

A characterization of the Grassmann embedding of $\mathbf{H}(q)$, with q even

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Abstract

In this note, we characterize the Grassmann embedding of $\mathbf{H}(q)$, q even, as the unique full embedding of $\mathbf{H}(q)$ in $\text{PG}(12, q)$ for which each ideal line of $\mathbf{H}(q)$ is contained in a plane. In particular, we show that no such embedding exists for $\mathbf{H}(q)$, with q odd. As a corollary, we can classify all full solid polarized embeddings of $\mathbf{H}(q)$ in $\text{PG}(12, q)$; they are necessarily Grassmann embeddings of $\mathbf{H}(q)$, with q even. (An embedding is called *solid* if the lines through a point are contained in a solid.)

1 Introduction

In the theory of finite geometries, full projective embeddings play a central role. Indeed, it attaches a field to a geometry, it provides a representation of the geometry which can be used to investigate certain properties, and in many cases, it tells something about the nature of collineations of that geometry. The basic question is always to try to classify all full projective embeddings for a certain class of geometries. For the class of finite generalized quadrangles, this has been done by Buekenhout & Lefèvre [1]. A similar result for finite generalized hexagons has not yet been established, and it seems that we are still very far from it. In particular, we even do not know yet an absolute and general upper bound on the dimension of the projective space. The additional assumption of being flat (which means that the lines of the hexagon through a point are contained in a plane of the projective space) allows to generalize some arguments of quadrangles to hexagons, and in fact shows that the dimension of the projective space is at most 7, see [5, 6]. But a complete classification of all flat full projective embeddings of finite generalized hexagons has not yet been obtained. Another condition that makes partial classification possible

is that of being polarized (which means that the set of points not opposite a given point in the hexagon does not span the whole projective space). The combination—flat and polarized—has been completely settled, even in the infinite case; see [5] and [4].

Another project is to investigate embeddings of the classical examples, which in the case of finite generalized hexagons, are all known examples. The central question is here to classify all full projective embeddings of the split Cayley hexagons, and to start with to determine the possible dimensions. In particular the question of an upper bound for the dimension shows up again. In this approach, one is already satisfied with a classification of embeddings in the case of maximal projective dimension (since projections usually behave much less nice and regular, and are much harder to characterize or to recover from general assumptions). Surprisingly, such an upper bound (namely, 13) has been determined for the finite *dual* split Cayley hexagons, and all full projective embeddings in that top dimensional space have been determined, see [7]. A similar result for the (finite) split Cayley hexagons seems out of reach. The standard conjecture here is that the known embeddings are the ones providing the upper bounds of the dimension. This would mean that the upper bound is 6 for all split Cayley hexagons over fields of characteristic unequal to 3, except for the smallest one (of order 2), which admits an embedding in 13-dimensional space. In characteristic 3, the upper bound is 13, as these hexagons are self-dual.

In this paper, we investigate an embedding of the split Cayley hexagons in characteristic 2 in 12-dimensional projective space, which we call the Grassmann embedding (because it is the image on the Grassmannian of all planes of $\text{PG}(5, q)$ of the set of planes containing $q + 1$ lines of the ordinary 5-dimensional embedding of the hexagon). This embedding is not so well known, and it does not appear explicitly in the literature, although implicitly, it is around in the theory of hyperplanes of dual polar spaces, see [2]; and it also has been discovered during the writing of this paper independently by Coolsaet (personal communication, unpublished). While we cannot prove yet that 12 is the upper bound for the dimension of any full projective embedding of the split Cayley hexagons in characteristic 2, and we can neither classify the full projective embeddings of these hexagons in 12-dimensional space, we are able to provide a natural characterization of these embeddings, which should ultimately be useful for a possible complete classification. In particular we show that this embedding is characterized by the dimension and the properties of being polarized and *solid* (all lines of the hexagon through a point are contained in a solid). This follows from a slightly stronger result, in which we only assume that *ideal lines* are contained in planes (see below for more details). Not insisting on this stronger result, we can state our main result as follows:

Main Result, first version. *If the split Cayley hexagon $H(q)$, $q > 2$, is embedded in $\text{PG}(d, q)$, with $d \geq 12$, such that the embedding is polarized and solid, then it is the*

Grassmann embedding.

