ON THE HALL-JANKO GRAPH WITH 100 VERTICES AND THE NEAR-OCTAGON OF ORDER (2, 4)

A. DE WISPELAERE AND H. VAN MALDEGHEM

Abstract. In this paper, we construct the Hall-Janko graph within the split Cayley hexagon $\mathsf{H}(4)$. Using this graph we then construct the near-octagon of order $(2, 4)$ as a subgeometry of the dual of $\mathsf{H}(4)$, with $\mathsf{J}_2 : 2$ as its automorphism group. These constructions are based on a lemma determining the possibilities for the structure of the intersection of two subhexagons of order 2 in $\mathsf{H}(4)$.

Keywords: Hall-Janko group, Hall-Janko graph, Generalized hexagon, Dickson's groups
MSC: 05C25, 51E12

1. Introduction

The sporadic simple Hall-Janko group J_2 is known to be the automorphism group of the Hall-Janko graph $\mathsf{HJ}(100)$, which is a strongly regular graph on 100 vertices and valency 36, acting rank 3 with subdegrees 36 and 63. Also, it is known that J_2 acts on a near-octagon $\mathsf{NO}(2, 4)$ of order $(2, 4)$. Moreover, the full automorphism group $\mathsf{J}_2 : 2$ of J_2 is a maximal subgroup of the Chevalley group $\mathsf{G}_2(4)$. The latter acts naturally on a generalized hexagon of order 4, the split Cayley hexagon $\mathsf{H}(4)$. Moreover, there exists a construction of the Hall-Janko graph using the split Cayley hexagon $\mathsf{H}(2)$ of order 2, which is also a subgeometry of both $\mathsf{H}(4)$ and $\mathsf{NO}(2, 4)$. This construction, however, is not homogeneous, in the sense that one vertex plays a special role in the construction and so it is not apparent that the thus constructed graph has even a transitive automorphism group. In the present paper, we explain these seemingly random coincidences by exhibiting the dual of $\mathsf{NO}(2, 4)$ inside $\mathsf{H}(4)$, as a subgeometry, and with stabilizer $\mathsf{J}_2 : 2$. Our construction will immediately imply that $\mathsf{NO}(2, 4)$ contains 100 subhexagons isomorphic to the dual of $\mathsf{H}(2)$, which form the vertex set of $\mathsf{HJ}(100)$. Adjacency will be given by intersecting in a subhexagon of order $(2, 1)$.

However, to achieve our goal, we must go in the opposite direction: first we find a set of 100 subhexagons of order 2 in $\mathsf{H}(4)$ on which

$\mathsf{J}_2 : 2$ acts rank 3, and hence which can be identified with the vertex set of $\mathsf{HJ}(100)$. We then characterize adjacency geometrically and show that the union of these subhexagons yields the dual of the unique near-octagon $\mathsf{NO}(2,4)$ order $(2,4)$. A central tool in all this is the determination of all possible structures of intersections of two subhexagons of order 2 in $\mathsf{H}(4)$, and this will take the largest part of the paper.

Our construction also completes the geometrical interpretations of all maximal subgroups of $\mathsf{G}_2(4)$. In the dissertation of the first author the corresponding problem for the group $\mathsf{G}_2(3)$ has been solved, and for $\mathsf{G}_2(4)$, only $\mathsf{J}_2 : 2$ remained unresolved. The main results of the present paper now show that all maximal subgroups of $\mathsf{G}_2(4)$ have an easy geometrical interpretation inside the generalized hexagon $\mathsf{H}(4)$. Indeed, they are either the stabilizer of a point, a line, a (Hermitian) spread, a distance-2 ovoid, or (the dual of) a sub-near-polygon (more exactly, a subhexagon of order $(1,4)$, a subhexagon of order 2, a non-thick subhexagons consisting of line regulus, its complement and all shortest paths between these, and the dual of sub-near-octagon of order (2,4)).

So our main result reads as follows.

Main Result. *The hexagon* $\mathsf{H}(4)$ *contains the dual of* $\mathsf{NO}(2,4)$ *as a subgeometry with stabilizer* $\mathsf{J}_2 : 2$*. The subhexagons isomorphic to the dual of* $\mathsf{H}(2)$ *contained in* $\mathsf{NO}(2,4)$ *form the vertex set of* $\mathsf{HJ}(100)$*, where two subhexagons are adjacent if they meet in a non-thick subhexagon.*

The paper is organized as follows. In Section 2, we introduce generalized hexagons, the split Cayley hexagons, some well-known results about these, the Hall-Janko graph, and some other notions that we will encounter in the course of our proofs. In Section 3 we prove our main lemma, namely, we determine the structure of the intersection of two arbitrary subhexagons of order 2 of $\mathsf{H}(4)$. In Section 4 we construct $\mathsf{HJ}(100)$ inside $\mathsf{H}(4)$, and in Section 5 we construct the near-octagon $\mathsf{NO}(2,4)$ inside the dual of $\mathsf{H}(4)$.

2. Preliminaries

2.1. **Generalized hexagons.** A *generalized hexagon* Γ, or briefly *hexagon*, is a bipartite graph with diameter 6 and girth 12. It is convenient to view one of the bipartitions of a generalized hexagon as point set and each element of the other bipartition as a line containing the points it is adjacent with. This way we obtain a *point-line geometry* (and adjacent elements x, y are then called *incident*, with symbols $x\mathtt{I}y$, or, if x is a point and y a line, $x \in y$), and the original graph is referred to as the *incidence graph*. Interchanging the names of "points" and "lines" gives us the *dual* of the geometry. If every vertex corresponding to a point has valency $t+1$ and every other vertex has valency $s+1$, then we say that the generalized hexagon *has order* (s,t), briefly *order* s if $s=t$. The definition implies that every vertex has valency at least 2 (this is a little exercise). If there are vertices of valency 2, then we call the hexagon *non-thick*; otherwise *thick*. If every vertex has valency at least 3, then the graph is automatically bi-regular (see 1.5.3 of [6]).

We will measure distances between the elements (points and lines) in the incidence graph, and we will use standard notation in graph theory such as $\Gamma(x)$ for the vertices adjacent with the vertex x. It will be convenient to denote vertices corresponding to points with lower case letters, and those corresponding to lines with upper case letter. We also use some specific terminology of incidence geometry, such as *collinear* points (which are points at distance 2), *concurrent* lines (lines at distance 2). The definition of a generalized hexagon implies that, given any two vertices a,b of Γ, either these elements are at distance 6 from one another, in which case we call them *opposite*, or there exists a unique shortest path (in the incidence graph) from a to b. If, in the latter case, $(a,\ldots,b_a,b)$ denotes this path, then the element b_a, also denoted by $\mathrm{proj}_b a$, is called the *projection of a onto b*. If two points x and y are at distance 4, then the unique point collinear to both is denoted by $x \bowtie y$. Every circuit of length 12 is called an *apartment*.

A *spread* of a generalized hexagon is a set of lines such that every line is at distance ≤ 2 from a unique element of the spread. It follows readily that all lines of a spread are at distance 6 from each other, and that there are $1+q^3$ elements in a spread if the generalized hexagon has order q. In other words, a spread is a perfect code in the incidence graph consisting of lines.

