Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A new characterization of projections of quadrics in finite projective spaces of even characteristic

F. De Clerck*, N. De Feyter

Department of Mathematics, Ghent University, Krijgslaan 281 - S22, B-9000 Gent, Belgium

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Affine partial linear spaces
Projections of quadrics
Antiflag types

ABSTRACT

We will classify, up to linear representations, all geometries fully embedded in an affine space with the property that for every antiflag $\{p, L\}$ of the geometry there are either $0, \alpha$, or q lines through p intersecting L. An example of such a geometry with $\alpha = 2$ is the following well known geometry HT_n. Let $\mathcal{Q}_{n+1}$ be a nonsingular quadric in a finite projective space $PG(n+1, q)$, $n \geq 3$, q even. We project $\mathcal{Q}_{n+1}$ from a point $r \notin \mathcal{Q}_{n+1}$, distinct from its nucleus if $n+1$ is even, on a hyperplane $PG(n, q)$ not through r. This yields a partial linear space HT_n whose points are the points p of $PG(n, q)$, such that the line $\langle p, r\rangle$ is a secant to $\mathcal{Q}_{n+1}$, and whose lines are the lines of $PG(n, q)$ which contain q such points. This geometry is fully embedded in an affine subspace of $PG(n, q)$ and satisfies the antiflag property mentioned. As a result of our classification theorem we will give a new characterization theorem of this geometry.

1. Introduction

A point set $\mathcal{K}$ in a point-line geometry is said to be a *set of class* $[m_1, \ldots, m_k]$, $0 \leq m_1 < \cdots < m_k \leq q+1$, with respect to a set $\mathcal{L}$ of lines if for every line L of $\mathcal{L}$, $|L \cap \mathcal{K}| = m_i$ for some $1 \leq i \leq k$. It is said to be a *set of type* $(m_1, \ldots, m_k)$ with respect to the set $\mathcal{L}$ of lines if every m_i actually occurs for some line L of $\mathcal{L}$. In this paper the set $\mathcal{K}$ will be a subset of the point set of a projective or affine space, while the set $\mathcal{L}$ will be a subset of lines of that projective or affine space.

A point p of a set S of points of $PG(n, q)$ or $AG(n, q)$ is *a singular point of* S if every line through p is either contained in S or intersects S in the point p only.

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *partial linear space* if every two points are incident with at most one line. It is said to be of order (s, t), if every line is incident with $s+1$ points, while every point is on $t+1$ lines. A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be of *antiflag class* $[\alpha_1, \ldots, \alpha_m]$ if for every antiflag $\{p, L\}$, i.e. for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$, not through p, the so-called *incidence number* $\alpha(p, L)$ of lines of $\mathcal{L}$ through p which intersect L is one of $\alpha_1, \ldots, \alpha_m$. We do not require that every incidence number actually occurs. If they do all occur then we say that the partial linear space is of *antiflag type* $(\alpha_1, \ldots, \alpha_m)$. A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be *fully embedded* in an affine space $AG(n, q)$, if $\mathcal{L}$ is a set of lines of $AG(n, q)$, $\mathcal{P}$ is the set of all affine points on the lines of $\mathcal{S}$ and I is the incidence of $AG(n, q)$. We also require that $\mathcal{P}$ spans $AG(n, q)$. As the embedding is defined by the set of lines $\mathcal{L}$, we will denote the geometry mostly as $\mathcal{S}(\mathcal{L})$ and call it an *affine partial linear space*. We will call $\mathcal{L}$ or $\mathcal{S}(\mathcal{L})$ singular (respectively non-singular) if $\mathcal{P}$ contains (respectively does not contain) a singular point.

We say that $\mathcal{L}$ is a *connected* line set if $\mathcal{S}(\mathcal{L})$ is connected. If $\mathcal{L}$ is not connected, then we call the line sets of the connected components of $\mathcal{S}(\mathcal{L})$ the connected components of $\mathcal{L}$. If $\mathcal{S}(\mathcal{L})$ is an affine partial linear space, then for every affine subspace U, let $\mathcal{L}_U$ be the set of lines of $\mathcal{L}$ in U, and let $\mathcal{S}(\mathcal{L})_U$ be the partial linear space $(\mathcal{P}_U, \mathcal{L}_U, \mathrm{I}_U)$, where $\mathcal{P}_U = \mathcal{P} \cap U$ and I_U is

* Corresponding author.
E-mail addresses: fdc@cage.ugent.be (F. De Clerck), nikiasdefeyter@yahoo.com (N. De Feyter).

0012-365X/$ – see front matter © 2010 Published by Elsevier B.V.
doi:10.1016/j.disc.2010.06.012

the incidence I restricted to $\mathcal{P}_U$ and $\mathcal{L}_U$. Note that $\mathcal{S}(\mathcal{L})_U$ is not the same as $\mathcal{S}(\mathcal{L}_U)$, as $\mathcal{S}(\mathcal{L})_U$ may contain isolated points, that is, points that are on no line, which is not the case for $\mathcal{S}(\mathcal{L}_U)$.

Let $\mathcal{Q}_{n+1}$ be a nonsingular quadric in a finite projective space $\mathrm{PG}(n+1, q)$, $n \geq 1$. Consider a point $r \notin \mathcal{Q}_{n+1}$, distinct from its nucleus if $n+1$ and q are even, and a hyperplane $\mathrm{PG}(n, q)$ not through r. Let $\mathcal{R}_n$ be the projection of the quadric $\mathcal{Q}_{n+1}$ from the point r on the hyperplane $\mathrm{PG}(n, q)$.

If q is odd, $\mathcal{R}_n$ is a point set of class $\left[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3), q+1\right]$ in $\mathrm{PG}(n, q)$. These sets have been classified for $q > 3$ in the case of $\mathrm{PG}(2, q)$, q odd [3] and for $q > 7$ in the case of $\mathrm{PG}(n, q)$, $n > 2$ and q odd [4].

If q is even then $\mathcal{R}_n$ is a point set of class $\left[1, \frac{1}{2}q+1, q+1\right]$ in $\mathrm{PG}(n, q)$. Point sets of class $[1, m, q+1]$ in $\mathrm{PG}(n, q)$, $q > 4$, have been classified. See [15,13,12,9] for more details. The case $q = 4$ is special, see also [10,11]. If $m = \frac{q}{2}+1$, $q > 4$, then a set of type $(1, m, q+1)$ is indeed the projection of a quadric, see Theorem 5.1 further on.

Let $n \geq 3$ and let $\mathcal{T}_n \subseteq \mathcal{R}_n$ be the set of points p of $\mathrm{PG}(n, q)$ such that the line $\langle p, r\rangle$ is a tangent to $\mathcal{Q}_{n+1}$ and let $\mathcal{P}_n = \mathcal{R}_n \setminus \mathcal{T}_n$. Let HT_n be the partial linear space whose points are the elements of $\mathcal{P}_n$ and whose lines are the lines of $\mathrm{PG}(n, q)$ which contain q points of $\mathcal{P}_n$. The geometry HT_n is a partial linear space of antiflag type $(0, 2, q)$. Moreover, if q is even, $\mathcal{T}_n$ is the set of points of a hyperplane Π_∞ of $\mathrm{PG}(n, q)$, hence the geometry HT_n is an affine partial linear space. If n is even, we write HT_n^+ if $\mathcal{Q}_{q+1}$ is a nonsingular hyperbolic quadric, and HT_n^- if $\mathcal{Q}_{n+1}$ is a nonsingular elliptic quadric.

The purpose of this paper is to characterize for q even the geometry HT_n as an affine partial linear space of antiflag type $(0, 2, q)$. The proof is based on earlier investigations of such geometries; we will summarize them in the next section.

