be a generalised quadrangle of order (s,s^2), and let $\infty = P^\perp$ where P is a point of $\mathcal{G}$. If $\mathcal{I}$ is an m-ovoid of $\mathcal{G}$ and $\mathcal{I} \setminus \infty$ is an intriguing set of $\mathcal{G} \setminus \infty$, then s is odd and $\mathcal{I}$ is a hemisystem ($m = (s+1)/2$) of $\mathcal{G}$.*

Proof. Let $\mathcal{I}$ be an m-ovoid of $\mathcal{G}$, with $0 < m < s+1$, and suppose that $\mathcal{I}$ is a negative intriguing set of $\mathcal{G} \setminus \infty$. Then by Lemma 4.2 and Theorem 5.2, $\mathcal{I}$ is intriguing at infinity with parameters (a_1, a_2), where

$$a_1 = \begin{cases} m(s+1) - s & P \notin \mathcal{I} \\ m(s+1) - 2s & P \in \mathcal{I} \end{cases}$$

and

$$a_2 = \begin{cases} m(s+1) & P \notin \mathcal{I} \\ m(s+1) - s & P \in \mathcal{I} \end{cases}.$$

Moreover

$$|\mathcal{I} \cap \infty| = \begin{cases} m(s^2+1) & P \notin \mathcal{I} \\ m(s^2+1) - s^2 & P \in \mathcal{I} \end{cases}.$$

First assume $P \in \mathcal{I}$, then $|\mathcal{I} \setminus \infty| = ms^3 - ms^2 + s^2$. Counting pairs $(Y,(Z,Z'))$ with $Y \in \mathcal{I} \setminus \infty$ and $Z, Z' \in \mathcal{I} \cap \infty$ with $Y \sim Z$, $Y \sim Z'$ and $Z \not\sim Z'$ we have

$$a_1(a_1-1)(ms^3 - ms^2 + s^2)/2 + a_2(a_2-1)(s^4 - ms^3 + ms^2 - s^2)/2 = s^2 x$$

where x denotes the number of pairs (Z,Z') as above. Hence $x = a_1(a_1-1)(ms - m + 1)/2 + a_2(a_2-1)(s^2 - ms + m - 1)/2$. On the other hand, since $\mathcal{I}$ is an m-ovoid of $\mathcal{G}$ and $P \in \mathcal{I}$ we also know that $x = (m-1)^2(s^2+1)s^2/2$. According to Table 4 we have that $a_1 = m(s+1) - 2s$ and $a_2 = a_1 + s$, in this case. Comparing the two values of x obtained, we have

$$2m^2 - 3(s+1)m + (s+1)^2 = 0$$

from which it follows that $m = (s+1)/2$.

Next assume that $P \notin \mathcal{I}$, then $|\mathcal{I} \setminus \infty| = ms^3 - ms^2$. Counting pairs $(Y,(Z,Z'))$ with $Y \in \mathcal{I} \setminus \infty$ and $Z, Z' \in \mathcal{I} \cap \infty$ with $Y \sim Z$, $Y \sim Z'$ and $Z \not\sim Z'$ we have

$$a_1(a_1-1)(ms^3 - ms^2)/2 + a_2(a_2-1)(s^4 - ms^3 + ms^2)/2 = s^2 x$$

where x denotes the number of pairs (Z,Z') as above. Hence $x = a_1(a_1-1)(ms-m)/2 + a_2(a_2-1)(s^2 - ms + m)/2$. On the other hand, since $\mathcal{I}$ is an m-ovoid of $\mathcal{G}$ and $P \notin \mathcal{I}$ we also know that $x = m^2(s^2+1)s^2/2$. Comparing the two values of x obtained, we have

$$m(2m - (s+1)) = 0$$

from which it follows that $m = (s+1)/2$. □

Moreover, for hemisystems we have

LEMMA 5.6 (Hemisystem ⟶ Negative Intriguing). *Let $\mathcal{I}$ be a hemisystem of a generalised quadrangle $\mathcal{G}$ of order (s, s^2), s odd. Let P be a point of $\mathcal{G}$. Then $\mathcal{I} \setminus P^\perp$ is a negative intriguing set of $\mathcal{G} \setminus P^\perp$ if and only if $|Y^\perp \cap \mathcal{I} \cap P^\perp|$ is a constant over all P and Y not both in $\mathcal{I}$.*

PROOF. Follows from Theorem 5.2. □

Open question: For every m-ovoid $\mathcal{O}$ of a generalised quadrangle $\mathcal{G}$ of order (s, s^2), does there exist a point P such that $\mathcal{O} \setminus P^\perp$ is an intriguing set of the associated partial quadrangle? (Compare with Theorem 5.5).

LEMMA 5.7 (Cone⟶ Negative Intriguing). *Let $\mathcal{G}$ be the generalised quadrangle of order (s, s^2) and let P be a point of $\mathcal{G}$. For every point $Z \in P^\perp$, the set of points $Z^\perp \setminus P^\perp$ is a negative intriguing set of $\mathcal{G} \setminus P^\perp$ with parameters $(s-1, s^2)$.*

PROOF. Let Z be a point of $P^\perp$ and let $\mathcal{I}$ be the set of points of $\mathcal{G} \setminus P^\perp$ contained in $Z^\perp$. (Clearly if $Z = P$ we get the empty set, so assume that $Z \neq P$). Let X be a point in $\mathcal{I}$. Then the only points of $Z^\perp$ collinear with X lie on the line XZ. Moreover, every point but Z on this line is not in $P^\perp$. So there are $s-1$ points collinear with X (and not equal to X) in $\mathcal{I}$. Now let Y be a point not in $\mathcal{I}$, but in $\mathcal{G} \setminus P^\perp$. Now Y is not collinear to Z, and we want to know how many points of $\mathcal{G}$ are collinear with both Y and Z, but not on the line ZP. This number is $\mu - 1 = s^2$. Therefore, $\mathcal{I}$ is a negative intriguing set of $\mathcal{G} \setminus P^\perp$ with parameters $(s-1, s^2)$. □

So from Lemma 5.6 and Lemma 5.7, we have two ways to obtain negative intriguing sets of a partial quadrangle which is a generalised quadrangle minus the perp of a point: namely, from unions of disjoint cones, and from hemisystems. We conjecture that these are the only two possible types of negative intriguing sets.