A *subhexagon* Γ' of a hexagon Γ is a subgeometry which is a generalized hexagon. A subgeometry Γ' is called *ideal* if $\Gamma'(x)=\Gamma(x)$, for every point x of Γ', and it is called *full* if $\Gamma'(L)=\Gamma(L)$ for every line L of Γ'.

Now let Γ be a generalized hexagon with point set $\mathcal{P}$ and line set $\mathcal{L}$, and suppose that Γ has order (s,t). A graph automorphism preserving $\mathcal{P}$ is called a *collineation*. If a collineation g of Γ fixes all elements incident with at least one element of a given path γ of length 4, then we call g a *root elation*, *γ-elation* or briefly an *elation*. We define an *axial elation* (also called a *axial collineation*) g as a collineation fixing all elements at distance at most 3 from a certain line L, which is then called the *axis* of g.

There are two kinds of — potential — elations, namely, a path of length 4 can start and end with a point, or with a line. The first type of elations will be referred to as a *point-elation*, the second as a *line-elation*. In a point-elation we shall speak of the *center* of this elation, by which we mean the point at distance 2 from both the beginning and ending point of the path γ.

If for all paths γ of length 4, the group of γ-elations has order s (if the middle element of γ is a point) or t (if the middle element of γ is a line), then we say that Γ is *Moufang*.

A *generalized homology* g is a collineation point-wise fixing an apartment, and also, for at least one element v of that apartment, fixing all elements incident with v.

Let us end these generalities by mentioning that generalized hexagons were introduced by Jacques Tits in his celebrated paper on trialities [5].

2.2. **Split Cayley hexagons.** The canonical examples of generalized hexagons are the *split Cayley hexagons* of order q arising from Dickson's simple groups $\mathsf{G}_2(q)$, with q any prime power. The split Cayley hexagons $\mathsf{H}(q)$, for any prime power q, can be constructed as follows (see Chapter 2 of [6], the construction is due to Jacques Tits [5]). Choose coordinates in the projective space $\mathsf{PG}(6,q)$ in such a way that the non-denerate quadric $\mathsf{Q}(6,q)$ of maximal Witt index has equation $X_0X_4+X_1X_5+X_2X_6=X_3^2$, and let the points of $\mathsf{H}(q)$ be all points of $\mathsf{Q}(6,q)$. The lines of $\mathsf{H}(q)$ are the lines on $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01},p_{02},\ldots,p_{56})$ satisfy the six relations $p_{12}=p_{34}, p_{56}=p_{03}, p_{45}=p_{23}, p_{01}=p_{36}, p_{02}=-p_{35}$ and $p_{46}=-p_{13}$.

To make the points and lines more concrete to calculate with, we will use the coordinatization of $\mathsf{H}(q)$ (see [6]). We apply it directly to our situation (q even), and obtain the labelling of points and lines of $\mathsf{H}(q)$ by i-tuples with entries in the field $\mathsf{GF}(q)$, and two 1-tuples (∞) and $[\infty]$, with $\infty\notin\mathsf{GF}(q)$, as given in Table 1.

While the assignment of coordinates might seem to make things a bit more complicated, the incidence relation becomes very easy. If we consider the 1-tuples (∞) and $[\infty]$ formally as 0-tuples (because they do not contain an element of $\mathsf{GF}(q)$), then a point, represented by an i-tuple, $0\le i\le 5$, is incident with a line, represented by a j-tuple, $0\le j\le 5$, if and only if either $|i-j|$=1 and the tuples coincide in the first $\min(i,j)$ coordinates, or $i=j=5$ and, with notation of Table 1,

POINTS

coordinates in $\mathsf{H}(q)$	coordinates in $\mathsf{PG}(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,k)$
(a,l,a')	$(l+aa',1,0,a,0,a^2,a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2+b'k)$
(a,l,a',l',a'')	$(al'+a'^2+a''l+aa'a'',a'',a,$ $a'+aa'',1,l+a^2a'',l'+a'a'')$

LINES

coordinates in $\mathsf{H}(q)$	coordinates in $\mathsf{PG}(6,q)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(l,1,0,a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(l+aa',1,0,a,0,a^2,a'),$ $(al'+a'^2,0,a,a',1,l,l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2+b'k),$ $(b'^2+k''b,b,0,b',1,k'',kk''+k')\rangle$

TABLE 1. Coordinatization of $\mathsf{H}(\mathsf{q})$, q even

$$\begin{cases} k'' = a^3k + l + a''a^2 + aa', \\ b' = a^2k + a', \\ k' = a^3k^2 + l' + kl + a^2a''k + a'a'' + aa''^2, \\ b = ak + a'', \end{cases}$$

or, equivalently,

$$\begin{cases} a'' = ak + b, \\ l' = a^3k^2 + k' + kk'' + a^2kb + bb' + ab^2, \\ a' = a^2k + b', \\ l = a^3k + k'' + ba^2 + ab'. \end{cases}$$

The group $\mathsf{G}_2(q)$ acts on $\Gamma := \mathsf{H}(q)$ in such a way that it turns $\mathsf{H}(q)$ into a Moufang hexagons. In particular, every line is the axis of a group of q axial elations. Also, $\mathsf{G}_2(q)$ acts transitively on the set of pairs of opposite points, and also the set of pairs of opposite lines. For every apartment Σ and every element $x \in \Sigma$, there is a group of q generalized homologies fixing $\Sigma \cup \Gamma(x)$.

We now introduce some more notation and terminology.

Let $\Gamma = \mathsf{H}(q)$ be the split Cayley generalized hexagon of order q. For an element u of Γ, we denote by $\Gamma_i(u)$ the set of points and lines of Γ at distance i from u. We fix the duality class of $\mathsf{H}(q)$ by requiring that all points of $\mathsf{H}(q)$ are *regular*, that is, for every three points x, y, z such that y and z are opposite x, the inequality $|\Gamma_i(x) \cap \Gamma_{6-i}(y) \cap \Gamma_{6-i}(z)| \geq 2$ implies $|\Gamma_i(x) \cap \Gamma_{6-i}(y) \cap \Gamma_{6-i}(z)| = q + 1$, for $i = 2, 3$ (see [4]).

The above regularity for $i = 3$ implies the following property (see [6], 1.9.17 and 2.4.15). Let x, y be two opposite points and let L, M be two (opposite) lines at distance 3 from both x, y. All points at distance 3 from both L, M are at distance 3 from all lines at distance 3 from both x, y. Hence we obtain a set $\mathcal{R}(x, y)$ of $q + 1$ points every member of which is at distance 3 from any member of a set $\mathcal{R}(L, M)$ of $q + 1$ lines. We call $\mathcal{R}(x, y)$ a *point regulus*, and $\mathcal{R}(L, M)$ a *line regulus*. Any regulus is determined by two of its elements. The two above reguli are said to be *complementary*, i.e. every element of one regulus is at distance 3 from every element of the other regulus. Every regulus has a unique complementary regulus. Two complementary reguli together with all shortest paths joining an element from one regulus with an element of the other form a non-thick subhexagon. We call two reguli *opposite* if every element of the first regulus is opposite every element of the second one.

