Codistances of 3-spherical buildings

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Abstract

We show that a 3-spherical building in which each rank 2 residue is connected far away from a chamber, and each rank 3 residue is simply 2-connected far away from a chamber, admits a twinning (i.e., is one half of a twin building) as soon as it admits a codistance, i.e., a twinning with a single chamber.

1 Introduction

Twin buildings have been introduced by M. A Ronan and J. Tits in the late 1980's. Their definition is motivated by the theory of Kac-Moody groups over fields. Kac-Moody groups are infinite-dimensional generalizations of Chevalley groups and the buildings associated with the latter are spherical. Spherical buildings have been classified by J. Tits in [Ti74]. This classification relies heavily on the fact that there is an opposition relation on the set of chambers of a spherical building. The idea in the definition of a twin building is to extend the notion of an opposition to non-spherical buildings: instead of taking one building, one starts with two buildings $\mathcal{B}_+, \mathcal{B}_-$ of the same type and defines an opposition relation between the chambers of the two buildings in question. Technically, this is done by requiring a *twinning function* between the chamber sets of the two buildings which takes its values in the Weyl group W. Two chambers x, y of $\mathcal{B}_+$ and $\mathcal{B}_-$ are then defined to be opposite, if their twinning is the idendity in W.

There are variations of the idea of a twinning. For instance, one can introduce 'by restriction' a twinning between one chamber of $\mathcal{B}_+$ and the building $\mathcal{B}_-$, seen as an application from the set of chambers of $\mathcal{B}_-$ to the Weyl group. A function from the set of chambers of a building $\mathcal{B}$ to its Weyl group and satisfying similar properties to those of this 'twinning to a chamber' will be called *a codistance on* $\mathcal{B}$. This idea occurs at various places in the literature (see for instance [Mu98] and [Ro08]). In particular, [Ro08] is dealing with the question to which extent the existence of a codistance of a building $\mathcal{B}$ restricts its structure. The main result of the present paper ensures that any 3-spherical building admitting a codistance and satisfying some local condition is in fact one 'half' of a twin building. In particular, it is already known if its diagram is simply laced and if each panel contains at least 4 chambers (see the second remark below).

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Here is the precise statement of our main result. For the definitions and notations we refer to Sections 2 and 3.

Main result: *Let $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be a thick building of 3-spherical type (W, S). Assume that the following two conditions hold.*

(lco) If R is a rank 2 residue of $\mathcal{B}_-$ containing a chamber c, then the set of chambers opposite c inside R is connected.

(lsco) If R is a rank 3 residue of $\mathcal{B}_-$ containing a chamber c, then the set of chambers opposite c inside R is simply 2-connected.

If there exists a codistance function $f : \mathcal{C}_- \to W$, then there exists a building $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$ and a mapping $\delta^* : (\mathcal{C}_- \times \mathcal{C}_+) \cup (\mathcal{C}_+ \times \mathcal{C}_-) \to W$ such that the following two statements hold.

a) $(\mathcal{B}_-, \mathcal{B}_+, \delta^*)$ is a twin building.

b) There exists a chamber $c \in \mathcal{C}_+$ such that $\delta^*(c, x) = f(x)$ for all $x \in \mathcal{C}_-$.

Remarks

On the conditions

By standard arguments there is no loss of generality if one restricts to the case where the building in question has irreducible type. In the following remarks this is always assumed.

3-sphericity: If we drop the 3-sphericity condition (together with conditions (lco) and (lsco)), the conclusion of our main result is not always true. Indeed it is fairly easy to construct examples of buildings admitting a codistance which cannot be realized as a 'half of a twin building'. For instance, it is a trivial fact that each thick building $\mathcal{B}_-$ of type $\tilde{A}_1$ admits a codistance f. Moreover, it can be shown that $\mathcal{B}_-$ can be realized as a 'half of a twin building' if and only if panels of the same type have the same cardinality (see [AB99], [RT99]).

It is an interesting question to wonder which buildings admitting a codistance can or not be realized as a 'half of a twin building'. It is most likely that all right-angled buildings admit a codistance, and that they can be realized as a 'half of a twin building' if and only if panels of the same type have the same cardinality. If there are finite entries different from 2 in the diagram, the question becomes more delicate. Nevertheless, we expect a behavior similar to the case of right-angled buildings if there are 'enough' infinities in the diagram. Hence, for the conclusion of our main result to hold, it is natural to assume that the diagram is *2-spherical* (i.e. there are no infinities in the diagram), in which case panels of the same type always have the same cardinality. By the following remarks, the conditions asked in addition to 3-sphericity are 'almost always' satisfied and therefore it remains to consider 2-spherical buildings which are not 3-spherical. We have no idea about what to expect in this case. On the one hand, the methods used in the proof of our main result completely fail in this more general context. On the other hand we could not manage to construct counter-examples in the $\tilde{A}_2$-case — a case which is well understood in a lot of respects. In the opposite direction goes the result of [MVM08] which shows that certain affine buildings do not admit any codistance.

Condition (lco): By the 3-sphericity assumption, all entries in the diagram are equal to 2, 3 or 4, if the rank is at least 3. It follows from an observation of Cuypers, see [Br93], that Condition (lco) is satisfied if there is no rank 2 residue isomorphic to the building associated with $B_2(2)$. In particular, Condition (lco) is satisfied if the diagram is *simply laced* (i.e. if all entries are 2 or 3).

Condition (lsco): It follows from [Ti86] Corrolaire 2 that Condition (lsco) is satisfied if the diagram is simply laced and if each panel contains at least 4 chambers. If there are subdiagrams of type B_2 we have to consider buildings of type B_3. For those the relevant results concerning Condition (lsco) may be found in [Ab96]. They imply that Condition (lsco) is satisfied if each residue of type B_3 comes from an embeddable polar space and if each panel contains at least 17 chambers. The first condition is equivalent to the fact that any A_2-residue corresponds to a desargesian projective plane, and it is very likely that it can be dropped. Moreover, it is expected that the bound 17 is not optimal.

In view of the remarks above, we have the following corollary of the main result:

Corollary 1: *Let $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be a thick, irreducible building of 3-spherical type (W, S) whose rank is at least 3. Then the conclusions of the main result hold as soon as one of the following conditions is satisfied:*

(1) (W, S) is simply laced and all panels contain at least 4 chambers.

(2) Any residue of type A_2 corresponds to a desarguesian projective plane and any panel contains at least 17 chambers.

An application to simply-laced buildings

Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be an irreducible 3-spherical twin building of rank at least 3 whose diagram is simply laced. Then it is known that $\mathcal{B}$ is Moufang (see for instance [AB08]) and therefore each of its spherical residues is Moufang. If we assume in addition that $\mathcal{B}$ is 3-spherical, then all its A_2-residues are (up to duality) all isomorphic to the building associated to a projective plane over a division ring K. If there is a D_4-subdiagram, then K is commutative and those buildings have been classified in [Mu99a]; in particular, they are of 'algebraic origin'. If there is no D_4-subdiagram, then $\mathcal{B}$ is of type A_n or $\tilde{A}_n$ for some $n \geq 3$. Those buildings are also known by [Ti74] and [Ti84] and of algebraic origin. Putting together all these informations we get the following corollary of our main result.

Corollary 2: *Let $\mathcal{B}_-$ be an irreducible, 3-spherical and simply laced building of rank at least 3 in which each panel contains at least 4 chambers. If $\mathcal{B}_-$ admits a codistance, then it is known and in particular of algebraic origin.*

Content

The paper is organized as follows. In Section 2, we collect the definitions, known results and preliminaries that we need. In Section 3, we prove some basic properties of a codistance; most properties are known to be valid for a twinning, but we need to reprove them here for a codistance. In Section 4, we show that, under the assumptions of our Main Result, the complex of chambers with codistance the identity is simply 2-connected (for *any* codistance!). In Section 5, we construct bijections between panels that are contained in a chamber of codistance the identity. These bijections will then be used in Section 6 to define codistances adjacent to a given codistance. Finally, in Section 7, we prove that all the codistances thus obtained constitute the second half of a twinning, the first half of which is the original building.

2 Preliminaries

In this section, we recall basic definitions and results.

Chamber systems

Let I be a set. A *chamber system* over I is a pair $\mathcal{C} = (C, (\sim_i)_{i\in I})$ where C is a set whose elements are called *chambers* and where $\sim_i$ is an equivalence relation on the set of chambers for each $i \in I$, such that if $c \sim_i d$ and $c \sim_j d$ then either $i = j$ or $c = d$.

We refer to [AB08, DM07] for the definitions of *i-adjacent chambers*, *galleries*, *J-galleries* *J-residues*, *i-panels*.

Two galleries $G = (c_0, \dots, c_k)$ and $H = (c'_0, \dots, c'_{k'})$ are said to be *elementary* 2-*homotopic* if there exist two galleries X, Y and two J-galleries G_0, H_0 for some $J \subset I$ of cardinality at most 2 such that $G = XG_0Y$, $H = XH_0Y$. Two galleries G, H are said to be 2-*homotopic* if there exists a finite sequence $G_0, G_1, \dots, G_l$ of galleries such that $G_0 = G, G_l = H$ and such that $G_{\mu-1}$ is elementary 2-homotopic to G_μ for all $1 \le \mu \le l$. The chamber system $\mathcal{C}$ is called *simply* 2-*connected* if it is connected and if each closed gallery is 2-homotopic to a trivial gallery.

Coxeter systems

A *Coxeter system* is a pair (W, S) consisting of a group W and a set $S \subset W$ such that $\langle S\rangle = W$, $s^2 = 1_W \neq s$ for all $s \in S$ and such that the set S and the relations $((st)^{o(st)})_{s,t\in S}$ constitute a presentation of W, where $o(g)$ denotes the order of g.

Let (W, S) be a Coxeter system. The matrix $M(S) := (o(st))_{s,t\in S}$ is called the *type* or the *diagram* of (W, S). For an element $w \in W$ we put $l(w) := \min\{k \in \mathbf{N} \mid w = s_1s_2 \dots s_k \text{ where } s_i \in S \text{ for } 1 \le i \le k\}$. The number $l(w)$ is called the *length* of w. For a subset J of S we put $W_J := \langle J\rangle$ and we call it *spherical* if W_J is finite.

The following proposition collects several basic facts on Coxeter groups which can be found in the usual standard references [Bo68] or [Hu90]. These facts will be used without reference throughout the paper.

Proposition 2.1.: *Let (W, S) be a Coxeter system.*

a) For $w \in W, s \in S$ we have $\{l(ws), l(sw)\} \subset \{l(w) - 1, l(w) + 1\}$.

b) *For* $w \in W, s, t \in S$ *with* $l(sw) = l(w) + 1 = l(wt)$ *we have* $l(swt) = l(w) + 2$ *or* $swt = w$*.*

c) *For* $J \subset S$ *the pair* (W_J, J) *is a Coxeter system and if* $l_J : W_J \to \mathbf{N}$ *is its length function, then* $l_J = l\,|_{W_J}$*.*

d) *Let* $w \in W$ *and* $J \subset S$*. Then there exists a unique element* $w_J \in wW_J$ *such that* $l(w_J t) = l(w_J) + 1$ *for all* $t \in J$*. Moreover, we have* $l(x) = l(w_J) + l_J(w_J^{-1}x)$ *for all* $x \in wW_J$*.*

e) *If* $J \subset S$ *is spherical, then there is a unique element* $r_J \in W_J$ *such that* $l(r_J w) + l(w) = l(r_J)$ *for all* $w \in W_J$*; the element* r_J *is a non-trivial involution if* $J \neq \emptyset$*.*

f) *Let* $w \in W$ *and let* $J \subset S$ *be spherical. Then there exists a unique element* $w^J \in wW_J$ *such that* $l(w^J t) = l(w^J) - 1$ *for all* $t \in J$ *and we have* $w^J = w_J r_J$*. Moreover we have* $l(x) = l(w^J) - l_J((w^J)^{-1}x)$ *for all* $x \in wW_J$*; in particular,* $l(w_J) + l(r_J) = l(w^J)$*.*

Buildings

Let (W, S) be a Coxeter system. A *building* of type (W, S) is a pair $\mathcal{B} = (C, \delta)$ where C is a set and where $\delta : C \times C \to W$ is a *distance function* satisfying the following axioms where $x, y \in C$ and $w = \delta(x, y)$:

(Bu 1) $w = 1$ if and only if $x = y$;

(Bu 2) if $z \in C$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;

(Bu 3) if $s \in S$, there exists $z \in C$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For a building $\mathcal{B} = (C, \delta)$ we define the chamber system $\mathbf{C}(\mathcal{B}) = (C, (\sim_s)_{s \in S})$ where two chambers $c, d \in C$ are defined to be s-adjacent if $\delta(c, d) \in \langle s \rangle$. The rank of a building $\mathcal{B}$ of type (W, S) is $|S|$.

In this paper all buildings are assumed to be of finite rank and *thick* (which means that for any $s \in S$ and any chamber $c \in C$ there are at least three chambers being s-adjacent to c).

For any two chambers x and y we set $l(x, y) = l(\delta(x, y))$. We say that a gallery $x_0, x_1, \ldots, x_n$ is *minimal* if $n = l(x_0, x_n)$.

In the following proposition we collect several basic facts about buildings. We refer to [Ro89] and [We03] for the details.