CONJECTURE 5.8. *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) and let P be a point of $\mathcal{G}$. If $\mathcal{I}$ is a negative intriguing set of the partial quadrangle $\mathcal{G} \setminus P^\perp$, then either:*

(i) *There exist points $Z_1, \ldots, Z_n$ of $P^\perp$ such that $\mathcal{I} = \bigcup_{i=1}^n \left(Z_i^\perp \setminus P^\perp\right)$, or*
(ii) *There exists a hemisystem $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{I} = \mathcal{H} \setminus P^\perp$.*

We are able to provide a partial answer to the above conjecture via the results remaining in this section. A proof of this conjecture would be a significant step in improving Thas' result that an m-ovoid of a generalised quadrangle of order (s, s^2), with s odd, is a hemisystem (see [**22**, Corollary 2]). Conjecture 5.8 aims to extend Theorem 5.5 to *weighted* m-ovoids. However, it seems this problem is very difficult.

Let A be the adjacency matrix of a strongly regular graph Γ. Let S be a set of points equipped with integral weights. Then the characteristic vector $\mathbb{1}_S$ is just a vector with integer entries. We say that S is a *weighted intriguing set* if there exist integers a and b such that $A\mathbb{1}_S = a\mathbb{1}_S + b\mathbb{1}$. Recall that there are three eigenspaces of A, namely V^0, V^+, V^-, where V^0 is one-dimensional and V^+ and V^- correspond with the positive and negative principal eigenvalues of A respectively. We say that a weighted intriguing set S is a *weighted m-ovoid* if $\mathbb{1}_S \in V^0 \perp V^-$. If Γ is the point graph of a generalised quadrangle, then this is equivalent to having the property that $\mathbb{1}_S \cdot \mathbb{1}_\ell = m$ for all lines ℓ. Consider a generalised quadrangle $\mathcal{G}$ of order (s, s^2), and let Z be a point of $\mathcal{G}$. Let S be the set of all points collinear with Z, and give every point of S the weight 1, except the point Z, which will have weight $-s+1$. Then S is a weighted 1-ovoid as $A\mathbb{1}_S = -s^2\mathbb{1}_S + s^2\mathbb{1}$. We can extrapolate this example by taking the perps of a set of noncollinear points Z_1, $Z_2, \ldots, Z_m$, a union of cones, which results in a weighted m-ovoid. We will show below (Theorem 5.10) that a negative intriguing set of $\mathcal{G}\setminus P^\perp$ extends to a weighted m-ovoid of $\mathcal{G}$, and therefore Conjecture 5.8 simplifies to the statement that if S is a weighted m-ovoid of $\mathcal{G}$ such that $S\setminus P^\perp$ is a (not weighted) negative intriguing set of $\mathcal{G}\setminus P^\perp$, then S is a hemisystem or a union of cones.

For the identity and "all-ones" matrices, we will sometimes use a subscript which describes the size of the matrix. For example, $I_{P^\perp}$ and $J_{P^\perp}$ denote the corresponding square matrices with $|P^\perp|$ rows and columns.

LEMMA 5.9. *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) and let P be a point of $\mathcal{G}$. Order the points of $\mathcal{G}$ so that the points of $P^\perp$ appear last, with P last of all. Let A be the adjacency matrix of the point-graph of $\mathcal{G}$, and let B be the adjacency matrix of the partial quadrangle $\mathcal{G}\setminus P^\perp$ such that*

$$A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}.$$

Let $\lambda = -s^2 - 1$. Then:

(a) *$D - \lambda I_{P^\perp}$ is invertible and moreover*

$$s^3(s^2+1)(s+1)(D-\lambda I)^{-1} = (s^4+s^3+s-1)I + J - (s^2+1)D - s(M+M^T) + s(s^2+s-1)E$$

where

$$M := \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \text{ and } E := \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

(b) $C(D-\lambda I)^{-1}\mathbb{1}_{P^\perp} = \mathbb{1}_{PQ}$.

(c) *If $\mathcal{I}$ is a negative intriguing set of $\mathcal{G}\setminus P^\perp$ with parameters (h_1', h_2'), then*

$$CC^T\mathbb{1}_{\mathcal{I}} = s^3\mathbb{1}_{\mathcal{I}} + s|\mathcal{I}|\mathbb{1}_{PQ}$$

and

$$C(D-\lambda I)^{-1}C^T\mathbb{1}_{\mathcal{I}} = s\mathbb{1}_{\mathcal{I}} + h_2'\mathbb{1}_{PQ}.$$

PROOF. **(a)** Since D is the adjacency matrix for $P^\perp$, the eigenvalues for D are $s-1$ and -1. Since $\lambda < -1$, it follows that $D - \lambda I_{P^\perp}$ is invertible. Now we apply $D - \lambda I_{P^\perp}$ to our proposed formula for the inverse $(D - \lambda I_{P^\perp})^{-1}$:

$$(D-\lambda I_{P^\perp})((s^4+s^3+s-1)I + J - (s^2+1)D - s(M+M^T) + s(s^2+s-1)E) =$$
$$(s^4+s^3+s-1)D + DJ - (s^2+1)D^2 - s(DM+DM^T) + s(s^2+s-1)DE -$$
$$\lambda(s^4+s^3+s-1)I - \lambda J + \lambda(s^2+1)D + \lambda s(M+M^T) - \lambda s(s^2+s-1)E.$$

Recall that the last row and column of D represent the point P. To compute the (i,j)-entry of D^2, we note that if $i,j \neq s^3+s+1$, then

$$D^2(i,j) = J(i,j) + (s-1)I(i,j) + (s-2)D(i,j).$$

So to complete the equation, we consider what happens when $i = j = s^3+s+1$. We then see that

$$D^2 = J + (s-1)I + (s-2)D + s^3E.$$

Also, it is not difficult to see that $DM = J - M - M^T - E$, $DE = M^T$, $DJ = sJ + s^3(M+E)$ and $DM^T = (s-1)M^T + (s^3+s)E$. So our equation simplifies to the following:

$$(D-\lambda I_{P^\perp})((s^4+s^3+s-1)I + J - (s^2+1)D - s(M+M^T) + s(s^2+s-1)E) = s^3(s^2+1)(s+1)I$$

from which the desired conclusion follows.

(b) Note that $C\mathbb{1}_{P^\perp} = (s^2+1)\mathbb{1}_{PQ}$, $CJ\mathbb{1}_{P^\perp} = (s^2+1)(s^3+s+1)\mathbb{1}_{PQ}$, $CM\mathbb{1}_{P^\perp} = CE\mathbb{1}_{P^\perp} = \underline{0}$ (the zero vector) and $CM^T\mathbb{1}_{P^\perp} = (s^2+1)\mathbb{1}_{PQ}$. Now

$$D\mathbb{1}_{P^\perp} = s\mathbb{1}_{P^\perp} + (0,\ldots,0,s^3)$$

and hence

$$CD\mathbb{1}_{P^\perp} = s(s^2+1)\mathbb{1}_{P^\perp} + C(0,\ldots,0,s^3) = s(s^2+1)\mathbb{1}_{PQ}.$$

A little calculation then shows that $C(D-\lambda I)^{-1}\mathbb{1}_{P^\perp} = \mathbb{1}_{PQ}$.