The case $q = 2$ is a true exception since there exists a (unique) polarized and solid embedding of $\mathbf{H}(2)$ in $\mathbf{PG}(13, 2)$. However, all embeddings of $\mathbf{H}(2)$ are classified; for an account on this, see [9].

2 Preliminaries

2.1 Generalized polygons

Generalized polygons were introduced by Jacques Tits in [11]. The claims below can be found in the monograph [12].

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a point-line geometry, where we view the elements of \mathcal{L} as sets of points. The *incidence graph* $\Gamma(\mathcal{S})$ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and $p \in \mathcal{P}$ is adjacent to $L \in \mathcal{L}$ if $x \in L$. Then we call \mathcal{S} a *generalized n -gon*, or *generalized polygon*, if the diameter of $\Gamma(\mathcal{S})$ is equal to n , and the girth of $\Gamma(\mathcal{S})$ is equal to $2n$ (the *girth* is the length of the smallest cycle in Γ). In fact, in the present paper, we will only need generalized 6-gons, or *generalized hexagons*. Another, more common example, is any projective plane, which is a generalized 3-gon.

It follows directly from the definition that elements of a generalized n -gon can have mutual distance at most n in the incidence graph. Elements at distance exactly n will be called *opposite*. If the valency of every vertex of $\Gamma(\mathcal{S})$ is at least 3, then we call \mathcal{S} *thick*; otherwise just *non-thick*. In a thick generalized polygon, the valency of a line is some constant $s + 1$, and the valency of a point is some (possibly different) constant $t + 1$. We then say that the order of the polygon is (s, t) . If $s = t$, we say that the order is s .

2.2 Linear and Veronesean embeddings

A plane oval \mathcal{O} of $\mathbf{PG}(d, \mathbb{K})$ is a set of points of $\mathbf{PG}(d, \mathbb{K})$ contained in a plane π , such that, for any point $x \in \mathcal{O}$, there is a unique line of π through x intersecting \mathcal{O} in just x , and all other lines through x intersect \mathcal{O} in exactly two points (including $x!$). Examples are conics.

Let again $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. A *linear (projective full) embedding* of \mathcal{S} in the projective space $\mathbf{PG}(d, \mathbb{K})$ over a field \mathbb{K} is an injective mapping $\alpha : \mathcal{P} \rightarrow \mathbf{PG}(d, \mathbb{K})$ such that the image of any line of \mathcal{S} is a (full) line of $\mathbf{PG}(d, \mathbb{K})$, and such that the image of

\mathcal{P} under α generates $\text{PG}(d, \mathbb{K})$. A *Veronesean (projective) embedding* of \mathcal{S} in the projective space $\text{PG}(d, \mathbb{K})$ over a field \mathbb{K} is an injective mapping $\alpha : \mathcal{P} \rightarrow \text{PG}(d, \mathbb{K})$ such that the image of any line is a plane oval, and such that the image of \mathcal{P} under α generates $\text{PG}(d, \mathbb{K})$.

We define equivalent projective embeddings in the usual way, i.e., two projective embeddings α and α' , be it Veronesean or linear, in $\text{PG}(d, \mathbb{K})$ and $\text{PG}(d, \mathbb{K})'$, respectively, are *equivalent* if there is a projectivity $\sigma : \text{PG}(d, \mathbb{K}) \rightarrow \text{PG}(d, \mathbb{K})'$ mapping p^α to $p^{\alpha'}$, for all points $p \in \mathcal{P}$.

For example, the ordinary (conic) Veronesean of $\text{PG}(2, \mathbb{K})$ yields a Veronesean embedding of $\text{PG}(2, \mathbb{K})$ in $\text{PG}(5, \mathbb{K})$. Conversely, every Veronesean embedding of $\text{PG}(2, \mathbb{K})$ into $\text{PG}(5, \mathbb{K})$, $q > 2$, is equivalent to the (conic) Veronesean of $\text{PG}(2, \mathbb{K})$, see [8].