Proposition 2.2.: *Let* (W, S) *be a Coxeter system and let* $\mathcal{B} = (C, \delta)$ *be a building of type* (W, S)*.*

a) *The chamber system* $\mathbf{C}(\mathcal{B}) = (C, (\sim_s)_{s \in s})$ *uniquely determines* $\mathcal{B}$*; in other words, the* s*-adjacency relations on* C *determine the distance function* δ*.*

b) *For* $c \in C$ *and* $J \subset S$ *we have* $R_J(c) = \{x \in C \mid \delta(c, x) \in W_J\}$*.*

c) *If* $d : C \times C \to \mathbf{N}$ *is the numerical distance between two chambers in* $(C, (\sim_s)_{s\in s})$*, then* $d = l$*.*

d) *Let* $c \in C$ *and let* $R \subset C$ *be a* J*-residue for some* $J \subset S$*. Then there exists a unique chamber* $x \in R$ *such that* $\delta(c,x) = (\delta(c,x))_J$*. Moreover, for all* $y \in R$ *one has* $\delta(c,y) = \delta(c,x)\delta(x,y)$ *and in particular,* $l(c,y) = l(c,x) + l(x,y)$*.*

Given $c \in C$ and a J-residue R of $\mathcal{B}$ as in Assertion d) of the previous proposition, then the chamber x of its statement is called the *projection of* c *onto* R and it is denoted by $\mathrm{proj}_R\, c$.

Given two residues R and R', we define $\mathrm{proj}_R\, R'$ by the set $\{\mathrm{proj}_R\, c | c \in R'\}$.

Two residues R_1 and R_2 of a building are called *parallel* if $\mathrm{proj}_{R_1} : R_2 \to R_1$ and $\mathrm{proj}_{R_2} : R_1 \to R_2$ are adjacency-preserving bijections inverse to each other.

The following proposition can be found in [DS87].

Proposition 2.3.: *Let* R, Q *be two residues of a building. Put* $\mathrm{proj}_R\, Q := \{\mathrm{proj}_R\, x \mid x \in Q\}$*. Then the following holds:*

a) $\mathrm{proj}_R\, Q$ *is a residue contained in* R*.*

b) *If* $R' := \mathrm{proj}_R\, Q$ *and* $Q' := \mathrm{proj}_Q\, R$ *are parallel.*

Let R be a spherical J-residue of a building of type (W, S). Two chambers x, y of R are *opposite* in R whenever $\delta(x, y) = r_J$. Two residues R_1 of type K_1 and R_2 of type K_2 in R are *opposite* in R if R_1 contains a chamber opposite to a chamber of R_2 and if $K_1 = r_J K_2 r_J$.

The following statement is an easy consequence of Theorem 3.28 of [Ti74].

Proposition 2.4.: *Let* R *be a spherical* J*-residue of a building of type* (W, S) *and let* R_1, R_2 *be two residues opposite in* R*. Then* R_1 *and* R_2 *are parallel.*

This last proposition is Lemma 2.6 in [CM06].

Proposition 2.5.: *Let* R_I, R_J, R_K *be residues of respective type* I, J, K *of a building of type* (W, S)*. Assume that* $R_I \subseteq R_J$*. Then we have* $\mathrm{proj}_{R_I} R_K = \mathrm{proj}_{R_I} \mathrm{proj}_{R_J} R_K$*.*

Twin buildings

Let $\mathcal{B}_+ = (C_+, \delta_+), \mathcal{B}_- = (C_-, \delta_-)$ be two buildings of the same type (W, S), where (W, S) is a Coxeter system. A *twinning* between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a mapping $\delta_* : (C_+ \times C_-) \cup (C_- \times C_+) \to W$ satisfying the following axioms, where $\epsilon \in \{+, -\}, x \in C_\epsilon, y \in C_{-\epsilon}$ and $w = \delta_*(x, y)$:

(Tw 1) $\delta_*(y, x) = w^{-1}$;

(Tw 2) if $z \in C_{-\epsilon}$ is such that $\delta_{-\epsilon}(y, z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x, z) = ws$;

(Tw 3) if $s \in S$, there exists $z \in C_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A *twin building of type* (W,S) is a triple $(\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ where $\mathcal{B}_+, \mathcal{B}_-$ are buildings of type (W,S) and where δ_* is a twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$.

Here is a lemma the proof of which is left to the reader (it follows directly from the definition above and an easy induction on the length of $\delta_+(x,y)$).

Lemma 2.6.: *Let $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta^*)$ be a twin building of type (W,S). Let $x, y \in \mathcal{C}_+$ and $z \in \mathcal{C}_-$ be such that $\delta_+(x,y) = \delta^*(x,z)$. Then y and z are opposite. In particular, if $x^{\mathrm{op}} = y^{\mathrm{op}}$, then $x = y$.*

3 Codistance

In this section, we take (W,S) a Coxeter system and $\mathcal{B} = (\mathcal{C}, \delta)$ a building of type (W,S).

Definition 3.1.: A *codistance* on $\mathcal{B}$ is a function $f : \mathcal{C} \to W$ such that, for all $s \in S$ and P an s-panel of $\mathcal{C}$, there exists $w \in W$ with $f(x) \in \{w, ws\}$ for all $x \in P$ and P contains a unique chamber with f-value the longest word of the two.

As an example, if $\mathcal{B}$ is half of a twin building and x is a chamber in the other half, the twinning to x is a codistance on $\mathcal{B}$.

Lemma 3.2.: *Let R be a J-residue of $\mathcal{B}$ and x be a chamber of R. Then the image of f restricted to R is $f(x)W_J$.*

Proof: It is obvious by the definition of f that the image is contained in $f(x)W_J$. Let w be a word of W_J written as a reduced word as $s_1 s_2 \dots s_k$. Using the fact that for all $s \in J$ and all chamber $y \in R$, there exists at least one chamber s-adjacent to y with f-value $f(y)s$, it is easy to prove (by induction on k) that there exists a chamber in R with f-value $f(x)w$. □

Proposition 3.3.: *Let R be a spherical J-residue. Then there exists a unique chamber c in R such that $l(f(c)) > l(f(y))$ for all $y \in R \setminus \{c\}$. This unique chamber will be denoted by $\operatorname{proj}_R f$. Moreover for all $y \in R$ we have $f(y) = f(c)\delta(c,y)$.*

Proof: Let y be a chamber in R and $w := f(y)$. By Lemma 3.2, f takes on R its values in wW_J. Since R is spherical, it is well-known that wW_J contains a unique longest word w^J. Moreover $l(x) = l(w^J) - l((w^J)^{-1}x)$ for all $x \in wW_J$. By Lemma 3.2, there exists a chamber $c \in R$ with $f(c) = w^J$.

Let y be a chamber in R. The distance $\delta(c,y)$ is in W_J and so can be written as a reduced word as $t_1 t_2 \dots t_k$. Therefore there is a gallery $c = y_0 \sim_{t_1} y_1 \sim_{t_2} \dots \sim_{t_k} y_k = y$. Using the fact that $l(w^J t_1 \dots t_i) = l(w^J) - i$, it is easy to prove by induction that $f(y) = w^J \delta(c,y)$. Therefore $l(f(y)) = l(w^J) - l(\delta(c,y)) = l(f(c)) - d(c,y) \leq l(f(c))$ with equality only for $y = c$. □

Proposition 3.4.: *Let f be a codistance, let R be a J-residue for some $J \subseteq S$. Put $l_f(R) := \min\{l(f(x)) \mid x \in R\}$ and $A_f(R) := \{x \in R \mid l(f(x)) = l_f(R)\}$.*

a) Let $x \in R$. Then $x \in A_f(R)$ if and only if $f(x)$ is the unique shortest word of $f(x)W_J$. Moreover, if $x, y \in A_f(R)$, then $f(x) = f(y)$.

b) Let $y \in R$. Then there exists $x \in A_f(R)$, such that $f(y) = f(x)\delta(x, y)$.

c) If J is spherical, then $A_f(R)$ is the set of all chambers opposite to $\mathrm{proj}_R f$ *in R.*

Proof: Let $y \in R$ and put $w := f(y)$.

By Lemma 3.2, $\{f(y) \mid y \in R\} = wW_J$. It is well known that there exists a unique shortest element $w_J \in wW_J$. Moreover $l(x) = l(w_J) + l(w_J^{-1}x)$ for all $x \in wW_J$. It follows that $A_f(R) = \{x \in R \mid f(x) = w_J\}$. This proves Part a) of the proposition.

Let now $t_1t_2 \dots t_k$ be a reduced representation of $w_J^{-1}w$ and let $x = y_0 \sim_{t_1} y_1 \sim_{t_2} \dots \sim_{t_k} y_k = y$ be a reduced gallery ending in y. Using the fact that $l(wt_kt_{k-1} \dots t_{i+1}) = l(w_Jt_1t_2 \dots t_i) = l(w_J) + i$, it follows from an easy induction on k that $x \in A_f(R)$. By construction $\delta(x, y) = t_1t_2 \dots t_k = w_J^{-1}w = f(x)^{-1}f(y)$. This finishes Part b).

Let J be spherical. Then $x \in A_f(R)$ if and only if $f(x) = w_J$. Let $c = \mathrm{proj}_R f$ so that $f(c) = w^J$ as in Proposition 3.3. We have seen that $f(x) = w^J\delta(c, x)$. Since $w^J = w_Jr_J$, where r_J is the unique longest word of W_J, we can conclude that $x \in A_f(R)$ if and only if $\delta(c, x) = r_J$, that is, if and only if x is opposite to c in R.

□

Definition 3.5.: For a codistance f, we denote by f^{op} the set of chambers of $\mathcal{C}$ with f-value 1_W.

Lemma 3.6.: *Let c be a chamber of $\mathcal{C}$. Then a shortest gallery from c to a chamber in f^{op} has length $l(f(c))$.*

Proof: It is obvious from the definition of codistance that no chamber at distance strictly less than $l(f(c))$ from c can be in f^{op}. Now by Proposition 3.4 with $J = S$, there exists $x \in A_f(\mathcal{C}) = f^{\mathrm{op}}$ such that $f(c) = \delta(x, c)$. Hence a minimal gallery from c to x will have length $l(f(c))$. □

For $c \in \mathcal{C}$, we define $f_c^{\mathrm{op}} = \{x \in f^{\mathrm{op}} | \delta(x, c) = f(c)\}$, that is the set of chambers of f^{op} closest to c, which is non-empty, by Lemma 3.6.

Lemma 3.7.: *The following statements are equivalent:*

a) the chamber x is in f_c^{op},

b) for any minimal gallery $x = x_0, x_1 \dots, x_n = c$ we have $l(f(x_i)) = i$ for all $0 \le i \le n$,

c) there exists a minimal gallery $x = x_0, x_1 \dots, x_n = c$ with $l(f(x_i)) = i$ for all $0 \le i \le n$.

Proof: Assume $x \in f_c^{\text{op}}$. Let $x = x_0, x_1 \dots, x_n = c$ be any minimal gallery from x to c. Since $\delta(x,c) = f(c)$, $n = l(f(c))$. By the axioms of codistance, the f-values of two adjacent chambers are either equal or have length difference one, hence we must have $l(f(x_i)) = l(f(x_{i-1})) + 1$ for all $1 \le i \le n$, which implies b).

Obviously b) implies c).

Assume that there exists a minimal gallery $x = x_0, x_1 \dots, x_n = c$ with $l(f(x_i)) = i$ for all $0 \le i \le n$. Then $l(f(x)) = 0$, so $f(x) = 1_W$. Assume that $f(x_i) = \delta(x, x_i)$, then $f(x_{i+1}) = \delta(x, x_{i+1})$. Indeed $x_i \sim_{s_i} x_{i+1}$ for some $s_i \in S$ and so $f(x_{i+1}) = f(x_i)$ or $s_i f(x_i)$. Since $l(f(x_{i+1})) \neq l(f(x_i))$, we are in the second case and $f(x_{i+1}) = s_i\delta(x, x_i) = \delta(x, x_{i+1})$. This proves by induction that $f(x_i) = \delta(x, x_i)$ for all $0 \le i \le n$, and so $f(c) = \delta(x, c)$, which yields a). □

Lemma 3.8.: *Let $x \in \mathcal{C}$ and $w \in W$ such that $l(f(x)w) = l(f(x)) + l(w)$. Then there exists a unique chamber c of $\mathcal{C}$ with $f(x)^{-1}f(c) = w = \delta(x, c)$.*

Proof: Let $s_1 s_2 \dots s_k$ be a reduced word for w. Since $l(f(x)w) = l(f(x)) + l(w)$, we have $l(f(x)s_1 s_2 \dots s_i) = l(f(x)) + i = 1 + l(f(x)s_1 s_2 \dots s_{i-1})$. Consider the s_1-panel on x, it follows from the axioms of codistance that this panel contains a unique chamber with f-value $f(x)s_1$, namely the projection of f on it. Continuing by induction on k, we can easily build a unique gallery $x = x_0 \sim_{s_1} x_1 \sim s_2 \dots \sim_{s_k} x_k = c$ such that $f(x_i) = f(x)s_1 s_2 \dots s_i$ for all i. Hence $w = \delta(x, c)$ and $f(c) = f(x)w$, and so c exists.