(c) First we prove that $CC^T\mathbb{1}_{\mathcal{I}} = s^3\mathbb{1}_{\mathcal{I}} + s|\mathcal{I}|\mathbb{1}_{PQ}$. Let P_i and P_j be the i-th and j-th points of the partial quadrangle. The (i,j) entry of CC^T is the number of points of $P^\perp$ which are collinear with both P_i and P_j. Now if P_i and P_j are collinear, that is $B(i,j) = 1$, then the only point of $P^\perp$ collinear to both P_i and P_j is the point of intersection of $P^\perp$ with the line joining P_i and P_j; so $CC^T(i,j) = 1$ in this case. Otherwise, if $B(i,j) = 0$, then there are $s+1$ points of $P^\perp$ collinear to all three points P, P_i and P_j (recall that this was a property of the ambient generalised quadrangle $\mathcal{G}$ for $\mathcal{G} \setminus P^\perp$ to be a partial quadrangle). Therefore, $CC^T = (s+1)J_{PQ} - sB + (s^2-s)I_{PQ}$ and hence

$$\begin{aligned} CC^T\mathbb{1}_\mathcal{I} &= (s+1)J_{PQ}\mathbb{1}_\mathcal{I} - sB\mathbb{1}_\mathcal{I} + (s^2-s)I_{PQ}\mathbb{1}_\mathcal{I} \\ &= (s+1)|\mathcal{I}|\mathbb{1}_{PQ} - s((-s^2+s-1)\mathbb{1}_\mathcal{I} + h_2'\mathbb{1}_{PQ}) + (s^2-s)\mathbb{1}_\mathcal{I} \\ &= s^3\mathbb{1}_\mathcal{I} + s|\mathcal{I}|\mathbb{1}_{PQ}. \end{aligned}$$

Now we list some formulae which can be worked out with some simple geometric arguments:

$$\begin{aligned} CMC^T &= CM^TC^T = CEC^T = 0, \\ CJC^T &= (s^2+1)^2 J_{PQ}, \\ CDC^T &= (s^2+1)J_{PQ} - CC^T. \end{aligned}$$

The last of these formulae will serve as a demonstration of how to compute all of them. The matrix DC^T measures the number of points of $P^\perp$ which are collinear with two points, one from $P^\perp$ and the other from the partial quadrangle. Upon applying C, we see see that $CDC^T = (s^2+1)J_{PQ} - CC^T$. From the above calculations, we arrive at

$$\begin{aligned} s^3(s^2+1)(s+1)C(D-\lambda I)^{-1}C^T\mathbb{1}_\mathcal{I} &= s(s^2+1)(s+1)CC^T\mathbb{1}_\mathcal{I} \\ &= s^3(s^2+1)(s+1)\left(s\mathbb{1}_\mathcal{I} + h_2'\mathbb{1}_{PQ}\right). \end{aligned}$$

Therefore, $C(D-\lambda I)^{-1}C^T\mathbb{1}_\mathcal{I} = s\mathbb{1}_\mathcal{I} + h_2'\mathbb{1}_{PQ}$. □

THEOREM 5.10 (Negative intriguing set $\longrightarrow$ weighted m-ovoid). *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2), s odd, and let P be a point of $\mathcal{G}$ such that $\mathcal{G} \setminus P^\perp$ is a partial quadrangle. Suppose $\mathcal{I}$ is a negative intriguing set of the partial quadrangle $\mathcal{G} \setminus P^\perp$. Then there is a subset $\mathcal{I}^*$ of points of $P^\perp$, equipped with integral weights, such that $\mathcal{I} \cup \mathcal{I}^*$ is a weighted m-ovoid of $\mathcal{G}$.*

PROOF. Suppose that $\mathcal{I}$ is a negative intriguing set of $\mathcal{G} \setminus P^\perp$ with parameters (h_1', h_2'). Let λ be the negative eigenvalue of A (i.e., $-s^2-1$), let h_2 be a positive integer, and let

$$v = (D - \lambda I_{P^\perp})^{-1}(-C^T\mathbb{1}_I + h_2\mathbb{1}_{P^\perp}).$$

We will show that there is a value of h_2 such that v is an integer valued vector and hence represents a weighted subset $\mathcal{I}^*$ of points of $P^\perp$. If we also show that v corresponds naturally to an eigenvector of A (see Lemma 2.1), then it will follow that $\mathcal{I} \cup \mathcal{I}^*$ is a weighted intriguing set in $\mathcal{G}$. We show first that $\mathbb{1}_\mathcal{I} + v - \alpha\mathbb{1}_{GQ}$ is an eigenvector of A with eigenvalue λ, where $\alpha = h_2/((s+1)(s^2+1))$. We apply A to our proposed eigenvector:

$$\begin{aligned} A\left(\mathbb{1}_\mathcal{I} + v - \alpha\mathbb{1}_{GQ}\right) =& A\mathbb{1}_\mathcal{I} + Av - \alpha s(s^2+1)\mathbb{1}_{GQ} \\ =& \begin{bmatrix} B\mathbb{1}_\mathcal{I} + Cv \\ C^T\mathbb{1}_\mathcal{I} + Dv \end{bmatrix} - \alpha s(s^2+1)\mathbb{1}_{GQ}. \end{aligned}$$

By Lemma 5.9,

$$\begin{aligned} Cv &= -C(D-\lambda I)^{-1}C^T\mathbb{1}_\mathcal{I} + h_2\mathbb{1}_{PQ} = -s\mathbb{1}_\mathcal{I} - h_2'\mathbb{1}_{PQ} + h_2\mathbb{1}_{PQ} \\ &= -\left((\lambda+s)\mathbb{1}_\mathcal{I} + h_2'\mathbb{1}_{PQ}\right) + \lambda\mathbb{1}_\mathcal{I} + h_2\mathbb{1}_{PQ} = -B\mathbb{1}_\mathcal{I} + \lambda\mathbb{1}_\mathcal{I} + h_2\mathbb{1}_{PQ} \end{aligned}$$

and hence $B\mathbb{1}_\mathcal{I} + Cv = \lambda\mathbb{1}_\mathcal{I} + h_2\mathbb{1}_{PQ}$. We also have

$$C^T\mathbb{1}_\mathcal{I} + Dv = C^T\mathbb{1}_\mathcal{I} + \lambda v - C^T\mathbb{1}_\mathcal{I} + h_2\mathbb{1}_{P^\perp} = \lambda v + h_2\mathbb{1}_{P^\perp}.$$