Some more terminology concerning linear embeddings of generalized hexagons: if in an embedding, the set of points not opposite any given point does not generate the whole projective space, then we call the embedding *polarized*; if the points collinear to any given point are contained in a plane or solid, then we call the embedding *flat* or *solid*, respectively.

2.3 Split Cayley hexagons

In the present paper, we will be concerned with the finite split Cayley hexagon $\text{H}(q)$ of order q , for each prime power q . It can be constructed as follows (and this construction is due to Jacques Tits [11]). Choose coordinates in the projective space $\text{PG}(6, q)$ in such a way that the parabolic quadric $\text{Q}(6, q)$ has equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. Then the points of $\text{H}(q)$ are all points of $\text{Q}(6, q)$. The lines of $\text{H}(q)$ are the lines on $\text{Q}(6, q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \dots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. This defines a linear embedding of $\text{H}(q)$ into $\text{PG}(6, q)$, which we call the *standard embedding of $\text{H}(q)$ in $\text{PG}(6, q)$* . It is linear, flat and polarized.

The lines of $\text{PG}(6, q)$ entirely contained in $\text{Q}(6, q)$, but which are not lines of $\text{H}(q)$, are called *ideal lines* of $\text{H}(q)$. Likewise, the planes of $\text{PG}(6, q)$ entirely contained in $\text{Q}(6, q)$ but not containing any line of $\text{H}(q)$, are called *ideal planes* of $\text{H}(q)$.

If q is even, then the projection of the standard embedding of $\text{H}(q)$ from the point $(0, 0, 0, 1, 0, 0, 0)$ of $\text{PG}(6, q)$ onto some hyperplane is faithful and yields a flat and polarized linear embedding of $\text{H}(q)$ in $\text{PG}(5, q)$, which we call the *standard embedding of $\text{H}(q)$ in $\text{PG}(5, q)$* . With each point x of $\text{PG}(5, q)$ corresponds a plane π_x of $\text{PG}(5, q)$ obtained by taking the span of the lines of $\text{H}(q)$ through x . Moreover, if x runs through

the set of points of a line L of $\mathbf{H}(q)$, then π_x runs through all planes of a certain solid ξ_L . Consequently, the composition θ of the map $x \mapsto \pi_x$ with the Grassmann embedding of all planes of $\mathbf{PG}(5, q)$ takes a point x of $\mathbf{H}(q)$ to a point x' of $\mathbf{PG}(19, q)$, and the set of points on a line L is taken to the set of points on a line L' of $\mathbf{PG}(19, q)$. Using the explicit form of the lines, it takes an elementary but tedious calculation to see that the image under θ of the point set of $\mathbf{PG}(5, q)$ spans a 12-dimensional subspace $\mathbf{PG}(12, q)$ of $\mathbf{PG}(19, q)$. Hence we obtain a linear embedding of $\mathbf{H}(q)$ in $\mathbf{PG}(12, q)$. We call this embedding the *Grassmann embedding of $\mathbf{H}(q)$* . We are not aware of an explicit mentioning in the literature of this embedding, although it is implicit in the work of B. De Bruyn [2], and it has independently been discovered by K. Coolsaet during the writing of this paper.

One interesting property of the Grassmann embedding is that every ideal line of $\mathbf{H}(q)$ is mapped onto a plane conic in $\mathbf{PG}(12, q)$. In particular, ideal lines in $\mathbf{PG}(12, q)$ are contained in planes.

We can now state our Main Result in some more detail.

Main Result, second version. *The Grassmann embedding of $\mathbf{H}(q)$ in $\mathbf{PG}(12, q)$, $q > 2$ and q even, is the unique embedding of $\mathbf{H}(q)$ in $\mathbf{PG}(d, q)$, $d \geq 12$, with the property that every ideal line is contained in a plane. For q odd, there is no embedding of $\mathbf{H}(q)$ with these properties.*

As a corollary one has immediately that this embedding is also the unique solid and polarized one in $\mathbf{PG}(d, q)$, $d \geq 12$, of $\mathbf{H}(q)$, $q > 2$. Hence the first version follows from the second one. Conversely, if every ideal line is contained in a plane, then the embedding is obviously solid, but in order to see that it is also polarized, we need a large part of our proof below. So the converse is not immediate.