Assume there exists another chamber c' with $f(x)^{-1}f(c') = w = \delta(x, c')$. Hence there exists a minimal gallery $x = x'_0, x'_1 \dots, x'_k = c'$ with $l(f(x_i)) = f(x) + i$ for all $0 \le i \le k$, of type $t_1, t_2, \dots, t_k$ where $t_1 t_2 \dots t_k = w$. On the other hand, since $\delta(x, c) = w$, there is a minimal gallery of type $t_1, t_2, \dots, t_k$ from x to c. Because f has to lengthen at each step, we see by induction that this gallery is exactly $x = x'_0, x'_1 \dots, x'_n = c'$, and so $c = c'$. □

Lemma 3.9.: *Let R be a residue of $\mathcal{B}$ and c a chamber of R. If $x \in f_c^{\text{op}}$ then $\text{proj}_R\, x \in A_f(R)$ and $l(\delta(x, \text{proj}_R\, x)) = l_f(R)$.*

Proof: Let J be the type of R and let $w = f(c) = \delta(x, c)$. We have $l(w) = l(w_J) + l(w_J^{-1}w)$. Hence, if $s_1 s_2 \dots s_k$ is a reduced word for w_J and $s_{k+1}s_{k+2}\dots s_n$ is a reduced word for $w_J^{-1}w \in W_J$, then $s_1 s_2 \dots s_n$ is a reduced word for w. Consider the gallery $x = x_0 \sim_{s_1} x_1 \sim_{s_2} \dots \sim_{s_n} x_n$ and such that $l(f(x_i)) = i$. Then $f(x_i) = s_1 s_2 \dots s_i$ and in particular $f(x_n) = w$. By construction we also have $\delta(x, x_n) = w$. A chamber satisfying those two conditions is unique by Lemma 3.8 and therefore $x_n = c$. As $w_J^{-1}w \in W_J$, $s_i \in W_J$ for $i \ge k+1$, and so $x_i \in R$ for $i \ge k$. Since $l(\delta(x, x_k)) = l(w_J)$, which is the shortest possible for a chamber in R, hence $x_k = \text{proj}_R\, x$ and $x_k \in A_f(R)$ by Proposition 3.4. Moreover $\delta(x, x_k) = f(x_k) = w_J = l_f(R)$. □

Lemma 3.10.: *The set f^{op} determines uniquely f.*

Proof: Assume there exists a codistance $f' \neq f$ on $\mathcal{B}$ with $f'^{\text{op}} = f^{\text{op}}$. Then consider c at minimal distance from f^{op} under the condition that $f'(c) \neq f(c)$. Of course, c is not in f^{op}. Let $c = c_0, c_1, \ldots, c_m$ be a shortest gallery from c to f^{op}. This minimal gallery has length $l(f(c))$ by Lemma 3.6. It is also a shortest gallery to f'^{op}, and so has length $l(f'(c))$. Therefore $l(f(c)) = l(f'(c))$. Now c_1 is closer to f^{op} than c, and so $f(c_1) = f'(c_1)$. By the definition of codistance $f(c) = f(c_1)$ or $f(c_1)t$ (where t is such that $c_0 \sim_t c_1$). Idem for f'. Since $l(f(c)) = l(f'(c))$, it implies that $f(c) = f'(c)$. This contradiction proves that $f = f'$. □

4 Simple connectedness of f^{op}

In this section we will apply a result proved in [DM07] using filtrations.

Let I be a set and let $\mathcal{C} = (C, (\sim_i)_{i\in I})$ be a chamber system over I. In the following we denote the set of non-negative integers by $\mathbf{N}$ and the set of positive integers by $\mathbf{N}_0$.

A *filtration* of $\mathcal{C}$ is a family $\mathcal{F} = (C_n)_{n\in\mathbf{N}}$ of subsets of C such that the following holds.

(F1) $C_n \subset C_{n+1}$ for all $n \in \mathbf{N}$,

(F2) $\bigcup_{n\in\mathbf{N}} C_n = C$,

(F3) for each $n > 0$ if $C_{n-1} \neq \emptyset$ then there exists an index $i \in I$ such that for each chamber $c \in C_n$ there exists a chamber $c' \in C_{n-1}$ which is i-adjacent to c.

A filtration $\mathcal{F} = (C_n)_{n\in\mathbf{N}}$ is called *residual* if for each $\emptyset \neq J \subset I$ and each J- residue R the family $(C_n \cap R)_{n\in\mathbf{N}}$ is a filtration of the chamber system $\mathcal{R} := (R, (\sim_j)_{j\in J})$.

For each $x \in C$ we put $|x| := \min\{\lambda \in \mathbf{N} \mid x \in C_\lambda\}$. For a subset X of C we put $|X| := \min\{|x| \mid x \in X\}$ and $\mathrm{aff}(X) := \{x \in X \mid |x| = |X|\}$. Note that $C_0 = \mathrm{aff}(C)$ if we assume that $C_0 \neq \emptyset$.

We say that a filtration satisfies *Condition (lco)* if for every rank 2 residue R, $\mathrm{aff}(R)$ is a connected subset of the chamber system $\mathcal{R}$.

We say that a filtration satisfies *Condition (lsco)* if for every rank 3 residue R, $\mathrm{aff}(R)$ is a simply 2-connected subset of the chamber system $\mathcal{R}$.

Theorem 4.1.:*[DM07] Suppose that the residual filtration $\mathcal{F} = (C_n)_{n\in\mathbf{N}}$ of the chamber system $\mathcal{C}$ satisfies (lco), (lsco) and $C_0 \neq \emptyset$. Then the following are equivalent:*

a) $\mathcal{C}$ is simply 2-connected;

b) $(C_n, (\sim_i)_{i\in I})$ is simply 2-connected for all $n \in \mathbf{N}$.

The filtration $\mathcal{F}_f$

We choose an injection $w \mapsto |w|$ from W into $\mathbf{N}$ such that $l(x) < l(y)$ implies $|x| < |y|$ for all $x, y \in W$ and such that $|1_W| = 0$. Such an injection exists because $\mathcal{B}$ is of finite rank. Let f be a codistance. We define C_n by setting $C_n := \{x \in C \mid |f(x)| \leq n\}$.

The goal of this subsection is to show the following proposition.

Proposition 4.2.: *With the definitions above, the family $\mathcal{F}_f := (C_n)_{n\in\mathbf{N}}$ is a residual filtration of the chamber system $\mathcal{C}$.*

Proof: It is obvious that $\mathcal{F}_f$ satisfies the axioms (F1) and (F2) and from this it follows that these axioms also hold 'residually'.

Let R be a J-residue of $\mathcal{C}$ with $J \neq \emptyset$ and let $|R| := \min\{k \mid C_k \cap R \neq \emptyset\}$. It follows from the definition of $\mathcal{F}_f$ and by Proposition 3.4 that $\mathrm{aff}(R) = C_{|R|} \cap R = A_f(R) = \{x \in R \mid f(x) = f(x)_J\}$.

Let $0 < n \in \mathbf{N}$ be such that $C_{n-1} \cap R \neq \emptyset$. We have to show that there is $t \in J$ with the property that each chamber x in $R \cap C_n$ is t-adjacent to a chamber $x' \in R \cap C_{n-1}$. If $C_n \cap R = C_{n-1} \cap R$ we can choose $t \in J$ arbitrarily and set $x' := x$ for each $x \in R \cap C_n$. Suppose now that $R \cap C_{n-1}$ is properly contained in $C_n \cap R$, choose $y \in R \cap C_n \setminus C_{n-1}$ and put $w := f(y)$. Since $| \cdot |$ injects W into $\mathbf{N}$, it follows from the definition of $\mathcal{F}_f$ that $f(y') = w$ for all $y' \in C_n \setminus C_{n-1}$. On the other hand, there exists $x \in A_f(R)$ such that $w = f(y) = f(x)\delta(x,y)$ by Assertion b) of Proposition 3.4. As $C_{n-1} \cap R \neq \emptyset$ it follows that $y \notin A_f(R)$ and hence $\delta(x,y) \in W_J \setminus \{1_W\}$. Let $t \in J$ be such that $l(\delta(x,y)t) = l(\delta(x,y)) - 1$. As $f(x) = f(x)_J = w_J$ and $\delta(x,y) \in W_J$ it follows that $l(wt) = l(w_J\delta(x,y)t) = l(w_J) + l(\delta(x,y)t) = l(w_J) + l(\delta(x,y)) - 1 = l(w) - 1$, by a property of w_J. For any chamber $z \in R \cap C_n$ we choose a chamber $z' \in R$ as follows. If $z \in C_{n-1}$ then we put $z' := z$. If $z \in C_n \setminus C_{n-1}$ then we know that $f(z) = w$ and we choose $z' \in R$ such that $z \sim_t z' \neq z$. In the first case, it is obvious that z' is in $R \cap C_{n-1}$; in the second case we have $f(z') = wt$ by the definition of f, as wt is shorter than w. It follows that $|wt| < |w| = n$ and therefore $z' \in C_{n-1}$. As $t \in J$ we have also $z' \in R$.

The case $J = S$ is a special case of the consideration above. This shows that $\mathcal{F}_f$ satisfies Axiom (F3). Hence $\mathcal{F}_f$ is a residual filtration. $\square$

Theorem 4.3.: *Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W,S) and f a codistance on $\mathcal{B}$. Suppose that the following conditions are satisfied:*

(3-sph.) If $J \subseteq S$ is of cardinality at most 3, then J is spherical.

(lco) If J is of cardinality 2, if $R \subset \mathcal{C}$ is a J-residue and if $x \in R$, then the chamber system $(\{y \in R \mid \delta(x,y) = r_J\}, (\sim_t)_{t\in J})$ is connected.

(lsco) If J is of cardinality 3, if $R \subset \mathcal{C}$ is a J-residue and if $x \in R$, then the chamber system $(\{y \in R \mid \delta(x,y) = r_J\}, (\sim_t)_{t\in J})$ is simply 2-connected.

Then the chamber system f^{op} is simply 2-connected.

Proof: Let $\mathcal{F}_f = (C_n)_{n\in\mathbf{N}}$ be the residual filtration of the previous subsection. Note first that $C_0 = f^{\mathrm{op}}$.

Given a spherical J-residue R of $\mathcal{B}$, then $\mathrm{aff}(R) = A_f(R)$ as we have proved above. By Assertion c) of Proposition 3.4, we have therefore $\mathrm{aff}(R) = \{x \in R \mid \delta(\mathrm{proj}_R f, x) = r_J\}$ for each such residue, where r_J is the longest word of W_J.

Now $\mathcal{F}_f$ satisfies (lco) and (lsco). As it is well-known that $\mathcal{C}$ is simply 2-connected (see for instance Theorem (4.3) in [Ro89]), the claim follows now from Theorem 4.1. $\square$

5 Bijections between panels

In this section, $\mathcal{B} = (\mathcal{C}, \delta)$ will be a building of type (W, S) satisfying the hypothesis of Theorem 4.3 and f will be a codistance on $\mathcal{B}$.

5.1 Preliminaries on panels

The following lemmas are easy.

Lemma 5.1.: *Two panels P_1 and P_2 are parallel if and only if* $\operatorname{proj}_{P_2} P_1 = P_2$.

Proof: The first implication is obvious. Assume $\operatorname{proj}_{P_2} P_1 = P_2$. By Proposition 2.3, the projections of two residues on one another are parallel, therefore $\operatorname{proj}_{P_1} P_2 = P_1$ and P_1 and P_2 are parallel. □

Lemma 5.2.: *For two parallel panels P_1 and P_2 of type s_1 and s_2 respectively, we must have $s_2 = w^{-1}s_1w$ where $w := \delta(x, \operatorname{proj}_{P_2} x)$ does not depend on the choice of x in P_1. Conversely, if $\delta(x, y) = w$, $s_2 = w^{-1}s_1w$ and $l(s_1w) = l(w) + 1$, then the s_1-panel on x is parallel to the s_2-panel on y.*

Proof: It is well known that for $w \in W$ and $s_1, s_2 \in S$ such that $l(s_1w) = l(w) + 1 = l(ws_2)$, we have $l(s_1ws_2) = l(w) + 2$ or $s_1ws_2 = w$. The result follows easily. □

The distance $\delta(x, \operatorname{proj}_{P_2} x)$ between two parallel panels P_1, P_2 will be denoted by $\delta(P_1, P_2)$.

Definition 5.3.: For $s \in S$, let $X_s := \{w \in W | w^{-1}sw \in S \text{ and } l(sw) = l(w) + 1\}$. For $w_1, w_2 \in X_s$, we say that $w_1 \prec w_2$ if and only if $l(w_1^{-1}w_2) = l(w_2) - l(w_1)$.

Lemma 5.4.: *For $w \in X_s$ and a given s-panel P, there exists an $w^{-1}sw$-panel P' parallel to P and with $\delta(P, P') = w$. Let J be a spherical subset of S containing s and let r_J be the longest word of W_J, then $x_J := sr_J$ is in X_s. Moreover, if $w \in W_J$ is in X_s, then $w \prec x_J$.*

Proof: The first statement is a corollary of Lemma 5.2. We have $x_J^{-1}sx_J = r_Jsr_J$ which has length $l(r_J) - l(sr_J) = 1$ and so is an element of S, and $l(sx_J) = l(r_J) = l(x_J) + 1$, hence the second statement. Finally $l(w^{-1}x_J) = l(w^{-1}sr_J) = l(r_Jsw) = l(r_J) - l(sw) = l(r_J) - l(w) - 1 = l(x_J) - l(w)$, hence the third statement. □

Definition 5.5.: Let Γ be the graph whose vertices are the panels of $\mathcal{B}$ with panels adjacent if there exists a rank 2 residue in which the two panels are opposite. For two adjacent panels P, Q, there exists a unique rank 2 residue containing P and Q, that will be denoted by $R(P, Q)$. A path $P_0, P_1, \ldots P_k$ (without repetitions) in Γ is called *compatible* if $\operatorname{proj}_{R(P_{i-1}, P_i)} P_0 = P_{i-1}$ for all $1 \leq i \leq k$. The number k is the *length* of that path.