The *generators* of $\mathsf{Q}(6, q)$ (i.e., the subspaces of $\mathsf{PG}(6, q)$ of maximal dimension contained in $\mathsf{Q}(6, q)$) are planes. Such a plane can either contain $q + 1$ hexagon lines through a point x or no hexagon lines at all. In the first case we call the plane a *hexagon plane*, and denote it by π_x. In the second case we call the plane an *ideal plane*. Note that all points of an ideal plane are mutually at distance 4 in the hexagon. The lines of an ideal plane (which are lines on $\mathsf{Q}(6, q)$) will be called *ideal lines*. Given an ideal line I, there exists a unique point p, which we will call the *focus* of I, that is collinear to all points on I. We will use the convention of denoting an ideal line with focus point p by I_p, while denoting the ideal line on two points x and y by I_{xy} (note that the line I_p is not uniquely determined by its notation, while I_{xy} is).

Let $\mathcal{H}$ be a hyperplane in $\mathsf{PG}(6, q)$. Then either $\mathcal{H}$ is a tangent hyperplane of $\mathsf{Q}(6, q)$ at some point x and the points of $\mathsf{H}(q)$ in $\mathcal{H}$ are the points not opposite this particular point x of $\mathsf{H}(q)$; or $\mathcal{H}$ intersects $\mathsf{Q}(6, q)$ in a non-degenerate elliptic quadric and the lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of a spread, called a *Hermitian* or *classical* spread of $\mathsf{H}(q)$; or $\mathcal{H}$ intersects $\mathsf{Q}(6, q)$ in a non-degenerate hyperbolic quadric and the lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of an ideal subhexagon $\mathsf{H}(1, q)$ of $\mathsf{H}(q)$ of order $(1, q)$, the points of which are those points of $\mathsf{H}(q)$ that are incident with exactly $q + 1$ lines of $\mathsf{H}(q)$ lying in $\mathcal{H}$ — and there are exactly $2(1+q+q^2)$ of them. This subhexagon is uniquely determined by any two opposite points x, y it contains and will be denoted by $\Delta(x, y)$. If, in $\Delta(x, y)$, collinearity of points is called adjacency, then we obtain the incidence graph of the Desarguesian projective plane $\mathsf{PG}(2, q)$ of order q. The lines of $\Delta(x, y)$ can be identified with the incident point-line pairs of that projective plane. The q^2+q+1 points of $\Delta(x, y)$ belonging to the same type of elements of $\mathsf{PG}(2, q)$ — points or lines — are the points of an ideal plane. Hence $\mathcal{H} \cap \mathsf{Q}(6, q)$ contains two projective planes π and π', the points of which are precisely the points of $\Delta(x, y)$, and which we call the *ideal twin planes* of $\mathcal{H}$ or of $\Delta(x, y)$.

Concerning Hermitian spreads, we mention the following property. Let L, M be 2 lines of the spread $\mathcal{S}$, then every line of $\mathsf{H}(q)$ in $\mathcal{R}(L, M)$ is contained in $\mathcal{S}$. This property implies that, since a line regulus spans a 3-space of $\mathsf{PG}(6, q)$, three of its lines that are not contained in a common regulus, uniquely determine any Hermitian spread in $\mathsf{H}(q)$. The stabilizer of every Hermitian spread inside $\mathsf{G}_2(q)$ is a group isomorphic to $\mathsf{SU}_3(q) : 2$, and all such subgroups are conjugate. It follows that there are precisely $\frac{1}{2}q^3(q^3-1)$ Hermitian spreads.

However, if q is equal to 2, then two opposite lines induce a unique Hermitian spread, as the following Fact states.

Fact 1. *If L and M are two opposite lines of* $\mathsf{H}(2)$*, then there exists a unique Hermitian spread $\mathcal{S}$ containing these two lines.*

Proof. By the transitivity on the set of pairs of opposite lines, there are a constant number C of Hermitian spreads containing two given opposite lines. Since, by the above, there are 28 Hermitian spreads, and since there are 63 lines and 32 lines opposite a given line, and since a Hermitian spread contains 9 lines, we obtain $C = \frac{28 \cdot 9 \cdot 8}{63 \cdot 32}$. □

As an immediate consequence of Fact 1, we have the following Corollary.

Corollary 2. *Any two Hermitian spreads of* $\mathsf{H}(2)$ *intersect in an unique line.*

Proof. Given a spread $\mathcal{S}$ and a line L in $\mathcal{S}$, there are $32-8$ ways to choose a line opposite L and not in $\mathcal{S}$. Hence, since by Fact 1 two opposite lines determine a spread, there are 3 spreads on L distinct of $\mathcal{S}$. In other words, there are $9 \cdot 3$ spreads which intersect $\mathcal{S}$ in a unique line. These 27 spreads, together with $\mathcal{S}$ itself, add up to a total of 28 spreads, which is the total number of spreads in $\mathsf{H}(2)$, and we are done. □

2.3. **Hall-Janko graph.** There is a strongly regular graph G with parameters $v = 100$, $k = 36$, $\lambda = 14$, $\mu = 12$. Uniqueness of the graph given the parameters only, is unknown. The graph was constructed by Hall and Janko. The full group of automorphisms is $\mathsf{J}_2 : 2$, acting rank 3, with point stabilizer $\mathsf{U}_3(3).2 \equiv \mathsf{G}_2(2)$.

One of the constructions of this graph starts with the dual H of the split Cayley hexagon of order 2. The vertices are an element ∞, the 36 subhexagons of order $(2, 1)$ of H, and the 63 points of H. The vertex ∞ is adjacent with every subhexagons and with no other vertex. Two subhexagons are adjacent when they have 4 lines in common, two points are adjacent when they have distance 4 and a subhexagon is adjacent to its points.

Dualizing this construction we obtain the Hall Janko graph within $\mathsf{H}(2)$.

Observation 3. *The Hall Janko graph on 100 vertices is constructed as a strongly regular graph on* $1+36+63$ *vertices, where the* 36 *represents the set of ideal non-thick subhexagons of order* $(1, 2)$ *of* $\mathsf{H}(2)$*, adjacent when they intersect in a point and all points collinear to this point (*4 *points and* 9 *lines), and the* 63 *corresponds to the set of* 63 *lines of* $\mathsf{H}(2)$*, adjacent when they have distance* 4*, and a subhexagon is adjacent to the lines in it.*

This construction is based on the point stabilizer, and hence is a non-homogeneous construction. We aim at a homogeneous construction starting from $\mathsf{J}_2 \leq \mathsf{G}_2(4)$. Moreover, the above construction will be a consequence of our results, which will be proved independently of the above construction.

2.4. **Near-polygons.** A *near-polygon* is a connected partial linear space of finite diameter satisfying the following property.

(NP) For each point p and every line L, there exists a unique point q on L nearest to p (with respect to the distance in the incidence graph)

One easily shows that the diameter is always even. If $2d$ is the diameter of the incidence graph, then the near-polygon is also called a *near-*$2d$*-gon.* For $d = 4$, we obtain a near-octagon.