The case $q = 2$ in the above is again a true exception, see [9].

Concerning the infinite case: the Grassmann embedding exists for the split Cayley hexagon over every field of characteristic 2 (perfect or not), but our methods of characterization only work in the finite case.

2.4 Some more properties of $\mathbf{H}(q)$

We need some additional properties of the split Cayley hexagon $\mathbf{H}(q)$.

First, ideal planes come in pairs. Indeed, more precisely, given an ideal plane \mathfrak{P} , then every point of \mathfrak{P} is at distance 4 (in the incidence graph) of every other point of \mathfrak{P} . Considering the set Ω of points of $\mathbf{H}(q)$ which are at distance 2 from at least 2 points of \mathfrak{P} , we obtain

a second ideal plane. Starting with Ω and carrying out the same construction yields \mathfrak{P} again. Hence we call the ideal planes \mathfrak{P} and Ω *twins*.

If \mathfrak{P} and Ω are twin ideal planes, then $\mathfrak{P} \cup \Omega$, together with all lines of $\mathbf{H}(q)$ that contain at least (and hence exactly) two points of $\mathfrak{P} \cup \Omega$, form a non-thick generalized hexagon (of order $(1, q)$), which we call a *non-thick ideal subhexagon* (the word *ideal* comes from the fact that all lines of $\mathbf{H}(q)$ through a point of that subhexagon are lines of that subhexagon).

Let x, y be two opposite points in $\mathbf{H}(q)$. Then the set \mathcal{R} of lines at distance 3 from both x and y is called a *regulus*, and it has the property that, whenever a point of $\mathbf{H}(q)$ is at distance 3 from at least two of its members, then this point is at distance 3 from all of its members. The points on the lines of such a regulus \mathcal{R} and at distance 2 from a point that is at distance 3 from all members of \mathcal{R} , form an ideal line (in $\text{PG}(6, q)$ and in $\text{PG}(5, q)$, q even, a regulus is the set of generators of one class of a hyperbolic quadric in some solid; the ideal lines on it constitute the other class of generators).

For a point x of $\mathbf{H}(q)$, we denote with x^\perp the set of points of $\mathbf{H}(q)$ collinear with x . We note that the union of all lines of $\mathbf{H}(q)$ through x in $\text{PG}(6, q)$ and in $\text{PG}(5, q)$, q even, is a plane of $\text{Q}(6, q)$ and $\text{PG}(5, q)$, respectively. Consequently, two ideal lines contained in x^\perp either coincide or meet in a unique point.

Finally, a line of $\text{PG}(5, q)$, q even, not belonging to $\mathbf{H}(q)$, and which is not an ideal line, will be called an *imaginary line*.

3 Proof of the Main Result

3.1 Projective planes

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be isomorphic to $\text{PG}(2, q)$ with $\mathcal{P} \subseteq \text{PG}(d, q)$, $\langle \mathcal{P} \rangle = \text{PG}(d, q)$, $d \geq 5$, and such that every member L of \mathcal{L} is a subset of points of a plane in $\text{PG}(d, q)$, which we denote by π_L if it is unique; if it is not unique, then π_L is the intersection of all such planes (and so π_L is the line of $\text{PG}(d, q)$ containing all points of L). We call such a representation of \mathcal{S} in $\text{PG}(d, q)$ a *generalized Veronesean embedding*.

We denote the line of $\text{PG}(d, q)$ spanned by two points $a, b \in \mathcal{P}$ by $\langle a, b \rangle$, while the line of \mathcal{S} through a, b is denoted by ab . More generally, we use the symbol $\langle A \rangle$ to denote the subspace of $\text{PG}(12, q)$ generated by the elements of A .

We will assume that $q > 2$.