Lemma 5.6.: *Let P, Q be two parallel panels of $\mathcal{B}$ and R be a residue containing Q. Then $\operatorname{proj}_R P$ is a panel parallel to P and to Q. Moreover, if $P = P_0, P_1, \ldots P_k = \operatorname{proj}_R P$ and $\operatorname{proj}_R P = T_0, T_1, \ldots T_l = Q$ are compatible paths in Γ, the second one contained in R, then $P = P_0, P_1, \ldots P_k = T_0, T_1, \ldots T_l = Q$ is a compatible path in Γ.*

Proof: The projection of a residue on a residue is a residue, so $P' := \operatorname{proj}_R P$ is either a chamber or a panel. Since $\operatorname{proj}_Q = \operatorname{proj}_Q \operatorname{proj}_R$ by Proposition 2.5, we have $Q = \operatorname{proj}_Q P = \operatorname{proj}_Q P'$, and so P' cannot be reduced to a chamber and is parallel to Q by Lemma 5.1. We must have P' parallel to $\operatorname{proj}_P R$ by Proposition 2.3. Since $\operatorname{proj}_P R \supseteq \operatorname{proj}_P Q = P$, we have $\operatorname{proj}_P R = P$ and P' is parallel to P.

We already have $\operatorname{proj}_{R(P_{i-1},P_i)} P_0 = P_{i-1}$ for all $1 \le i \le k$ by hypothesis. For all $1 \le i \le l$, we have $\operatorname{proj}_{R(T_{i-1},T_i)} P = \operatorname{proj}_{R(T_{i-1},T_i)} \operatorname{proj}_R P = T_{i-1}$ by Proposition 2.5 and because $T_0, T_1, \ldots, T_l$ is a compatible path. This concludes the proof. □

Lemma 5.7.: *Two panels are parallel if and only if there exists a compatible path in Γ from one to the other.*

Proof: The right to left implication will be proved by an induction on the length of the path. If the path has length one, the result is obvious since opposite panels in a residue are parallel. Assume we have proved the result for all paths of length strictly less than k, and assume $P = P_0, P_1, \ldots P_k = Q$ is a compatible path in Γ. By induction P is parallel to P_{k-1}. We have $\operatorname{proj}_Q = \operatorname{proj}_Q \operatorname{proj}_{R(P_{k-1},P_k)}$ by Proposition 2.5, and so $\operatorname{proj}_Q P = \operatorname{proj}_Q P_{k-1}$ which is equal to Q since P_{k-1} and Q are parallel. By Lemma 5.1, that means P and Q are parallel.

The left to right implication will be proved by an induction on the distance between the two panels. Let P, Q be two parallel panels. If $l(\delta(P,Q)) = 0$ then $P = Q$ and the trivial path $P = P_0 = Q$ is compatible. Suppose $l(\delta(P,Q)) = l > 0$ and the result is proved for all parallel panels at distance strictly less than l. Choose $c \in P$ and let $d = \operatorname{proj}_Q c$. There exists a chamber e adjacent to d such that $l(\delta(c,d)) = l(\delta(c,e)) + 1$. Let R be the unique rank 2 residue containing Q and e. By Lemma 5.6, $\operatorname{proj}_R P = Q'$ is a panel parallel to P and to Q. Since there is a chamber in R closer to P than d, Q cannot be equal to Q' and so they are opposite in R (property of rank 2 residues). Moreover $l(\delta(P,Q')) < l(\delta(P,Q))$. By induction, there exists a compatible path $P = P_0, P_1, \ldots P_k = Q'$. Since $R = R(Q', Q)$, the path $P = P_0, P_1, \ldots P_k, Q$ is compatible. □

Lemma 5.8.: *Let R be a spherical rank 3 residue in $\mathcal{B}$ and P, Q be two parallel panels in R. If there is more than one compatible path contained in R from P to Q, then P and Q are opposite in R and there are exactly two such paths. Moreover these two paths have the same length.*

Proof: Let $P = P_0, P_1, \ldots P_k = Q$ and $P = P'_0, P'_1, \ldots P'_l = Q$ be two compatible paths in R, and let $Q' = P_{k-i} = P'_{l-i}$ such that $P_{k-j} = P'_{l-j}$ for all $0 \le j \le i$ and $P_{k-i-1} \neq P'_{l-i-1}$. Therefore $R(P_{k-i-1}, P_{k-i}) \neq R(P'_{l-i-1}, P'_{l-i})$.

Choose $c \in P$ and let $d = \operatorname{proj}_{Q'} c$. Suppose that P and Q' are not opposite in R, so that there exists a chamber e not in Q' adjacent to d and such that $l(\delta(c,e)) = l(\delta(c,d)) + 1$.

Since there are only two rank 2 residues in R containing a given panel, the rank 2 residue R' containing Q' and e must be either $R(P_{k-i-1}, P_{k-i})$ or $R(P'_{l-i-1}, P'_{l-i})$, without loss of generality we can assume $R' = R(P_{k-i-1}, P_{k-i})$. Then $\text{proj}_{R'} P = P_{k-i-1}$ is opposite to Q' in R', in contradiction with the existence of e. Therefore P and Q' are opposite. Since Q' cannot be the projection of P on any rank 2 residue containing it, Q' must be equal to Q. Using this and Lemma 5.7 in the building R, we conclude that for two non-opposite parallel panels of R, there is exactly one compatible path in R from one to the other.

Let P and Q be opposite in R and let R' be a rank 2 residue in R containing Q. Then there is exactly one compatible path $P = P_0, P_1, \dots P_k = Q$ such that $R' = R(P_{k-1}, P_k)$. Indeed $P_{k-1} = \text{proj}'_R P$ is determined and there is only one compatible path between P and P_{k-1} since they are not opposite. Since there are two rank 2 residues containing Q in R, there are exactly two compatible path in R from P to Q.

A spherical residue of rank 3 is of type A_3, C_3, H_3, $A_1 \oplus A_1 \oplus A_1$ or $A_1 \oplus I_n$. Knowing the distance between two opposite panels in R and in all rank 2 residues of R, it is easy to determine the length of compatible paths between opposite panels. A case by case analysis easily yields that the two compatible paths between two opposite panels have the same length. □

Lemma 5.9.: *Let P, Q be two parallel panels of $\mathcal{B}$. Then all compatible paths from P to Q have the same length.*

Proof: We will prove this by induction on $l(\delta(P, Q))$.

If $l(\delta(P, Q)) = 0$, then $P = Q$ and the trivial path $P = P_0 = Q$ is the only compatible path from P to Q. Assume $l(\delta(P, Q)) = L > 0$ and we have proved the result for all parallel panels at distance strictly less than L. Take two compatible paths from P to Q: $P = P_0, P_1, \dots, P_k = Q$ and $P = P'_0, P'_1, \dots, P'_l = Q$. If $P_{k-1} = P'_{l-1} = Q'$, then $l(\delta(P, Q')) < L$ and so $k - 1 = l - 1$ and we can conclude.

Assume now $P_{k-1} \neq P'_{l-1}$, so that $R(P_{k-1}, P_k) \neq R(P'_{l-1}, P'_l)$, and let R be the rank 3 residue containing those two rank 2 residues. Let Q' be the projection of P on R. By Lemma 5.6, Q' is parallel to P and Q. Since P_{k-1} and P'_{l-1} are not opposite Q', by Lemma 5.8, there is exactly one compatible path in R from Q' to P_{k-1}, resp. P'_{l-1}, they will be denoted respectively by $Q' = T_0, T_1, \dots, T_m = P_{k-1}$ and $Q' = T'_0, T'_1, \dots, T'_n = P'_{l-1}$. We have $P_{k-1} = \text{proj}_{R(P_{k-1}, P_k)} P = \text{proj}_{R(P_{k-1}, P_k)} \text{proj}_R P = \text{proj}_{R(P_{k-1}, P_k)} Q'$, and so $Q' = T_0, T_1, \dots, T_m, P_k = Q$ is a compatible path. By similar arguments, $Q' = T'_0, T'_1, \dots, T'_n, P'_l = Q$ is also a compatible path. By Lemma 5.8, these two paths in R must have the same length, and so $m = n$.

By Lemma 5.7, there is a compatible path from P to Q', denoted by $P = S_0, S_1, \dots, S_j = Q'$. By Lemma 5.6, the paths $P = S_0, S_1, \dots, S_j = Q' = T_0, T_1, \dots, T_m = P_{k-1}$ and $P = S_0, S_1, \dots, S_j = Q' = T'_0, T'_1, \dots, T'_m = P_{l-1}$ are both compatible of length $j + m$. On the other hand $P = P_0, P_1, \dots, P_{k-1}$ and $P = P'_0, P'_1, \dots, P'_{l-1}$ are also compatible paths. Since $l(\delta(P, P_{k-1})) < L$ and $l(\delta(P, P'_{l-1})) < L$, we can use the hypothesis of induction, and so $k - 1 = j + m$ and $l - 1 = j + m$. We conclude that $k = l$. □

Definition 5.10.: We define the compatible distance between two parallel panels P and Q to be the length of a compatible path joining them. It will be denoted by $l_c(P,Q)$. Note, that the previous lemma shows that the compatible distance between two parallel panels is well defined.

By standard arguments using convex hulls and apartments (which are however a bit lengthy) one can prove the following.

Lemma 5.11.: *Let $w \in X_s$ and let P, P' be s-panels and Q, Q' be $w^{-1}sw$-panels such that $\delta(P,Q) = w = \delta(P',Q')$. Then $l_c(P,Q) = l_c(P',Q')$.*

Definition 5.12.: Let $w \in X_s$. Then we define its compatible length, denoted by $l_c(w)$ to be the compatible distance between an s-panel P and an $w^{-1}sw$-panel Q such that $\delta(P,Q) = w$.

5.2 Bijections

Definition 5.13.: We will say that a residue R is *in* f^{op}, resp. f_c^{op}, if it contains a chamber in f^{op}, resp. f_c^{op}. For $s \in S$, let $\mathcal{P}_s^{\text{op}}(f)$, resp. $\mathcal{P}_{s,c}^{\text{op}}(f)$, be the set of all s-panels in f^{op}, resp. f_c^{op}.

Notice that all chambers of a panel in f^{op} are in f^{op} except for one, namely $\text{proj}_P f$.

Proposition 5.14.: *Let $P \in \mathcal{P}_s^{\text{op}}(f)$, $w \in X_s$ and $t = w^{-1}sw$. Let P' be a t-panel with $\delta(P,P') = w$. Then the following conditions are equivalent:*

a) P' contains a chamber with f-value w;

b) $f(x) \in \{w, wt\}$ for $x \in P'$ and exactly one chamber of P' has f-value wt;

c) $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$ for all chambers x of P';

d) $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$ for some chamber x of P';

There exists exactly one panel P' satisfying those conditions, and it will be denoted by $\pi(P,w)$.

Proof: Conditions a) and b) are equivalent by the definition of a codistance. Assume P' satisfies b). Let x be a chamber with f-value w in P'. Then $\delta(\text{proj}_P x, x) = w = f(x)$. Since $\text{proj}_P x$ cannot be equal to $\text{proj}_P f$ (otherwise the chamber with f-value wt would be at distance $l(w)$ from a chamber in f^{op}), $\text{proj}_P x \in f_x^{\text{op}}$ and $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$. Now let $z = \text{proj}_{P'} f$ be the unique chamber with f-value wt. If y is any chamber of P in f_{op}, $\delta(y,z) = wt$, so $y \in f_z^{\text{op}}$ and $P \in \mathcal{P}_{s,z}^{\text{op}}(f)$. Obviously c) implies d). Now assume P' satisfies d), then P contains $y \in f^{\text{op}}$ and $\delta(y,x) = f(x)$. If $y = \text{proj}_P x$, then $\delta(y,x) = w$, otherwise $\delta(y,x) = sw = wt$. In both cases, P' contains a chamber with f-value w.

We now prove the existence of such a panel. Let p be the unique chamber of P not in f^{op} (so $f(p) = s$). As $l(f(p)w) = l(sw) = l(w) + 1$, by Lemma 3.8, there exists a unique chamber c with $sf(c) = w = \delta(p, c)$. Let P' be the t-panel on c. By Lemma 5.2, P' is parallel to P and $\delta(P, P') = w$. It obviously satisfies a).

Now we want to show that P' is unique. Let Q be a t-panel with $\delta(P, Q) = w$ satisfying b). Let x be the chamber of Q with f-value $wt = sw = f(p)w$. Obviously $\operatorname{proj}_P x = p$, so $\delta(p, x) = w$. By Lemma 3.8, a chamber with that property is unique. Therefore $x = c$ and $Q = P'$. □

Lemma 5.15.: *Let Q be a t-panel of $\mathcal{B}$ and let w be the shortest word of $\{f(x)|x \in Q\}$. Suppose $wtw^{-1} := s \in S$. Then there exists an s-panel $P \in \mathcal{P}_s^{\text{op}}(f)$ such that $Q = \pi(P, w)$.*

Proof: Since $w^{-1}sw = t$ and $l(sw) = l(wt) = l(w) + 1$, we have $w \in X_s$. Let x be a chamber of Q with $f(x) = w$. Let $y \in f_x^{\text{op}}$ so that $\delta(y, x) = w = f(x)$. Let P be the s-panel on y. By construction P is a panel in $\mathcal{P}_{s,x}^{\text{op}}(f)$ which is parallel to Q by Lemma 5.2. Moreover $\delta(P, Q) = w$, hence by Proposition 5.14 $Q = \pi(P, w)$. □

Definition 5.16.: For $P, Q \in \mathcal{P}_s^{\text{op}}(f)$ and $w \in X_s$, we put $P \equiv_w Q$ if and only if $\pi(P, w) = \pi(Q, w)$. This is an equivalence relation on $\mathcal{P}_s^{\text{op}}(f)$. For $P \equiv_w Q$, we put $\beta(P, Q, w)$ the bijection from P to Q defined by $\operatorname{proj}_Q \operatorname{proj}_{\pi(P,w)}$.