3. A theorem on intersections of subhexagons

3.1. **Statement of the result.** In this section, we determine the intersection of two subhexagons of order 2 of $\mathsf{H}(4)$. We will prove the following theorem.

Theorem 4. *Let* Γ' *be an order* 2 *subhexagon of* $\mathsf{H}(4)$*. Then Table* 2 *captures*

- A *notation of a possible intersection* S *of* Γ' *with another order* 2 *subhexagon* Γ''*;*
- B *a description of* S*;*
- C *the number* χ_S *of order* 2 *subhexagons* Γ'' *which intersect* Γ' *accordingly and finally*
- D *the number of such configurations within* Γ'*.*

The lower index l *in the labelling* S_l^p *of these configurations denotes the number of lines they contain, while the upper index* p *denotes the number of points.*

A	*B*	*C*	*D*
S_{63}^{63}	Γ'	*1*	*1*
S_{21}^{14}	*weak subhexagon of order* $(1,2)$	*72*	*36*
S_9^7	*lines concurrent with an ideal line* I_p*, together with all points in* π_p	*252*	*252*
S_8^9	*points on two opposite lines, together with the three connecting paths*	*2016*	*1008*
S_7^{15}	*lines concurrent to a given line* L*, together with all incident points*	*63*	*63*
S_7^3	*lines concurrent to a given line* L	*126*	*63*
S_5^6	*lines incident to two collinear points,* x *and* y *together with the points on an ideal line on* x *and likewise on* y	*1512*	*756*
S_4^3	*path of length* 6*, starting with a line*	*6048*	*3024*
S_3^3	*all lines on a point, together with all points incident with one of these lines*	*378*	*189*
S_1^1	*a point and a single incident line*	*3024*	*189*
S_1^2	*two collinear points*	*3024*	*189*
S_1^3	*3 points on a line*	*252*	*63*
S_0^0	*the empty set*	*4032*	*1*

TABLE 2. Intersections of two order 2 subhexagons in $\mathsf{H}(4)$

3.2. **Preliminary lemmas.** Before embarking on the proof of this theorem, we state a known fact and separate some lemmas from the proof.

Fact 5 (1.8.5 of [6]). *Let Γ' be a subhexagon of order (s, t), $s, t \geq 1$, of a generalized hexagon Γ. Then Γ' is uniquely determined by an apartment Σ, and by two neighbor sets $\Gamma'(p)$ and $\Gamma'(L)$ of Γ' with p incident with L, and both belonging to Σ. In particular, if $\Gamma'(p) = \Gamma(p)$ and $\Gamma'(L) = \Gamma(L)$, then Γ' coincides with Γ.*

If Γ is a split Cayley generalized hexagon of even order, we can say more.

Lemma 6. *Let Γ' be a subhexagon of order 2 of a split Cayley generalized hexagon Γ of even order. Then Γ' is uniquely determined by an apartment Σ and two neighbor sets $\Gamma'(p)$ and $\Gamma'(L)$ of Γ', with p and L elements of Σ. In particular, if $\Gamma'(p) = \Gamma(p)$ and $\Gamma'(L) = \Gamma(L)$, then Γ' coincides with Γ.*

Proof. First of all, if p is incident with L, then this lemma is equivalent to Fact 5. Secondly, if L is a line of Σ at distance 5 from p, then the projections of the points in $\Gamma'(L) \setminus \Sigma$ onto one of the lines on p, that are opposite L, gives us a point row and line pencil as in Fact 5 to conclude the lemma. Hence we may assume that p and L are distance 3 apart. Without loss of generality we may coordinatize Γ such that the apartment Σ contains the points $(\infty), (0), (0,0), (0,0,0), (0,0,0,0)$ and $(0,0,0,0,0)$, with $p = (\infty)$ and $L = [0,0]$. We show that there is a unique subhexagon Γ' of order 2 containing the point $(0,0,1)$ of $\Gamma(L)$ and the line $[1] \in \Gamma(p)$. There is certainly at least one such hexagon, by restricting the coordinates to $\mathsf{GF}(2)$. Now let Γ' be such a subhexagon and let (a) be a point of Γ' on $[\infty]$, with $a \neq 0$. We show that $a = 1$.

In the following we will use that, if three concurrent lines L_1, L_2, L_3 belong to Γ', and two points $x_1 \in L_1$ and $x_2 \in L_2$, with $x_i \neq L_1 \cap L_2$, then also $I_{x_1x_2} \cap L_3$ belongs to Γ'. This follows readily from the regularity of Γ'.

Since (a) belongs to Γ', so do the point $(a,0,0,0,0)$ and the line $[a,0,0,0]$ (projection onto $[0,0,0,0]$). The unique line on $(1,0)$ (which belong to Γ') at distance 4 from $[a,0,0,0]$ is, after an easy calculation, equal to $[1,0,a^3]$. Also, projecting $(a,0,0,0,0)$ onto the latter yields $(1,0,a^3,a^2) \in \Gamma'$. Now we find $(1,0,0,a^2)$ as the intersection point of $[1,0,0]$ (which belongs to Γ') and the ideal line on (∞) and $(1,0,a^3,a^2)$. But there are only three points of Γ' on $[1,0,0]$, and these are $(1,0)$, $(1,0,0)$ and $(1,0,1)$ (as projection of $(0,0,1)$). Consequently, as $a \neq 0$ by assumption, $a^2 = 1$ and so $a = 1$ is uniquely determined. □

Concerning the intersection of two order 2 subhexagons in $\mathsf{H}(4)$, we have the following three lemmas.

Lemma 7 (Regulus Condition)**.** *Let Γ' and Γ'' be two order 2 subhexagons of $\Gamma \cong \mathsf{H}(4)$. Let x, y, z be three points of $\Gamma' \cap \Gamma''$ that are on an ideal line I.*

(i) *If $\Gamma'(x) = \Gamma''(x)$ and $|\Gamma'(y) \cap \Gamma''(y)| > 1$, then both $\Gamma'(y) = \Gamma''(y)$ and $\Gamma'(z) = \Gamma''(z)$.*
(ii) *If $|\Gamma'(x) \cap \Gamma''(x)| = |\Gamma'(y) \cap \Gamma''(y)| = 1$, then $\Gamma'(z) = \Gamma''(z)$.*
(iii) *If $|\Gamma'(x) \cap \Gamma''(x)| = |\Gamma'(y) \cap \Gamma''(y)| = 2$, then $|\Gamma'(z) \cap \Gamma''(z)| = 2$ as well.*

Proof. Let p denote the focus of the ideal line I. Note that if two opposite lines L, M as well as three points a, b, c, with $a \in L$, $b \in M$ and a, b, c on an ideal line of Γ, belong to the intersection of two order 2 subhexagons, then so does the unique line through c of the line regulus through L, M. We will refer to the previous argument as the *regulus property.* First of all, a repeated use of this regulus property proves (i).