2. Known results

In [5] we started the investigation of the affine partial linear spaces of antiflag class $[0, \alpha, q]$. We have given in that paper more or less a complete classification of the partial linear spaces of antiflag class $[0, \alpha, q]$ in $\mathrm{AG}(2, q)$ and $\mathrm{AG}(3, q)$. Actually, as one might easily construct affine geometries of antiflag type $(0, 1, q)$ as well as ones of antiflag type $(0, q-1, q)$, we restrict ourselves from now on to the case $1 < \alpha < q-1$. In that paper we also explained that if the partial linear space is not the linear presentation of a set K of class $[0, 1, \alpha+1, q+1]$ in the hyperplane Π_∞ of the affine space $\mathrm{AG}(n, q)$, $n \geq 4$, then $\alpha = 2$ and q is even. For the definition of a linear representation and for more details we refer to [5]. Finally from the results of that paper, it follows that we may assume in this paper that the partial linear space is non-singular and connected. Hence, from now on we can restrict ourselves to the embedding of non-singular connected partial linear spaces of antiflag class $[0, 2, q]$, in $\mathrm{AG}(n, q)$, $n > 3$ and q even. We will need the following classification in the planar case.

Theorem 2.1 ([5]). *Let $\mathcal{L}$ be a set of lines of an affine plane π, which is of antiflag class $[0, 2, q]$, $q = 2^h$, $h > 1$. Then one of the following cases occurs.*

Type I. *$\mathcal{L}$ is the empty set.*
Type II. *$\mathcal{L}$ consists of a number of parallel lines.*
Type III. *$\mathcal{L}$ is the set of lines of a (Bruck) net of order q and degree 3.*
Type IV. *$\mathcal{L}$ is a dual oval.*
Type V. *$\mathcal{L}$ consists of all lines of π.*

Line sets of $\mathrm{AG}(n, q)$ of antiflag type $(0, \alpha)$ are completely classified, see [6] for an overview. The most difficult part of the classification is the case $\alpha = 2$, see [8], in which case $q = 2^h$, $h > 1$ and the geometry $\mathcal{S}(\mathcal{L})$ is called an affine $(0, 2)$-geometry. Assuming the affine $(0, 2)$-geometry is not a linear representation then the following cases can occur (because of Theorem 2.1, we may assume $n > 2$).

1. The geometry HT_3 which is fully embedded in $\mathrm{AG}(3, q)$ and the geometry HT_4^- which is fully embedded in $\mathrm{AG}(4, q)$. In both cases, every affine plane is of Type I, II or IV. So there are no planes of Type III.
2. A $(0, 2)$-geometry $\mathcal{A}(O_\infty)$ fully embedded in $\mathrm{AG}(3, q)$, which is constructed as follows. Let O_∞ be an oval of Π_∞ with nucleus n_∞. Choose a basis such that $\Pi_\infty : X_3 = 0$, $n_\infty(1, 0, 0, 0)$ and $(0, 1, 0, 0), (0, 0, 1, 0), (1, 1, 1, 0) \in O_\infty$. Let f be the o-polynomial such that
$$O_\infty = \{(\rho, f(\rho), 1, 0) \mid \rho \in \mathrm{GF}(q)\} \cup \{(0, 1, 0, 0)\},$$
and for every affine point $p(x, y, z, 1)$ let
$$O_\infty^p = \{(y + zf(\rho) + \rho, f(\rho), 1, 0) \mid \rho \in \mathrm{GF}(q)\} \cup \{(z, 1, 0, 0)\}.$$
Let S_p be the set of lines through p and a point of O_∞^p. Let $\mathcal{L}$ be the union of the sets S_p, for all affine points p. If O_∞ is not a conic then $\mathcal{S}(\mathcal{L})$ is connected [7] and we put $\mathcal{A}(O_\infty) = \mathcal{S}(\mathcal{L})$. If O_∞ is a conic then $\mathcal{S}(\mathcal{L})$ consists of two connected components, both of which are projectively equivalent with the geometry HT_3 [7]. Therefore we put $\mathcal{A}(O_\infty) = \mathrm{HT}_3$ if O_∞ is a conic. In either case $\mathcal{A}(O_\infty)$ is a $(0, 2)$-geometry with $s = q-1$, $t = q$, fully embedded in $\mathrm{AG}(3, q)$. Every affine plane is a plane of Type I, II or IV. So there are no planes of Type III.
3. A $(0, 2)$-geometry $\mathcal{I}(n, q, e)$ fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, which is constructed as follows. Let U be a hyperplane of $\mathrm{AG}(n, q)$. Choose a basis such that $\Pi_\infty : X_n = 0$ and $U : X_{n-1} = 0$. Let $e \in \{1, 2, \ldots, h-1\}$ be such that $\gcd(e, h) = 1$, and let φ be the collineation of $\mathrm{PG}(n, q)$ such that
$$\varphi : p(x_0, x_1, \ldots, x_{n-1}, x_n) \mapsto p^\varphi(x_0^{2^e}, x_1^{2^e}, \ldots, x_n^{2^e}, x_{n-1}^{2^e}).$$

Put $U_\infty = U \cap \Pi_\infty$ and let $\mathcal{K}_\infty$ be the set of points of U_∞ fixed by φ. Then $\mathcal{K}_\infty$ is the point set of a projective geometry $\mathrm{PG}(n-2, 2) \subseteq U_\infty$. Let $\mathcal{L}$ be the set of affine lines L such that either $L \subseteq U$ and $L \cap \Pi_\infty \in \mathcal{K}_\infty$, or L intersects U in an affine point p and L intersects Π_∞ in the point p^φ. Then $\mathcal{I}(n, q, e) = \mathcal{S}(\mathcal{L})$ is a $(0, 2)$-geometry with $s = q - 1$, $t = 2^{n-1} - 1$, fully embedded in $\mathrm{AG}(n, q)$.
The hyperplane U has the property that, for every affine plane π containing two intersecting lines of $\mathcal{I}(n, q, e)$, π is of Type III if $\pi \subseteq U$ and π is of Type IV if $\pi \not\subseteq U$. In particular, if $n = 3$, then U is the only plane of Type III.

In [5] the following results have been proved.

Theorem 2.2 ([5]). *Let $\mathcal{L}$ be a connected line set of $\mathrm{AG}(n, q)$ of antiflag class $[0, \alpha, q]$, with $1 < \alpha < q - 1$, such that there are no planes of Type IV, and such that the lines of $\mathcal{L}$ span $\mathrm{AG}(n, q)$. Then $\mathcal{L}$ is the line set of a linear representation $T^*_{n-1}(\mathcal{K}_\infty)$ of a point set $\mathcal{K}_\infty$ of class $[0, 1, \alpha + 1, q + 1]$.*

Theorem 2.3 ([5]). *Let $\mathcal{L}$ be a non-singular connected line set of $\mathrm{AG}(3, q)$ of antiflag class $[0, 2, q]$, $q = 2^h$, $h > 1$. If $\mathcal{L}$ is not the set of all lines of $\mathrm{AG}(3, q)$, and if $\mathcal{S}(\mathcal{L})$ is not a linear representation then either $\mathcal{S}(\mathcal{L})$ is the geometry $\mathcal{A}(O_\infty)$ or the geometry $\mathcal{I}(3, q, e)$.*

In the rest of this paper we will treat the case $n > 3$. If $\mathcal{L}_U$ is connected and is the line set of a linear representation then, following the notation of [8], we say that the subspace U is of type **C**, and we let $P_\infty(U)$ denote the set of points at infinity of the lines of $\mathcal{L}_U$. Notice that since $\mathcal{L}_U$ is connected, $P_\infty(U)$ spans $U \cap \Pi_\infty$. If $\mathcal{L}_U$ is the set of all affine lines of U, then we say that U is of type **E**. Notice that, by definition, a subspace of type **E** is also a subspace of type **C**.
We say that U is of type **C*** if U is of type **C** and $P_\infty(U)$ contains a point p_∞ and a line L_∞ not through p_∞, such that the number of lines through p_∞ which are contained in $P_\infty(U)$ and intersect L_∞, is not equal to 1 or $q + 1$.
If π is a plane of Type III or V, then analogously $P_\infty(\pi)$ denotes the set of points at infinity of the lines of $\mathcal{L}_\pi$.