Hence,

$$\begin{aligned}A(\mathbb{1}_{\mathcal{I}}+v-\alpha\mathbb{1}_{GQ}) &= A\mathbb{1}_{\mathcal{I}}+Av-\alpha s(s^2+1)\mathbb{1}_{GQ} = \begin{bmatrix} B\mathbb{1}_{\mathcal{I}}+Cv \\ C^T\mathbb{1}_{\mathcal{I}}+Dv\end{bmatrix} - \alpha s(s^2+1)\mathbb{1}_{GQ} \\ &= \begin{bmatrix}\lambda(\mathbb{1}_{\mathcal{I}}-\alpha(s+1)\mathbb{1}_{PQ}) \\ \lambda(v-\alpha(s+1)\mathbb{1}_{P^\perp})\end{bmatrix} + \alpha s\lambda\mathbb{1}_{GQ} \\ &= \lambda(\mathbb{1}_{\mathcal{I}}+v-\alpha\mathbb{1}_{GQ}).\end{aligned}$$

So $\mathbb{1}_{\mathcal{I}}+v-\alpha\mathbb{1}_{GQ}$ is an eigenvector of A with eigenvalue λ. Note that this is true no matter what choice we make for the value of h_2. We will show that there exist values of h_2 such that v is integer valued. Suppose h_2 is divisible by s^2+1 and let $X = h_2/(s^2+1)$. A tedious calculation shows that if we restrict v to the points of $P^\perp \setminus \{P\}$, then

$$v|_{P^\perp\setminus\{P\}} = \frac{-C^T\mathbb{1}_{\mathcal{I}}}{s^2} + X\mathbb{1}_{P^\perp\setminus\{P\}}.$$

We then look at the action of v on the point P to derive

$$v|_{\{P\}} = \frac{|\mathcal{I}|}{s^2} + X(1-s).$$

Recall from Theorem 5.2 that $|\mathcal{I}|$ is divisible by s^2, so $v|_{\{P\}}$ is an integer. By simple geometric arguments, we know that

$$DC^T\mathbb{1}_{\mathcal{I}} = -C^T\mathbb{1}_{\mathcal{I}} + |\mathcal{I}|\left(\mathbb{1}_{P^\perp\setminus\{P\}} + (s^2+1)\mathbb{1}_P\right).$$

So

$$(D-\lambda I_{P^\perp})C^T\mathbb{1}_{\mathcal{I}} = s^2C^T\mathbb{1}_{\mathcal{I}} + |\mathcal{I}|(D-\lambda I_{P^\perp})\mathbb{1}_P$$

and hence

$$C^T\mathbb{1}_{\mathcal{I}} = s^2(D-\lambda I_{P^\perp})^{-1}C^T\mathbb{1}_{\mathcal{I}} + |\mathcal{I}|\mathbb{1}_P$$

as $D-\lambda I_{P^\perp}$ is invertible. Hence every entry of $C^T\mathbb{1}_{\mathcal{I}}$ is divisible by s^2, thus proving that v is integer valued. □

Corollary 5.11 (Negative intriguing set of the right size $\longleftrightarrow$ Hemisystem). *Let $\mathcal{G}$ be a generalised quadrangle of order (s,s^2), s odd, and let P be a point of $\mathcal{G}$ such that $\mathcal{G}\setminus P^\perp$ is a partial quadrangle. Suppose $\mathcal{I}$ is a negative intriguing set of the partial quadrangle $\mathcal{G}\setminus P^\perp$ such that $|\mathcal{I}|$ is either*

$$s^2(s^2-1)/2 \quad \text{or} \quad s^2(s^2+1)/2.$$

Then there is a subset $\mathcal{I}^*$ of points of $P^\perp$ such that $\mathcal{I}\cup\mathcal{I}^*$ is a hemisystem of $\mathcal{G}$.

Proof. Suppose that $\mathcal{I}$ is a negative intriguing set of $\mathcal{G}\setminus P^\perp$ with parameters (h_1', h_2'). Let λ be the negative eigenvalue of A (i.e., $-s^2-1$), let h_2 be a positive integer, and let

$$v = (D-\lambda I_{P^\perp})^{-1}(-C^T\mathbb{1}_I + h_2\mathbb{1}_{P^\perp}).$$

By Theorem 5.10, if h_2 is divisible by s^2+1, then v is integer valued. Recall that

$$v|_{P^\perp\setminus\{P\}} = \frac{-C^T\mathbb{1}_{\mathcal{I}}}{s^2} + \frac{h_2}{s^2+1}\mathbb{1}_{P^\perp\setminus\{P\}} \quad \text{and} \quad v|_{\{P\}} = \frac{|\mathcal{I}|}{s^2} + \frac{h_2}{s^2+1}(1-s).$$

Let $X = h_2/(s^2+1)$. Then $\sum v = X(s^3+1) - |\mathcal{I}|$ and

$$\begin{aligned} v\cdot v &= \frac{\mathbb{1}_{\mathcal{I}}CC^T\mathbb{1}_{\mathcal{I}}^T}{s^4} - 2X\frac{|\mathcal{I}|(s^2+1)}{s^2} + X^2(s^3+s) + \left(\frac{|\mathcal{I}|}{s^2} + X(1-s)\right)^2 \\ &= \frac{|\mathcal{I}|}{s} + \frac{(s+1)|\mathcal{I}|^2}{s^4} - 2X\frac{|\mathcal{I}|(s+1)}{s} + X^2(s^3+s^2-s+1).\end{aligned}$$

So

$$v \cdot v - \sum v = \frac{|\mathcal{I}|}{s} + \frac{(s+1)|\mathcal{I}|^2}{s^4} - 2X\frac{|\mathcal{I}|(s+1)}{s} + X^2(s^3+s^2-s+1) - X(s^3+1) + |\mathcal{I}|$$
$$= X^2(s^3+s^2-s+1) + X\left(\frac{-2|\mathcal{I}|(s+1)}{s} - s^3 - 1\right) + \frac{|\mathcal{I}|}{s} + \frac{(s+1)|\mathcal{I}|^2}{s^4} + |\mathcal{I}|.$$