Notice that $\beta(Q, P, w)\beta(P, Q, w) = 1_P$ and that, by construction, $\beta(P, Q, w)$ maps $\operatorname{proj}_P f$ onto $\operatorname{proj}_Q f$ via $\operatorname{proj}_{\pi(P,w)} f$.

Proposition 5.17.: *Let $w_1, w_2 \in X_s$ with $w_1 \prec w_2$ and $P, Q \in \mathcal{P}_s^{\text{op}}(f)$ with $P \equiv_{w_1} Q$. Then $P \equiv_{w_2} Q$ and $\beta(P, Q, w_1) = \beta(P, Q, w_2)$.*

Proof: Let $s_1 s_2 \dots s_l$ be a reduced word for w_1 and $k := l(w_2)$. Since $l(w_1^{-1}w_2) = l(w_2) - l(w_1)$, we can write $w_1^{-1}w_2$ as the reduced word $s_{l+1} \dots s_k$. Hence the word $s_1 s_2 \dots s_l s_{l+1} \dots s_k$ is a word for w_2 and since it has length k, it is reduced. Let p, resp. q, be the unique chamber of P, resp. Q, not in f^{op}. Looking up the proof of Proposition 5.14, we see that $\pi(P, w_2)$ is uniquely determined by a chamber c with $f(c) = sw_2$ and $\delta(p, c) = w_2$. In the process, we built a gallery $p = x_0 \sim_{s_1} x_1 \sim s_2 \dots \sim_{s_k} x_k = c$ such that $f(x_i) = ss_1 s_2 \dots s_i$. Notice that x_l is the unique chamber at distance w_1 from p with f-value sw_1. Similarly we can build a gallery $q = x'_0 \sim_{s_1} x'_1 \sim s_2 \dots \sim_{s_k} x'_k = c'$ to determine $\pi(Q, w_2)$, and x'_l is the unique chamber at distance w_1 from q with f-value sw_1. Since $P \equiv_{w_1} Q$, we must have $x_l = x'_l$ and so $x_i = x'_i$ for all $i \geq l$. Therefore $c = c'$ and so $\pi(P, w_2) = \pi(Q, w_2)$. This proves the first statement.

¿From each chamber x of P, there exists a gallery of type $s_1 s_2 \dots s_l$ to $\operatorname{proj}_{\pi(P,w_1)} x$ and a gallery of type $s_1 s_2 \dots s_k$ to $\operatorname{proj}_{\pi(P,w_2)} x$. In both galleries, the i-th chamber is the unique one with f-value $s_1 s_2 \dots s_i$ if $f(x) = 1_W$ and of f-value $ss_1 s_2 \dots s_i$ otherwise. Therefore the first gallery is the beginning of the second one. We can use the same argument for Q. Therefore we must have $\beta(P, Q, w_1) = \beta(P, Q, w_2)$. □

Theorem 5.18.: *For any two s-panels P, Q in $\mathcal{P}_s^{\text{op}}(f)$, we can define a bijection $\beta(P, Q)$ from P to Q in such a way that the following hold for all $P, Q, R \in \mathcal{P}_s^{\text{op}}(f)$:*

a) $\beta(P, P) = 1_P$;

b) $\beta(Q,P)\beta(P,Q) = 1_P$*;*

c) $\beta(Q,R)\beta(P,Q) = \beta(P,R)$*;*

d) $\beta(P,Q)(\mathrm{proj}_P f) = \mathrm{proj}_Q f$*.*

Proof:

For $P, Q \in \mathcal{P}_s^{\mathrm{op}}(f)$, we say that they are t-adjacent, denoted by $P \sim_t Q$, if there exist $p \in P \cap f^{\mathrm{op}}$ and $q \in Q \cap f^{\mathrm{op}}$ with $p \sim_t q$. Let $P \sim_t Q$, both in $\mathcal{P}_s^{\mathrm{op}}(f)$. Let $J = \{s, t\}$ and R be the J-residue containing P and Q. Since R is spherical by the hypothesis on $\mathcal{B}$, $x_J := sr_J \in X_s$. Notice that $\pi(P, x_J)$ is a panel opposite to P in R which contains a chamber with f-value r_J. By Proposition 3.3, R contains only one such chamber, namely $\mathrm{proj}_R f$. Hence $\pi(P, x_J)$ is the unique panel of type $x_J^{-1} s x_J = r_J^{-1} s r_J$ containing $\mathrm{proj}_R f$. Similarly $\pi(Q, x_J)$ is the same panel. Therefore $P \equiv_{x_J} Q$ and we put $\beta(P,Q) := \beta(P,Q,x_J)$. If $s = t$, $x_J = 1_W$, $P = Q = \pi(P, x_J)$ and $\beta(P,Q) = 1_P$.

As noticed above, $\beta(P,Q)$ maps $\mathrm{proj}_P f$ onto $\mathrm{proj}_Q f$.

Let $P, Q \in \mathcal{P}_s^{\mathrm{op}}(f)$ and choose $p \in P \cap f^{\mathrm{op}}$ and $q \in Q \cap f^{\mathrm{op}}$. By Theorem 4.3, f^{op} is connected, and so there exists a gallery γ from p to q contained in f^{op}. If $\gamma = (x_0 = p, x_1, x_2, \ldots, x_n = q)$, let X_i be the s-panel containing x_i. By definition those panels are in $\mathcal{P}_s^{\mathrm{op}}(f)$ and $X_i \sim_{t_i} X_{i+1}$ for some $t_i \in S$. We define $\beta(\gamma, P, Q) := \beta(X_{n-1}, Q) \ldots \beta(X_1, X_2)\beta(P, X_1)$. By the above comment $\beta(\gamma, P, Q)$ maps $\mathrm{proj}_P f$ onto $\mathrm{proj}_Q f$.

We will now show that if γ_1 and γ_2 are two galleries in f^{op} from a chamber of P to a chamber of Q, then $\beta(\gamma_1, P, Q) = \beta(\gamma_2, P, Q)$. As $\beta(P,P) = 1_P$, we can assume γ_1 and γ_2 start and finish with the same chamber. Hence this is equivalent to showing that for a closed gallery γ in f^{op}, $\beta(\gamma, P, P) = 1_P$.

By Theorem 4.3, f^{op} is simply 2-connected. Therefore there exists a finite sequence of elementary homotopies from the closed gallery γ to a trivial gallery based in $p \in P$ such that all intermediate galleries are contained in f^{op}. Since two galleries differing by an elementary homotopy are equal except in a rank 2 residue, it is enough to show that $\beta(\gamma, P, P) = 1_P$ for γ in a rank 2 residue in order to prove it in general.

Let $\gamma = (x_0, x_1, x_2, \ldots, x_n = x_0)$ be a closed gallery in f^{op} contained in a rank 2 residue R of type $\{t, u\}$ (where t or u could be equal to s). Let the X_i's be defined as above. Consider two consecutive chambers x_i and x_{i+1}. They are v-adjacent, where $v \in \{t, u\}$. Let $J := \{s, v\}$ and $K = \{s, t, u\}$, which are spherical by the hypothesis on $\mathcal{B}$. Let R be the K-residue containing x_0. We have $X_i \sim_v X_{i+1}$ and, as seen above $X_i \equiv_{x_J} X_{i+1}$ and $\beta(X_i, X_{i+1}) = \beta(X_i, X_{i+1}, x_J)$. By Lemma 5.4, $x_J \prec x_K$, and so, by Proposition 5.17, $X_i \equiv_{x_K} X_{i+1}$ and $\beta(X_i, X_{i+1}) = \beta(X_i, X_{i+1}, x_K)$. For all $j = 0, 1, \ldots, n-1$, $\pi(X_j, x_K)$ is the panel of type $x_K^{-1} s x_K$ containing $\mathrm{proj}_R f$, which is opposite to X_j in R, we will denote it by $\pi(R)$. We have

$$\begin{aligned}\beta(\gamma, X_0, X_0) &= \beta(X_{n-1}, X_0)\dots\beta(X_1, X_2)\beta(X_0, X_1)\\ &= \beta(X_{n-1}, X_0, x_K)\dots\beta(X_1, X_2, x_K)\beta(X_0, X_1, x_K)\\ &= \operatorname{proj}_{X_0}\operatorname{proj}_{\pi(R)}\dots\operatorname{proj}_{X_2}\operatorname{proj}_{\pi(R)}\operatorname{proj}_{X_1}\operatorname{proj}_{\pi(R)}\\ &= \operatorname{proj}_{X_0}\operatorname{proj}_{\pi(R)} = 1_{X_0}\end{aligned}$$

because $\operatorname{proj}_{\pi(R)}\operatorname{proj}_{X_i} = 1_{\pi(R)}$. This completes the proof that $\beta(\gamma_1, P, Q) = \beta(\gamma_2, P, Q)$ for γ_1 and γ_2 two galleries in f^{op} from a chamber of P to a chamber of Q. We put $\beta(P, Q) := \beta(\gamma, P, Q)$ for γ any gallery in f^{op} from a chamber of P to a chamber of Q.

It is obvious that a), b) and c) are satisfied. Since d) is satisfied for adjacent panels, it will be satisfied, by induction, for any two panels.

□

Theorem 5.19.: *Let $P, P' \in \mathcal{P}_s^{\mathrm{op}}(f)$ with $P \equiv_w P'$ for $w \in X_s$. Then $\beta(P, P') = \beta(P, P', w)$.*

Proof: This is obvious if $w = 1_W$, so assume it is not the case. We will prove the result by induction on $l_c(w)$. Let $Q := \pi(P, w) = \pi(P', w)$ be a t-panel (i.e. $w^{-1}sw = t$). By Lemma 5.11, $l_c(P, Q) = l_c(P', Q) = l_c(w)$.

Assume first that $l_c(w) = 1$, so that P and Q are opposite in a rank 2 J-residue R. The panels P' and Q are also opposite in the same residue R. Hence $w = x_J = sr_J$. By the hypothesis on $\mathcal{B}$, $\operatorname{aff}(R) = A_f(R) = R \cap f^{\mathrm{op}}$ is connected. Choose $p \in P \cap f^{\mathrm{op}}$ and $p' \in P' \cap f^{\mathrm{op}}$. There exists a gallery γ from p to p' contained in $\operatorname{aff}(R)$. If $\gamma = (x_0 = p, x_1, x_2, \dots, x_n = p')$, let X_i be the s-panel containing x_i. Those panels are in $\mathcal{P}_s^{\mathrm{op}}(f)$ and in R, and $X_i \sim_{t_i} X_{i+1}$ for some $t_i \in J$. By definition, $\beta(P, P') = \beta(\gamma, P, P') := \beta(X_{n-1}, P')\dots\beta(X_1, X_2)\beta(P, X_1)$. Moreover $\beta(X_i, X_{i+1}) = \beta(X_i, X_{i+1}, x_J)$ if $t_i \neq s$ and $\beta(X_i, X_{i+1}) = 1_{X_i}$ if $t_i = s$. Therefore $\beta(P, P') = \beta(P, P', x_J)$.

Now assume $l_c(w) = k > 1$ and assume the result is proved for all $w' \in X_s$ with $l_c(w') < k$. Let $P = P_0, P_1, \dots, P_k = Q$ be a compatible path from P to Q, which exists by Lemma 5.7, and let $P' = P'_0, P'_1, \dots, P'_k = Q$ be a compatible path from P' to Q with residues $R(P_i, P_{i+1})$ and $R(P'_i, P'_{i+1})$ of the same type for all $0 \le i \le k-1$.

Let R be the rank 3 residue containing $R(P_{k-1}, P_k) = R(P'_{k-1}, P'_k)$, $R(P_{k-2}, P_{k-1})$ and $R(P_{k-2}, P_{k-1})$. Let J be the type of R, which contains t. Let $T = \operatorname{proj}_R P$ and $T' = \operatorname{proj}_{R'} P'$, which are panels by Lemma 5.6. Let $c \in Q$. By Proposition 5.14, $P \in \mathcal{P}_{s,c}^{\mathrm{op}}(f)$, so there exists $x \in P \cap f_c^{\mathrm{op}}$. By Lemma 3.9, $\operatorname{proj}_R x \in A_f(R)$. That means T contains chambers in $A_f(R)$, so whose f-value is w_J. We also have $\delta(x, \operatorname{proj}_T x) = w_J$, so that $T = \pi(P, w_J)$. By the same argument, $T' = \pi(P', w_J)$ and so T' contains chambers in $A_f(R)$. Therefore T and T' are s'-panels where $s' = w_J^{-1}sw_J \in J$.