3. On the non-existence of a special class of affine connected line sets

Lemma 3.1. *Let $\mathcal{L}$ be a connected line set of $\mathrm{AG}(n, q)$, $n \geq 4$, $q = 2^h$, $h > 1$, of antiflag type $(0, 2, q)$, such that there is a hyperplane U of type **C** and a plane π of Type III which intersects U in a line $L \in \mathcal{L}$, such that the point $p_\infty = L \cap \Pi_\infty$ is not a singular point of $P_\infty(U)$. Then there is no plane of Type IV.*

Proof. Note that by Theorem 2.3, the theorem holds in the case $n = 3$. We will prove the theorem by using induction and hence we assume that the theorem holds for all $m < n$, $n \geq 4$.
Let $U_\infty = U \cap \Pi_\infty$. Since p_∞ is not a singular point of $P_\infty(U)$, there is a line $L_\infty \subseteq U_\infty$ through p_∞ which contains exactly three points of $P_\infty(U)$. Using the fact that $P_\infty(U)$ is a point set of type $(0, 1, 3, q + 1)$ which spans U_∞, it is easy to show that there must also be a second line $L'_\infty \neq L_\infty$ through p_∞ which contains exactly three points of $P_\infty(U)$. Let V_∞ be an $(n - 3)$-space through p_∞ such that $L'_\infty \subseteq V_\infty$, $L_\infty \not\subseteq V_\infty$ and $P_\infty(U) \cap V_\infty$ spans V_∞. Let $\pi' = \langle L, L_\infty\rangle$ and let $V = \langle L, V_\infty\rangle$. Then π' is a plane of Type III, V is an $(n - 2)$-space of type **C** (respectively a plane of Type III if $n = 4$), $\pi' \cap V = L$ and $p_\infty = L \cap \Pi_\infty$ is not a singular point of $P_\infty(V)$.
Let $U' \neq U$ be a hyperplane of $\mathrm{AG}(n, q)$ parallel to U. Let $W = \langle \pi, V\rangle$ and $V' = U' \cap W$, and let $X = \langle \pi, \pi'\rangle$ and $\pi'' = U' \cap X$. Then by the induction hypothesis, there are no planes of Type IV in W or X. By Theorem 2.2, W and X are of type **C**. Hence V' is of type **C** (respectively of Type III if $n = 4$) with $P_\infty(V') = P_\infty(V)$, and π'' is of Type III with $P_\infty(\pi'') = P_\infty(\pi')$. Now $\pi'' \cap V'$ is a line $L' \in \mathcal{L}$, and $p_\infty = L' \cap \Pi_\infty$ is not a singular point of $P_\infty(V')$. By the induction hypothesis, there are no planes of Type IV in U'. By Theorem 2.2, U' is of type **C**.
We prove that $P_\infty(U') = P_\infty(U)$. Let $p'_\infty \neq p_\infty$ be a point of $P_\infty(U)$. Let $\pi_1 = \langle L, p'_\infty\rangle$, $X_1 = \langle \pi_1, \pi\rangle$ and $\pi'_1 = X_1 \cap U'$. Then π_1 is of Type III or V. As π is of Type III, Theorem 2.3 implies that X_1 is of type **C**. Hence π'_1 is of the same type as π_1, and $p'_\infty \in P_\infty(\pi'_1)$. Since U' is of type **C**, $p'_\infty \in P_\infty(U')$. It follows that $P_\infty(U) \subseteq P_\infty(U')$. Analogously $P_\infty(U') \subseteq P_\infty(U)$, so $P_\infty(U') = P_\infty(U)$.
We conclude that every hyperplane U' parallel to U is of type **C** and has $P_\infty(U') = P_\infty(U)$. Suppose that there is a plane π' of Type IV. Clearly π' is not parallel to U. Let $L' = \pi' \cap U$. Since π' is of Type IV, $\mathcal{L}_{\pi'}$ contains exactly one line L'' parallel to L'. Let U' be the hyperplane parallel to U which contains L''. Then $p'_\infty = L'' \cap \Pi_\infty \in P_\infty(U')$. Hence $p'_\infty \in P_\infty(U'')$ for all hyperplanes U'' parallel to U. But now $\mathcal{L}_{\pi'}$ contains every line parallel to L', a contradiction. So there are no planes of Type IV. □

Theorem 3.2. *A connected line set of $\mathrm{AG}(n, q)$, $n \geq 4$, $q = 2^h$, $h > 1$, of antiflag type $(0, 2, q)$, such that there is a hyperplane of type **C*** and a plane of Type IV, does not exist.*

Proof. Suppose that such a line set $\mathcal{L}$ does exist. Let U be a hyperplane of type **C***. By Lemma 3.1, we must only prove that there is a plane of Type III which intersects U in a line L of $\mathcal{L}$, and that $L \cap \Pi_\infty$ is not a singular point of $P_\infty(U)$.
Let $U_\infty = U \cap \Pi_\infty$. Since U is of type **C***, there is a point $p_\infty \in P_\infty(U)$ and a line $L_\infty \subseteq P_\infty(U)$ not through p_∞ such that the number of lines through p_∞ intersecting L_∞ which are completely contained in $P_\infty(U)$, is not equal to 1 or $q + 1$. Since $P_\infty(U)$ is a point set of type $(0, 1, 3, q + 1)$, it follows that through every point of L_∞ there is at least one line in the plane $\langle p_\infty, L_\infty\rangle$ which contains exactly three points of $P_\infty(U)$. It follows that no point of L_∞ is a singular point of $P_\infty(U)$.
Let π be a plane of U such that $\pi \cap \Pi_\infty = L_\infty$, and let p be an affine point of π. Then π is a plane of Type V. Since the set θ_p of points at infinity of the lines of $\mathcal{L}$ through p spans Π_∞, there is a line $L \in \mathcal{L}$ which intersects U in the point p.

Please cite this article in press as: F. De Clerck, N. De Feyter, A new characterization of projections of quadrics in finite projective spaces of even characteristic, Discrete Mathematics (2010), doi:10.1016/j.disc.2010.06.012

Consider the 3-space $V = \langle \pi, L\rangle$ and the connected component $\mathcal{L}'$ of $\mathcal{L}_V$ which contains the line L and the affine lines of π. Since $\mathcal{L}'$ has a plane of Type V, Theorem 2.3 implies that either $\mathcal{L}'$ is singular, or $\mathcal{L}'$ is nonsingular and the line set of a linear representation. In the last case, there is a plane of Type III which intersects π (and U) in a line M of $\mathcal{L}$. As the point $M \cap \Pi_\infty$ is on L_∞, it is not a singular point of $P_\infty(U)$.

