Intersections of Apartments

Peter Abramenko   Hendrik Van Maldeghem*

Abstract

We show that, if a building is endowed with its complete system of apartments, and if each panel is contained in at least four chambers, then the intersection of two apartments can be any convex subcomplex contained in an apartment. This combinatorial result is particularly interesting for lower dimensional convex subcomplexes of apartments, where we definitely need the assumption on the four chambers per panel in the building. The corresponding statement is not true anymore for arbitrary systems of apartments, and counter-examples for infinite convex subcomplexes exist for any type of buildings. However, when we restrict to finite convex subcomplexes, the above remains true for arbitrary systems of apartments if and only if every finite subset of chambers of the standard Coxeter complex is contained in the convex hull of two chambers.

1 Introduction

Buildings were defined by Jacques Tits in the sixties as the natural geometric structures related to groups of Lie type — providing natural permutation modules for these groups. In this setting, only the spherical buildings are important, and a full classification for irreducible rank at least three exists, see the seminal monograph [6]. However, it became clear later that also non-spherical buildings (e.g. affine buildings) play a very important role in group theory, and we will not make any restrictions concerning the type of the building in this paper. In the original definition of buildings apartments play a crucial role. In fact, the two main axioms of buildings in [6] are about apartments. One axiom states that every pair of chambers is contained in an apartment, and the other that

---

two apartments can be isomorphically mapped onto each other by an isomorphism fixing two chosen simplices in the intersection and all their faces. Naturally, one can ask which convex subcomplexes of apartments can be realized as intersections of two apartments. We provide a complete answer to this question for thick buildings in the present paper. Note that this result goes a little bit in the opposite direction of the main use of apartments. Indeed, usually one reduces a problem concerning buildings to a problem in an apartment — a Coxeter complex. Here, we reduce potential problems involving the intersection of roots in an apartment (which is by definition a convex subcomplex of that apartment) to the intersection of just two apartments, but in the whole building.

In the eighties, a local approach to buildings [7], again initiated by Jacques Tits, led the latter to introduce buildings without mentioning apartments, but using chamber systems. But around the same time, or even a little bit later, Jacques Tits completed the classification of affine buildings of irreducible rank at least four, and there the apartments play a prominent role [8]. In fact, they were so important that the class of objects to which the classification could be applied (this class is slightly larger than that of the affine buildings) was called *systems of apartments* by Tits. Later, in the nineties, yet another definition of buildings was introduced, again by Jacques Tits, this time for the benefit of *twin buildings*, which are natural generalizations of spherical buildings, and which are the natural geometric structures for Kac-Moody groups [9] over fields. That new definition interprets buildings as W-metric spaces, a point of view which will also be important in parts of the present paper (see e.g. Lemma 3.7). It again does not explicitly mention apartments which, however, remain to be an important notion also in this approach . For instance, together with twin buildings, the notion of a *generalized Moufang building* became important. This notion, which was first introduced for spherical buildings in the appendix of [6], uses in a crucial way so-called *roots* in a building, which are nothing other than . . . half-apartments.

So it is clear that the apartments of a building are of vital importance in the whole theory. Although one sometimes removes them from the definition, in order to make things simpler, they have remained crucial in the results and the theory. Therefore, the question of what the intersection of two apartments can be is a fundamental one for all types of buildings. We answer this question in the present paper. The result is neither surprising nor difficult to prove if this intersection contains chambers. However, the problem becomes much more difficult in the case of lower dimensional convex subcomplexes of apartments. First of all, it is clear that then not every convex subcomplex is an intersection of two apartments if the building has only three chambers per panel. It is quite surprising that this is in fact the only obstacle, i.e., in a building with at least four chambers per panel, *any* convex subcomplex of an apartment is an intersection of two apartments. In this

context 'apartment' means 'member of the complete system of apartments'. If one also considers arbitrary apartment systems, then the answer becomes more involved, and we have also dealt with that situation. The simple case of two adjacent chambers — which is for every building and every apartment system a convex set contained in an apartment — already has a nice application, as we will show.

The paper is organized as follows. In Section 2 we introduce some basic notions and fix some notation. We also state our main results and an application. In the rest of the paper, we then proceed to prove these results. Along the way, we have phrased some lemmas and propositions slightly more generally then necessary for our purposes, because we think that these results can be of independent use. For instance, Section 3 deals with constructing apartments under various conditions — obviously important in this paper, but potentially also useful in many other situations. In Section 4, we consider the case of a complete system of apartments, while Section 5 looks at the other situations.

2   Preliminaries and Main Results

We assume that the reader is familiar with basic concepts from building theory, as presented in [2], [5] and [6]. In order to fix some notation and avoid confusion, we will repeat some of these concepts here. We mainly take the simplicial point of view. So a building Δ will be a chamber complex which admits a system of apartments satisfying the usual axioms (see e.g. [6], Chapter 3, or [2], Section 4.1). However, on the one hand, we shall *not* always assume that Δ is thick, but state this condition explicitly whenever needed. On the other hand, we *do* assume that Δ is finite-dimensional. As usual, the top-dimensional simplices of Δ are called *chambers*, the simplices of codimension 1 are called *panels*, and Δ is called thick if each panel is contained in at least three chambers. We denote by $\mathcal{C} = \mathrm{Ch}(\Delta)$ the set of chambers of Δ, and, for any chamber subcomplex κ of Δ, by $\mathrm{Ch}(\kappa)$ the set of chambers of κ. Two chambers are called *adjacent* if they are different (note that some authors do not require this) and share a common panel. A *gallery (of length n)* is a sequence $(C_0, C_1, \ldots, C_n)$ of $n + 1$ chambers in which C_{i-1} and C_i are adjacent, for all $1 \leq i \leq n$. Such a gallery is called *minimal* if there is no gallery of length $< n$ starting in C_0 and ending in C_n. In this case we also say that n is the *(gallery) distance* between C_0 and C_n. If two chambers $C, D \in \mathcal{C}$ are at distance n, we write $\mathrm{d}(C, D) = n$.

A chamber subcomplex κ of Δ is called *convex* if for any two chambers C, D of κ, every gallery $(C_0, C_1, \ldots, C_n)$ in Δ with $C_0 = C$, $C_n = D$ and $n = \mathrm{d}(C, D)$ is contained in κ. An arbitrary subcomplex κ of Δ is called *convex* if it is the intersection of convex chamber subcomplexes of Δ. This is the definition of convexity given by Tits in [6], Section 1.5,

but differs from the one given in [2], Section 4.11. However, if κ contains a chamber or is contained in an apartment of Δ, these two definitions agree, and in this paper we will only consider subcomplexes of apartments. For any set $\mathcal{S}$ of simplices of Δ, the *convex closure* or *convex hull* of $\mathcal{S}$ is defined as the intersection of all convex subcomplexes containing $\mathcal{S}$.

Recall that all apartments of Δ are Coxeter complexes which are isomorphic to each other. If they are isomorphic to the standard Coxeter complex $\Sigma(W, S)$ (as defined in [2], Section 3.1), where (W, S) is a Coxeter system, we say that Δ is of type (W, S). Each building admits a maximal apartment system $\widetilde{\mathcal{A}}$ (see for instance [2], Section 4.5), which is called the *complete* system of apartments of Δ. A subset $\mathcal{A}$ of $\widetilde{\mathcal{A}}$ is a system of apartments of Δ if and only if any two chambers of Δ are contained in some element of $\mathcal{A}$. If Δ is of type (W, S), the elements of $\widetilde{\mathcal{A}}$ are precisely the simplicial subcomplexes of Δ which are isomorphic to $\Sigma(W, S)$ (see [2], Proposition 4.59). Whenever we talk about apartments of Δ without further specification, we mean elements of the complete system $\widetilde{\mathcal{A}}$ of apartments of Δ.

We fix an apartment Σ of a building Δ. As a preparation towards our First Main Result, we shall prove in Section 4:

Proposition 2.1 *If Δ is thick and κ is a convex chamber subcomplex of Σ, then there exists an apartment Σ' of Δ such that $\kappa = \Sigma \cap \Sigma'$.*

Remark 2.2 Assume there exists a panel P in Σ which is contained in exactly 3 chambers of Δ. Then it is obvious that the wall M of Σ containing P cannot be written as $M = \Sigma \cap \Sigma'$ for some apartment Σ' since $\Sigma \cap \Sigma'$ has to contain at least one of the three chambers containing P.