Now suppose $|\mathcal{I}| = s^2(s^2-1)/2$. Then it turns out that

$$v \cdot v - \sum v = \left((s^3+s^2-s+1)X - \frac{s+1}{2}(s^3+s^2-s+1)\right)\left(X - \frac{(s-1)(s^3+3s^2+s-1)}{2(s^3+s^2-s+1)}\right)$$

and hence v is a zero vector if and only if $X = (s+1)/2$. That is, if we let $h_2 = (s^2+1)(s+1)/2$, then v is a characteristic function for a subset $\mathcal{I}^*$ of $P^\perp$, and the union of $\mathcal{I}$ with $\mathcal{I}^*$ forms a negative intriguing set of the generalised quadrangle $\mathcal{G}$. By Theorem 5.5, $\mathcal{I} \cup \mathcal{I}^*$ is a hemisystem of $\mathcal{G}$. A similar argument holds for the case $|\mathcal{I}| = s^2(s^2+1)/2$. □

5.3. Examples of intriguing sets of $\mathsf{Q}^-(5,q)$. Segre [**21**] proved that for $q = 3$, there is just one hemisystem up to equivalence, and it was long thought to be the only example of such an object. However, Cossidente and Penttila [**11**] constructed an infinite family of hemisystems of $\mathsf{Q}^-(5,q)$ admitting $\mathsf{P}\Omega^-(4,q)$, together with a special example for $q = 5$ admitting the triple cover of A_7.

There are many tight sets of $\mathsf{Q}^-(5,q)$, simply because there are many partial spreads of $\mathsf{Q}^-(5,q)$. However, some interesting examples arise from Cameron-Liebler line classes. A set of lines $\mathcal{L}$ of $\mathsf{PG}(3,q)$ is said to be a *Cameron-Liebler line class* if there exists a constant i such that $\mathcal{L}$ meets every (regular) line spread of $\mathsf{PG}(3,q)$ in i elements. Such a set of lines gives rise to an i-tight set of $\mathsf{Q}^-(5,q)$ as follows: first note that every spread of the symplectic generalised quadrangle $\mathsf{W}(3,q)$ is a spread of $\mathsf{PG}(3,q)$, and so the set of lines of $\mathcal{L}$ in $\mathsf{W}(3,q)$ meets each spread of $\mathsf{W}(3,q)$ in i elements. Hence, by dualising, we obtain an i-tight set of $\mathsf{Q}(4,q)$ (see [**3**, Corollary 4.11]). By embedding, we produce an i-tight set of $\mathsf{Q}^-(5,q)$. The Cameron-Liebler line classes can only give rise to tight sets of $\mathsf{Q}^-(5,q)$ which consist of a line, a pair of skew lines, or a complement of one of these [**9**]. However, much more is known about the existence and non-existence of Cameron-Liebler line classes, and so we refer the interested reader to [**18**] or [**12**] for more on this topic. Finally we note that a $\mathsf{Q}(4,q)$ embedded in $\mathsf{Q}^-(5,q^2)$ (subfield embedding) is $(q+1)$-tight, and the points of $\mathsf{Q}(4,q^2)$ which are collinear but not equal to their conjugate forms a $q(q^2-1)$-tight set of $\mathsf{Q}^-(5,q^2)$ (see [**2**, Theorem 8]).

6. Partial quadrangles obtained from a hemisystem

Recall that a *hemisystem* $\mathcal{H}$ of a generalised quadrangle $\mathcal{G}$ of order (s,s^2), s odd, is a set of $(s^3+1)(s+1)/2$ points of $\mathcal{G}$ such that every line of $\mathcal{G}$ is incident with exactly $(s+1)/2$ elements of $\mathcal{H}$. From $\mathcal{H}$, we construct a partial quadrangle $\mathsf{PQ}(\mathcal{H})$ as follows:

POINTS	the points of $\mathcal{H}$
LINES	the lines of $\mathcal{G}$.

The parameters are thus $((s-1)/2, s^2, (s-1)^2/2)$. Since the complement of a hemisystem is again a hemisystem, we may regard this construction as removing "infinity", where "infinity" is a hemisystem.

LEMMA 6.1. *Let $\mathcal{G}$ be a generalised quadrangle of order (s,s^2) (s odd), let $\mathcal{H}$ be a hemisystem of $\mathcal{G}$, and let $\mathcal{I}$ be an intriguing set of the partial quadrangle $\mathsf{PQ}(\mathcal{H})$ with intersection numbers (h_1', h_2'). Then we have the following information:*

Case	*Eigenvalue*	*Size*
Negative intriguing set	$(-s^2+s-2)/2$	$h_2'(s+1)$
Positive intriguing set	$s-1$	$h_2'(s^3+1)/(s-1)^2$
Point set	$(s-1)(s^2+1)/2$	$(s+1)(s^3+1)/2$

TABLE 5. Eigenvalues and sizes of intriguing sets of $\mathsf{PQ}(\mathcal{H})$.

THEOREM 6.2. *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2), s odd, and let $\mathcal{H}$ be a hemisystem of $\mathcal{G}$. Let $\mathcal{I}$ be an intriguing set of $\mathcal{G}$ with parameters (h_1, h_2). If $\mathcal{I} \setminus \mathcal{H}$ is an intriguing set of the partial quadrangle $\mathcal{G} \setminus \mathcal{H}$, then we have the following possibilities for the intersection numbers (a_1, a_2) at infinity:*

	$a_1 - a_2$	a_2	$\lvert\mathcal{I} \setminus \mathcal{H}\rvert$
$- \to -$	$-(s^2+s)/2$	–	$(m(s^2+1)-a_2)(s+1)$
$- \to +$	$-(s^2+s)$	–	$(m(s^2+1)-a_2)(s^3+1)/(s-1)^2$
$+ \to -$	$(s^2+s)/2$	$i/2$	$i(s+1)/2$

TABLE 6. Possibilities for intersection numbers (a_1, a_2).

PROOF. The positive eigenvalues for $\mathcal{G}$ and $\mathcal{G} \setminus \mathcal{H}$ are both equal to $s-1$. However, the negative eigenvalues differ: $-s^2-1$ for $\mathcal{G}$ and $(-s^2+s-2)/2$ for $\mathcal{G} \setminus \mathcal{H}$. Now we must have that $h_1 - h_2$ and $(h_1 - a_1) - (h_2 - a_2)$ are eigenvalues for the respective geometries:

Case		$h_1 - h_2$	$(h_1 - a_1) - (h_2 - a_2)$	$a_1 - a_2$	a_2	$\lvert\mathcal{I} \setminus \mathcal{H}\rvert$
(i)	$- \to -$	$-s^2-1$	$(-s^2+s-2)/2$	$-(s^2+s)/2$	–	$(m(s^2+1)-a_2)(s+1)$
(ii)	$- \to +$	$-s^2-1$	$s-1$	$-(s^2+s)$	–	$(m(s^2+1)-a_2)(s^3+1)/(s-1)^2$
(iii)	$+ \to -$	$s-1$	$(-s^2+s-2)/2$	$(s^2+s)/2$	$i/2$	$i(s+1)/2$
(iv)	$+ \to +$	$s-1$	$s-1$	0	$i/2\frac{s^2-1}{s^2-s+1}$	$i(s+1)/2$

TABLE 7. Details on the intersection numbers.