Suppose that $\mathcal{L}'$ is singular. Then there is a plane π' of Type V in V which intersects π (and U) in an affine line L'. Let $p'_\infty = L' \cap \Pi_\infty$. Since $p'_\infty \in L_\infty$, there is a line L'_∞ through p'_∞ in the plane $\langle p_\infty, L_\infty\rangle$ which contains exactly three points of $P_\infty(U)$. Hence the plane $\pi'' = \langle L', L'_\infty\rangle$ is a plane of Type III. Let V' be the 3-space $\langle \pi', \pi''\rangle$, and let $\mathcal{L}''$ be the connected component of $\mathcal{L}_{V'}$ which contains the lines of $\mathcal{L}$ in the planes π' and π''. By Theorem 2.3, $\mathcal{L}''$ is either singular with vertex a point and base a planar net, or $\mathcal{L}''$ is nonsingular and the line set of a linear representation. In either case, there is a plane π_0 of Type III in V' which intersects U in the line L' of $\mathcal{L}$. Again, since $p'_\infty = L' \cap \Pi_\infty$ is on the line L_∞, it is not a singular point of $P_\infty(U)$. □

Theorem 3.3. *A connected line set of* $\mathrm{AG}(n, q)$, $n > 4$, $q = 2^h$, $h > 1$, *of antiflag type* $(0, 2, q)$, *such that there is a 3-space of type* $\mathbf{C}^*$ *and a plane of Type* IV, *does not exist.*

Proof. We will prove the theorem by induction on the dimension of the affine space. Hence suppose that such a line set $\mathcal{L}$ does exist in $\mathrm{AG}(n, q)$, but does not exist in $\mathrm{AG}(m, q)$, $m < n$.

Let W be a 3-space of type $\mathbf{C}^*$, and let p be an affine point of W, then θ_p spans Π_∞ (see [5]). Let U be a hyperplane containing W such that the lines of $\mathcal{L}$ in U through p, span U. Let $\mathcal{L}'$ be the connected component of $\mathcal{L}_U$ containing the lines of $\mathcal{L}_W$ and the lines of $\mathcal{L}_U$ through p. By the induction hypothesis, $\mathcal{L}'$ does not have any planes of Type IV. By Theorem 2.2, $\mathcal{L}'$ is the line set of a linear representation, and so U is of type $\mathbf{C}$. Since W is of type $\mathbf{C}^*$ and $W \subseteq U$, U is a hyperplane of type $\mathbf{C}^*$. Now by Theorem 3.2, we are done. □

4. The Shult spaces at infinity

A *Shult space* $\mathcal{S}$ (see for instance [2]) is a partial linear space of order (s, t), with the property that for every antiflag $\{p, L\}$, the incidence number $\alpha(p, L)$ is either 1 or $s + 1$. If the case $s + 1$ does not occur, then $\mathcal{S}$ is a *generalized quadrangle*. The *radical* of a Shult space $\mathcal{S}$ is the set of points of $\mathcal{S}$ which are collinear with all points of $\mathcal{S}$. A Shult space is said to be *degenerate* if its radical is not empty.

In order to finish our classification it will become clear that we need a classification of the Shult spaces, fully embedded in a projective space, in other words of projective Shult spaces. Projective generalized quadrangles were classified by Buekenhout and Lefèvre [1], while general projective Shult spaces were classified by Lefèvre-Percsy [14]. We summarize the results in the next theorem.

Theorem 4.1 ([1,14]). *Let $\mathcal{S}$ be a Shult space fully embedded in* $\mathrm{PG}(n, q)$. *Then one of the following cases occurs.*

1. *$\mathcal{S}$ is the geometry of all points and all lines of* $\mathrm{PG}(n, q)$. *The radical is* $\mathrm{PG}(n, q)$ *itself.*
2. *The point set of $\mathcal{S}$ is the union of k subspaces of dimension $m + 1$ through a given m-space U, $k > 1$, $0 \le m \le n - 2$. The line set is the set of all lines in these $(m + 1)$-spaces. The radical of $\mathcal{S}$ is U.*
3. *$\mathcal{S}$ is formed by the points and lines of a quadric $\mathcal{Q}$ (of projective index at least 1) of* $\mathrm{PG}(n, q)$, $n \ge 3$. *The radical of $\mathcal{S}$ is the space of all singular points of $\mathcal{Q}$.*
4. *q is a square and $\mathcal{S}$ is formed by the points and the lines of a Hermitian variety $\mathcal{H}$ (of projective index at least 1) of* $\mathrm{PG}(n, q)$, $n \ge 3$. *The radical of $\mathcal{S}$ is the space of all singular points of $\mathcal{H}$.*
5. *The points of $\mathcal{S}$ are the points of* $\mathrm{PG}(n, q)$. *There is an m-space U and an $(n - m - 1)$-space W skew to U, with $m \ge -1$, $n - m - 1 \ge 3$ and odd, and a symplectic polarity β in W, such that the line set is the set of all lines in the $(m + 2)$-spaces joining U to a line of W which is totally isotropic with respect to β. The radical of $\mathcal{S}$ is U.*

Theorem 4.2. *Let $\mathcal{L}$ be a connected line set of* $\mathrm{AG}(n, q)$, $n \ge 4$, $q = 2^h$, $h > 1$, *of antiflag type* $(0, 2, q)$, *such that there are no 3-spaces of type* $\mathbf{C}^*$. *Then for every point p of $\mathcal{S}(\mathcal{L})$, either the set θ_p of points at infinity of the lines of $\mathcal{L}$ through p does not contain any lines, or the incidence structure of points and lines contained in θ_p, is a Shult space fully embedded in Π_∞.*

Proof. Let p be a point of $\mathcal{S}(\mathcal{L})$, and suppose that θ_p contains a point p_∞ and a line L_∞ such that $p_\infty \notin L_\infty$. Let $\pi_\infty = \langle p_\infty, L_\infty\rangle$ and let U be the 3-space $\langle p, \pi_\infty\rangle$. Consider the connected component $\mathcal{L}'$ of $\mathcal{L}_U$ which contains $\langle p, p_\infty\rangle$ and the affine lines of the plane $\langle p, L_\infty\rangle$. By Theorem 2.3, $\mathcal{L}'$ is either nonsingular and the line set of a linear representation, or $\mathcal{L}'$ is singular with vertex either a point, or the plane π_∞. If $\mathcal{L}'$ is singular, then either $\theta_p \cap \pi_\infty$ is the union of two or three concurrent lines of π_∞, or $\pi_\infty \subseteq \theta_p$. So the number of lines through p_∞ which intersect L_∞ and are contained in θ_p, is either 1 or $q + 1$. If $\mathcal{L}'$ is nonsingular and the line set of a linear representation, then since U is not of type $\mathbf{C}^*$, the same conclusion holds. □

Lemma 4.3. *Let $\mathcal{L}$ be a connected line set of* $\mathrm{AG}(n, q)$, $n \ge 4$, $q = 2^h$, $h > 1$, *of antiflag type* $(0, 2, q)$. *Let p be a point of $\mathcal{S}(\mathcal{L})$. A point $p_\infty \in \Pi_\infty$ is a singular point of θ_p if and only if it is a singular point of $\mathcal{L}$.*

Proof. Let $L = \langle p, p_\infty\rangle$. Notice that p_∞ is a singular point of θ_p if and only if $p_\infty \in \theta_p$ and every plane through L is of Type II or V. If p_∞ is a singular point of $\mathcal{L}$, then it is easily seen that p_∞ is a singular point of θ_p.

Suppose that p_∞ is a singular point of θ_p. We prove that p_∞ is a singular point of θ_r for every point r of $\mathcal{S}(\mathcal{L})$, whence p_∞ is a singular point of $\mathcal{L}$. By connectedness, we only have to consider the case that r is collinear to p. So assume that $r \neq p$ and r is collinear to p. If $r \in L$ then $p_\infty \in \theta_r$ and every plane through L is of Type II or V. Hence p_∞ is a singular point of θ_r.