So it is clear that Proposition 2.1 cannot be true in general for (lower dimensional) convex subcomplexes of Σ. Amazingly enough, it turns out to be true if we only add the (obviously necessary) condition that each panel is contained in at least 4 chambers of Δ.

Theorem 2.3 (First Main Result) *Suppose every panel of the building Δ is contained in at least 4 chambers. Then for every convex subcomplex κ of Σ, there exists an apartment Σ' of Δ such that $\kappa = \Sigma \cap \Sigma'$.*

Now a natural question arises: What can we say if the building Δ is *not* endowed with its complete set of apartments? We will prove that, whenever a convex subcomplex of an apartment has infinitely many vertices, then Theorem 2.3 does not hold in general

(when the system of apartments is not complete). In fact, in this case, such a convex subcomplex might not even be the intersection of *all* apartments in which it is contained. Hence we are lead to consider only the case where our convex subcomplex has a finite number of simplices. Then we will show that every such convex subcomplex (contained in an apartment of an arbitrarily given apartment system $\mathcal{A}$ of Δ) is the intersection of two apartments of $\mathcal{A}$ if and only if the building is a direct product of spherical and affine buildings. Here is the precise result:

Theorem 2.4 (Second Main Result) *Let Δ be a thick building of type (W, S). Then the following are equivalent.*

(I) *For any apartment system $\mathcal{A}$ of Δ, the convex closure of two given chambers C and D is the intersection of two apartments in $\mathcal{A}$.*

(II) *For any apartment system $\mathcal{A}$ of Δ, every convex finite set κ of chambers contained in some member of $\mathcal{A}$ is the intersection of two apartments in $\mathcal{A}$.*

(III) *For every finite subset F of chambers of the Coxeter complex $\Sigma(W, S)$, there exist two chambers $C, D \in \Sigma(W, S)$ such that F is contained in the convex closure of C and D.*

If every panel of Δ is contained in at least four chambers, and Condition (III) is satisfied, then every finite convex subcomplex contained in an apartment is the intersection of two apartments (with respect to any apartment system $\mathcal{A}$).

Now it is proved by Caprace in [4] that Assertion (III) is equivalent to $\Sigma(W, S)$ having only spherical and affine components.

We shall also prove the following related result for pairs of adjacent chambers.

Proposition 2.5 *Let Δ be a thick building endowed with an arbitrary system $\mathcal{A}$ of apartments. Then the set of two adjacent chambers and all their faces is always the intersection of all apartments in $\mathcal{A}$ in which they are both contained.*

As an application of this proposition we shall prove in Section 5:

Proposition 2.6 *Every map between the chamber sets of a thick building Δ and an arbitrary building Δ' (endowed with arbitrary systems of apartments $\mathcal{A}$ and $\mathcal{A}'$, respectively) which bijectively maps the set of chambers of any apartment $\Sigma \in \mathcal{A}$ to the set of chambers of some apartment $\Sigma' \in \mathcal{A}'$ is injective and preserves adjacency of chambers. Hence it induces a simplicial isomorphism of Δ onto a thick subbuilding of Δ' in case Δ is 2-spherical.*

3   Constructing apartments

In the course of the proof of our main results, we shall make use of some well-known facts, and some lemmas that might be of independent interest. We collect these assertions in this section. Most of them are concerned with the construction of apartments containing certain sets of simplices and satisfying certain conditions.

We start by repeating some more notions from the simplicial theory of buildings. If Σ is a Coxeter complex, a *root* of Σ is the image of a folding of Σ (see [6], Chapter 2, or [2], Section 3.4). It is well known that a subcomplex κ of Σ is convex if and only if it is an intersection of roots of Σ. If α is a root of Σ, the subcomplex of Σ generated by all chambers of Σ which are not in α is again a root of Σ, denoted by $-\alpha$ (if Σ is understood) and called the root *opposite* α in Σ. The subcomplex $\alpha \cap (-\alpha)$ is called the *boundary* or *wall* of α (or of $-\alpha$); it is the subcomplex of Σ generated by all panels in Σ which are contained in precisely one chamber of α. If Δ is a building, a *root* or *wall* in Δ is a root or wall in one of its apartments.

In a Coxeter complex Σ, the *support* supp$A$ of a simplex A is the intersection of all walls of Σ containing A. We repeat the following result from [1]:

Proposition 3.1 *If κ is a convex subcomplex of a Coxeter complex Σ, A is a maximal simplex of κ, and M is a wall in Σ containing A, then $\kappa \subseteq M$. Hence, $\kappa \subseteq$ supp$A$.*

Proof. See Proposition 1 (ii) of [1].

In the following, Δ denotes a building of type (W, S), and $\mathcal{C}$ is the set of chambers of Δ. The following property of apartments is well known if $\Sigma \cap \Sigma'$ contains chambers, and is proved in general in [2]:

Proposition 3.2 *Let Σ, Σ' be two apartments of Δ. Then there exists an isomorphism $\varphi : \Sigma \to \Sigma'$ such that φ fixes each simplex in $\Sigma \cap \Sigma'$.*

Proof. See Proposition 4.101 of [2]

Before we continue, we remind the reader that Δ comes equipped with a *Weyl distance function* $\delta : \mathcal{C} \times \mathcal{C} \to W$ (see [2], Section 4.8). This function δ is an important tool in the theory of buildings. The latter can, in fact, be based completely on the properties of the Weyl distance function (see [2], Chapter 5). If Δ and Δ' are two buildings of type (W, S) with chamber sets $\mathcal{C}$ and $\mathcal{C}'$, and with Weyl distance functions δ and δ', respectively, then

two subsets $\mathcal{M} \subseteq \mathcal{C}$ and $\mathcal{M}' \subseteq \mathcal{C}'$ are called W-*isometric* (or just *isometric*) if there exists a bijective map $f : \mathcal{M} \to \mathcal{M}'$ such that $\delta'(f(C), f(D)) = \delta(C, D)$, for all $C, D \in \mathcal{M}$. One of the fundamental facts in this context is that a subset $\mathcal{M} \subseteq \mathcal{C}$ is contained in $\mathrm{Ch}(\Sigma)$, for some apartment Σ of Δ, if and only if $\mathcal{M}$ is isometric to a subset of $W = \mathrm{Ch}(\Sigma(W, S))$, see [5], Theorem 3.6, or [2], Theorem 5.73.

This will be used in the next two propositions.

Proposition 3.3 *Given a root α of Δ with boundary wall M, a panel $P \in M$ and a chamber C containing P but not contained in α, there exists an apartment Σ containing $\alpha \cup \{C\}$.*

Proof. Consider any apartment Σ' containing α and let D be the chamber in Σ' containing P and not contained in α. Then it is clear that $\mathrm{Ch}(\alpha) \cup \{C\}$ is W-isometric to $\mathrm{Ch}(\alpha) \cup \{D\}$. The assertion now follows from Theorem 3.6 in [5]. $\qquad \square$

Proposition 3.4 *A subset $\mathcal{M}$ of $\mathcal{C} = \mathrm{Ch}(\Delta)$ is contained in the set of chambers of some apartment of Δ if and only if*

$(\star)$ $\delta(C, E) = \delta(C, D)\delta(D, E)$*, for all $C, D, E \in \mathcal{M}$.*

Proof. This is contained in [2], see Exercise 5.77. We provide a proof for completeness' sake.

The necessity of Condition $(\star)$ is clear by standard properties of apartments. To verify sufficiency, first fix a chamber $C_0 \in \mathcal{M}$ and define a map $f : \mathcal{M} \to W$ by $f(C) = \delta(C_0, C)$ for all $C \in \mathcal{M}$. Due to the Condition $(\star)$, this is an isometry. Indeed, denoting the natural Weyl distance on W by δ_W, we have

$$\begin{aligned} \delta_W(f(C), f(D)) = f(C)^{-1} f(D) &= \delta(C_0, C)^{-1} \delta(C_0, D) \\ &= \delta(C, C_0)\delta(C_0, D) = \delta(C, D). \end{aligned}$$

Slightly abusing notation, we also denote the induced map $\mathcal{M} \to f(\mathcal{M})$ by f. Then f is a bijective isometry, so that $f^{-1} : f(\mathcal{M}) \to \mathcal{M}$ is an isometry. Now, by Theorem 3.6 of [5], f^{-1} can be extended to an isometry $g : W \to \mathcal{C}$, and hence $\mathcal{M}$ is contained in $g(W)$, which is the set of chambers of an apartment of Δ. $\qquad \square$

We now prove some lemmas. Their proofs use projections, a concept which we briefly recall now (for details, we refer to [6], Chapter 3, or [2], Sections 4.9 and 5.3). Let A, B be

simplices of the building Δ. If A and B are chambers, we denote by $\mathrm{d}(A, B)$ the gallery distance between A and B. In general, we define $\mathrm{d}(A, B) = \min\{\mathrm{d}(C, D) \mid C, D \in \mathcal{C}, A \subseteq C$ and $B \subseteq D\}$. Then there is a unique simplex P containing A with the property that, for any chamber C containing A, one has $\mathrm{d}(C, B) = \mathrm{d}(A, B)$ if and only if $P \subseteq C$. This simplex P is called the *projection of B onto A* and denoted by $P = \mathrm{proj}_A B$. If B is a chamber, then also $\mathrm{proj}_A B$ is a chamber, and it has the "gate property" $\mathrm{d}(C, B) = \mathrm{d}(C, \mathrm{proj}_A B) + \mathrm{d}(\mathrm{proj}_A B, B)$, for all chambers C containing A. Moreover, $\delta(C, B) = \delta(C, \mathrm{proj}_A B)\delta(\mathrm{proj}_A B, B)$ in this case (see [2], Proposition 5.34).