Lemma 1 *If L, M are two distinct lines of \mathcal{S} , meeting in the point $z \in \mathcal{P}$, and $x \in \mathcal{P}$ is a point off $L \cup M$ not contained in $\langle \pi_L, \pi_M \rangle$, then every point $y \in \mathcal{P}$ off xz is contained in the space $W := \langle \pi_L, \pi_M, x \rangle$.*

Proof The line xy meets $L \cup M$ in two distinct points u, v , with $u \in L$ and $v \in M$. So the space π_{xy} has a line in common with $\langle \pi_L, \pi_M \rangle$ and contains x . Hence $\pi_{xy} = \langle x, u, v \rangle \subseteq W$. \square

Lemma 2 *If $d \geq 5$ and if a proper subspace H of $\text{PG}(d, q)$ contains all points of \mathcal{S} off a certain line $L \in \mathcal{L}$, then either*

- (A) *$d = 5$, the points of \mathcal{S} off L form an affine plane $\text{AG}(2, q)$ in $\text{PG}(5, q)$, and L defines a plane π_L skew to the projective completion of $\text{AG}(2, q)$, or*
- (B) *all points of \mathcal{S} except exactly one are contained in H .*

Proof Suppose some proper subspace H contains all points of \mathcal{S} off a line $L \in \mathcal{L}$. We may assume that H is spanned by the points of \mathcal{S} off L . Then there is a least one point $x \in L$ not contained in H . For each line $M \in \mathcal{L}$ containing x , with $M \neq L$, the space π_M is a plane and intersects H in a line. It follows that all points of M except x are contained in a line L_M of H . If there is a second point $x' \in L$ not contained in H , then we obtain likewise for each line $M' \in \mathcal{L}$ through x' , with $M' \neq L$, a line $L'_{M'}$ in H containing all points of M' except for x' . Each of the lines L_M meets each of the lines $L'_{M'}$.

First assume that no third point of L lies outside H . As $q \geq 3$, this implies that at least two points of L are contained in H , and so π_L is a plane meeting H in a line. This implies that H is a hyperplane. From the above, we see that, if at least two points of L are not contained in H , then H is at most 3-dimensional, a contradiction. Hence in such a case exactly one point x of L is not contained in H .

Next, assume that at least three points x, x', x'' of L lie outside H . In H , there arise three sets of q lines, and two lines of different sets always intersect. It follows that all points of \mathcal{S} off L lie in a plane, which by definition coincides with H . Hence, deleting L and its points, there arises an affine plane $\text{AG}(2, q)$ in H . As \mathcal{S} generates a d -dimensional space, with $d \geq 5$, π_L is a plane which is disjoint from the plane H , and $d = 5$. \square

These two lemmas have now the following important consequence.

Corollary 3 *We have $d \leq 5$. Also, if $d = 5$, then for every line $L \in \mathcal{L}$, the space π_L is a plane. Moreover, if $d = 5$ and we do not have Case (A) above, then for every two distinct lines $L, M \in \mathcal{L}$, the planes π_L and π_M meet in a unique point.*

Proof Suppose $d > 5$. Take two arbitrary lines $L, M \in \mathcal{L}$. Then $W := \langle \pi_L, \pi_M \rangle$ is at most 4-dimensional. Let d' be the dimension of W . Hence there are two points $x, y \in \mathcal{P}$ with $\langle W, x, y \rangle$ a $(d' + 2)$ -dimensional space. From Lemma 1 follows that xy contains the intersection point z of L and M , and that all points of \mathcal{S} off xy are contained in both $\langle W, x \rangle$ and $\langle W, y \rangle$, hence in W . This now contradicts Lemma 2.

If π_L were a line, then $d' \leq 3$ and the argument of the previous paragraph can be copied and leads to a contradiction with $d = 5$. Hence π_L is a plane, and so is π_M . If these planes generated a subspace of dimension at most 3, then again the same argument leads to a contradiction.