Suppose that $r \notin L$. Since the line $L' = \langle p, r \rangle$ is a line of $\mathcal{L}$, the plane $\pi = \langle L', p_\infty \rangle$ is a plane of Type V. So $M = \langle r, p_\infty \rangle \in \mathcal{L}$, in other words, $p_\infty \in \theta_r$. Suppose that some plane $\pi' \neq \pi$ containing M is not of Type II. Then by Theorem 2.1, it is of Type III, IV or V. In any case, $\mathcal{L}_{\pi'}$ contains a line M' through r, distinct from the line M. Now $\alpha(p, M') > 0$, so there is a line $L'' \neq L'$ of $\mathcal{L}$ through p which intersects M' in an affine point. Since p_∞ is a singular point of θ_p, the plane $\pi'' = \langle L'', p_\infty \rangle$ is a plane of Type V.

Consider the 3-space $U = \langle \pi, \pi' \rangle$. Then $\pi'' \subseteq U$ and the affine lines of the planes π and π'' and the line M' are all in the same connected component $\mathcal{L}'$ of $\mathcal{L}_U$. As $\mathcal{L}'$ has two distinct planes of Type V, namely π and π'', Theorem 2.3 implies that $\mathcal{L}'$ is either singular with vertex the point p_∞ or the plane $\pi_\infty = U \cap \Pi_\infty$, or $\mathcal{L}'$ is nonsingular and the line set of a linear representation. But since p_∞ is a singular point of θ_p, it is also a singular point of $\theta_p \cap \pi_\infty$. So $\mathcal{L}'$ cannot be a nonsingular linear representation. Hence $\mathcal{L}'$ is singular and p_∞ is a singular point of $\mathcal{L}'$. So the plane $\pi' = \langle M', p_\infty \rangle$ is a plane of Type V. We conclude that every plane through the line $M = \langle r, p_\infty \rangle$ is a plane of Type II or of Type V. Hence p_∞ is a singular point of θ_r. □

Lemma 4.4. *Let $\mathcal{L}$ be a nonsingular connected line set of* $\mathrm{AG}(n, q)$, $n \geq 4$, $q = 2^h$, $h > 1$, *of antiflag type* $(0, 2, q)$, *such that there are no* 3*-spaces of type* $\mathbf{C}^*$. *Then for any two points p, p' of $\mathcal{S}(\mathcal{L})$, the projective index $g(\theta_p)$ of θ_p, that is, the maximal dimension of a subspace contained in θ_p, equals the projective index $g(\theta_{p'})$ of $\theta_{p'}$.*

Proof. We give the proof in the case p and p' are distinct and collinear. The theorem then follows by connectedness. Let $p_\infty = \langle p, p' \rangle$. Then $p_\infty \in \theta_p$. If θ_p does not contain any lines, then $U_\infty = p_\infty$ is a subspace of dimension $g(\theta_p)$ through p_∞, contained in θ_p. Suppose that θ_p contains a line. By Theorem 4.2 and Lemma 4.3, θ_p is the point set of a nondegenerate Shult space fully embedded in Π_∞. By Theorem 4.1, there is a subspace $U_\infty \subseteq \Pi_\infty$ of dimension $g(\theta_p)$ through p_∞, contained in θ_p.

Let $U = \langle p, U_\infty \rangle$. Then U is of type $\mathbf{E}$ and contains p', so $U_\infty \subseteq \theta_{p'}$. It follows that $g(\theta_{p'}) \geq g(\theta_p)$. Analogously, one proves that $g(\theta_{p'}) \leq g(\theta_p)$, and hence we may conclude that $g(\theta_{p'}) \leq g(\theta_p)$. □

Theorem 4.5. *Let $\mathcal{L}$ be a nonsingular connected line set of* $\mathrm{AG}(n, q)$, $n \geq 4$, $q = 2^h$, $h > 1$, *of antiflag type* $(0, 2, q)$, *such that there is a plane of Type* IV *and a plane of Type* V, *but no* 3*-spaces of type* $\mathbf{C}^*$. *Then for every point p of $\mathcal{S}(\mathcal{L})$, θ_p is a nonsingular quadric in Π_∞ of projective index at least one. Moreover, if $n - 1$ is odd, then all quadrics θ_p are of the same character.*

Proof. Since there is a plane of Type V, there is a point r of $\mathcal{S}(\mathcal{L})$ such that θ_r contains a line. Hence by Lemma 4.4, for every point p of $\mathcal{S}(\mathcal{L})$, the set θ_p contains a line. By Theorem 4.2, for every point p of $\mathcal{S}(\mathcal{L})$, the incidence structure of points and lines contained in θ_p, which will be denoted by $\mathcal{S}_p$, is a Shult space fully embedded in Π_∞. Theorem 4.1 tells us exactly what $\mathcal{S}_p$ looks like. Notice that, since every line of Π_∞ which is contained in the point set of $\mathcal{S}_p$, is a line of $\mathcal{S}_p$, $\mathcal{S}_p$ cannot be of symplectic polarity type. By Lemma 4.3 and the remark preceding it, $\mathcal{S}_p$ is nondegenerate. So by Theorem 4.1, for every point p of $\mathcal{S}(\mathcal{L})$, either θ_p is a nonsingular Hermitian variety in Π_∞ of projective index at least 1 (so q is a square), or θ_p is a nonsingular quadric in Π_∞ of projective index at least 1.

Suppose that q is a square and there is a point p of $\mathcal{S}(\mathcal{L})$ such that θ_p is a nonsingular Hermitian variety in Π_∞. Then θ_p is a point set of type $(1, \sqrt{q} + 1, q + 1)$. By Theorem 2.1, every line of Π_∞ intersects θ_p in 0, 1, 2, 3 or $q + 1$ points. So $q = 4$.

Let π_∞ be a plane of Π_∞ such that $\pi_\infty \cap \theta_p$ is a nonsingular Hermitian curve, and let $W = \langle p, \pi_\infty \rangle$. Let $\mathcal{L}'$ be the connected component of $\mathcal{L}_W$ containing the lines through p. Let L^1_∞ and L^2_∞ be distinct lines of π_∞ which intersect θ_p in three points. Then by Theorem 2.1, $\pi_1 = \langle p, L^1_\infty \rangle$ and $\pi_2 = \langle p, L^2_\infty \rangle$ are distinct planes of Type III with respect to $\mathcal{L}'$. By Theorem 2.3, and since the geometry $\mathcal{A}(O_\infty)$ has no planes of Type III, and the geometry $\mathcal{I}(3, q, e)$ has only one plane of Type III, W is of type $\mathbf{C}$ and $P_\infty(W) = \theta_p \cap \pi_\infty$ is a nonsingular Hermitian curve.

Suppose that there is an affine m-space V, $3 \leq m \leq n - 1$, such that $p \in V$, $V_\infty = V \cap \Pi_\infty$ intersects θ_p in a nonsingular Hermitian variety, and V is of type $\mathbf{C}$. Let $U_\infty \subseteq \Pi_\infty$ be an m-space containing V_∞, such that $U_\infty \cap \theta_p$ is a nonsingular Hermitian variety, and let $U = \langle p, U_\infty \rangle$. Let $p_\infty \in P_\infty(V) = V_\infty \cap \theta_p$. Then there is a line $L_\infty \subseteq U_\infty$ such that $L_\infty \cap V_\infty = p_\infty$ and $|L_\infty \cap \theta_p| = 3$. The plane $\pi = \langle p, L_\infty \rangle$ is a plane of Type III which intersects V in the line $L = \langle p, p_\infty \rangle \in \mathcal{L}$. Furthermore $p_\infty = L \cap \Pi_\infty$ is not a singular point of $P_\infty(V)$, as $P_\infty(V)$ has no singular points. By Lemma 3.1, the connected component $\mathcal{L}'$ of $\mathcal{L}_U$ containing the lines of $\mathcal{L}_V$ and $\mathcal{L}_\pi$ has no planes of Type IV. If $m < n - 1$, then Theorem 2.2 implies that U is of type $\mathbf{C}$. We recall that U_∞ intersects θ_p is a nonsingular Hermitian variety. If $m = n - 1$, then $U = \mathrm{AG}(n, q)$ and $\mathcal{L}' = \mathcal{L}$ by connectedness. So there are no planes of Type IV at all, a contradiction.