(i) Suppose that $\mathcal{I}$ is an m-ovoid and $\mathcal{I} \setminus \mathcal{H}$ is negative intriguing. Then $|\mathcal{I}| = m(s^3+1)$ and hence

$$|\mathcal{I} \setminus \mathcal{H}| = \frac{m(s^2+1)-a_2}{(s-1)(s^2+1)-(-s^2+s-2)}(s+1)(s^3+1) = (m(s^2+1)-a_2)(s+1).$$

(ii) Suppose that $\mathcal{I}$ is an m-ovoid and $\mathcal{I} \setminus \mathcal{H}$ is positive intriguing. Again we have $|\mathcal{I}| = m(s^3+1)$, but now we obtain

$$|\mathcal{I} \setminus \mathcal{H}| = \frac{m(s^2+1)-a_2}{(s-1)^2}(s^3+1).$$

(iii) Suppose that $\mathcal{I}$ is an i-tight set and $\mathcal{I} \setminus \mathcal{H}$ is negative intriguing. Then

$$|\mathcal{I} \setminus \mathcal{H}| = \frac{i-a_2}{(s-1)(s^2+1)-(-s^2+s-2)}(s^3+1)(s+1) = (i-a_2)(s+1).$$

Since $\mathcal{H}$ is a hemisystem, we have that $|\mathcal{H} \cap \mathcal{I}| = (s+1)i/2$. So $|\mathcal{I} \setminus \mathcal{H}| = |\mathcal{I}| - |\mathcal{H} \cap \mathcal{I}| = (s+1)i/2$. This gives $(i-a_2)(s+1) = (s+1)i/2$ and hence $a_2 = i/2$.

(iv) Suppose that $\mathcal{I}$ is an i-tight set and $\mathcal{I} \setminus \mathcal{H}$ is positive intriguing. Then $|\mathcal{I}| = i(s+1)$ and hence

$$|\mathcal{I} \setminus \mathcal{H}| = \frac{i-a_2}{(s-1)(s^2+1)-2(s-1)}(s^3+1)(s+1) = \frac{i-a_2}{(s-1)^2}(s^3+1).$$

Since $\mathcal{H}$ is a hemisystem, we have $|\mathcal{H} \cap \mathcal{I}| = (s+1)i/2$. So $|\mathcal{I} \setminus \mathcal{H}| = |\mathcal{I}| - |\mathcal{H} \cap \mathcal{I}| = (s+1)i/2$ and we obtain $\frac{i-a_2}{(s-1)^2}(s^3+1) = (s+1)i/2$ and therefore $a_1 = a_2 = i/2\frac{s^2-1}{s^2-s+1}$. If we now compare with Lemma 6.1, we arrive at the equation

$$2h_2'(s^3+1) = i(s-1)(s^2-1).$$

However, since we know a_2, and $h'_2 = h_2 - a_2$, we can substitute h'_2 in the above equation and we obtain

$$2(1 - \frac{s^2-1}{2(s^2-s+1)})(s^3+1) = (s-1)(s^2-1)$$

which implies that $2s = 0$; a contradiction.

□

Corollary 6.3. *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2), s odd, and let $\mathcal{H}$ be a hemisystem of $\mathcal{G}$. Both an $(s+1)$-tight set and an (s^2+1)-tight set of $\mathcal{G}$ never yield intriguing sets of $\mathsf{PQ}(\mathcal{H})$. Furthermore, $\mathsf{Q}^+(3, q)$ and $\mathsf{Q}(4, q)$ embedded in $\mathsf{Q}^-(5, q)$, never yield intriguing sets of $\mathsf{PQ}(\mathcal{H})$.*

Proof. A $\mathsf{Q}^+(3, q)$ embedded in $\mathsf{Q}^-(5, q)$ is a $(q+1)$-tight set and a $\mathsf{Q}(4, q)$ section is a (q^2+1)-tight set. Suppose $\mathcal{I}$ is an $(s+1)$-tight set of $\mathcal{G}$ such that $\mathcal{I}\setminus\mathcal{H}$ is a negative intriguing set of $\mathsf{PQ}(\mathcal{H})$ (the only case allowed by Theorem 6.2). Then we observe immediately a contradiction because $h_1 - a_1$ is negative. In the case that $\mathcal{I}$ is an (s^2+1)-tight set of $\mathcal{G}$, the parameter a_1 is equal to $s^2 + (s+1)/2$. Now a_1 is the number of points of $\mathcal{I} \cap \mathcal{H}$ which are collinear with an arbitrary point of $\mathcal{I} \setminus \mathcal{H}$. Therefore, a_1 is divisible by $(s+1)/2$, which implies that $s+1$ divides $2s^2 + s + 1 = (2s-1)(s+1) + 2$; a contradiction. So an $(s+1)$-tight set and an (s^2+1)-tight set of $\mathcal{G}$ never induce intriguing sets of $\mathsf{PQ}(\mathcal{H})$. □

Examples exist for the first and third cases of Theorem 6.2 which we demonstrate in what follows. For the second case of Theorem 6.2, we do not have any examples when the generalised quadrangle is $\mathsf{Q}^-(5, s)$. In this generalised quadrangle, an m-ovoid is a hemisystem and we believe that only negative intriguing sets can arise in the partial quadrangle.

Conjecture 6.4. *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2), s odd, and let $\mathcal{H}$ be a hemisystem of $\mathcal{G}$. Let $\mathcal{I}$ be another hemisystem of $\mathcal{G}$. Then $\mathcal{I} \setminus \mathcal{H}$ is a negative intriguing set of the partial quadrangle $\mathcal{G} \setminus \mathcal{H}$.*

The authors are not aware of a situation in which the above situation is violated, and a proof of this fact would be a surprising result on the nature of hemisystems.