Lemma 3.5 *Let α_1, α_2 be two roots in Δ having the same wall M as their boundary. Then $\alpha_1 \cup \alpha_2$ is an apartment of Δ if and only if there exists a panel $P \in M$ such that the chamber $C_1 \in \alpha_1$ with $P \subseteq C_1$ is different from the chamber $C_2 \in \alpha_2$ with $P \subseteq C_2$.*

Proof. Obviously, if $C_1 = C_2$, then $\alpha_1 \cup \alpha_2$ cannot be an apartment. So let us assume now that $C_1 \neq C_2$. We first claim that this implies $\alpha_1 \cap \alpha_2 = M$. Indeed, let x be any simplex in α_1 with $x \notin M$. Then $\mathrm{proj}_P x \neq P$, because otherwise there would be a minimal gallery between P and x in some apartment Σ containing α_1, starting with a chamber outside α_1, which is impossible since α_1 is convex and every chamber of Σ containing x has to be in α_1. Hence P is strictly contained in $\mathrm{proj}_P x$. But the only simplex in α_1 strictly containing P is C_1, implying $\mathrm{proj}_P x = C_1$. By assumption, $C_1 \notin \alpha_2$. Hence x cannot belong to α_2. Similarly, $y \notin \alpha_1$ for any $y \in \alpha_2 \setminus M$. Therefore, $\alpha_1 \cap \alpha_2 = M$.

We now consider apartments $\Sigma_1, \Sigma_2 \in \mathcal{A}$ containing α_1, α_2, respectively. Denote by β_i the root opposite α_i in Σ_i, $i = 1, 2$. By Proposition 3.2, there exists a simplicial isomorphism $\varphi : \Sigma_1 \to \Sigma_2$ which fixes $\Sigma_1 \cap \Sigma_2$ pointwise. So in particular φ is the identity on M. This implies that $\varphi(\alpha_1)$ has again M as its bounding wall, i.e., $\varphi(\alpha_1) = \alpha_2$ or $\varphi(\alpha_1) = \beta_2$. We may assume $\varphi(\alpha_1) = \beta_2$ as otherwise we simply compose φ with the reflection in Σ_2 about the wall M, and this reflection interchanges α_2 with β_2 (and note that this composite map still fixes M pointwise). We now define a simplicial map $\rho : \alpha_1 \cup \alpha_2 \to \Sigma_2$ by $\rho_{|\alpha_2} = \mathrm{id}_{\alpha_2}$ and $\rho_{|\alpha_1} = \varphi_{|\alpha_1}$.

Note that ρ is well defined since $\alpha_1 \cap \alpha_2 = M$ and $\varphi_{|M} = \mathrm{id}_M$. It is also clear that ρ is a simplicial morphism. Next we define $\sigma : \Sigma_2 \to \alpha_1 \cup \alpha_2$ by $\sigma_{|\alpha_2} = \mathrm{id}_{\alpha_2}$ and $\sigma_{|\beta_2} = \varphi_{|\beta_2}^{-1}$. Again, σ is a well defined simplicial morphism (here we use $\alpha_2 \cap \beta_2 = M$), and it is obvious that σ and ρ are inverse to each other. Hence $\sigma : \Sigma_2 \to \alpha_1 \cup \alpha_2$ is a simplical isomorphism. It follows that $\alpha_1 \cup \alpha_2 = \sigma(\Sigma_2)$ is an apartment of Δ, by Proposition 4.59 in [2] (note that the application of the latter does not require σ to be type-preserving, although it is easy to see that it is here). $\square$

We mention a consequence of this lemma, thereby slightly improving Lemma 2.2 of [3]. We denote the set of chambers containing a given panel P by $\mathcal{C}_P$.

Corollary 3.6 *Given a panel P and a wall M in Δ with $P \in M$, there exists a family $(\alpha_C)_{C \in \mathcal{C}_P}$ of roots with the following properties.*

(1) *$C \in \alpha_C$, for all $C \in \mathcal{C}_P$;*

(2) *M is the boundary wall of α_C;*

(3) *$\alpha_C \cup \alpha_D$ is an apartment of Δ, for all $C, D \in \mathcal{C}_P$ with $C \neq D$.*

Proof. Let Σ_0 be any apartment containing M, and let $C_0, D_0 \in \mathcal{C}_P$ be the two chambers in Σ_0 containing P, with $C_0 \neq D_0$. Let $\alpha_{C_0}, \alpha_{D_0}$ be the roots in Σ_0 which are bounded by the wall M and contain the chambers C_0, D_0, respectively.

Now, for any $C \in \mathcal{C}_P \setminus \{C_0, D_0\}$, there exists an apartment Σ_C containing $\alpha_{C_0} \cup \{C\}$ (direct by Proposition 3.3). Let α_C be the root in Σ_C with bounding wall M (note that $M \subseteq \alpha_{C_0} \subseteq \Sigma_C$) and containing C. Given two distinct elements $C, D \in \mathcal{C}_P$, the two roots α_C and α_D have the wall M in common, but they do not share the same chambers C and D, respectively, through $P \in M$. So we can apply Lemma 3.5 and infer that $\alpha_C \cup \alpha_D$ is an apartment of Δ. □

The following technical lemma will be needed in the proof of the Main Lemma in Section 4. Before we state it, we recall the definition of links and stars. If Ω is an arbitrary simplicial complex and A is a simplex in Ω, we define the *link of A in Ω* as the subcomplex

$$\mathrm{Lk}_\Omega(A) = \{B \in \Omega \mid A \cup B \text{ is a simplex in } \Omega \text{ and } A \cap B = \emptyset\},$$

and the *star of A in Ω* as

$$\mathrm{St}_\Omega(A) = \{B \in \Omega \mid A \cup B \text{ is a simplex in } \Omega\}.$$

Recall that for $A \in \Delta$, $\mathrm{Lk}_\Delta(A)$ is again a building (with apartments $\mathrm{Lk}_\Sigma(A)$ for all apartments Σ of Δ containing A).

Lemma 3.7 *Let M be a wall in Δ, let A be a simplex of M and let $L_A := \mathrm{Lk}_\Delta(A)$ be its link in Δ. Then for every apartment Σ_A of the building L_A with $M \cap L_A \subseteq \Sigma_A$, there exists an apartment Σ of Δ such that $\Sigma_A = \Sigma \cap L_A$ and $M \subseteq \Sigma$.*

9

Proof. In this proof, we will have to combine the simplicial view with the W-metric approach to buildings. This requires some additional notation.

We set $\mathcal{C}_A = \{C \in \mathcal{C} \mid A \subseteq C\}$. If $J \subseteq S$ denotes the cotype of A, then $\mathcal{C}_A$ is a *J-residue* in $\mathcal{C}$, and $(\mathcal{C}_A, \delta_{|\mathcal{C}_A \times \mathcal{C}_A})$ is a W-metric building of type (W_J, J) which corresponds to the simplicial building $L_A = \mathrm{Lk}_\Delta(A)$. To the apartment Σ_A of L_A corresponds the apartment (set of chambers) $\mathcal{C}'_A := \{A \cup B \mid B$ is a chamber of $\Sigma_A\}$ of $\mathcal{C}_A$. Note that, therefore, $\mathcal{C}'_A$ satisfies Condition $(\star)$ of Proposition 3.4.