Repetition of the above reasoning leads to a contradiction. So there is no point p of $\mathcal{S}(\mathcal{L})$ such that θ_p is a nonsingular Hermitian variety. It follows that for all points p of $\mathcal{S}(\mathcal{L})$, θ_p is a nonsingular quadric in Π_∞ of projective index at least 1. If $n - 1$ is odd, then by Lemma 4.4 all the quadrics θ_p are of the same character. □

5. Characterization of the line set of HT_n

In this section, we complete the characterization of the line set of the geometry HT_n. We rely heavily on the following result, due to Hirschfeld and Thas [13].

Theorem 5.1 ([13]). *If $\mathcal{K}$ is a nonsingular point set of type $\left(1, \frac{1}{2}q + 1, q + 1\right)$ in* $\mathrm{PG}(n, q)$ *with $n \geq 4$ and $q = 2^h$, $h > 2$, then $\mathcal{K} = \mathcal{R}_n$. For $q = 4$ the same conclusion holds if there is no plane intersecting $\mathcal{K}$ in a unital or a Baer subplane.*

Please cite this article in press as: F. De Clerck, N. De Feyter, A new characterization of projections of quadrics in finite projective spaces of even characteristic, Discrete Mathematics (2010), doi:10.1016/j.disc.2010.06.012

Lemma 5.2. *Consider the geometry* HT_n *in* $\mathrm{AG}(n, q)$, $n \geq 3$, $q = 2^h$, $h > 1$. *A plane of Type* IV *does not contain any isolated points of* HT_n.

Proof. Consider the nonsingular quadric $\mathcal{Q}_{n+1}$ in $\mathrm{PG}(n+1, q)$ and the point r of $\mathrm{PG}(n+1, q)$ such that $\mathcal{R}_n$ is the projection of $\mathcal{Q}_{n+1}$ from r onto a hyperplane of $\mathrm{PG}(n+1, q)$ containing $\mathrm{AG}(n, q)$ as an affine subgeometry. Let π be a plane of Type IV. Then the 3-space $\langle r, \pi \rangle$ intersects $\mathcal{Q}_{n+1}$ in a nonsingular hyperbolic quadric $\mathrm{Q}^+(3, q)$. As every point of $\mathrm{Q}^+(3, q)$ is on a line of $\mathrm{Q}^+(3, q)$, every point of HT_n in π is on a line of HT_n in π. □

Lemma 5.3. *Let* $\mathcal{L}$ *be a nonsingular connected line set of* $\mathrm{AG}(n, q)$, $n \geq 4$, $q = 2^h$, $h > 1$, *of antiflag type* $(0, 2, q)$, *such that for some point* p *of* $\mathcal{S}(\mathcal{L})$, θ_p *is a nonsingular quadric. Then for all points* r *of* $\mathcal{S}(\mathcal{L})$, θ_r *is a nonsingular quadric. If there is no plane of Type* V, *then* $n = 4$ *and* $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_4^-$.

Proof. Let p be a point of $\mathcal{S}(\mathcal{L})$ such that θ_p is a nonsingular quadric. Clearly $\mathcal{S}(\mathcal{L})$ is not a linear representation. By Theorems 2.2 and 3.3, there is a plane of Type IV but no 3-space of type $\mathbf{C}^*$. If there is a plane of Type V, then we are done by Theorem 4.5.

Suppose that there is no plane of Type V. Then $\mathcal{S}(\mathcal{L})$ is a $(0, 2)$-geometry fully embedded in $\mathrm{AG}(n, q)$ and hence, $\mathcal{S}(\mathcal{L}) = \mathcal{I}(n, q, e)$, or $n = 4$ and $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_4^-$. Since θ_p is a nonsingular quadric, $\mathcal{S}(\mathcal{L}) \neq \mathcal{I}(n, q, e)$. Note that in the case $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_4^-$, θ_r is indeed a nonsingular elliptic quadric in Π_∞, for every point r of $\mathcal{S}(\mathcal{L})$. □

Theorem 5.4. *Let* $\mathcal{L}$ *be a nonsingular connected line set of* $\mathrm{AG}(n, q)$, $n \geq 4$, $q = 2^h$, $h > 1$, *of antiflag type* $(0, 2, q)$, *such that for every point* p *of* $\mathcal{S}(\mathcal{L})$, θ_p *is a nonsingular quadric in* Π_∞. *If* n *is odd then* $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_n$, *if* n *is even then* $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_n^+$ *if* $\theta_p = \mathrm{Q}^+(n-1, q)$ *and* $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_n^-$ *if* $\theta_p = \mathrm{Q}^-(n-1, q)$.

Proof. We will prove the theorem by induction. So assume that the theorem holds for all $m < n$. By Lemmas 4.4 and 5.3, we may assume that for every point p of $\mathcal{S}(\mathcal{L})$, θ_p is a nonsingular quadric of projective index at least one.

Step 1. *Intersection with hyperplanes*. Let U be an affine hyperplane, and let $U_\infty = U \cap \Pi_\infty$. Let p be a point of $\mathcal{S}(\mathcal{L})$ in U, and let $\mathcal{L}'$ be the connected component of $\mathcal{L}_U$ containing the lines through p. Since θ_p is a nonsingular quadric in Π_∞, $\theta_p \cap U_\infty$ is either a nonsingular quadric or a cone with vertex a point p_∞ and base a nonsingular quadric in an $(n-3)$-space V_∞ of Π_∞ not containing p_∞. If θ_p is a nonsingular quadric, then by Lemma 5.3 and the induction hypothesis, $\mathcal{S}(\mathcal{L}') = \mathrm{HT}_{n-1}$.

Suppose that $\theta_p \cap U_\infty$ is a cone with vertex a point p_∞ and base a nonsingular quadric in an $(n-3)$-space V_∞ of Π_∞ not containing p_∞. By Lemma 4.3, p_∞ is a singular point of $\mathcal{L}'$. Let $V = \langle p, V_\infty \rangle$ and let $\mathcal{L}''$ be the connected component of $\mathcal{L}_V$ which contains the lines through p. If $n = 4$ then $\theta_p = \mathrm{Q}^+(3, q)$. So U_∞ is a tangent plane to θ_p, and V_∞ is a line containing two points of θ_p. By Theorem 2.1, V is a plane of Type IV. So $\mathcal{L}''$ is a dual oval and $\mathcal{L}'$ is the singular line set with vertex p_∞ and base the dual oval $\mathcal{L}''$ in the plane V. If $n \geq 5$, then by Lemma 5.3 and the induction hypothesis, $\mathcal{S}(\mathcal{L}'') \simeq \mathrm{HT}_{n-2}$. So $\mathcal{L}'$ is the singular line set with vertex p_∞ and base $\mathcal{L}''$.