Since P_{k-2} is in a compatible path from P, it is parallel to P. By Lemma 5.6, T is a panel parallel to P and P_{k-2}. Moreover there exists a compatible path from P to P_{k-2} containing T. Since all compatible paths between two given panels have the same length, the length of a compatible path from P to T is less or equal to $k-2$. Hence $l_c(w_J) \le l_c(w) - 2$.

Choose $p \in T \cap A_f(R)$ and $p' \in T' \cap A_f(R)$. Because of the hypothesis on $\mathcal{B}$, there exists a gallery γ from p to p' contained in $A_f(R)$. If $\gamma = (p = x_0, x_1, x_2, \ldots, x_n = p')$, let X_i be the s'-panel containing x_i and $X_i \sim_{t_i} X_{i+1}$ for some $t_i \in J$ for $0 \leq i \leq n-1$. By Lemma 5.15, there exists an s-panel $Q_i \in \mathcal{P}_s^{\text{op}}(f)$ such that $X_i = \pi(Q_i, w_J)$ for all $0 \leq i \leq n$. Of course we take $Q_0 = P$ and $Q_n = P'$. Since $\delta(Q_i, X_i) = w_J$, we have $\text{proj}_R Q_i = X_i$ for all $0 \leq i \leq n$. By Property c) of Theorem 5.18, $\beta(P, P') = \beta(Q_{n-1}, Q_n) \ldots \beta(Q_1, Q_2)\beta(Q_0, Q_1)$.

Let $J_i = \{s', t_i\} \subset J$, let R_i be the J_i-residue containing X_i and X_{i+1} and let $w_i := w_J s' r_{J_i}$. We have $w_i^{-1} s w_i = r_{J_i} s' w_J^{-1} s w_J s' r_{J_i} = r_{J_i} s' r_{J_i} = t_i \in J_i$, and $l(s w_J s' r_{J_i}) = l(w_J r_{J_i}) = l(w_J s' r_{J_i}) + 1$ since $w_J r_{J_i} = w_{J_i} r_{J_i}$ is the longest word of $w_J W_{J_i}$. Therefore $w_i \in X_s$ and $\pi(Q_i, w_i) = \pi(Q_{i+1}, w_i)$ is the t_i-panel containing the only chamber of R_i with f-value $w_{J_i} r_{J_i}$, that is $\text{proj}_{R_i} f$. Therefore $Q_i \equiv_{w_i} Q_{i+1}$.

Since $\text{proj}_{R_i} Q_i = \text{proj}_{R_i} \text{proj}_R Q_i = \text{proj}_{R_i} X_i = X_i$, a compatible path from Q_i to X_i (of length $l_c(w_J) \leq k-2$) completed by the panel $\pi(Q_i, w_i)$ is a compatible path of length $l_c(w_i) \leq k-1$. By induction, this means that $\beta(Q_i, Q_{i+1}) = \beta(Q_i, Q_{i+1}, w_i)$.

Let $\tilde{w} := w_J s' r_J$. By a similar argument to the one for w_i, $\tilde{w} \in X_s$ and $\pi(Q_i, \tilde{w})$ is the $\tilde{w}^{-1} s \tilde{w}$-panel containing $\text{proj}_R f$. Moreover $w_i \prec \tilde{w}$ for all $0 \leq i \leq n-1$. Indeed $l(w_i^{-1}\tilde{w}) = l(r_{J_i} s' w_J^{-1} w_J s' r_J) = l(r_{J_i} r_J) = l(r_J) - l(r_{J_i}) = l(s' r_J) - l(s' r_{J_i}) = l(w_J) + l(s' r_J) - (l(w_J) + l(s' r_{J_i})) = l(\tilde{w}) - l(w_i)$. By Proposition 5.17, we have $Q_i \equiv_{\tilde{w}} Q_{i+1}$ and $\beta(Q_i, Q_{i+1}, w_i) = \beta(Q_i, Q_{i+1}, \tilde{w})$. Therefore

$$\begin{aligned}\beta(P, P') &= \beta(Q_{n-1}, Q_n) \ldots \beta(Q_1, Q_2)\beta(Q_0, Q_1) \\ &= \beta(Q_{n-1}, Q_n, \tilde{w}) \ldots \beta(Q_1, Q_2, \tilde{w})\beta(Q_0, Q_1, \tilde{w}) \\ &= \beta(P, P', \tilde{w}).\end{aligned}$$

We have $l(w^{-1}\tilde{w}) = l(w^{-1} w_J s' r_J) = l(r_J s' w_J^{-1} w) = l(r_J) - l(s' w_J^{-1} w)$ because $s' w_J^{-1} w \in W_J$. Moreover $l(s' w_J^{-1} w) = l(w_J^{-1} s w) = l(w_J^{-1} w t) = l(wt) - l(w_J) = l(w) + 1 - l(w_J)$ because $wt \in wW_J$. Hence $l(w^{-1}\tilde{w}) = l(r_J) + l(w_J) - 1 - l(w)$. On the other hand, $l(\tilde{w}) - l(w) = l(w_J s' r_J) - l(w) = l(w_J) + l(s' r_J) - l(w) = l(w_J) + l(r_J) - 1 - l(w)$ since $w_J s' r_J \in wW_J$. Therefore $w \prec \tilde{w}$, and so $\beta(P, P', w) = \beta(P, P', \tilde{w})$. This concludes the proof.

□

Corollary 5.20.: *Let R be a rank 2 residue of $\mathcal{B}$ such that $l_f(R) = \min\{l(f(x)) \mid x \in R\} \in X_s$ and $l_f(R)^{-1} s l_f(R) = t \in \text{typ}(R)$. Let $c \in R$ and $P, P' \in \mathcal{P}_{s,c}^{\text{op}}(f)$. Then $\beta(P, P')(\text{proj}_P c) = \text{proj}_{P'} c$.*

Proof: Let $w = f(c)$, J the type of R, so $l_f(R) = w_J$. Let $d = \text{proj}_R f$, so that $f(d) = w^J$, where w^J is the unique longest word of wW_J. From the hypothesis on w_J, we easily get that $sw^J \in X_s$ and $(sw^J)^{-1} s (sw^J) = u \in J$, hence there exists a panel Q through d, of type u, which is parallel to both P and P'. Since $\delta(P, Q) = sw^J = w^J u = \delta(P', Q)$ and Q contains a chamber with f-value $w^J u$, we have $Q = \pi(P, sw^J) = \pi(P', sw^J)$. Therefore $P \equiv_{sw^J} P'$ and, by Theorem 5.19, $\beta(P, P') = \beta(P, P', sw^J) = \text{proj}_{P'} \text{proj}_Q$.

There exist $x \in P \cap f_c^{\text{op}}$ and $x' \in P' \cap f_c^{\text{op}}$. By Lemma 3.7, there exist minimal galleries $x = x_0, x_1, \ldots, x_n = c$ and $x' = x'_0, x'_1, \ldots, x'_n = c$ with $l(f(x_i)) = i = l(f(x'_i))$

for all $0 \leq i \leq n$ and going through $\mathrm{proj}_P c$, resp.$\mathrm{proj}_{P'} c$. Obviously $\mathrm{proj}_P c = x_1$ if $x_1 \in P$ and x_0 otherwise. By Proposition 3.3, $f(c) = f(d)\delta(d,c)$; moreover $l(f(c)) = l(w) = l(w^J) - l(\delta(d,c)) = l(f(d)) - l(\delta(d,c))$. Hence there is a minimal gallery $c = y_0, y_1, \ldots, y_m = d$ with $l(f(y_i)) = l(f(c)) + i$ for all $0 \leq i \leq m$ and going through $\mathrm{proj}_Q c$. Obviously $\mathrm{proj}_Q c = y_{m-1}$ if $y_{m-1} \in Q$ and $y_m = d$ otherwise. We have that $x = x_0, x_1, \ldots, x_n = y_0, y_1, \ldots, y_m = d$ is a minimal gallery, and so there is a minimal gallery (which is a subgallery of the previous one) from $\mathrm{proj}_P c$ to $\mathrm{proj}_Q c$ going through c. Therefore $\mathrm{proj}_P c = \mathrm{proj}_P \mathrm{proj}_Q c$. By a similar argument, $\mathrm{proj}_{P'} c = \mathrm{proj}_{P'} \mathrm{proj}_Q c$.

Putting all together, $\beta(P,P')(\mathrm{proj}_P c) = \mathrm{proj}_{P'} \mathrm{proj}_Q \mathrm{proj}_P \mathrm{proj}_Q c = \mathrm{proj}_{P'} \mathrm{proj}_Q c = \mathrm{proj}_{P'} c$, because P and Q are parallel.

□

Theorem 5.21.: *Let c be a chamber of $\mathcal{B}$ and $P, P' \in \mathcal{P}^{\mathrm{op}}_{s,c}(f)$. Then $\beta(P,P')(\mathrm{proj}_P c) = \mathrm{proj}_{P'} c$.*

Proof: We will prove this by induction on $l(f(c))$.

Assume $l(f(c)) = 0$. If $x \in f^{\mathrm{op}}_c$, then $\delta(x,c) = f(c) = 1_W$, so $f^{\mathrm{op}}_c = \{c\}$. Hence $P = P'$ contains c, and the statement is obvious since $\beta(P,P') = 1_P$.

Assume $l(f(c)) = 1$. If $f(c) = s$, then f^{op}_c consists of all chambers s-adjacent to c (except for c itself). Hence $P = P'$ contains c, and the statement is again obvious. We now consider the case $f(c) = t \neq s$. Let R be the $\{s,t\}$-residue containing c. Then $l_f(R) = 1_W$ satisfies the conditions of Corollary 5.20, and so we are done.

Assume now $l(f(c)) = l \geq 2$ and the theorem is proved for all chambers c' with $l(f(c')) < l$. Let u, t be the last 2 letters in a reduced word for $f(c)$, so that $l(f(c)ut) = l(f(c)) - 2$. Let R be the $\{u,t\}$-residue containing c. Therefore $l_f(R) \leq l(f(c)) - 2$. If $l_f(R)$ satisfies the conditions of Corollary 5.20, we are done, so we will assume it does not. If $\mathrm{proj}_R P$ was a panel, it would be parallel to P and $l_f(R)$ would satisfy the above conditions. Hence $\mathrm{proj}_R P$ is a chamber p. Similarly $\mathrm{proj}_R P'$ is a chamber p'. Since P contains a chamber x in f^{op}_c and $\mathrm{proj}_R P = \mathrm{proj}_R x$, we have by Lemma 3.9 that $p \in A_f(R)$, and similarly $p' \in A_f(R)$. Moreover, there exists a minimal gallery from x to c going through p, and so $x \in f^{\mathrm{op}}_p$ by Lemma 3.7. Similarly $x' \in f^{\mathrm{op}}_{p'}$. By the hypothesis on $\mathcal{B}$, there exists a gallery $p = p_0, p_1, \ldots, p_n = p'$ (without repetitions) entirely contained in $A_f(R)$. For all $1 \leq j \leq n$, let Q_j be the unique panel containing p_{j-1} and p_j, and let $z_j = \mathrm{proj}_{Q_j} f$. Since $l(f(z_j)) = l_f(R) + 1$, we have $l(f(z_j)) < l$ for all $1 \leq j \leq n$. For each $1 \leq j \leq n-1$, we can choose $x_j \in f^{\mathrm{op}}_{p_j}$ and P_j the s-panel through x_j. We also put $x_0 = x$, $P_0 = P$, $x_n = x'$ and $P_n = P'$.

By Lemma 3.7, there exists a gallery from x_j to p_j with f getting strictly longer at each step for all $1 \leq j \leq n-1$. Since $l(f(z_j)) = l(f(p_j)) + 1 = l(f(z_{j+1}))$, $z_j, p_j \in Q_j$ and $z_{j+1}, p_j \in Q_{j+1}$, by adding a chamber at the end of the previous gallery, we get two minimal galleries from x_j to z_j and from x_j to z_{j+1}, both with f getting strictly longer at each step. Hence $x_j \in f^{\mathrm{op}}_{z_j}$ and $x_j \in f^{\mathrm{op}}_{z_{j+1}}$ for all $1 \leq j \leq n-1$. Therefore $P_j \in \mathcal{P}^{\mathrm{op}}_{s,z_j}(f)$ and $P_j \in \mathcal{P}^{\mathrm{op}}_{s,z_{j+1}}(f)$ for all $1 \leq j \leq n-1$. For a similar reason $P = P_0 \in \mathcal{P}^{\mathrm{op}}_{s,z_1}(f)$ and

$P' = P_n \in \mathcal{P}^{\text{op}}_{s,z_n}(f)$. We conclude that $P_{i-1}, P_i \in \mathcal{P}^{\text{op}}_{s,z_i}(f)$ for all $1 \leq i \leq n$. Hence, by induction, $\beta(P_{i-1}, P_i)(\text{proj}_{P_{i-1}} z_i) = \text{proj}_{P_i} z_i$ for all $1 \leq i \leq n$.

Since $\text{proj}_R P_i$ is a chamber for all $1 \leq i \leq n$ and projections of residues on one another are parallel, $\text{proj}_{P_i} R$ is also a chamber, hence $\text{proj}_{P_i} z_i = \text{proj}_{P_i} R = \text{proj}_{P_i} c$ and $\text{proj}_{P_{i-1}} z_i = \text{proj}_{P_{i-1}} R = \text{proj}_{P_{i-1}} c$. Therefore we have $\beta(P_{i-1}, P_i)(\text{proj}_{P_{i-1}} c) = \text{proj}_{P_i} c$.