We now embark on the proof of the lemma. We first choose a (simplicial) root $\widehat{\alpha}$ in Δ which has M as boundary and set $\alpha := \mathrm{Ch}(\widehat{\alpha})$. We then choose a panel $P \in M$ with $A \subseteq P$ and $P \setminus A \in \Sigma_A$ (recall that $M \cap L_A \subseteq \Sigma_A$), and denote by C_1, C_2 the two chambers in $\mathcal{C}'_A$ which contain P. Set

$$\alpha_i := \{C \in \mathcal{C}'_A \mid \mathrm{d}(C, C_i) < \mathrm{d}(C, C_{3-i})\},$$

for $i = 1, 2$. These sets α_1 and α_2 are the sets of chambers corresponding to the two roots in Σ_A with boundary wall $M \cap L_A$ (which is contained in Σ_A by assumption).

STEP I. We first prove the assertion in the special case that $\alpha_1 \subseteq \alpha$. In this case we will show that there exists an apartment Σ of Δ with $\alpha \cup \alpha_2 \subseteq \mathrm{Ch}(\Sigma)$.

We want to verify Condition $(\star)$ for $\alpha \cup \alpha_2$. To that aim, we choose three chambers $C, D, E \in \alpha \cup \alpha_2$ and distinguish some (non-disjoint) cases.

(1) $C, D, E \in \alpha$. Condition $(\star)$ is satisfied since α is part of an apartment.

(2) $C, D, E \in \alpha_1 \cup \alpha_2$. As remarked above, $\alpha_1 \cup \alpha_2 = \mathcal{C}'_A$ satisfies Condition $(\star)$.

(3) $C \in \alpha$; $D, E \in \alpha_1 \cup \alpha_2$. First note that the simplicial roots in L_A corresponding to $\mathcal{C}_A \cap \alpha$ and α_1 have the same wall $M \cap L_A$. Since $\alpha_1 \subseteq \mathcal{C}_A \cap \alpha$, they must be equal. Now set $C' := \mathrm{proj}_A C \in \mathcal{C}_A \cap \alpha = \alpha_1$. Then, using the standard property of the projection mapping that $\delta(C, C')\delta(C', X) = \delta(C, X)$, for all chambers $X \in \mathcal{C}_A$ (see Proposition 5.34(2) in [2]), we obtain

$$\delta(C, D)\delta(D, E) = \delta(C, C')\delta(C', D)\delta(D, E) = \delta(C, C')\delta(C', E) = \delta(C, E),$$

where we also used the fact that $\mathcal{C}'_A$ satisfies Condition $(\star)$.

(4) $C, D \in \alpha$; $E \in \alpha_1 \cup \alpha_2$.

Set $C' := \text{proj}_A C$ and $D' := \text{proj}_A D$. As in (3), we have $C', D' \in \alpha_1$. Then, again using the above mentioned standard property of projections, we obtain

$$
\begin{aligned}
\delta(C,D)\delta(D,E) &= \delta(C,C')\delta(C',D)\delta(D,E) && \text{since } C, C', D \in \alpha, \\
&= \delta(C,C')\delta(C',D)\delta(D,D')\delta(D',E) && \text{by (3),} \\
&= \delta(C,C')\delta(C',D')\delta(D',E) && \text{since } C', D, D' \in \alpha, \\
&= \delta(C,C')\delta(C',E) && \text{since } C', D', E \in \alpha_1 \cup \alpha_2, \\
&= \delta(C,E) && \text{by (3).}
\end{aligned}
$$

Now the other cases all follow easily:

(5) $C \in \alpha_1 \cup \alpha_2$; $D, E \in \alpha$. The claim follows from (4) by taking inverses.

(6) $C, E \in \alpha$; $D \in \alpha_2$. By (4), we have $\delta(C,D) = \delta(C,E)\delta(E,D)$, hence
$$\delta(C,D)\delta(D,E) = \delta(C,E)\delta(E,D)\delta(D,E) = \delta(C,E).$$

(7) $C, D \in \alpha_1 \cup \alpha_2$; $E \in \alpha$. Follows from (3) by taking inverses.

(8) $C, E \in \alpha_1 \cup \alpha_2$; $D \in \alpha$. Using (3), we have
$$\delta(C,D)\delta(D,E) = \delta(C,D)\delta(D,C)\delta(C,E) = \delta(C,E).$$

So, by Proposition 3.4, there is an apartment Σ of Δ with $\alpha \cup \alpha_2 \subseteq \text{Ch}(\Sigma)$. Hence Σ contains M (the boundary of $\widehat{\alpha}$) and also Σ_A (since $\mathcal{C}'_A \subseteq \text{Ch}(\Sigma)$). Therefore, $\Sigma \cap L_A$ contains Σ_A. However, since $\Sigma \cap L_A$ is also an apartment of L_A (because $A \in \Sigma$), we must have $\Sigma \cap L_A = \Sigma_A$, and the assertion follows.

STEP II. We now reduce the general case to the case handled in Step I.

We first observe that $\widehat{\alpha} \cap L_A$ is a root in L_A with $M \cap L_A$ as bounding wall. Recall that this is also the common wall of the roots $\widehat{\alpha}_1, \widehat{\alpha}_2$ in L_A corresponding to α_1, α_2, respectively. Let C be the chamber of α which contains the panel P introduced in the beginning of this proof. Without loss of generality, we may assume that $C \neq C_1$ (otherwise we interchange the roles of α_1 and α_2). Then, by Lemma 3.5, the union of the two roots $\widehat{\alpha} \cap L_A$ and $\widehat{\alpha}_1$ is an apartment $\widetilde{\Sigma}_A$ of L_A.

We can now apply Step I to α and to this apartment $\widetilde{\Sigma}_A$. Thus we obtain an apartment $\widetilde{\Sigma}$ with $\alpha \cup \alpha_1 \subseteq \text{Ch}(\widetilde{\Sigma})$. Let $\widetilde{\alpha}$ be the root opposite $\widehat{\alpha}$ in $\widetilde{\Sigma}$. Then M is also the boundary wall of $\widetilde{\alpha}$. Note that $\widehat{\alpha}_1$ is a root in $\widetilde{\Sigma} \cap L_A$, and that $\widehat{\alpha}_1$ shares the boundary $M \cap L_A$ with the two opposite roots $\widetilde{\alpha} \cap L_A$ and $\widehat{\alpha} \cap L_A$. Therefore, $\widehat{\alpha}_1 = \widetilde{\alpha} \cap L_A$ or $\widehat{\alpha}_1 = \widehat{\alpha} \cap L_A$. Since $C_1 \neq C$, it follows that $\widehat{\alpha}_1 = \widetilde{\alpha} \cap L_A$, and hence $\alpha_1 \subseteq \text{Ch}(\widetilde{\alpha})$. Now we can apply Step I to $\text{Ch}(\widetilde{\alpha})$ and Σ_A, and the lemma is proved. $\square$

4   Complete systems of apartments

In this section, we prove our First Main Result. We denote by Δ an arbitrary thick building, and $\mathcal{A}$ is the complete apartment system of Δ.

The proof of our First Main Result will be by an induction the first step of which is in fact Proposition 2.1. Hence we first prove Proposition 2.1, which we restate here. Note that this proposition is contained in [2] in the form of the exercises 5.83 and 5.84.

Proposition 4.1 *If κ is a nonempty convex chamber subcomplex of an apartment Σ of Δ, then there exists an apartment Σ' of Δ such that $\kappa = \Sigma \cap \Sigma'$.*

Proof. We may assume $\kappa \neq \Sigma$ (otherwise set $\Sigma' = \Sigma$). Let $\mathcal{M}$ be the set of walls of Σ containing some panel $P \in \kappa$ with the property that κ contains exactly one chamber through P. The set $\mathcal{M}$ is not empty since Σ is connected and $\kappa \neq \Sigma$. We choose an index set J such that $\mathcal{M} = \{M_j \mid j \in J\}$ and, for each $j \in J$, a panel P_j contained in $M_j \cap \kappa$ so that P_j is contained in precisely one chamber of κ. Furthermore, we denote by C_j, D_j, $j \in J$, the two chambers of Σ containing P_j, with $C_j \in \kappa$ and consequently $D_j \notin \kappa$. Let α_j, $j \in J$, be the root of Σ containing C_j but not D_j, which implies that M_j is the boundary of α_j. Since κ is a convex chamber subcomplex of the apartment Σ, we easily see that $\kappa \subseteq \alpha_j$, for all $j \in J$. Hence

$$\kappa \subseteq \bigcap_{j \in J} \alpha_j.$$

Now suppose, by way of contradiction, that some simplex X is contained in α_j, for all $j \in J$, but not in κ. By considering a minimal gallery joining X to κ, and taking into account that the intersection of all α_j, $j \in J$, is convex itself, we see that we may assume that X is a chamber adjacent to some chamber X' of κ. By construction, there exists $i \in J$ such that $X \cap X' \subseteq M_i$. We now obtain the contradiction $X, X' \in \alpha_i$.