So every point p of $\mathcal{S}(\mathcal{L})$ in U is on a line of a connected component $\mathcal{L}'$ of $\mathcal{L}_U$, such that $\mathcal{L}'$ is either the line set of HT_{n-1} or the singular line set with vertex a point $p_\infty \in U_\infty$ and base a dual oval if $n = 4$ or the line set of HT_{n-2} if $n \geq 5$. One verifies that only the following possibilities can occur.

1. $\mathcal{L}_U$ consists of only one connected component which is of one of the described types.
2. $\mathcal{L}_U$ consists of two nonsingular connected components $\mathcal{L}_1$ and $\mathcal{L}_2$. If n is even then $\mathcal{L}_1$ and $\mathcal{L}_2$ are both line sets of a geometry HT_{n-1}. If n is odd then $\mathcal{S}(\mathcal{L}_1) = \mathrm{HT}_{n-1}^+$ and $\mathcal{S}(\mathcal{L}_2) = \mathrm{HT}_{n-1}^-$. In both cases, the point sets of $\mathcal{S}(\mathcal{L}_1)$ and $\mathcal{S}(\mathcal{L}_2)$ partition the set of all affine points of U.
3. $n \geq 5$ and $\mathcal{L}_U$ consists of two singular connected components $\mathcal{L}_1$ and $\mathcal{L}_2$ with vertex the same point $p_\infty \in U_\infty$ and base nonsingular line sets $\mathcal{L}_1'$ and $\mathcal{L}_2'$, respectively. If n is odd then $\mathcal{L}_1'$ and $\mathcal{L}_2'$ are both line sets of a geometry HT_{n-2}. If n is even then $\mathcal{S}(\mathcal{L}_1') = \mathrm{HT}_{n-2}^+$ and $\mathcal{S}(\mathcal{L}_2') = \mathrm{HT}_{n-2}^-$. In both cases, the point sets of $\mathcal{S}(\mathcal{L}_1)$ and $\mathcal{S}(\mathcal{L}_2)$ partition the set of all affine points of U.

In particular, the set of points of $\mathcal{S}(\mathcal{L})$ in U is a point set of type $(0, \frac{1}{2}q, q)$. Let $\mathcal{P}$ be the point set of $\mathcal{S}(\mathcal{L})$. Then for every affine hyperplane U, $\mathcal{P} \cap U$ is a point set of type $(0, \frac{1}{2}q, q)$. Hence $\mathcal{R} = \mathcal{P} \cup \Pi_\infty$ is a point set of type $\left(1, \frac{1}{2}q + 1, q + 1\right)$ in $\mathrm{PG}(n, q)$.

Step 2. *If* $\mathcal{R}$ *is nonsingular, then* $\mathcal{S}(\mathcal{L}) = \mathbf{HT}_n$. Suppose that $\mathcal{R}$ is nonsingular. Then by Theorem 5.1, $\mathcal{R} = \mathcal{R}_n$ (notice that if $q = 4$, no plane intersects it in a Baer subplane or a unital, since every affine plane intersects Π_∞ in a line). Since the lines of HT_n are precisely the affine lines which are contained in $\mathcal{R}_n$, $\mathcal{L}$ must be a subset of the line set of HT_n. Using the fact that for every point $p \in \mathcal{P}$, θ_p is a nonsingular quadric, it easily follows that $\mathcal{L}$ is the line set of HT_n, and assuming n is even, that $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_n^+$ if $\theta_p = \mathrm{Q}^+(n-1, q)$ and $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_n^-$ if $\theta_p = \mathrm{Q}^-(n-1, q)$.

Step 3. *If* $\mathcal{R}$ *is singular, then* $\mathcal{P}$ *is the set of all affine points*. Suppose that $\mathcal{R}$ is singular, with singular point p. Let U be an affine hyperplane containing p. Then $\mathcal{R} \cap U$ is also a singular point set of type $\left(1, \frac{1}{2}q + 1, q + 1\right)$ with singular point p. Above, we have deduced how $\mathcal{L}_U$ looks like, and hence what the set $\mathcal{P} \cap U$ looks like. We stress that since $n \geq 4$, for every point p of $\mathcal{P}$ in U, θ_p has a nonempty intersection with $U_\infty = U \cap \Pi_\infty$. So every point of $\mathcal{P} \cap U$ is on a line of a connected component of $\mathcal{L}_U$.

Since $\mathcal{R} \cap U$ must be a singular set, it follows that either every affine point of U is a point of $\mathcal{P}$, or $\mathcal{L}_U$ consists of one connected component which is singular. Suppose that we are in the last case. Then clearly $p = p_\infty$ is in Π_∞ and is the vertex of $\mathcal{L}_U$.

Please cite this article in press as: F. De Clerck, N. De Feyter, A new characterization of projections of quadrics in finite projective spaces of even characteristic, Discrete Mathematics (2010), doi:10.1016/j.disc.2010.06.012

Let r be a point of $\mathcal{P} \cap U$. Since $\mathcal{L}_U$ is singular, $\theta_r \cap U_\infty$ must be a singular quadric with vertex the point p_∞. Hence U_∞ is the tangent $(n-2)$-space to the nonsingular quadric θ_r at p_∞.

Let U' be an affine hyperplane containing p_∞ and r, such that $U'_\infty = U' \cap \Pi_\infty \neq U_\infty$. Then $\mathcal{R} \cap U'$ must be a singular set with singular point p_∞. So analogously, either every affine point of U' is a point of $\mathcal{P}$, or $\mathcal{L}_{U'}$ consists of one connected component which is singular, with singular point p_∞. But the latter would again imply that U'_∞ is the tangent $(n-2)$-space to the nonsingular quadric θ_r at p_∞. So $U'_\infty = U_\infty$, a contradiction. It follows that for every affine hyperplane U' containing the points p_∞ and r such that $U'_\infty \neq U_\infty$, every affine point of U' is a point of $\mathcal{P}$. But then every affine point of U is a point of $\mathcal{P}$, a contradiction. We conclude that for every affine hyperplane U containing the singular point p of $\mathcal{R}$, every affine point of U is a point of $\mathcal{P}$. So $\mathcal{P}$ is the set of all affine points of $\mathrm{AG}(n, q)$.

Step 4. *$\mathcal{P}$ cannot be the set of all affine points*. Suppose that $\mathcal{P}$ is the set of all affine points. Let p be a point of $\mathcal{P}$ and U_∞ an $(n-2)$-space in Π_∞ such that $\theta_p \cap U_\infty$ is a nonsingular quadric, and let $U = \langle p, U_\infty \rangle$. Then the connected component $\mathcal{L}'$ of $\mathcal{L}_U$ which contains the lines through p is the line set of HT_{n-1}. Hence $\mathcal{L}_U$ consists of two connected components $\mathcal{L}'$ and $\mathcal{L}''$, each of which is the line set of a geometry HT_{n-1}.

Let $(p_0, \ldots, p_k)$ be a path in the point graph of $\mathcal{S}(\mathcal{L})$, such that p_0 is a point of $\mathcal{S}(\mathcal{L}')$, p_k is a point of $\mathcal{S}(\mathcal{L}'')$, and $p_1, \ldots, p_{k-1} \notin U$ (such a path exists by connectedness). We show that, if $k \geq 3$, then we can find a shorter path with the same properties. For $0 \leq i \leq k-1$, let $L_i = \langle p_i, p_{i+1} \rangle$. Since $k \geq 3$, $p_{k-2} \notin U$. Suppose that the line L_{k-2} intersects U in an affine point r. If r is a point of $\mathcal{S}(\mathcal{L}'')$, then $(p_0, \ldots, p_{k-2}, r)$ is the required path. On the other hand if r is a point of $\mathcal{S}(\mathcal{L}')$, then (r, p_{k-1}, p_k) is the required path. Suppose that L_{k-2} is parallel to U. Since $\alpha(p_{k-2}, L_{k-1}) > 0$, there is a line $M \neq L_{k-2}$ of $\mathcal{L}$ through p_{k-2} which intersects L_{k-1} in an affine point p'. If $p' \in U$, then $p' = L_{k-1} \cap U = p_k$, and $(p_0, \ldots, p_{k-2}, p_k)$ is the required path. If $p' \in U$, then we can apply the argument used above on the path $(p_0, \ldots, p_{k-2}, p', p_k)$ to find the required path.