Secondly, let X_i, Y_i and Z_i, with $i = 1, \ldots, 4$, denote the four lines of $\Gamma \setminus \Gamma_1(p)$ on x, y and z, respectively, and suppose the first two indices indicate the lines of Γ'. By the regulus property X_1, X_2 and Y_1, Y_2 determine the lines Z_1, Z_2 of Γ' on z. However, this implies that the regulus on Z_1 and X_3 ought to contain one of the two lines Y_3, Y_4, as does the regulus on Z_1 and X_4. Consequently Z_1, and in the same way Z_2, belongs to Γ'', and we are done for (ii).

And finally, the regulus property yields, under the assumptions of (iii), that $|\Gamma'(z) \cap \Gamma''(z)| \geq 2$. However, if $\Gamma'(z)$ were equal to $\Gamma''(z)$, then (i) of this lemma would lead to a contradiction. □

Lemma 8. *If two order 2 subhexagons of $\mathsf{H}(4)$ share a line L and all of the lines concurrent to L, then either all of the points at distance 3 from L belong to the intersection or none of them do.*

Proof. Let Γ' and Γ'' denote two such order 2 subhexagons of $\mathsf{H}(4)$ — and we assume that $\Gamma' \neq \Gamma''$ — and denote the points on L by l_i, $i \in \{1, 2, 3\}$. Considering regularity and the ideal lines of $\mathsf{H}(4)$, it is clear that, since Γ' and Γ'' have L and all of its points in common, the points l_i are collinear with exactly 4 or no additional common points.

Furthermore we claim that, if one of the points on L is collinear to 4 additional common points, then no line at distance 4 from L on one of these particular points is not contained in the intersection of our two subhexagons. Suppose, by way of contradiction, that l_1 is collinear to 4 additional common points, one of which we denote by a, and that M, with $d(L, M) = 4$, is a common line on a. Denote the unique third point of I_{al_i} in Γ' by a_i, for $i = 1, 2, 3$. Then, by the previous lemma, all lines on a_2 and a_3, and consequently also those on a and a_1, are common to Γ' and Γ''. Hence, Γ' and Γ'' share all lines at distance at most 3 from l_1. If we coordinatize this situation such that $l_1 = (\infty)$, and Γ' is obtained from this coordinatization by restricting coordinates to $\mathsf{GF}(2)$, then $[0, 0, 0, x, 0]$, with $x \in \mathsf{GF}(4)\setminus\mathsf{GF}(2)$, belongs to $\Gamma''\setminus\Gamma'$, as it is the projection of the line $[0, 0]$ onto the point $(0, 0, 0, x)$ (if a point on $[0, 0, 0] = M$ belonged to both hexagons then by Fact 5 these two would coincide). By assumption the projection of the line $[0, 0, 0, x, 0]$ onto the point (1) has to be a line in Γ'. However, given the incidence relations of $\mathsf{H}(4)$ as listed in Section 2.2, any point with coordinates $(1, l, a', l', a'')$ on $[0, 0, 0, x, 0]$ has $l = x \notin \mathsf{GF}(2)$ in its second entry, a contradiction. Hence the claim.

To complete the proof of the lemma we now suppose, again by way of contradiction, that in the intersection of Γ' with Γ'', l_1 is collinear to 6 and l_2 is collinear to only 2 common points. Once again, we coordinatize Γ' with entries in $\mathsf{GF}(2)$ in such a way that $L = [\infty]$, $l_1 = (\infty)$ and $l_2 = (0)$. By the previous claim, there are no common lines on $(0, 0)$. Hence $[0, 0, x]$, with $x \in \mathsf{GF}(4) \setminus \mathsf{GF}(2)$, belongs to Γ'' and not to Γ'. The projection of $(0, 0, x)$, which by assumption belongs to $\Gamma'' \setminus \Gamma'$, onto this particular line implies that $[0, 0, x, x, 0]$ belongs to Γ''. However, since $[1, x]$ is the unique line on the point (1) at distance 4 from $[0, 0, x, x, 0]$, this yields a contradiction (as all lines on the point (1) should belong to $\Gamma' \cap \Gamma''$) and we are done. □

Lemma 9. *Let Γ' and Γ'' be two order 2 subhexagons of $\mathsf{H}(4)$. Suppose Γ' and Γ'' share two collinear points x and y. If x is incident with exactly 2 common lines, then so is y, while if x is incident with exactly 3 common lines, then y is either incident with a unique common line or y is incident with 3 of them.*

Proof. We offer a group theoretic proof, using elations and generalized homologies. In fact, we will only use the following facts:

- (*i*) $\mathsf{H}(4)$ is Moufang,
- (*ii*) elations have order two,
- (*iii*) if a generalized homology fixes an apartment Σ and all lines trough one point of Σ, then it fixes all lines through all points of Σ (follows from the explicit description of generalized homologies as given in Proposition 4.5.11 in [6], noting that $x^3 = 1$ for every $x \in \mathsf{GF}(4) \setminus \{0\}$),
- (*iv*) $\mathsf{G}_2(4)$ acts transitively on the subhexagons of order 2, and the stabilizer of such subhexagon induces $\mathsf{G}_2(2)$ in it.

Label the lines of Γ' on x by X_0, X_1, X_2 and those on y by Y_0, Y_1, Y_2, with $X_0 = Y_0 = xy$.

Suppose first that Γ'' contains X_1, X_2 and Y_1, but not Y_2. Using (iv) above, we see that there exists $g \in \mathsf{G}_2(4)$ fixing X_i, $i = 0, 1, 2$, and Y_1, and mapping X_2 to Y_2. Composing g with suitable elations, we obtain a generalized homology $h \in \mathsf{G}_2(4)$ fixing X_i, $i = 0, 1, 2$, and Y_1, and mapping X_2 to Y_2. This contradicts (ii) above.

Now suppose that Γ'' contains Y_1, but not X_1, X_2, Y_2. There is an elation e fixing all elements of $\Gamma(y) \cup \Gamma(X_0)$ and mapping X_1 onto a line different from X_1 and X_2. Since e is involutive, X_2 is also mapped onto a line different from X_1 and X_2. It follows that the image of Γ'' under e shares X_1, X_2 and Y_1, but not Y_2 with Γ', contradicting the previous paragraph.

The lemma is proved. □

3.3. **Proof of Theorem 4.** To prove Theorem 4 we use the following strategy. For each type of intersection, we count the number of such configurations in a fixed subhexagon Γ' of order 2, and we determine a lower bound on the number of subhexagons of order 2 intersecting Γ' in that particular configurations by first exhibiting that number of subhexagons containing that configuration, and then proving that each such subhexagon indeed intersects Γ' exactly in that configuration. If every one of these lower bounds is as in Table 2, then we have obtained all possible subhexagons of order 2 and the lower bounds all become exact numbers. Moreover, no other intersection configuration is possible. This follows from the fact that the sum of all numbers in Column C of Table 2 equals the total number of order 2 subhexagons of $\mathsf{H}(4)$, which equals 20800, by the orbit counting formula applied to a fixed subhexagon Γ' of order 2 and $\mathsf{G}_2(4).2$ (the stabilizer of Γ' is isomorphic to $\mathsf{G}_2(2).2$).

We will deal with the various configurations in the order as they are written down in Table 2 (we hereby skip the trivial configuration S_{63}^{63}).