Hence we have shown that κ is the intersection of all roots α_j, for j ranging over J. Since Δ is thick, we can choose, for all $j \in J$, a chamber $D'_j \supseteq P_j$, with $C_j \neq D'_j \neq D_j$, and we consider the sets

$$L = \kappa \cup \{D_j \mid j \in J\}, \text{ and } L' = \kappa \cup \{D'_j \mid j \in J\}.$$

We now claim that the mapping $\varphi : \mathcal{C}(L) \to \mathcal{C}(L')$, which is the identity on $\mathcal{C}(\kappa)$ and maps D_j onto D'_j, for all $j \in J$, is a W-isometry. To that aim, we consider for each pair

$i, j \in J$, $i \neq j$, a minimal gallery γ_{ij} joining C_i with C_j (and entirely contained in κ). We extend this to a gallery

$$\lambda_{ij} = (D_i, \underbrace{C_i, \dots, C_j}_{\gamma_{ij}}, D_j).$$

Since $M_i \neq M_j$, and since neither M_i nor M_j cross the gallery γ_{ij} (because κ is contained in α_i and α_j), we see that λ_{ij} is a gallery in Σ crossing every wall of Σ at most once. This implies that λ_{ij} is a minimal gallery. If we put $s_i = \delta(C_i, D_i)$, for all $i \in J$, then $\delta(D_i, D_j) = s_i \delta(C_i, C_j) s_j$, for all pairs $i, j \in J$, $i \neq j$.

Now the gallery

$$\lambda'_{ij} = (D'_i, \underbrace{C_i, \dots, C_j}_{\gamma_{ij}}, D'_j)$$

has the same type as λ_{ij} and hence is also reduced. Hence $\delta(D'_i, D'_j) = s_i \delta(C_i, C_j) s_j = \delta(D_i, D_j)$. It now follows rather easily that φ is a W-isometry (the proof of $\delta(D_i, X) = \delta(D'_i, X)$, for all $i \in I$ and all $X \in \mathrm{Ch}(\kappa)$, is similar to that of $\delta(D_i, D_j) = \delta(D'_i, D'_j)$ above). Since L is contained in an apartment, namely in Σ, Theorem 3.6 of [5], or equivalently, Proposition 3.4, implies that L' is contained in some apartment $\Sigma' \in \mathcal{A}$. Define $\kappa' := \Sigma \cap \Sigma'$. Then, by the foregoing, $\kappa \subseteq L \cap L' \subseteq \Sigma \cap \Sigma' = \kappa'$. Hence κ' is a convex subcomplex of Σ containing κ, in particular it contains C_i, for all $i \in J$. But it does not contain D_i, because $D_i \notin \Sigma'$. Consequently $\kappa' \subseteq \alpha_i$, for all $i \in J$. Hence

$$\kappa' \subseteq \bigcap_{i \in J} \alpha_i = \kappa$$

and so $\kappa' = \kappa$.

The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

From now on, for the rest of this section, our standing hypothesis is that each panel of Δ is contained in at least 4 chambers.

Main Lemma 4.2 *For every apartment $\Sigma \in \mathcal{A}$ and every wall M in Δ, there exists an apartment $\Sigma' \in \mathcal{A}$ containing M and satisfying $\Sigma' \cap \Sigma = M \cap \Sigma$.*

Proof. The proof has two main steps: the case $M \cap \Sigma = \emptyset$ is treated separately, and then the general case is reduced to the first step using Lemma 3.7.

Case I: $M \cap \Sigma = \emptyset$.

We choose an arbitrary panel P in M and four distinct chambers C_1, C_2, C_3, C_4 containing P. Then, by Corollary 3.6, there exist four roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with boundary wall M containing C_1, C_2, C_3, C_4, respectively, such that $\Sigma_{ij} := \alpha_i \cup \alpha_j$ is an apartment, for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

The apartment Σ meets every root α_i, $i \in \{1, 2, 3, 4\}$, in a convex subcomplex. If there are two distinct $k, \ell \in \{1, 2, 3, 4\}$ such that $\Sigma \cap \alpha_k = \Sigma \cap \alpha_\ell = \emptyset$, then the assertion follows by setting $\Sigma' = \Sigma_{k\ell}$. Suppose now that for some $j \in \{1, 2, 3, 4\}$ the intersection $\Sigma \cap \alpha_j$ has dimension at least 1, and that for some $i \in \{1, 2, 3, 4\} \setminus \{j\}$, the intersection $\Sigma \cap \alpha_i$ is nonempty. Consider the convex subcomplex $\Theta := \Sigma_{ij} \cap \Sigma$ of Σ_{ij} of dimension at least 1. Then Θ is a chamber complex in its own right (see Proposition 1(ii) in [1]), and, in particular, Θ is a connected simplicial complex (since $\dim \Theta > 0$). By assumption there exists a vertex x in $\Sigma \cap \alpha_i$, and a vertex y in $\Sigma \cap \alpha_j$. So, we find a path $x = x_0, x_1, \ldots, x_n = y$ in Θ connecting x and y (all $\{x_\ell, x_{\ell+1}\}$ are edges in Θ). Then at least one of the vertices x_k is in M. Indeed, if $y \in \alpha_i$, then $y = x_n \in \alpha_i \cap \alpha_j = M$. If y is not in α_i, then there exists a k with $x_k \in \alpha_i$ and $x_{k+1} \notin \alpha_i$. In particular, x_{k+1} is not in M, and hence it is in the interior of α_j, implying $\{x_k, x_{k+1}\} \in \alpha_j$. Hence $x_k \in \alpha_i \cap \alpha_j = M$. However, we assumed $M \cap \Sigma = \emptyset$. So $M \cap \Theta \neq \emptyset$ is not possible, and we obtain a contradiction.

So we may assume that for every $i \in \{1, 2, 3, 4\}$, $\Sigma \cap \alpha_i$ is either 0-dimensional or empty. We may also assume that $\Sigma \cap \alpha_1$ is 0-dimensional. Let x be a vertex in $\Sigma \cap \alpha_1$. Then, for all $i \in \{2, 3, 4\}$, $\Sigma \cap \Sigma_{1i}$, being a 0-dimensional convex subcomplex of Σ, is contained in the support of x in Σ by Proposition 3.1. Now, by Proposition 1(iii) of [1], this support, which is also a 0-dimensional convex subcomplex of Σ, can have at most two vertices. Hence there is at most one $i \in \{2, 3, 4\}$ with $\Sigma \cap \alpha_i \neq \emptyset$ since $\Sigma \cap \alpha_i$ is contained in the support of x in Σ and the intersections $\Sigma \cap \alpha_j$, for $j \in \{1, 2, 3, 4\}$, are pairwise disjoint (because $M \cap \Sigma = \emptyset$). Hence we can put $\Sigma' = \alpha_k \cup \alpha_\ell$, with k and ℓ distinct in $\{2, 3, 4\}$ such that $\Sigma \cap \alpha_k = \Sigma \cap \alpha_\ell = \emptyset$.

Case II: $M \cap \Sigma \neq \emptyset$.