We conclude that there exists a point $r \notin U$ which is collinear to a point p' of $\mathcal{S}(\mathcal{L}')$ and a point p'' of $\mathcal{S}(\mathcal{L}'')$. Suppose that, when n is odd, the point $p_\infty = \langle p', p'' \rangle \cap \Pi_\infty$ is not the nucleus of the nonsingular parabolic quadric $\theta_{p'}$. Then there is a line L_∞ of U_∞ through p_∞ which is secant to $\theta_{p'}$. By Theorem 2.1, the plane $\pi = \langle p', L_\infty \rangle$ is a plane of Type IV. The point p'' is in the plane π but not on a line of $\mathcal{L}_\pi$, since p'' is a point of $\mathcal{S}(\mathcal{L}'')$.

Let V be the 3-space $\langle \pi, r \rangle$, let $\pi_\infty = V \cap \Pi_\infty$ and let $\mathcal{L}'''$ be the connected component of $\mathcal{L}_V$ which contains the lines of $\mathcal{L}_\pi$ and the lines $\langle p', r \rangle$, $\langle r, p'' \rangle$. Then $\theta_{p'} \cap \pi_\infty$ is either a nonsingular conic or the union of two distinct lines. In the first case, $\mathcal{S}(\mathcal{L}''') = \mathrm{HT}_3$ by Lemma 5.3. But the plane π is of Type IV and contains an isolated point p'', a contradiction to Lemma 5.2. So $\theta_{p'} \cap \pi_\infty$ is the union of two lines, which implies that $\mathcal{L}'''$ is singular [5]. As π is of Type IV, $\mathcal{L}'''$ is the singular line set with vertex a point $r_\infty \in \pi_\infty$ and base the dual oval $\mathcal{L}_\pi$. But then π does not contain any isolated points, a contradiction.

Suppose that n is odd and p_∞ is the nucleus of the nonsingular parabolic quadric $\theta_{p'}$. Let L be a line of $\mathcal{L}'$ through p' and let $r' \neq p'$ be a point of L which is collinear to r. Then we can repeat the argument above with the points r', p'' and r to obtain again a contradiction. We conclude that $\mathcal{P}$ cannot be the set of all affine points. □

6. Conclusion

From the results in [5] where we have treated the case $n \leq 3$ and from the results in this paper we may conclude that we have the complete classification of nonsingular affine line sets of antiflag class $[0, \alpha, q]$, $1 < \alpha < q-1$, up to linear representations. We recall that the singular case was solved in [5], and that for the cases $\alpha = 1$ and $\alpha = q-1$ highly irregular examples exist and are easy to construct.

Theorem 6.1. *Let $\mathcal{L}$ be a nonsingular connected line set of* $\mathrm{AG}(n, q)$, $q > 3$, *of antiflag class* $[0, \alpha, q]$, $1 < \alpha < q-1$. *Then one of the following cases occurs.*

1. $\alpha = 2$, $q = 2^h$ *and* $\mathcal{S}(\mathcal{L}) = \mathrm{HT}_n$.
2. $\alpha = 2$, $q = 2^h$ *and* $\mathcal{S}(\mathcal{L}) = \mathcal{I}(n, q, e)$.
3. $\mathcal{S}(\mathcal{L}) = T^*_{n-1}(\mathcal{K}_\infty)$, *with $\mathcal{K}_\infty$ a nonsingular point set of class* $[0, 1, \alpha+1, q+1]$ *in Π_∞ which spans Π_∞.*
4. $\alpha = 2$, $n = 2$, $q = 2^h$ *and $\mathcal{L}$ is a dual oval.*
5. $\alpha = 2$, $n = 3$, $q = 2^h$ *and* $\mathcal{S}(\mathcal{L}) = \mathcal{A}(O_\infty)$.

Acknowledgement

The research of the second author was supported by a BOF ("Bijzonder Onderzoeksfonds") grant at Ghent University.

References

[1] F. Buekenhout, C. Lefèvre, Generalized quadrangles in projective spaces, Arch. Math. (Basel) 25 (1974) 540–552.
[2] F. Buekenhout, E. Shult, On the foundations of polar geometry, Geom. Dedicata 3 (1974) 155–170.
[3] F. De Clerck, N. De Feyter, A characterization of the sets of internal and external points of a conic, European J. Combin. 28 (2006) 1910–1921.
[4] F. De Clerck, N. De Feyter, Projections of quadrics in finite projective spaces of odd characteristic, Innov. Incidence Geom. 3 (2006) 51–80.
[5] F. De Clerck, N. De Feyter, On connected line sets of antiflag class $[0, \alpha, q]$ in $\mathrm{AG}(n, q)$, European J. Combin. 29 (2008) 1427–1435.

Please cite this article in press as: F. De Clerck, N. De Feyter, A new characterization of projections of quadrics in finite projective spaces of even characteristic, Discrete Mathematics (2010), doi:10.1016/j.disc.2010.06.012

[6] F. De Clerck, N. De Feyter, J.A. Thas, Affine embeddings of $(0, \alpha)$-geometries, European J. Combin. 27 (2006) 74–78.
[7] N. De Feyter, The embedding in $AG(3, q)$ of $(0, 2)$-geometries with no planar nets, J. Combin. Theory Ser. A 109 (1) (2005) 1–23.
[8] N. De Feyter, The embedding of $(0, 2)$-geometries and semipartial geometries in $AG(n, q)$, Adv. Geom. 5 (2005) 279–292.
[9] D.G. Glynn, On the characterization of certain sets of points in finite projective geometry of dimension three, Bull. Lond. Math. Soc. 15 (1) (1983) 31–34.
[10] J.W.P. Hirschfeld, X. Hubaut, Sets of even type in $PG(3, 4)$, alias the binary $(85, 24)$ projective geometry code, J. Combin. Theory Ser. A 29 (1) (1980) 101–112.
[11] J.W.P. Hirschfeld, X. Hubaut, J.A. Thas, Sets of type $(1, n, q + 1)$ in finite projective spaces of even order q, C. R. Math. Rep. Acad. Sci. Canada 1 (3) (1978–1979) 133–136.
[12] J.W.P. Hirschfeld, J.A. Thas, Sets of type $(1, n, q + 1)$ in $PG(d, q)$, Proc. Lond. Math. Soc. (3) 41 (2) (1980) 254–278.
[13] J.W.P. Hirschfeld, J.A. Thas, The characterization of projections of quadrics over finite fields of even order, J. Lond. Math. Soc. (2) 22 (2) (1980) 226–238.
[14] C. Lefèvre-Percsy, Sur les semi-quadriques en tant qu'espaces de Shult projectifs, Acad. Roy. Belg. Bull. Cl. Sci. (5) 63 (2) (1977) 160–164.
[15] M. Tallini Scafati, Caratterizzazione grafica delle forme hermitiane di un $S_{r,q}$, Rend. Mat. Appl. (5) 26 (1967) 273–303.

Please cite this article in press as: F. De Clerck, N. De Feyter, A new characterization of projections of quadrics in finite projective spaces of even characteristic, Discrete Mathematics (2010), doi:10.1016/j.disc.2010.06.012