By Theorem 5.18, $\beta(P, P') = \beta(P', P_{n-1}) \ldots \beta(P_1, P_2)\beta(P, P_1)$. We easily conclude that $\beta(P, P') \,\text{proj}_P c = \text{proj}_{P'} c$.

□

6 Adjacent codistances

Definition 6.1.: Two codistances f and g on $\mathcal{B}$ are called *s-adjacent* if $\mathcal{P}^{\text{op}}_s(f) = \mathcal{P}^{\text{op}}_s(g)$. We denote it by $f \sim_s g$.

Lemma 6.2.: *Let f, g be two codistances on a building $\mathcal{B}$. Let R be a spherical J-residue in f^{op}. Let $s \in J$. If f and g are s-adjacent, then* $\text{proj}_R f$ *and* $\text{proj}_R g$ *are $r_J s r_J$-adjacent in $\mathcal{B}$.*

Proof: Suppose f and g are s-adjacent. Then $\mathcal{P}^{\text{op}}_s(f) = \mathcal{P}^{\text{op}}_s(g)$, which means that the s-panels of R in f^{op} and in g^{op} coincide. The $r_J s r_J$-panel P containing $d := \text{proj}_R f$ is opposite in R to all s-panels of R in f^{op}. Suppose there is another panel P' opposite to those same s-panels. Of course P' is also of type $r_J s r_J$. Let $d' := \text{proj}_{P'} f$. Then there exists a minimal gallery from d to d' which can be extended to a minimal gallery from d to a chamber c opposite to d, that is a chamber in f^{op}. The s-panel containing c is in f^{op}, and so should be opposite to both P and P', which is not possible. Similarly, the panel of type $r_J s r_J$ containing $\text{proj}_R g$ is the only panel of R opposite in R to all s-panels in g^{op}. Therefore these 2 panels of type $r_J s r_J$ coincide, and so $\text{proj}_R f$ and $\text{proj}_R g$ are $r_J s r_J$-adjacent. □

Lemma 6.3.: *Let f be a codistance on a building $\mathcal{B}$, and let g be a codistance s-adjacent to f. Let R be a J-residue in f^{op}, with $s \in J$. Then R is in g^{op}.*

Proof: Since R is in f^{op}, R contains a chamber x in f^{op}. The s-panel containing x is in f^{op}, and so by hypothesis, it is in g^{op}. Since this panel is in R, it means R is in g^{op}. □

Lemma 6.4.: *Let f be a codistance on a k-spherical building $\mathcal{B}$ such that f^{op} is connected, and let g be a codistance s-adjacent to f. Let R be a J-residue of rank $\leq k-1$ in f^{op}, with $s \in J$. Then* $\text{proj}_R g$ *determines g uniquely.*

Proof: Suppose that g_1 and g_2 are two codistances s-adjacent to f with $\text{proj}_R g_1 = \text{proj}_R g_2$. By hypothesis, $\mathcal{P}^{\text{op}}_s(f) = \mathcal{P}^{\text{op}}_s(g_1) = \mathcal{P}^{\text{op}}_s(g_2)$.

We claim that $g_1^{\text{op}} \subseteq g_2^{\text{op}}$. Let $x \in g_1^{\text{op}}$, then the J-residue R_x containing x is in g_1^{op}. By Lemma 6.3, R_x is also in f^{op}. Since f^{op} is connected, there is a gallery $x_0, x_1, \ldots, x_n$ in

f^{op} with $x_0 \in R$ and $x_n \in R_x$. We will show by induction on n that $\operatorname{proj}_{R_x} g_1 = \operatorname{proj}_{R_x} g_2$. If $n = 0$, then $P = Q$ and the result is obvious. Assume that we have shown that for every J-residue at "distance" (in the sense described above) at most $n-1$ of R, the projections of g_1 and g_2 coincide. Let R' be the J-residue containing x_{n-1}. By the induction hypothesis, $\operatorname{proj}_{R'} g_1 = \operatorname{proj}_{R'} g_2$. We have $x_{n-1} \sim_t x_n$. If $t \in J$ then $R_x = R'$ and we are done. So assume $t \notin J$, let $K = J \cup \{t\}$ (which is spherical by hypothesis) and let $\tilde{R}$ be the K-residue containing R_x and R'. Then the residue of type $\operatorname{op}_K(J) := \{r_K u r_K | u \in J\}$ containing $\operatorname{proj}_{\tilde{R}} f$ is the only residue of $\tilde{R}$ opposite in $\tilde{R}$ to all J-residues of $\tilde{R}$ in f^{op}, by similar arguments as in Lemma 6.2. Similarly, the $\operatorname{op}_K(J)$-residue containing $\operatorname{proj}_{\tilde{R}} g_i$ is the only residue of $\tilde{R}$ opposite in $\tilde{R}$ to all J-residues in g_i^{op}, for $i = 1, 2$. By Lemma 6.3, the sets of J-residues in f^{op} and in g_i^{op} $(i = 1, 2)$ coincide. Therefore these three $\operatorname{op}_K(J)$-residues coincide, let us name it T. As $f(y) = f(\operatorname{proj}_{\tilde{R}} f)\delta(\operatorname{proj}_{\tilde{R}} f, y)$ for $y \in \tilde{R}$, by Lemma 3.3, we have $l(f(y)) = r_K - l(\delta(\operatorname{proj}_{\tilde{R}} f, y))$ for $y \in \tilde{R}$, and so $\operatorname{proj}_{R'} f = \operatorname{proj}_{R'} \operatorname{proj}_{\tilde{R}} f$. Similarly for g_1, g_2 and for R_x. We have $\operatorname{proj}_{R'} g_1 = \operatorname{proj}_{R'} g_2$, and so $\operatorname{proj}_{R'} \operatorname{proj}_{\tilde{R}} g_1 = \operatorname{proj}_{R'} \operatorname{proj}_{\tilde{R}} g_2$, with $\operatorname{proj}_{\tilde{R}} g_1, \operatorname{proj}_{\tilde{R}} g_2 \in T$. Since R' and T are parallel, it means $\operatorname{proj}_{\tilde{R}} g_1 = \operatorname{proj}_{\tilde{R}} g_2$, which implies $\operatorname{proj}_{R_x} g_1 = \operatorname{proj}_{R_x} g_2$ by similar arguments. This finishes the proof by induction. Since $x \in g_1^{\mathrm{op}}$ and, for any $y \in R_x$, $g_1(y) = g_1(\operatorname{proj}_{R_x} g_1)\delta(\operatorname{proj}_{R_x} g_1, y)$ by Lemma 3.3, we have $1_W = r_J \delta(\operatorname{proj}_{R_x} g_1, x)$. Therefore $r_J = \delta(\operatorname{proj}_{R_x} g_1, x) = \delta(\operatorname{proj}_{R_x} g_2, x)$, which implies that $x \in g_2^{\mathrm{op}}$. By symmetry, we get $g_1^{\mathrm{op}} = g_2^{\mathrm{op}}$. We now conclude by Lemma 3.10.

□

Proposition 6.5.: *Let $\tilde{\mathcal{C}}$ be the set of all codistances on a 3-spherical building $\mathcal{B}$. Then $(\tilde{\mathcal{C}}, (\sim_s)_{s \in S})$ is a chamber system.*

Proof: It follows from the definition that $\sim_s$ is an equivalence relation on $\tilde{\mathcal{C}}$ for all $s \in S$. Suppose $f \sim_s g$ and $f \sim_t g$ for $s, t \in S$ and $f \neq g$. Let $J = \{s, t\}$, which is spherical. Let R be a J-residue in f^{op}. By Lemma 6.4, $\operatorname{proj}_R f$ and $\operatorname{proj}_R g$ are distinct. By Lemma 6.2, the chambers $\operatorname{proj}_R f$ and $\operatorname{proj}_R g$ are $r_J s r_J$-adjacent and also $r_J t r_J$-adjacent in $\mathcal{B}$. Since the chambers of $\mathcal{B}$ form a chamber system, it means $r_J s r_J = r_J t r_J$, and hence $s = t$. □

¿From now on, we again assume that $\mathcal{B} = (\mathcal{C}, \delta)$ is a 3-spherical building of type (W, S) satisfying (lco) and (lsco) and that f is a codistance on $\mathcal{B}$. Let $\mathcal{B}^* = (\mathcal{C}^*, (\sim_s)_{s \in S})$ be the chamber system on the connected component of f.

Fix $s \in S$ and $\tilde{P}$ in $\mathcal{P}_s^{\mathrm{op}}(f)$. For each chamber p of $\tilde{P}$ in f^{op}, we will define another codistance on $\mathcal{B}$. Let $\beta(p) := \{\beta(\tilde{P}, Q)(p) | Q \in \mathcal{P}_s^{\mathrm{op}}(f)\}$. By Theorem 5.18, this set contains exactly one chamber in each panel of $\mathcal{P}_s^{\mathrm{op}}(f)$, none of which being the projection of f on it.

Theorem 6.6.: *For $c \in \mathcal{B}$, choose $P \in \mathcal{P}_{s,c}^{\mathrm{op}}(f)$, and put*

$$g(c) = \begin{cases} sf(c) & \textit{if } \operatorname{proj}_P c \in \{\operatorname{proj}_P f, \beta(p) \cap P\} \\ f(c) & \textit{otherwise.} \end{cases}$$

Then g is a codistance on $\mathcal{B}$. Moreover g is s-adjacent to f and, for $P \in \mathcal{P}_s^{\mathrm{op}}(f)$, $\operatorname{proj}_P g = \beta(p) \cap P$.

Proof: The function $g : \mathcal{C} \to W$ is well defined by Theorem 5.21 and by statement d) of Theorem 5.18. Let Q be a t-panel, so that $f(x) \in \{w, wt\}$ for all $x \in Q$ and Q contains a unique chamber $q := \text{proj}_Q f$ with f-value the longest word of the two, which we can assume to be wt.

Assume $w^{-1}sw = t$. Then $w \in X_s$ and there exists $P \in \mathcal{P}_s^{\text{op}}(f)$ parallel to Q with $\delta(P, Q) = w$. By Proposition 5.14, $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$ for all chambers x of Q. Since P and Q are parallel, proj_P and proj_Q are inverse bijections between P and Q. Hence $g(x) = f(x)$ for all $x \in Q$, except for q whose g-value is $sf(q) = swt = w$ and for $\text{proj}_Q(\beta(p) \cap P)$ whose g-value is $sf(\text{proj}_Q(\beta(p) \cap P)) = sw = wt$. Hence $g(x) \in \{w, wt\}$ and Q contains a unique chamber with g-value wt.

Now assume $w^{-1}sw \neq t$. We claim that either $g(x) = f(x)$ for all $x \in Q$ or $g(x) = sf(x)$ for all $x \in Q$. In the first case, it is obvious Q will satisfy the codistance condition for g. Suppose we are in the second case. Then $g(x) \in \{sw, swt\}$ for all $x \in Q$ and Q contains a unique chamber with g-value swt. We just need to show that $l(swt) = l(sw) + 1$ to get that Q satisfies the codistance condition for g. If $l(sw) = l(w) + 1$, it is known that either $l(swt) = l(w) + 2$ or $swt = w$. Since the second case is excluded, we have $l(swt) = l(w) + 2 = l(sw) + 1$. If $l(sw) = l(w) - 1$ and $l(swt) = l(sw) - 1$, then $l(swt) = l(w) - 2 = l(wt) - 3$, and we get a contradiction, hence if $l(sw) = l(w) - 1$ we also get $l(swt) = l(sw) + 1$.

We now prove the claim. Let $x \in Q$ with f-value w, $y \in f_x^{\text{op}}$ and P the s-panel containing y, so that $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$. If we add the chamber q to a minimal gallery from y to x, we get a minimal gallery from y to q with the required condition on f, and so, by Lemma 3.7, $y \in f_q^{\text{op}}$ and $P \in \mathcal{P}_{s,q}^{\text{op}}(f)$. Let x' be another chamber of Q with f-value w and $P' \in \mathcal{P}_{s,x'}^{\text{op}}(f)$. By the same argument, $P' \in \mathcal{P}_{s,q}^{\text{op}}(f)$. By Theorem 5.21, it means $\beta(P, P')(\text{proj}_P q) = \text{proj}_{P'} q$. Since P and Q (resp. P' and Q) are not parallel, $\text{proj}_P Q$ and $\text{proj}_{P'} Q$ are chambers, and so $\text{proj}_P q = \text{proj}_P x$ and $\text{proj}_{P'} q = \text{proj}_{P'} x'$. Therefore $\beta(P, P')(\text{proj}_P x) = \text{proj}_{P'} x'$, and so $\text{proj}_P x \in \{\text{proj}_P f, \beta(p) \cap P\}$ if and only if $\text{proj}_{P'} x' \in \{\text{proj}_{P'} f, \beta(p) \cap P'\}$. Moreover we also have $\text{proj}_P x \in \{\text{proj}_P f, \beta(p) \cap P\}$ if and only if $\text{proj}_P q \in \{\text{proj}_P f, \beta(p) \cap P\}$. Therefore the claim is proved.