The corollary is completely proved. □

Remark 4 All the above holds for arbitrary (finite) projective planes. An interesting question is whether one can classify all generalized Veronesean embeddings (of all projective planes of order q) in $\text{PG}(5, q)$. This has been solved and will appear in a different paper, along with other, related, results. Notice that not every generalized Veronesean embedding is a conic Veronesean embedding, and there are two classes of counterexamples. For more information, we refer to [10]

3.2 The split Cayley hexagon $\mathbf{H}(q)$

From now on we assume that $\mathbf{H}(q)$, $q > 2$, is fully embedded in $\text{PG}(d, q)$, $d \geq 12$, and that the points of every ideal line are contained in a (not necessarily unique) plane. Then Corollary 3 implies that the points of a given ideal plane \mathfrak{P}_1 span a subspace S_1 of dimension $d_1 \leq 5$. The same token shows that the points of its twin, say \mathfrak{P}_2 , span a subspace S_2 of dimension $d_2 \leq 5$. Let \mathfrak{H} be the non-thick ideal subhexagon with point set $\mathfrak{P}_1 \cup \mathfrak{P}_2$. Then it is well known (and, in fact, easy to see), that for $q > 2$ the set \mathfrak{P} of points of $\mathbf{H}(q)$ incident with some line of \mathfrak{H} is a geometric hyperplane of $\mathbf{H}(q)$, whose complement induces a connected subgeometry of $\mathbf{H}(q)$. Hence, it follows that $12 \geq d_1 + d_2 + 1 + 1 \geq d \geq 12$. Consequently $d_1 = d_2 = 5$, $d = 12$, and the points of every ideal line span a unique plane of $\text{PG}(12, q)$ (use Corollary 3). It also follows that $S_1 \cap S_2 = \emptyset$ (in particular, the points of two ideal lines, one of \mathfrak{P}_1 and one of \mathfrak{P}_2 , span a subspace of dimension 5).

Let L_1, M_1 be two ideal lines of \mathfrak{P}_1 , and let x_1 be a point in \mathfrak{P}_1 not on $L_1 \cup M_1$. Let X_2 be the ideal line in \mathfrak{P}_2 all of whose points are collinear to x_1 (in $\mathbf{H}(q)$). Then each point of L_1 and M_1 is collinear in $\mathbf{H}(q)$ with a unique point of X_2 and the connecting lines between L_1 and X_2 form a line-regulus \mathcal{R}_L ; similarly for M_1 and X_2 , we obtain the

line-regulus \mathcal{R}_M . By a previous observation, the points on each of these line-reguli span a 5-dimensional space, say S_L and S_M , respectively. Let K be a third ideal line all of whose points are contained in \mathcal{R}_L . Let π_K be the plane generated by the points of K . Projecting X_2 in S_L from π_K onto the plane generated by the points of L_1 , we see that \mathcal{R}_L induces the restriction of an isomorphism φ_L from the plane generated by the points of X_2 onto the plane generated by the points of L_1 , and so φ_L maps X_2 onto L_1 . Likewise, we obtain an isomorphism φ_M mapping X_2 onto M_1 . We now easily see that the product $\varphi_L^{-1}\varphi_M$ maps L_1 isomorphically onto M_1 and the image of a point p_1 on L_1 is the point q_1 of M_1 with the property that x_1, p_1, q_1 are collinear in the ideal plane \mathfrak{P}_1 (i.e., $\varphi_L^{-1}\varphi_M$ corresponds with a perspectivity from L_1 to M_1 with center x_1). Considering appropriate compositions of such maps, we see that L_1 is invariant under a group isomorphic to the group of projectivities of L_1 , which is $\mathrm{PGL}_2(q)$, where, moreover, this (permutation) group is a subgroup of the projective group $\mathrm{PGL}_3(q)$ of the plane π_{L_1} . By the 3-transitivity of this group, it follows easily that no 3 points of L_1 are contained in a line of $\mathrm{PG}(12, q)$. Hence the points of L_1 form a plane oval. Using [8], this now implies that each ideal plane is isomorphic to a conic Veronesean and that all ideal lines are conics.