Lemma 6.5 (Nice Cone⟶ Negative Intriguing). *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) (s odd), let $\mathcal{H}$ be a hemisystem of $\mathcal{G}$, and let Z be a point of $\mathcal{H}$. If the complement of $\mathcal{H}$ is intriguing at infinity for the partial quadrangle $\mathcal{G} \setminus Z^\perp$, then $Z^\perp \setminus \mathcal{H}$ is a negative intriguing set with parameters $((s-1)/2, (s^2+1)/2)$ of the partial quadrangle $\mathcal{G} \setminus \mathcal{H}$*

Proof. Let $X \in Z^\perp \setminus \mathcal{H}$. The points of $Z^\perp$ collinear with X lie on the line ZX, and this line meets $\mathcal{H}'$ in $(s+1)/2$ points. So X is collinear with precisely $h'_1 = (s-1)/2$ other points of $Z^\perp \setminus \mathcal{H}$. Now suppose that $X \notin Z^\perp \setminus \mathcal{H}$. So in particular X is not in $Z^\perp$ and hence we can use the fact that $\mathcal{G} \setminus Z^\perp$ is a partial quadrangle. Since $\mathcal{H}'$ is intriguing at infinity for the partial quadrangle $\mathcal{G} \setminus Z^\perp$, there exists a constant a_2 such that $|X^\perp \cap \mathcal{H}' \cap Z^\perp| = a_2$. Now by Lemma 5.6, this value of a_2 is $(s^2+1)/2$. □

If we have a partial spread $\mathcal{S}$ of a generalised quadrangle $\mathcal{G}$, and a point X not covered by any line of $\mathcal{S}$, then X is collinear with exactly one point of each member of $\mathcal{S}$. If half of these points of collinearity are contained in a hemisystem, then we might obtain an intriguing set of the associated partial quadrangle.

Lemma 6.6 (Nice Partial Spread⟶ Negative Intriguing). *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2), s odd, and let $\mathcal{H}$ be a hemisystem of $\mathcal{G}$. Let $\mathcal{I}$ be a set of points covered by a partial spread of c lines of $\mathcal{G}$ where c is even. If $\mathcal{I}$ is intriguing at infinity, then for every point X not in $\mathcal{I}$, half of the c points of $\mathcal{I}$ collinear with X are contained in $\mathcal{H}$, and $\mathcal{I}\setminus\mathcal{H}$ is a negative intriguing set of the partial quadrangle $\mathcal{G} \setminus \mathcal{H}$ with parameters $((c - s^2 + s)/2 - 1, c/2)$.*

Proof. It follows from Theorem 6.2. □

Such partial spreads as those described in Lemma 6.6 have been found by computer for small q in $\mathsf{Q}^-(5,q)$. Therefore we have examples in which a tight set of the generalised quadrangle induces negative intriguing sets of the partial quadrangle arising from a hemisystem.

REMARK 6.7. A positive intriguing set with $h_2' = 1$ has size $(s^3+1)/(s-1)^2$ which is only an integer when $s = 2, 3$. Now let $\mathcal{O}$ be a maximal partial ovoid of a generalised quadrangle $\mathcal{G}$ of order (s, s^2). If $\mathcal{O}$ is an intriguing set with parameters (h_1', h_2') of $\mathsf{PQ}(\mathcal{H})$, then $h_1' = 0$ and hence $\mathcal{O}$ is a negative intriguing set with $h_2' = (s^2-s+2)/2$. In this case, we would have $|\mathcal{O}| = (s^2-s+2)(s+1)/2 = (s^3+s+2)/2$, which happens to be the theoretical upper-bound for the size of a partial ovoid of $\mathsf{Q}^-(5,s)$ given by De Beule, Klein, Metsch, and Storme [**13**]. In $\mathsf{Q}^-(5,3)$, there is a set of points of size 16 that is a negative intriguing set of $\mathsf{PQ}(\mathcal{H})$ where $\mathcal{H}$ is Segre's hemisystem. It happens to be the unique maximal partial ovoid of $\mathsf{Q}^-(5,3)$ first discovered by Ebert and Hirschfeld [**15**].

7. Partial quadrangles that have a linear representation

A *k-cap* of a projective space $\mathsf{PG}(n,q)$ is a set of k points with no three collinear. Calderbank [**6**] proved using number-theoretic arguments that if a partial quadrangle is a linear representation then $q \geqslant 5$ or it is isomorphic to the linear representation of one of the following: (i) An ovoid $\mathcal{O}$ of $\mathsf{PG}(3,q)$; (ii) A Coxeter 11-cap of $\mathsf{PG}(4,3)$; (iii) A Hill 56-cap of $\mathsf{PG}(5,3)$; (iv) A 78-cap of $\mathsf{PG}(5,4)$; (v) A 430-cap of $\mathsf{PG}(6,4)$. Tzanakis and Wolfskill [**23**] then proved that if $q \geqslant 5$, we must be in the first case. The examples in the first case are equivalent to the partial quadrangles obtained from removing the special point ∞ from a *Tits generalised quadrangle* $\mathsf{T}_3(\mathcal{O})$ of order (q, q^2), where $\mathcal{O}$ is an ovoid of $\mathsf{PG}(3,q)$. Hence we have just three known *exceptional partial quadrangles* arising from (i) the Coxeter 11-cap (yielding a $\mathsf{PQ}(2,10,2)$), (ii) the Hill 56-cap (yielding a $\mathsf{PQ}(2,55,20)$) and (iii) the so-called Hill 78-cap (yielding a $\mathsf{PQ}(3,77,14)$). (It is still an open problem whether there exists a 430-cap of $\mathsf{PG}(6,4)$ or not.) For more details on these caps, we refer the reader to Hill's paper [**19**].

LEMMA 7.1 (Hyperplane $\longrightarrow$ Intriguing). *Let Γ be a partial quadrangle with a linear representation in $\mathsf{PG}(n,q)$, corresponding to a cap $\mathcal{K}$ of $\mathsf{PG}(n,q) \subset \mathsf{PG}(n+1,q)$. Let π be the set of affine points in some hyperplane of $\mathsf{PG}(n+1,q)$ different from $\pi_\infty = \mathsf{PG}(n,q)$. Then π is an intriguing set of Γ with parameters*

$$((q-1)|\pi \cap \mathcal{K}|, |\mathcal{K} \setminus \pi|).$$

PROOF. Let P be a point of π. Then for every point Q of $\pi \cap \mathcal{K}$, there are $q-1$ affine points on QP, other than P, which are collinear with P. Hence in total we have $(q-1)|\pi \cap \mathcal{K}|$ other points of the partial quadrangle collinear with P. Now suppose P is not in π. Then clearly a point of $\mathcal{K} \cap \pi$ is not on a line connecting P with a point of π. Since every line not in π must meet π in a point, it follows that the intersection number is $|\mathcal{K} \setminus \pi|$ in this case. □

The example in the lemma above could either be a negative or positive intriguing set depending on the intersection of the given hyperplane with the cap.