Let $P \in \mathcal{P}_s^{\text{op}}(f)$. Then P contains chambers in f^{op} and one chamber p with $f(p) = s$. Obviously $P \in \mathcal{P}_{s,p}^{\text{op}}(f)$, hence $\text{proj}_P p = p$ and so $g(p) = sf(p) = 1_W$. Hence $p \in g^{\text{op}}$ and $P \in \mathcal{P}_s^{\text{op}}(g)$. Let P be a s-panel not in $\mathcal{P}_s^{\text{op}}(f)$. Then $f(x) \in \{w, ws\}$ for $x \in P$ with $s \neq w \neq 1$. Hence $g(x) \in \{w, sw, ws, sws\}$ for $x \in P$. Since $1_W \notin \{w, sw, ws, sws\}$, no chamber of P is in g^{op}, and so $P \notin \mathcal{P}_s^{\text{op}}(g)$. This proves that $f \sim_s g$.

Finally, let $P \in \mathcal{P}_s^{\text{op}}(f)$. Then for any $c \in P$, $P \in \mathcal{P}_{s,c}^{\text{op}}(f)$, therefore $g(c) = f(c)$ unless $c \in \{\text{proj}_P f, \beta(p) \cap P\}$. Hence the only chamber of P with g-value s is $\beta(p) \cap P$.

□

Proposition 6.7.: *Let $\mathcal{B}$ be a 3-spherical building of type (W, S) satisfying (lco) and (lsco). Let $J \subseteq S$ be spherical, and f be a codistance on $\mathcal{B}$. Let R be a J-residue of $\mathcal{B}$ in f^{op} and $\tilde{R}$ be the J-residue containing f in $\mathcal{B}^*$. Then $\alpha : \tilde{R} \to R : g \to \text{proj}_R g$ is a bijection such that:*

(i) $\forall g_1, g_2 \in \tilde{R}, s \in J$, *we have* $g_1 \sim_s g_2$ *if and only if* $\alpha(g_1) \sim_{r_J s r_J} \alpha(g_2)$,
(ii) $\forall g \in \tilde{R}, c \in R$, *we have* $c \in g^{\mathrm{op}}$ *if and only if* $\delta(\alpha(g), c) = r_J$.

Proof:

By Lemma 6.4, α is injective. Let $d = \mathrm{proj}_R f$. We will show by induction on the distance $l(\delta(x,d))$ that α is surjective. Notice first that $\alpha(f) = d$ so it is true if $l(\delta(x,d)) = 0$. Suppose we have proved that there exists $g \in \tilde{R}$ with $\alpha(g) = x$ for all x satisfying $l(\delta(x,d)) < l$ and suppose $l(\delta(y,d)) = l$. Let $y = y_0, y_1, \ldots, y_l = d$ be a minimal gallery. By hypothesis, there exists $g_1 \in \tilde{R}$ with $\alpha(g_1) = y_1$. Let T be the t-panel containing y_0 and y_1 for some $t \in S$. Let $s = \mathrm{op}_J(t) = r_J t r_J \in J$. By Lemma 6.3 and an easy induction, R is in g^{op} for any $g \in \tilde{R}$ and so in particular for g_1. Therefore there exists $c \in R \cap g_1^{\mathrm{op}}$ and the s-panel P containing c is in $\mathcal{P}_s^{\mathrm{op}}(g_1)$. By construction P and T are opposite hence parallel. Let $p := \mathrm{proj}_P\, y$. Using Theorem 6.6, we can construct a codistance g s-adjacent to g_1 with $\mathrm{proj}_P\, g = p$. By Lemma 6.2, $\mathrm{proj}_R\, g$ and $\mathrm{proj}_R\, g_1$ are t-adjacent, and so $\mathrm{proj}_R\, g \in T$. Since $\mathrm{proj}_P\, g = \mathrm{proj}_P \mathrm{proj}_R\, g$ (by an argument used above), we must have $\mathrm{proj}_R\, g = y$. Therefore $\alpha(g) = y$ and α is surjective.

By Lemma 6.2, if g_1 and g_2 are s-adjacent in $\tilde{R}$, then $\mathrm{proj}_R\, g_1$ and $\mathrm{proj}_R\, g_2$ are $r_J s r_J$-adjacent in R.

Now assume g_1 and g_2 are codistances in $\tilde{R}$ with $\mathrm{proj}_R\, g_1 \sim_{r_J s r_J} \mathrm{proj}_R\, g_2$ for some $s \in J$. Let P be the $r_J s r_J$-panel containing them and $e := \mathrm{proj}_P\, d$. As α is surjective, there exists $g \in \tilde{R}$ with $\alpha(g) = e$. We have shown above that there exist codistances g_1' and g_2', both s-adjacent to g, with $\mathrm{proj}_R\, g_1' = \mathrm{proj}_R\, g_1$ and $\mathrm{proj}_R\, g_2' = \mathrm{proj}_R\, g_2$. By the injectivity of α, $g_1' = g_1$ and $g_2' = g_2$, and so g_1 and g_2 are both s-adjacent to g. Since $\mathcal{B}^*$ is a chamber system, this means $g_1 \sim_s g_2$. This proves (i).

We now prove (ii). Let $g \in \tilde{R}$. By Lemma 3.3, for all $c \in R$, $g(c) = g(\alpha(g))\delta(\alpha(g), c)$. Since $R \in g^{\mathrm{op}}$ as noticed above, g takes on R its values in W_J, and so $g(\alpha(g)) = r_J$. Hence $c \in g^{\mathrm{op}} \iff g(c) = 1_W \iff \delta(\alpha(g), c) = r_J$. □

Corollary 6.8.: *The chamber system* $\mathcal{B}^*$ *has the same diagram as* $\mathcal{B}$.

Proof. Let M be the diagram of $\mathcal{B}$, which mean that each rank 2 J-residue is a generalized M_J-gon. Let $\tilde{R}$ be a J-residue of rank 2 of $\mathcal{B}^*$.Let g be a codistance in $\tilde{R}$ and let R be a J-residue in g^{op}. Then, by Proposition 6.7, $\tilde{R}$ is a building of the same type as R, hence a generalized M_J-gon. Therefore $\mathcal{B}^*$ has diagram M. □

7 Construction of the twinning

In order to construct a twinning we apply the main result of [Mu98] which we recall below and whose statement requires some preparation.

Let (W, S) be a Coxeter system and let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of type (W, S). An *opposition relation* between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a non-empty subset $\mathcal{O}$ of $\mathcal{C}_+ \times \mathcal{C}_-$ such that there exists a twinning δ^* of $\mathcal{B}_+$ and $\mathcal{B}_-$ with the property that $\mathcal{O} = \{(x, y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta^*(x, y) = 1_W\}$.

A *local opposition relation* between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a non-empty subset $\mathcal{O}$ of $\mathcal{C}_+ \times \mathcal{C}_-$ such that for each $(x, y) \in \mathcal{O}$ and each subset $J \subseteq S$ of cardinality at most 2 the set $\mathcal{O} \cap (R_J(x) \times R_J(y))$ is an opposition relation between the J-residues of x and y. Note that the definition of a local opposition relation makes perfect sense for two chamber systems of type (W, S) as well.

Here is the main result of [Mu98].

Theorem 7.1.: *Let (W, S) be a Coxeter system and let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two thick buildings of type (W, S) and let $\mathcal{O}$ be a non-empty subset of $\mathcal{C}_+ \times \mathcal{C}_-$. Then $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$ if and only if it is a local opposition relation between the two buildings.*

The following corollary of the previous theorem has been proved in [Mu99, p.28], we paraphrase that proof here.

Corollary 7.2.: *Let (W, S) be a Coxeter system and let $(\mathcal{C}_+, (\sim_s)_{s\in S}), (\mathcal{C}_-, (\sim_s)_{s\in S})$ be two connected, thick chamber systems of type (W, S) whose universal 2-covers are buildings. Suppose that there exists a local opposition relation $\mathcal{O} \subseteq (\mathcal{C}_+ \times \mathcal{C}_-)$ between them. Then the chamber systems are buildings. In particular, there exist unique distances $\delta_+ : \mathcal{C}_+ \times \mathcal{C}_+ \to W$, $\delta_- : \mathcal{C}_- \times \mathcal{C}_- \to W$ and $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ such that $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta^*)$ is a twin building of type (W, S) and such that $\mathcal{O} = \{(x, y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta^*(x, y) = 1_W\}$.*

Proof: Let $\overline{\mathcal{B}}_\epsilon = (\overline{\mathcal{C}}_\epsilon, (\sim_s)_{s\in S})$ be the universal 2-cover of $(\mathcal{C}_\epsilon, (\sim_s)_{s\in S})$, which is a building by hypothesis, with covering morphism $\phi_\epsilon : \overline{\mathcal{C}}_\epsilon \to \mathcal{C}_\epsilon$, for $\epsilon = +, -$. Let $\overline{\mathcal{O}} = \{(x, y) \in \overline{\mathcal{C}}_+ \times \overline{\mathcal{C}}_- | (\phi_+(x), \phi_-(y) \in \mathcal{O}\}$. Obviously $\overline{\mathcal{O}}$ is a local opposition relation between $\overline{\mathcal{B}}_+$ and $\overline{\mathcal{B}}_-$. By the previous theorem, this means that $\overline{\mathcal{O}}$ is the opposition relation of a twin building $(\overline{\mathcal{B}}_+, \overline{\mathcal{B}}_-, \overline{\delta}_*)$.

Let $\overline{x} \neq \overline{y} \in \overline{\mathcal{C}}_-$. By Lemma 2.6, $\overline{x}^{\mathrm{op}} \neq \overline{y}^{\mathrm{op}}$. Hence there exists $\overline{z} \in \overline{\mathcal{C}}_+$ such that $(\overline{z}, \overline{x}) \in \overline{\mathcal{O}}$ but $(\overline{z}, \overline{y}) \notin \overline{\mathcal{O}}$. If $\phi_-(\overline{x}) = v = \phi_-(\overline{y})$, then we have both $(\phi_+(\overline{z}), v) \in \mathcal{O}$ and $(\phi_+(\overline{z}), v) \notin \mathcal{O}$, a contradiction. This shows that ϕ_- is injective and hence is the identity. The same argument shows that ϕ_+ is the identity. Therefore $\overline{\mathcal{C}}_\epsilon = \mathcal{C}_\epsilon$ for $\epsilon = +, -$ and the result follows. □

In order to apply the corollary above we need the following lemma.

Lemma 7.3.: *Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of spherical type (W, S), let $r \in W$ be the longest element in W and let $\mathcal{O}$ be a non-empty subset of $\mathcal{C}_+ \times \mathcal{C}_-$. Then the following are equivalent.*

a) $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$.

b) There exists a bijection $\alpha : \mathcal{C}_+ \to \mathcal{C}_-$ such that the following two conditions are satisfied:

(i) For all $x, y \in \mathcal{C}_+$ and all $s \in S$ we have $x \sim_s y$ if and only if $\alpha(x) \sim_{rsr} \alpha(y)$;

(ii) $\mathcal{O} = \{(x, y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_-(\alpha(x), y) = r\}$.

Proof: Suppose $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$. Then there exists a twinning between them. Define $\alpha : \mathcal{C}_+ \to \mathcal{C}_-$ by $\alpha(x) = \text{proj}_{\mathcal{C}_-} x$, that is the unique chamber y of $\mathcal{C}_-$ with $\delta^*(x, y) = r$. It is easily checked that α is a bijection and satisfies (i) and (ii).

Now suppose there exists a bijection α satisfying (i) and (ii). We define a mapping δ^* from $(\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$ into W by $\delta(x, y) := r\delta_-(\alpha(x), y)$ and $\delta(y, x) := \delta(x, y)^{-1}$, for $x \in \mathcal{C}_+$ and $y \in \mathcal{C}_-$. Using the axioms of buildings, it can easily be checked that δ^* is a twinning. Moreover $\mathcal{O} = \{(x, y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta^*(x, y) = 1_W\}$, so $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$.

□

Proof of the main result

Let $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be a thick building of type (W, S) satisfying all necessary properties and let $f : \mathcal{C}_- \to W$ be a codistance.

Consider the chamber system of all codistances of $\mathcal{B}_-$ which is a chamber system over S. Let $\mathcal{C}_+$ be the connected component containing f and consider the chamber system $(\mathcal{C}_+, (\sim_s)_{s\in S})$ which is a connected chamber system of type (W, S) by Corollary 6.8. It readily follows from Proposition 6.7 that all J-residues of rank at most 3 are spherical buildings and in particular that $(\mathcal{C}_+, (\sim_s)_{s\in S})$ is thick . By a result of Tits [Ti81] it follows that the universal 2-cover of this chamber system is a building.

We define $\mathcal{O} \subseteq \mathcal{C}_+ \times \mathcal{C}_-$ by setting $\mathcal{O} := \{(g, c) \in \mathcal{C}_+ \times \mathcal{C}_- \mid g(c) = 1_W\}$. Using the lemma above and Proposition 6.7 we see that $\mathcal{O}$ is a local opposition between the chamber systems $(\mathcal{C}_+, (\sim_s)_{s\in S})$ and $(\mathcal{C}_-, (\sim_s)_{s\in S})$, which are both thick chamber systems of type (W, S) whose universal covers are buildings. Therefore, Corollary 7.2 yields the twin building.

Now it is easy to see that $f' := \delta^*(f, .)$ is a codistance on $\mathcal{B}_-$ with

$$f'^{\text{op}} = \{c \in \mathcal{C}_- | \delta^*(f, c) = 1_W\} = \{c \in \mathcal{C}_- | (f, c) \in \mathcal{O}\} = \{c \in \mathcal{C}_- | f(c) = 1_W\} = f^{\text{op}}.$$

By Lemma 3.10, we have $f' = f$ and so $\delta^*(f, x) = f(x)$ for all $x \in \mathcal{C}_-$.

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