Configuration S_{21}^{14}. First of all, let Δ be one of the $\frac{63 \cdot 32}{14 \cdot 4}$ non-thick subhexagons of order $(1, 2)$ contained in Γ'. By Fact 5, the choice of an external point (that is, a point not in Γ') on any line in Δ, determines another order 2 subhexagon. We thus obtain two additional order 2 subhexagons. A repeated use of Lemma 9 shows that both order 2 subhexagons in fact intersect Γ' exactly in all points and lines of Δ.

Configuration S_9^7. The number of configuration of type S_9^7 in Γ' is $63 \cdot 4$, as we can choose π_p in 63 ways and π_p contains 4 ideal lines. The lines concurrent to such an ideal line, say I_p, can contain no further common points as otherwise, by Fact 5, the thus defined subhexagon Γ'' would coincide with Γ'. Denote the lines of Γ' incident with p by M_0, M_2, M_4 and those at distance 3 from p that are concurrent to both I_p and M_i by L_i and L_{i+1}, for $i = 0, 2, 4$. Let Σ be the apartment containing the point p, an external point on L_0 and the line L_2. By Fact 5, this apartment together with the points of $\Gamma'(M_0)$ and the line M_4 uniquely determines an order 2 subhexagon Γ''. We now claim that Γ'' intersects Γ' in the points of π_p and all lines incident with I_p. Indeed, by Lemma 9, the lines on both points in $I_p \cap \Sigma$ are common lines and hence, by the regulus property, so are those on the third point of this ideal line. Moreover, none of these lines at distance 3 from p contain intersection points. Finally, if a point p' distinct from p in $\pi_p \setminus I_p$ would be on another common line, next to M_i, then, using Lemma 9, we first obtain $\Gamma'(p') = \Gamma''(p')$, and secondly, we obtain a contradiction to Lemma 8.

Configuration S_8^9. Inside Γ' we can choose two opposite lines, say L and M, in $63 \cdot 16$ ways. By definition of this particular configuration, Γ'' has to be an order 2 subhexagon on L and M containing all three points of Γ' on these lines. As the paths from one to the other are fixed, Fact 5 states that an additional line on one of the points on L or M will determine Γ'' completely. Not wanting Γ'' to be equal to Γ', we have two remaining choices for such a line, giving a total of $63 \cdot 16 \cdot 2$ order 2 subhexagons that intersect Γ' in S_8^9, as we will show. The fact that the points in this intersection are on no further common lines, is an easy consequence of Lemma 9. If, on the other hand, one of the 6 lines connecting L to M would contain 3 intersection points, then one obtains an order $(2, 1)$ subhexagon within $\Gamma' \cong \mathsf{H}(2)$, a contradiction and we are done.

Configuration S_7^{15}. The number of corresponding configurations is clearly equal to the number of lines L of Γ'. It is clear that the orbit of Γ' under the group of axial collineations with axis L has size 2, and that the subhexagon in this orbit distinct from Γ' meets Γ' exactly the elements of Γ' at distance at most 3 from L. So we have 63 configurations and 63 order 2 subhexagons.

Configuration S_7^3. There are again 63 ways to choose the line L and all lines concurrent with it in Γ'. Now let Σ be an apartment containing L, and let x, y be the points of Σ on L. Let g and h be γ-elations, where γ is contained in Σ and has middle point x and y, respectively, and which do not preserve Γ'. If z is the third point on L in Γ', then, considering $(\Gamma')^g$, it follows from Lemma 8 that both g and h map

$\Gamma'(z) \setminus \{L\}$ onto $\Gamma(z) \setminus \Gamma'(z)$. Hence gh preserves $\Gamma'_2(L)$ and maps Γ' onto a subhexagon Γ'' with the right intersection. By the previous configuration, there is an order 2 subhexagon Γ''' intersecting Γ''' in $\{L\} \cup \Gamma''_1(L) \cup \Gamma''_2(L) \cup \Gamma''_3(L)$. Hence also Γ''' has the right intersection with Γ'.

Configuration S_5^6. Configuration S_5^6 is determined by choosing inside Γ' a line L, two of its points, say x and y, an ideal line on x in π_y, say I_y, and likewise on y in π_x, say I_x. Hence there are $63 \cdot 3 \cdot 2^2$ such configurations in Γ'. As we want Γ'' to contain x, y, $\Gamma'(I_x)$ and $\Gamma'(I_y)$ and no more, the choice of a point on L that is not in Γ', fixes Γ'' completely. Let M and N be the lines in the defining apartment Σ, concurrent with I_y and I_x and at distance 3 from x and y, respectively. Obviously, the points of Γ'' in $x^\perp$ and $y^\perp$ that are not on I_x and I_y do not belong to Γ' (since the third point of Γ'' on L is no point of Γ' either). Furthermore, as a direct consequence of first Lemma 9 and then Lemma 7 the lines of Γ'' incident with the points on $I_x \setminus \{y\}$ and $I_y \setminus \{x\}$ are no lines of Γ'. In other words, Γ' and Γ'' indeed share only those 5 lines and 6 points and one can choose Γ'' in $63 \cdot 3 \cdot 2^3$ ways.

Configuration S_4^3. One readily counts the number of length 6 paths, starting and ending with a line, within Γ' to be equal to $63 \cdot 3 \cdot 2^4$. Let γ be such a path and denote the end lines by L and M. As we want to define Γ'' in such a way that the intersection with Γ' contains only those points and lines that are in γ, we have no further choice in the points on L and M. Hence, by Fact 5, Γ'' is determined by the choice of a third line, which ought to be in $\mathsf{H}(4) \setminus \Gamma'$, on the unique common point on one of these two lines. As such, Lemma 9 leads to the fact that the 3 points in this path are on precisely two common lines, that is, they are on the lines of the path and on no more. To prove that the intersection of Γ' with Γ'' equals γ, it now suffices to show that the two lines of γ that are concurrent to L and M contain only the points of the path itself. However, if one of these lines were to contain 3 intersection points, then, looking in the hexagon plane spanned by the first 2 or last two lines of the path γ, this would imply that two projective planes of order 2 inside $\mathsf{PG}(2,4)$ intersect in exactly two lines, one of which contains three common points and one of which contains a unique intersection point, and we claim that this is impossible. Indeed, coordinatize $\mathsf{PG}(2,4)$ such that those two common lines are the lines $X_0 = 0$ and $X_1 = 0$. Suppose that the points $(0,0,1)$, $(0,1,0)$ and $(0,1,1)$ belong to both order 2 subplanes, while $(1,0,0)$ and $(1,0,1)$, and $(1,0,x)$ and $(1,0,x^2)$ belong to F_1 and F_2, respectively. Since

$(1,1,1)$ belongs to F_1 and $(1,1,x^2)$ belongs to F_2, the line on $(0,0,1)$ and $(1,1,1)$ belongs to the intersection as well, a contradiction and the claim follows.

Since we had two choices for the defining line of Γ'', there are $63\cdot 3\cdot 2^5$ order 2 subhexagons in Γ which intersect Γ' in such a path of length 6.