It also follows that $\mathfrak{P}_1 \cup \mathfrak{P}_2$ is projectively unique. Moreover, the lines of $\mathbf{H}(q)$ joining a point of \mathfrak{P}_1 to a point of \mathfrak{P}_2 induce an anti-isomorphism from \mathfrak{P}_1 to \mathfrak{P}_2 , both endowed with a projective plane structure. It follows that the set \mathfrak{P} of points of $\mathbf{H}(q)$ contained in a line of $\mathbf{H}(q)$ meeting $\mathfrak{P}_1 \cup \mathfrak{P}_2$ in two points, is unique in $\mathrm{PG}(12, q)$, up to a semilinear automorphism.

We can now prove that q must be even.

Lemma 5 *q is even.*

Proof Consider a point x of $\mathbf{H}(q)$, and an arbitrary ideal line L in x^\perp . Since we may think of x belonging to \mathfrak{P}_1 and L to \mathfrak{P}_2 , we see that $\langle x, L \rangle$ is a solid S in $\mathrm{PG}(12, q)$, and x^\perp is a quadratic cone \mathcal{Q} . Consider now two distinct ideal lines L_1, L_2 in x^\perp such that no point of x^\perp is contained in $L_1 \cap L_2 \cap L$. We have $|L_1 \cap L_2| = |L \cap L_1| = |L \cap L_2| = 1$ (see Subsection 2.4). So the conics L_1 and L_2 share the tangent line at their common point x_{12} ; similarly for L and L_1 (common point x_1), and for L and L_2 (common point x_2). It easily follows that these tangent lines share a common point \tilde{x} . Clearly, the points x_{12}, x_1, x_2 lie on three different generators of \mathcal{Q} . Projecting \mathcal{Q}, \tilde{x} and the three previous tangent lines from the vertex x of \mathcal{Q} , we now obtain three concurrent tangent lines to a plane conic. Consequently q is even. \square

Let x be a point of $\mathbf{H}(q)$ and let ξ_x be the solid spanned by the lines of $\mathbf{H}(q)$ through x ; the first part of this section implies that x^\perp is the set of all points of $\mathbf{H}(q)$ in ξ_x . From the

foregoing proof it follows that there is a unique point \tilde{x} contained in every plane spanned by an ideal line contained in ξ_x . This point is the nucleus of every conic corresponding to an ideal line in ξ_x .

Let's denote the point set of $\mathbf{H}(q)$ by \mathcal{P} .

Lemma 6 *The set of points $\tilde{\mathcal{P}} = \{\tilde{x} \mid x \in \mathcal{P}\}$ is the set of points of a 5-dimensional subspace \tilde{W} of $\mathbf{PG}(12, q)$. Also, with the lines of $\mathbf{H}(q)$ correspond the lines of a standard embedding of $\mathbf{H}(q)$ in \tilde{W} .*

Proof Let I be an imaginary line of $\mathbf{H}(q)$. The set $\tilde{I} = \{\tilde{x} \mid x \in I\}$ is the set of nuclei of conics contained in a line set \mathcal{L} of lines of $\mathbf{H}(q)$ joining points of one conic C_1 to another conic C_2 in two planes spanning a 5-dimensional subspace; remark that the elements of \mathcal{L} are maximal spaces of a Segre variety $\mathcal{S}_{1;2}$ of a line and a plane, see §25.5 of [3]. Hence \tilde{I} is the point set of a line of $\mathbf{PG}(12, q)$. Likewise, if C is an ideal line, then the point set $\tilde{C} = \{\tilde{x} \mid x \in C\}$ is a line in the nucleus plane of the conic Veronesean corresponding to the twin ideal plane of any ideal plane containing C . If the distance of the points x and y of \mathcal{P} (measured in the incidence graph of $\mathbf{H}(q)$) is either 4 or 6, then the foregoing implies that $\tilde{x} \neq \tilde{y}$; if x and y are collinear, but distinct, then, considering ideal lines through x and y in twin ideal planes, it follows that $\tilde{x} \neq \tilde{y}$. Hence the mapping $x \mapsto \tilde{x}$ is injective.