Choose a maximal simplex A in $M \cap \Sigma$ and set $L_A := \mathrm{Lk}_\Delta(A)$, $M_A := M \cap L_A$ and $\Sigma_A := \mathrm{Lk}_\Sigma(A) = \Sigma \cap L_A$. Then M_A is a wall in L_A, Σ_A is an apartment of L_A and $M_A \cap \Sigma_A = \emptyset$ since A is maximal in $M \cap \Sigma$. Applying Case I to M_A and Σ_A, we find an apartment Σ'_A of L_A containing M_A and satisfying $\Sigma'_A \cap \Sigma_A = \emptyset$. In view of Lemma 3.7 we can find $\Sigma' \in \mathcal{A}$ with $\Sigma'_A = \Sigma' \cap L_A$ and $M \subseteq \Sigma'$. This implies that A is maximal in $\Sigma' \cap \Sigma$ (noting $A \subseteq B \in \Sigma' \cap \Sigma$ implies $B \setminus A \in \Sigma'_A \cap \Sigma_A = \emptyset$). And $\Sigma' \cap \Sigma$ is a convex subcomplex of Σ'. Then $\Sigma' \cap \Sigma \subseteq M$ by Proposition 3.1, because $A \in M$. Hence $\Sigma' \cap \Sigma = \Sigma' \cap \Sigma \cap M = (\Sigma' \cap M) \cap \Sigma = M \cap \Sigma$. $\square$

Theorem 4.3 *Let Δ be a building with the property that each of its panels is contained in at least four chambers, and let κ be a convex subcomplex of an apartment Σ of Δ. Then there exists an apartment Σ' of Δ such that $\Sigma' \cap \Sigma = \kappa$.*

Proof. The proof goes by induction on the codimension $\mathrm{codim}(\kappa)$. The case $\mathrm{codim}(\kappa) = 0$ is settled by Proposition 4.1. So let us assume $\mathrm{codim}(\kappa) > 0$. Choose a maximal simplex A of κ. Recall that κ is a chamber complex (see [1], Proposition 1(ii)). This implies in particular $\dim A = \dim \kappa < \dim \Sigma$. So there exists a panel P in Σ with $A \subseteq P$. Denote by M the wall of Σ containing P. Then $\kappa \subseteq M$ by Proposition 3.1.

Let C_1, C_2 be the two chambers of Σ which contain P. Let x_1 be the vertex in $C_1 \setminus P$, i.e. $C_1 = P \cup \{x_1\}$, and set $A_1 := A \cup \{x_1\}$. Note that $x_1 \notin M$ and hence $A_1 \notin M$. Now let κ_1 be the convex hull of $\kappa \cup \{A_1\}$ in Σ. Then $\dim \kappa_1 = \dim A_1$. Indeed, let M' be any wall of Σ containing A_1. Then M' contains κ by Proposition 3.1 and hence also κ_1 since M' is convex and $\kappa \cup \{A_1\} \subseteq M'$. Since this is true for any wall M' of Σ containing A_1, we see that κ_1 is contained in the intersection $\mathrm{supp}A_1$ of all these walls (and which we have called the *support* of A_1 in Σ above). We now claim that A_1 is maximal in $\mathrm{supp}A_1$. Indeed, if A_1' is any element of Σ properly containing A_1, we choose a chamber C' of Σ containing A_1' and a panel P' of C' with $A_1 \subseteq P'$ and $A_1' \nsubseteq P'$. Then the wall through P' contains A_1 but not A_1', since it does not contain $C' = P' \cup \{A_1'\}$. Hence the claim. Therefore $\dim \kappa_1 = \dim(\mathrm{supp}A_1) = \dim A_1 = 1 + \dim A$.

In particular, we can apply the induction hypothesis to κ_1, which gives us an apartment Σ_1 of Δ satisfying $\Sigma_1 \cap \Sigma = \kappa_1$.

Now let D_1 be a chamber of Σ_1 containing A_1, and let P_1 be the panel $P_1 = D_1 \setminus \{x_1\}$. So P_1 contains A but not A_1. Denote by M_1 the wall of Σ_1 containing P_1. We see that M_1 also contains A but not A_1. By Proposition 3.1, this implies first of all $\kappa \subseteq M_1$ (note that κ is also a convex subcomplex of Σ_1 since κ is convex in Δ). Secondly, A is maximal in $M_1 \cap \kappa_1$. Indeed, if not, then $M_1 \cap \kappa_1$ would contain some simplex strictly bigger than A and hence of dimension $\dim A + 1 = \dim \kappa_1$. But then Proposition 3.1 would imply $\kappa_1 \subseteq M_1$, contradicting $A_1 \notin M_1$. Hence $\dim(M_1 \cap \kappa_1) = \dim A = \dim \kappa$.

We now claim that $M_1 \cap \kappa_1 = \kappa$. The inclusion $\kappa \subseteq M_1 \cap \kappa_1$ is clear (we verified $\kappa \subseteq M_1$ above). Now let β be any root of Σ_1 containing κ. If β contains x_1, it also contains $A_1 = A \cup \{x_1\}$ and hence κ_1. If β does not contain x_1, then A, which is in β and which is joinable to x_1, must be contained in the wall N which bounds β. However, we showed above that A is maximal in $M_1 \cap \kappa_1$. Hence $A \in N$ implies $M_1 \cap \kappa_1 \subseteq N$ by Proposition 3.1. So for *any* root β of Σ_1 which contains κ we have $M_1 \cap \kappa_1 \subseteq \beta$. Hence $M_1 \cap \kappa_1$ is contained in the intersection of all these roots β, and this intersection is equal to κ because κ is a convex subcomplex of Σ_1. So we have also proved $M_1 \cap \kappa_1 \subseteq \kappa$, and the claim follows.

Recall that $\kappa_1 = \Sigma_1 \cap \Sigma$. Hence the previous claim immediately implies $M_1 \cap \Sigma = M_1 \cap \Sigma_1 \cap \Sigma = M_1 \cap \kappa_1 = \kappa$. Applying Lemma 4.2, we find an apartment Σ' of Δ which contains M_1 and satisfies $\Sigma' \cap \Sigma = M_1 \cap \Sigma = \kappa$. $\square$

Corollary 4.4 *If κ is a convex subcomplex of Σ with $\dim \kappa < \dim \Sigma$, then there exists a wall M in Σ containing κ, and for each such wall M there exists some wall M' in Δ such that $\kappa = M \cap M'$.*

Proof. Choose the apartment Σ' as in Theorem 4.3 such that $\Sigma' \cap \Sigma = \kappa$. Let A be a maximal simplex of κ and choose walls M, M' in Σ, Σ', respectively, with $A \in M$ and $A \in M'$ (this is possible since $\dim A = \dim \kappa < \dim \Sigma = \dim \Sigma'$). Then by Proposition 3.1, $\kappa \subseteq M$ and $\kappa \subseteq M'$. So we have

$$\kappa \subseteq M \cap M' \subseteq \Sigma \cap \Sigma' = \kappa,$$

and the assertion follows. $\square$

5   Incomplete systems of apartments

In this section we show our Second Main Result.

We start with a proposition that motivates our restriction to consider solely *finite* convex subcomplexes in the sequel.

Proposition 5.1 *Let Δ be a thick building and let Σ be an apartment (in the complete apartment system $\mathcal{A}$ of Δ). Let κ be an infinite subset of Σ. Then the family $\mathcal{A}^* = \{\Sigma^* \in \mathcal{A} \mid \kappa \not\subseteq \Sigma^*\}$ is a system of apartments for Δ. In particular, $\mathcal{A}' = \mathcal{A}^* \cup \{\Sigma\}$ is also a system of apartments for Δ, and κ is contained in a unique member of $\mathcal{A}'$. Hence if κ is any infinite proper convex subcomplex of Σ, it cannot be the intersection of all apartments of $\mathcal{A}'$ in which it is contained.*

Proof. Let C, D be two chambers of Δ. Then the convex hull H of C, D is contained in some apartment $\Sigma_H \in \mathcal{A}$. Since H is convex, Proposition 4.1 implies the existence of an apartment $\Sigma'_H \in \mathcal{A}$ such that $\Sigma_H \cap \Sigma'_H = H$. Since H is finite, we have $\kappa \not\subseteq H$ and so κ cannot be contained in both of Σ_H and Σ'_H. Hence at least one of these is a member of $\mathcal{A}^*$.

The assertion now follows easily. $\square$

Theorem 5.2 *Let Δ be a thick building of type (W, S). Then the following are equivalent.*

(I) *For any apartment system $\mathcal{A}$ of Δ, the convex closure of two given chambers C and D is precisely the intersection of two apartments in $\mathcal{A}$ containing C and D.*

(II) *For any apartment system $\mathcal{A}$ of Δ, every finite convex chamber subcomplex κ contained in some member of $\mathcal{A}$ is precisely the intersection of two apartments in $\mathcal{A}$ containing κ.*

(III) *For every finite subset F of chambers of the standard Coxeter complex $\Sigma(W, S)$, there exist two chambers $C, D \in \Sigma(W, S)$ such that F is contained in the convex closure of C and D.*

(IV) *For every triplet $\{X, Y, Z\}$ of chambers of the standard Coxeter complex $\Sigma(W, S)$, there exist two chambers $C, D \in \Sigma(W, S)$ such that $\{X, Y, Z\}$ is contained in the convex closure of C and D.*

Proof. We show (IV)$\Rightarrow$(III)$\Rightarrow$(II)$\Rightarrow$(I)$\Rightarrow$(IV), where the last implication requires most of the work.