Configuration S_3^3. Configuration S_3^3 is constructed by taking a line, all of its points, and all lines incident with one of these points, which can be done in $63\cdot 3$ ways. Denote the line by L, its special point by l_0 and the lines on l_0 by M and N. Inside the still-to-be-constructed Γ'' we now know all lines concurrent to L, that is, M and N on l_0 and those that are not in Γ' on the other two points of L. Moreover, we know that the points on M and N ought to be points off Γ'. By Lemma 8 the points on all of the lines concurrent to L are thereby fixed. Moreover, the proof of Configuration S_7^{15} shows that we have two choices for such a subhexagon of order 2, and clearly these two subhexagons meet Γ' in the right configuration (otherwise the projection of an additional common element onto either L or l_0 produces a common element either concurrent with L or collinear with l_0 distinct from the ones inside the configuration).

Configuration S_1^1. Obviously, there are $63\cdot 3$ ways to choose an incident point-line pair (x,L). The lines on x and the points on L are fixed as those that are not contained in Γ'. Denote these points and lines by l_i and M_i, with $i=0,1,2$ and $M_0=L$ and $l_0=x$. To fix Γ'' in terms of Fact 5, we need a point m_1 on M_1, a line N_1 on l_1, a line on m_1 and a point on N_1. Summarized this yields $63\cdot 3\cdot \frac{4^4}{2^4}$ such subhexagons Γ'' and, by construction, it may be clear that Γ'' shares no further points and lines with Γ'.

Configuration S_1^2. This configuration is completely similar to the previous one. Inside Γ' we consider a line L, two of its points l_0 and l_1 and fix the lines on these points as those that are not contained in Γ', say M_0, M_1 and N_0, N_1. Now Γ'' is determined by a point m_0 on M_0, a line on m_0, a point on N_0 and finally, a point on L, that is not in Γ'. Considering the lines through l_0, Lemma 9 yields that there are no further common lines on l_1. In conclusion, there are $63\cdot 3\cdot \frac{4^4}{2^4}$ subhexagons that intersect Γ' in a line and two of its points.

Configuration S_1^3. Let L denote the line of Γ'. The lines of Γ'' concurrent with L are determined. And since these lines can be obtained from $\Gamma_2'(L)$ by applying the composition of a suitable point elation (with center a point x on L and fixing the lines through a point $y\neq x$ on L)

and an axial elation (with axis in Γ' concurrent with L through the point z of Γ' on L distinct from x, y), there is at least one subhexagon with the desired intersection. The configurations S_7^3 and S_7^{15} show that there are at least 4 such. Hence, in total, there are $63 \cdot 4$ subhexagons intersecting Γ' in a configuration of type S_1^3.

Configuration S_0^0. This is probably the most involved case, as the empty set gives a single configuration in Γ', but there are many more subhexagons intersecting in the empty set. Our approach is based on the construction of Hermitian spreads in [7].

Step 1: Construction of Γ''. Consider a Hermitian spread $\mathcal{S}$ of Γ' and let L and M be two of its lines. If p, p' and q, q' are the points of $\mathsf{H}(4) \setminus \Gamma'$ on L and M, respectively, and suppose p is opposite q then, by [7], $\Delta(p, q)$ is a non-thick subhexagon of order $(1, 4)$ of $\mathsf{H}(4)$ which intersects Γ' precisely in the lines of $\mathcal{S}$. Hence, none of the points of Γ' is in one of the two ideal twin planes, π and π', of $\Delta(p, q)$. In other words, every line of Γ' either belongs to $\mathcal{S}$ or intersects the lines of $\Delta(p, q)$ in a point off the weak subhexagon. Now suppose g is a group element of the automorphism group of $\Delta(p, q)$ that maps $\mathcal{S}$ to a set of 9 lines which is disjoint from $\mathcal{S}$ and denote the image of $\mathcal{S}$ under g by $\mathcal{S}'$. We then claim that $\mathcal{S}'$ uniquely determines an order 2 subhexagon of $\mathsf{H}(4)$, say Γ'', which shares no points nor lines with Γ'. Indeed, let L' and M' be two lines of $\mathcal{S}'$ and define Γ'' on these two lines by taking all points on these lines that are not contained in π nor in π'. Note that there is a unique third line of $\mathcal{R}(L', M')$ in $\mathcal{S}'$ and hence, by Lemma 6, the subhexagon Γ'' is fixed and the claim is proven. Moreover, again by [7], the lines of Γ'' in $\Delta(p, q)$ are just those of $\mathcal{S}'$, and all lines of Γ'' either belong to $\mathcal{S}'$ or are concurrent to one of these lines in a point off $\pi \cup \pi'$. Hence it may be clear that Γ' and Γ'' share no lines and consequently also no points (otherwise two lines of $\Delta(p, q)$, one in $\mathcal{S}$ and one in $\mathcal{S}'$, would intersect in a point outside $\pi \cup \pi'$, a contradiction).

Step 2.a: Upper bound on the cardinality. We will now count the number of subhexagons, disjoint from Γ', that are obtained in this way. First of all, within a subhexagon of order $(1, 4)$, there are $\frac{105 \cdot 64 \cdot 3}{9 \cdot 8} = 280$ such sets of 9 lines, which for sake of simplicity we will call *spreads*. Moreover, one easily counts that there are 24 spreads on a single line and 3 of them on a pair of opposite lines. Hence, since Fact 1 states that inside an order 2 hexagon two opposite lines uniquely determine a spread, there are $280 - 1 - 2 \cdot \binom{9}{2} - 9 \cdot (23 - 2 \cdot 8) = 144$ spreads that are disjoint from the given spread $\mathcal{S}$ in $\Delta(p, q)$. Therefore, the number of order 2 subhexagons disjoint from Γ' that arise in this way is at most equal to $28 \cdot 144 = 4032$, as there are 28 spreads within Γ'.

Step 2.b: Lower bound on the cardinality. We now claim that this number is also a lower bound for the cardinality of this set (in other words, no two subhexagons constructed above coincide), which will complete the proof of Theorem 4. Indeed, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two spreads in Γ' and denote the corresponding non-thick ideal subhexagons by Δ_1 and Δ_2. Each of these two spreads now determines a set of 144 order 2 subhexagons disjoint from Γ'. We denote the respective sets by Ω_1 and Ω_2. Our aim is now to prove that the intersection of these two sets is the empty set.

It suffices to show that if $\mathcal{S}_1'$ in Δ_1 determines Γ'', there is no $\mathcal{S}_2'$ in Δ_2 which is disjoint from $\mathcal{S}_2$ and determines Γ''. Suppose by way of contradiction that there exists such an $\mathcal{S}_2'$ in Δ_2. If g denotes the element of $\mathrm{Aut}(\Gamma')$ that maps $\mathcal{S}_1$ onto $\mathcal{S}_2$ and consequently also Δ_1 onto Δ_2, then obviously $\mathcal{S}_2'^{g^{-1}}$ is a spread in Δ_1 which is disjoint from $\mathcal{S}_1$. We will denote this spread by $\mathcal{S}''$ and we will show that the lines of $\mathcal{S}''^g = \mathcal{S}_2'$ cannot be contained in Γ''.