Now let x_1, x_2, x_3 be three collinear points of $\mathbf{H}(q)$, and let p_1, p_2 be two further points on the line x_1x_2 of $\mathbf{H}(q)$. In p_1^\perp , we can choose three concurrent ideal lines, say through the common point x , containing respectively x_1, x_2, x_3 , and one ideal line L not containing x to see that \tilde{x}_1, \tilde{x}_2 and \tilde{x}_3 lie in a common plane π_1 , which contains the “tildes” of at least $q^2 + 1$ points of p_1^\perp (namely, all points on ideal lines containing x and meeting L). Likewise, using p_2 , the points $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ lie in a common plane π_2 containing the “tildes” of at least $q^2 + 1$ points of p_2^\perp . It follows that $\pi_1 \neq \pi_2$ and that $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are contained in the intersection line of π_1 and π_2 . We easily deduce from this that, for any line M of $\mathbf{H}(q)$, the set $\{\tilde{x} \mid x \in M\}$ forms a line of $\mathbf{PG}(12, q)$. Hence $\tilde{\mathcal{P}}$ is the set of points of a 5-dimensional subspace \tilde{W} of $\mathbf{PG}(12, q)$. With the set of all lines of $\mathbf{H}(q)$ corresponds the set of all lines of a flatly embedded generalized hexagon, isomorphic to $\mathbf{H}(q)$, in \tilde{W} . The assertion now follows from the Main Result (ii) and (iii) of [6]. \square

Since for any two points x, y of $\mathbf{H}(q)$, $x \neq y$, with $d(x, y) \neq 4$, we can choose ideal lines in x^\perp and y^\perp in a such a way that they are contained in a common pair of twin ideal planes, we see that \tilde{x} is never contained in ξ_y ; for x, y in $\mathbf{H}(q)$, with $d(x, y) = 4$, consider an ideal plane containing x and y to see that \tilde{x} is not contained in ξ_y . It follows that the projection

of \mathcal{P} from \widetilde{W} is a flat embedding (say, α) of $\mathbf{H}(q)$ in some $\text{PG}(6, q)$. Hence this embedding is uniquely determined. It follows immediately that our original embedding is polarized. We now show that our original embedding is also unique. Indeed, first we remark that \widetilde{W} is uniquely determined by \mathfrak{P} , as it is the span of the nucleus planes of the twin conic Veroneseans in \mathfrak{P} . Now we choose a point x of $\mathbf{H}(q)$ outside \mathfrak{P} . Then the configuration $\mathfrak{P} \cup \{x\}$ is still projectively unique. Let y be any point of $\mathbf{H}(q)$, not belonging to $\mathfrak{P} \cup \{x\}$. By connectivity of the complement of \mathfrak{P} in $\mathbf{H}(q)$, we may assume that y is collinear with x . Let L be the line incident with x and y . Let z be the unique point on L belonging to \mathfrak{P} . Clearly z does not belong to $\mathfrak{P}_1 \cup \mathfrak{P}_2$ (since all lines of $\mathbf{H}(q)$ through points of this set are contained in \mathfrak{P}). Hence there are unique points $z_i \in \mathfrak{P}_i$, $i = 1, 2$ such that z, z_1, z_2 are collinear. We may suppose that $\alpha(y)$ is known (it is any point of $\alpha(\mathbf{H}(q)) \setminus \alpha(x)$ collinear in $\alpha(\mathbf{H}(q))$ with $\alpha(x)$). Then $\alpha(z), \alpha(z_1), \alpha(z_2)$ are uniquely defined. The point z is uniquely determined by the intersection of the line $z_1 z_2$ in $\text{PG}(12, q)$ with the hyperplane spanned by the points of $\alpha(\mathbf{H}(q))$ not opposite $\alpha(z)$ and \widetilde{W} . Hence also L is determined, and so is y , as the intersection of L with the hyperplane spanned by the points of $\alpha(\mathbf{H}(q))$ not opposite $\alpha(y)$ and \widetilde{W} .

The proof of our Main Result is now complete.

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