(1) (IV)$\Rightarrow$(III). We use induction on $|F| \geq 3$. For $|F| = 3$, this is precisely (IV). Now let $|F| > 3$ and choose $Z \in F$ arbitrary. Then, by the induction hypothesis, there exist two chambers X, Y in $\Sigma(W, S)$ such that $F \setminus \{Z\}$ is contained in the convex closure of X and Y. Applying (IV) to X, Y, Z gives us the desired pair C, D of chambers.

(2) (III)$\Rightarrow$(II). Denote by $\widetilde{\mathcal{A}}$ the complete apartment system of Δ. Suppose $\kappa \subseteq \Sigma \in \mathcal{A}$. By Proposition 4.1, there exists a member $\widetilde{\Sigma}$ of $\widetilde{\mathcal{A}}$ such that $\kappa = \Sigma \cap \widetilde{\Sigma}$. We may assume $\widetilde{\Sigma} \notin \mathcal{A}$. Define
$$\widetilde{\kappa} := \{E \in \mathrm{Ch}(\widetilde{\Sigma}) \mid E \text{ has a panel in } \kappa\}.$$
Since $\widetilde{\kappa}$ is clearly a finite set, (III) implies that $\widetilde{\kappa}$ is contained in the convex hull H of two chambers X, Y of $\widetilde{\Sigma}$. Let now Σ' be a member of $\mathcal{A}$ containing X, Y. We claim that $\kappa = \Sigma \cap \Sigma'$. Indeed, we clearly have $\kappa \subseteq \Sigma \cap \Sigma'$, so assume by way of contradiction that $\Sigma \cap \Sigma' \not\subseteq \kappa$. Then there are adjacent chambers C, D with $C \in \kappa$, $D \notin \kappa$ and $C, D \in \Sigma \cap \Sigma'$. By definition of $\widetilde{\kappa}$, each panel P contained in κ is contained in precisely two chambers of $\widetilde{\kappa}$. Since $\widetilde{\kappa} \subseteq H \subseteq \Sigma'$, these are the two chambers in Σ' which contain P. So the two chambers of Σ' containing P are in fact contained in $\widetilde{\kappa}$. Applied to $P = C \cap D$, we obtain $C, D \in \widetilde{\kappa}$. Hence $D \in \Sigma \cap \widetilde{\Sigma} = \kappa$, a contradiction. The claim is proved, and so is Assertion (II).

(3) (II)⇒(I). This is obvious since the convex closure of two chambers always contains a finite number of chambers.

(4) (I)⇒(IV). Given three chambers X, Y, Z in some apartment Σ of Δ, we show that there exist two chambers C, D in Σ such that X, Y, Z are contained in the convex hull of C and D. This is trivial if Z is contained in the convex closure of X and Y. So we may assume that Z is not contained in that convex closure. We first prove (IV) in the special case that Z is adjacent to some chamber E of the convex closure of X and Y. So assume, by way of contradiction, that there are three chambers C_1, C_2, C_3 in some apartment Σ, with C_3 adjacent to some chamber E of the convex closure θ of C_1 and C_2, but C_3 does not belong to θ, such that the convex closure of any two chambers C, D of Σ does not contain all of them.

Let $\widetilde{\mathcal{A}}$ be the complete system of apartments for Δ, and let $\mathcal{A}$ be the subset of $\widetilde{\mathcal{A}}$ consisting of those apartments that either do not contain $\{C_1, C_2\}$ or contain $\{C_1, C_2, C_3\}$. We show that $\mathcal{A}$ is a system of apartments for Δ. It suffices to show that any two chambers U, V are contained in a member of $\mathcal{A}$. Let κ be the convex closure of U and V. Choose an apartment $\Sigma' \in \widetilde{\mathcal{A}}$ with $U, V \in \Sigma'$ and hence $\kappa \subseteq \Sigma'$. If $\{C_1, C_2\} \not\subseteq \kappa$, then we choose, using Proposition 4.1, another apartment $\Sigma'' \in \widetilde{\mathcal{A}}$ such that $\kappa = \Sigma' \cap \Sigma''$. Then not both of Σ' and Σ'' contain $\{C_1, C_2\}$, and hence at least one of Σ' and Σ'' is in $\mathcal{A}$.

Suppose now that $\{C_1, C_2\} \subseteq \kappa$. Let E' be the chamber of Σ' which contains the panel $P = E \cap C_3$ and is distinct from E. We first show that $E' \notin \kappa$. For this, let $\varphi : \Sigma' \to \Sigma$ be the isomorphism fixing $\Sigma' \cap \Sigma$ (and hence θ) pointwise (see Proposition 3.2). Then the convex hull $\varphi(\kappa)$ of $\varphi(U)$ and $\varphi(V)$ contains $\varphi(\theta) = \theta$. If $E' \in \kappa$, then $\varphi(\kappa)$ also contains $\varphi(E')$. Since $\varphi(E')$ is a chamber of Σ with $\varphi(E') \neq \varphi(E) = E$ and $P = \varphi(P) \subseteq \varphi(E')$, we must have $\varphi(E') = C_3$. So the convex closure of $\varphi(U)$ and $\varphi(V)$ contains $\varphi(\{C_1, C_2, E'\}) = \{C_1, C_2, C_3\}$, contradicting our assumption. Consequently we must have $E' \notin \kappa$.

Now let α be the root in Σ' containing E but not E'. Since κ is convex, $E \in \kappa$ and $E' \notin \kappa$, it follows that $\kappa \subseteq \alpha$. Since we assume $\{C_1, C_2\} \subseteq \kappa$, also $\theta \subseteq \alpha$. By Proposition 3.3, there exists an apartment $\Sigma'' \in \widetilde{\mathcal{A}}$ with $\alpha \cup \{C_3\} \subseteq \Sigma''$. Since $\{C_1, C_2, C_3\} \subseteq \Sigma''$, we have $\Sigma'' \in \mathcal{A}$. This completes the proof that $\mathcal{A}$ is a system of apartments for Δ.

However, the convex closure of C_1 and C_2 can not be equal to the intersection of two apartments of $\mathcal{A}$ containing both of C_1 and C_2 since the former does not contain C_3 and the latter always does. This contradicts Assumption (I). So C_1, C_2, C_3 as described cannot exist. Consequently, if Z is adjacent to a chamber of the convex closure of X and Y, then $\{X, Y, Z\}$ is contained in the convex closure of two chambers of Σ.

We now prove the general case by induction on the gallery distance n from Z to the convex closure of X, Y. If $n = 1$, then this is the foregoing. If $n > 1$, then there is a chamber Z' adjacent to Z and at distance $n - 1$ from the convex closure of X, Y. The induction hypothesis implies the existence of two chambers C', D' in Σ such that X, Y, Z' are contained in the convex closure of C', D'. Applying the above to $\{C', D', Z\}$, the assertion is now clear.

Since every apartment of Δ is isomorphic to the standard Coxeter complex $\Sigma(W, S)$, the equivalence of (I) up to (IV) is proved completely. $\hspace{1cm}\square$

For the classification of Coxeter complexes satisfying Condition (III) of the above theorem, we refer to [4], Theorem 7.2. Necessary and sufficient is that each irreducible component is affine or spherical. Since spherical buildings are automatically endowed with the complete system of apartments, we may restrict to the irreducible affine case to state the following proposition.