LEMMA 7.2. *Let $\mathcal{I}$ be an intriguing set with intersection numbers (h_1', h_2') of one of the three exceptional partial quadrangles. Then we have the following information:*

Case	*Coxeter* 11-*cap* *Eigenvalue*	*Size*	*Hill* 56-*cap* *Eigenvalue*	*Size*	*Hill* 78-*cap* *Eigenvalue*	*Size*
Negative intriguing set	-5	$9h_2'$	-23	$(27/5)h_2'$	-22	$16h_2'$
Positive intriguing set	4	$(27/2)h_2'$	4	$(27/4)h_2'$	10	$(128/7)h_2'$
Point set	22	243	112	729	234	4096

TABLE 8. Eigenvalues and sizes for intriguing sets of the exceptional partial quadrangles.

By Lemma 7.1, the affine points in a hyperplane will have associated eigenvalue $q|\pi \cap \mathcal{K}| - |\mathcal{K}|$ and so $|\pi \cap \mathcal{K}|$ is 2 or 5 for the Coxeter 11-cap, 11 or 20 for the Hill 56-cap, and 14 or 22 for the Hill 78-cap.

LEMMA 7.3 (Nice Secundum $\longrightarrow$ Positive intriguing). *Let Γ be a partial quadrangle with a linear representation in $\mathsf{PG}(n,q)$, corresponding to a cap $\mathcal{K}$ of $\mathsf{PG}(n,q) \subset \mathsf{PG}(n+1,q)$. Let S be a secundum of $\mathsf{PG}(n+1,q)$ such that every hyperplane π containing S meets $\mathcal{K}$ in a constant number of points. Then the affine points of S form an intriguing set of Γ with parameters*

$$((q-1)|S \cap \mathcal{K}|, |S \cap \pi| - |S \cap \mathcal{K}|).$$

PROOF. Let X be a point of S, and let C be a point of $\mathcal{K}$. If $C \notin S$, then there are no affine points of S incident with XC, but if $C \in S$, then the affine points on the line XC are all in S. So regardless of how S meets the cap $\mathcal{K}$, it is clear that there are $1 + (q-1)|\mathcal{K} \cap S|$ points collinear with X in the associated partial quadrangle. So our first parameter is $(q-1)|\mathcal{K} \cap S|$. Now we look to the case that X is not a point of S, and again, let C be a point of $\mathcal{K}$. Clearly XC is not contained in S, but it may be disjoint from S or meet S in a point. Let π be the hyperplane XS. Now if $C \in S$, then XC cannot meet S in another point since otherwise XC would be contained in S. If C were not in π, then the unique point of intersection of XC with π would be X, and hence XC would not contain any points of S. So suppose $C \in \pi \setminus S$. Now XC is a line of π, and S is a hyperplane of π, thus XC meets S in a point. Moreover, it is clear that this point of intersection is an affine point, and so the lines XC which meet S in a point are precisely those for which $C \in (\pi \cap \mathcal{K}) \setminus (S \cap \mathcal{K})$. Hence, the affine points of S form an intriguing set with second parameter equal to $|S \cap \pi| - |S \cap \mathcal{K}|$. □

We remark that there are secunda of $\mathsf{PG}(5,3)$ which meet the Coxeter 11-cap in 3 points, and hence every hyperplane containing such a secundum must meet the Coxeter 11-cap in 5 points. Similarly, there are secunda of $\mathsf{PG}(6,3)$ which meet the Hill 56-cap in 8 points, and such that every incident hyperplane meets this cap in 20 points. Finally, we also have secunda of $\mathsf{PG}(6,4)$ for the Hill 78-cap which satisfy the hypotheses of Lemma 7.3. Below we give some other examples which were found by computer.

7.1. Coxeter 11-cap. The permutation group induced on the Coxeter 11-cap is M_{11}, and the full stabiliser of the cap in $\mathsf{PGL}(6,3)$ is $3^5 : (M_{11} \times 2)$. We note that this group is also the full automorphism group of the associated partial quadrangle. There were many negative intriguing sets found by computer, and we report on those which were deemed interesting. There is a negative intriguing set of size 45 admitting M_{10}, and it is thus far, the only known negative intriguing set of this size. Similarly, there are only two known negative intriguing sets of size 54, admitting groups of size 108 and 864 respectively. There is an intriguing set of size 81 which is the complement of the union of three hyperplanes (with stabiliser of size 648). There are at least two copies of $M_9 : 2$ in the automorphism group; one meets the normal elementary abelian subgroup 3^5 trivially, the other in a subgroup of order 3^2. These two groups give rise to intriguing sets of size 63 and 108, the former is the complement of the disjoint union of negative intriguing sets of size 45.

There is a positive intriguing set of size 27 which is the complement of the union of 11 hyperplanes, each meeting the cap in 5 points. Its stabiliser is $D_{18} \times S_3$. As noted above, there are solids of $\mathsf{PG}(5,3)$ meeting the cap in 3 points, and hence we have positive intriguing sets of size 27. All known examples arise from a sequence of unions and complements of elements in the orbits of these two examples of size 27.

7.2. Hill 56-cap. The permutation group induced on the Hill 56-cap is $\mathsf{PSL}(3,4).2$, and the full stabiliser of the cap in $\mathsf{PGL}(7,3)$ is $3^6 : (2.\mathsf{PSL}(3,4).2)$. We note that this group is also the full automorphism group of the associated partial quadrangle. The only known negative intriguing set found so far is the set of affine points contained in a hyperplane meeting the cap in 11 points. As for positive intriguing sets, we have hyperplanes on 20 cap points, solids on 4 cap points, and thousands of other examples which are too numerous to list here. Most of these had stabilisers of order 27 or 54.

7.3. Hill 78-cap. The permutation group induced on the Hill 78-cap is $(13 : 6) \times C_3$, and the full stabiliser of the cap in $\mathsf{P\Gamma L}(8,4)$ is $3^7 : ((C_{117} : C_3) : 2)$. We note that this group is also the full automorphism group of the associated partial quadrangle. Probably due to the fact that this partial quadrangle has less symmetry than the other examples above, there were many intriguing sets found, and none believed to be particularly interesting to the authors. The smallest negative intriguing set found had size 512 (so with parameters $(10, 32)$), and the smallest positive intriguing set had size 128 (parameters $(17, 7)$) and hence attains the minimum size. Most of the intriguing sets found had their full stabiliser acting regularly on them.

8. Concluding remarks

We introduced the definition of an intriguing set via strongly regular graphs, and although much of the interest so far has been on intriguing sets of generalised quadrangles and partial quadrangles, it may perhaps also be interesting to investigate the intriguing sets of other particular families of strongly regular graphs.

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