Proposition 5.3 *Let Δ be an irreducible building of affine type with given apartment system $\mathcal{A}$, and suppose each panel is contained in at least four chambers. Then every finite convex subcomplex κ contained in some apartment $\Sigma \in \mathcal{A}$ is the intersection of Σ with some other member $\Sigma' \in \mathcal{A}$.*

Proof. Let $\widetilde{\mathcal{A}}$ be the complete system of apartments. By Theorem 4.3, there exists $\widetilde{\Sigma} \in \widetilde{\mathcal{A}}$ such that $\kappa = \Sigma \cap \widetilde{\Sigma}$. Now denote by $\mathrm{Vert}(\kappa)$ the set of vertices of κ and define

$$\widetilde{\kappa} := \bigcup_{x \in \mathrm{Vert}(\kappa)} \mathrm{St}_{\widetilde{\Sigma}}(x).$$

Then $\widetilde{\kappa}$ is the union of a finite set of chambers of $\widetilde{\Sigma}$. Hence $\widetilde{\kappa}$ is contained in the convex hull of two chambers, and there exists $\Sigma' \in \mathcal{A}$ with $\widetilde{\kappa} \subseteq \Sigma'$. For every $x \in \mathrm{Vert}(\kappa)$, the links $\mathrm{Lk}_{\widetilde{\Sigma}}(x)$ and $\mathrm{Lk}_{\Sigma'}(x)$ are two apartments in $\mathrm{Lk}_\Delta(x)$ with $\mathrm{Lk}_{\widetilde{\Sigma}}(x) \subseteq \mathrm{Lk}_{\Sigma'}(x)$, since $\mathrm{St}_{\widetilde{\Sigma}}(x) \subseteq \Sigma'$. Hence $\mathrm{Lk}_{\widetilde{\Sigma}}(x) = \mathrm{Lk}_{\Sigma'}(x)$, and consequently also $\mathrm{St}_{\widetilde{\Sigma}}(x) = \mathrm{St}_{\Sigma'}(x)$, implying

$$\widetilde{\kappa} = \bigcup_{x \in \mathrm{Vert}(\kappa)} \mathrm{St}_{\Sigma'}(x).$$

We claim that $\Sigma \cap \Sigma' = \kappa$. Indeed, since κ is certainly contained in both of Σ and Σ', we may assume by way of contradiction that some vertex y is contained in $\Sigma \cap \Sigma'$ but not in κ. By connectivity of $\Sigma \cap \Sigma'$ (recall that apartments in affine buildings are *geodesically convex*), this implies that there is some edge $\{x, x'\}$, with $x \in \kappa$, and with $x' \in \Sigma \cap \Sigma'$

19

and not in κ. Hence $x' \in \mathrm{St}_{\Sigma'}(x)$ and so $x' \in \widetilde{\kappa} \subseteq \widetilde{\Sigma}$. Since x' also belongs to Σ, we obtain the contradiction $x' \in \Sigma \cap \widetilde{\Sigma} = \kappa$.

The proposition is proved. $\qquad\square$

Remark 5.4 If κ contains a chamber, the above proof yields (referring to Proposition 4.1 instead of Theorem 4.3) the same result without the assumption that each chamber of the thick building Δ is contained in at least four chambers.

We now prove Proposition 2.5.

Proposition 5.5 *For every thick building Δ and every apartment system $\mathcal{A}$ of Δ, the subcomplex formed by two adjacent chambers is always the intersection of all members of $\mathcal{A}$ containing both of those chambers.*

Proof. Let C and D be two adjacent chambers of Δ. Let E be a chamber in the intersection of all apartments containing both of C and D, and assume, by way of contradiction that $E \notin \{C, D\}$. By convexity, we may assume that E is adjacent to one of C, D, and without loss of generality, we may assume E is adjacent to D, and hence not to C. Let E' be a third chamber containing the panel $D \cap E$, $E' \notin \{D, E\}$. Then any apartment containing C and E' contains D (since (C, D, E') is a minimal gallery) and does not contain E, a contradiction.

The proposition is proved completely. $\qquad\square$

We end this paper with an application of the last result.

Proposition 5.6 *Let Δ be a thick building, and let Δ' be an arbitrary building. Let Δ and Δ' be endowed with arbitrary systems of apartments $\mathcal{A}$ and $\mathcal{A}'$, respectively. Let φ be a map from $\mathrm{Ch}(\Delta)$ to $\mathrm{Ch}(\Delta')$. Suppose that φ bijectively maps the set of chambers of any apartment $\Sigma \in \mathcal{A}$ to the set of chambers of some apartment $\Sigma' \in \mathcal{A}'$. Then φ is injective and preserves adjacency of chambers (i.e., any two chambers of Δ are adjacent if and only if their images under φ are). Furthermore, $\varphi(\mathrm{Ch}(\Delta))$ is the set of chambers of a thick subbuilding of Δ'. Hence φ induces a simplicial isomorphism of Δ onto a thick subbuilding of Δ' in case Δ is 2-spherical.*

Proof. First note that φ is injective. Indeed, this follows from the fact that φ is injective when restricted to the set of chambers of any apartment, and the fact that every pair of

chambers is contained in at least one apartment. Secondly, note that the image $\varphi(\mathrm{Ch}(\Delta))$ is the chamber set of a subbuilding of Δ' with $\varphi(\mathcal{A})$ (obvious notation) as apartment system. Indeed, we only need to check that every pair $\varphi(C), \varphi(D)$ of chambers of $\varphi(\mathrm{Ch}(\Delta))$ is contained in an element of $\varphi(\mathcal{A})$ (with C, D chambers of Δ). But that follows immediately from our assumptions. Hence, from now on, we may assume that $\varphi(\mathrm{Ch}(\Delta))$ coincides with $\mathrm{Ch}(\Delta')$, and that $\mathcal{A}'$ is the set of all images under φ of the members of $\mathcal{A}$.

We now show that Δ' is thick. Let P be a panel of Δ' contained in the two chambers $\varphi(C)$ and $\varphi(D)$. Then, considering any chamber of Δ adjacent to C and different from D, we can use Proposition 5.5 to see that there exists some apartment $\Sigma \in \mathcal{A}$ containing C but not D. Let E' be the unique chamber of $\varphi(\mathrm{Ch}(\Sigma))$ adjacent to $\varphi(C)$ and containing P (recall that $\varphi(\mathrm{Ch}(\Sigma)) = \mathrm{Ch}(\Sigma')$, for some $\Sigma' \in \mathcal{A}'$). Since $D \notin \Sigma$, it follows from global injectivity that $E' \neq \varphi(D)$. This shows thickness of Δ'.

Now we show that φ preserves adjacency. Let C, D be adjacent in Δ. Then the intersection of all apartments in $\mathcal{A}'$ containing $\varphi(C)$ and $\varphi(D)$ contains every chamber of the convex hull H of $\varphi(C)$ and $\varphi(D)$. Let $\varphi(E)$ be a chamber in H. Then E is contained in every apartment in $\mathcal{A}$ containing C, D. By Proposition 5.5, $E \in \{C, D\}$, and hence $\varphi(E) \in \{\varphi(C), \varphi(D)\}$. So we have proved that $\varphi(C)$ and $\varphi(D)$ are the only chambers in H, showing that $\varphi(C)$ and $\varphi(D)$ are adjacent. Interchanging the roles of Δ and Δ', we also see that φ^{-1} maps adjacent chambers to adjacent chambers.

In view of Proposition 3.21 of [6], the proposition is completely proved. $\qquad\square$

References

[1] P. Abramenko, Walls in Coxeter complexes, *Geom. Dedicata* **49** (1994), 71–84.

[2] P. Abramenko and K. Brown, *Buildings: Theory and Applications*, Graduate Texts in Mathematics **248**, Springer, 2008.

[3] P. Abramenko and H. Van Maldeghem, Maps between buildings that preserve a given Weyl distance, *Indag. Math.* **15** (2004), 305–319.

[4] P.-E. Caprace, *"Abstract" homomorphisms of split Kac-Moody groups*, Mem. Amer. Math. Soc. **198**, 2009.

[5] M. A. Ronan, *Lectures on Buildings*, in: Perspect. Math. **7**, Academic Press, San Diego, CA, 1989.

[6] J. Tits, *Buildings of Spherical Type and Finite BN-pairs*, Lect. Notes **386**, Springer, 1974.

[7] J. Tits, A local approach to buildings, **in** *The Geometric Vein. The Coxeter Festschrift* (ed. D. Chandler *et al.*), Springer (1981), 519–547.

[8] J. Tits, Immeubles de type affine, **in** *Buildings and the Geometry of Diagrams*, Como 1984 (ed. L. Rosati), Lect. Notes **1181**, Springer (1986), 157–190.

[9] J. Tits, Twin buildings and groups of Kac-Moody type, **in** *Groups, Combinatorics and Geometry*, Durham 1990 (ed. M. Liebeck & J. Saxl), London Math. Soc. Lecture Notes Ser. **165**, Cambridge University Press, Cambridge (1992), 249–286.

Addresses of the authors:

Peter Abramenko
Department of Mathematics
University of Virginia
P.O.Box 400137
Charlottesville, VA 22904
USA
`pa8e@virginia.edu`

Hendrik Van Maldeghem
Department of Pure Mathematics and Computer Algebra
Ghent University
Krijgslaan 281, S22
9000 Gent
Belgium
`hvm@cage.UGent.be`