First of all, we recall that by Corollary 2 any two spreads of Γ' intersect in a unique line. Secondly, as there are 12 line reguli in a spread, every point of the ideal twin planes, that is not on a line of a certain spread $\mathcal{S}$, is at distance 3 from exactly 3 lines of $\mathcal{S}$ (as there are $12 + 9$ points in each of those planes and hence every one of the points not on a spread line belongs to the opposite regulus of such a line regulus). Below we shall refer to this property as the points-at-distance-3 property.

We now consider $\mathcal{S}_1$ and $\mathcal{S}''$ within Δ_1 and determine the image of both sets under g. Since $\mathcal{S}_1$ and $\mathcal{S}_1^g = \mathcal{S}_2$ are two spreads of Γ', they share a line L. Let M denote the line of $\mathcal{S}_1$ such that L is the image of M under g and note that M can coincide with the line L itself. Denote the points of L and M in π and π' (the ideal twin planes of Δ_1) by x and x', and y and y', respectively. By definition of a spread, either x or x' is not on a line of $\mathcal{S}''$. Suppose x is this point, then by the point-at-distance-3 property three out of the five points on I_x, the ideal line containing all points of Δ_1 collinear to x, are on a line of $\mathcal{S}''$. Now considering the action of the element g, the point x can be mapped onto the point y or onto the point y' (note that if $M = L$, then $y = x$ and $x' = y'$). In any case, the ideal line I_x will be mapped onto the ideal line corresponding to the point x^g, being y or y'. Either I_{x^g} intersects $\pi \cup \pi'$ in the point x^g or it belongs to the hexagon twin plane not containing the point x^g.

In the former case, at least two of those three lines of $\mathcal{S}''$ that are concurrent to I_x are mapped onto lines which do not belong to Δ_1. Since these lines should belong to Γ'', they have to intersect the lines of Δ_1 in the lines of $\mathcal{S}'_1$. However, these image lines intersect Δ_1 in lines incident with x^g, a contradiction.

In the latter case, the lines of $\mathcal{S}''$ concurrent to I_x are mapped onto lines of Δ_1. By definition these image lines belong to Γ''. In other words, those three lines of $\mathcal{S}''$ are mapped onto lines of $\mathcal{S}'_1$, as this is the intersection of Δ_1 and Γ''. However, since $\mathcal{S}''^g = \mathcal{S}'_2$ and since $\mathcal{S}'_2$ determines $\Delta_2 \neq \Delta_1$, $\mathcal{S}'_2$ and $\mathcal{S}'_1$ are two distinct spreads of Γ'' which intersect in at least two lines, a contradiction and hence the claim.

4. Construction of the graph

It is well known that (see Atlas of Finite Simple Groups [2])

$$\mathsf{G}_2(2) \leq \mathsf{J}_2 : 2 \leq \mathsf{G}_2(4).2$$

with $\mathsf{G}_2(4).2 \cong \mathrm{Aut}(\mathsf{H}(4)) =: G$. Moreover, one of the orbits of $\mathsf{G}_2(2)$ on the point set $\mathcal{P}$ of $\mathsf{H}(4)$ will be isomorphic with $\mathsf{H}(2)$. We now consider such a maximal subgroup $\mathsf{J}_2 : 2$ of G and the subgroup $\mathsf{G}_2(2)$ as a maximal subgroup of $\mathsf{J}_2 : 2$. To simplify notation, we will denote by H_2 the order 2 subhexagon stabilized by the latter group. By the orbit counting formula one now readily checks that the orbit of H_2 under the group action of $\mathsf{J}_2 : 2$ has size 100. Let Ω be the set of the 100 order 2 subhexagon obtained as such. It may be clear that, fixing H_2 within Ω, the number of orbits of $\mathsf{G}_2(2)$ on $\Omega \setminus \{\mathsf{H}_2\}$ will be determined by the number of distinct intersections of the elements of this set with H_2 – with respect to the substructures in H_2. Indeed, suppose

$$(\Omega \setminus \{\mathsf{H}_2\})^{\mathsf{G}_2(2)} = \Omega_1 \cup \Omega_2 \cup \ldots \Omega_k$$

and let ω_1 denote an element of Ω_1 for which $\mathsf{H}_2 \cap \omega_1 = S$. Then for every such a substructure S' in H_2, i.e. for every substructure such that there exists an element $g \in \mathsf{G}_2(2)$ for which $S^g = S'$, there is a corresponding element of Ω_1 that intersects H_2 in S'. In other words, 99 must be the sum of multiples of numbers of Column D of Table 2, except for the number in the first row, of course.

As a direct consequence of Theorem 4, there are only five possible intersections of an element $\mathsf{H}'_2 \in \Omega$ with H_2, being S^0_0, S^{14}_{21}, S^3_7, S^{15}_7 and S^3_1. First, we rule out type S^0_0. Indeed, inside Ω this type of intersection with H_2 would occur a multiple of 28 times, which is non-combinable with the other possible intersections (the latter are all multiples of 9, as is the total number of subhexagons, 99, that we are looking for).

As the substructures corresponding to the last three types arises 63 times inside H_2, S_{21}^{14} necessarily has to be one of the possible intersections (as the sum should be 99). Now suppose the second (and final) type of intersection is of type S_1^3, that is 3 points on a line, and consider two concurrent lines L and M in H_2. If H_2' is the hexagon that intersects H_2 in the points of L and H_2'' is the one corresponding to M, then $\mathsf{H}_2' \cap \mathsf{H}_2''$ contains $x = L \cap M$ and the lines of $\mathsf{H}(4) \setminus \mathsf{H}_2$ on x. In other words, there are exactly two lines on x within this intersection, which is impossible both in S_{21}^{14} and in S_1^3.

Also intersections of type S_7^{15} lead to a contradiction as follows. Let L and M be the defining lines of two such intersections with L opposite M and denote the associated hexagons by H_2' and H_2''. Then the three points of H_2 at distance 3 from both L and M belong to $\mathsf{H}_2' \cap \mathsf{H}_2''$. Also, the six lines of $\mathsf{H}(4)$ through these three points not in H_2 must belong to both H_2' and H_2''. So the intersection $\mathsf{H}_2' \cap \mathsf{H}_2''$ contains, by the regularity of $\mathsf{H}(4)$, a configuration isomorphic to S_8^9. But this cannot be contained in an S_7^{15} configuration, nor in an S_{21}^{14} configuration.

We now have the following theorem:

Theorem 10. *Let $\Upsilon = (V, E)$ be the graph with vertex set $V = \Omega = \mathsf{H}_2^{\mathsf{J}_2:2}$, where two vertices are adjacent whenever the corresponding order 2 subhexagons intersect in a configuration S_{21}^{14}, while two vertices are non-adjacent if the corresponding order 2 subhexagons intersect in a configuration S_7^3. Then Υ is isomorphic to the Hall-Janko graph.*

Proof. We have shown above that Υ has the same parameters as $\mathsf{HJ}(100)$. Since it is also clearly rank 3, it must be isomorphic to $\mathsf{HJ}(100)$. Alternatively, this also follows from Observation 3, by identifying each member $of \Omega$ that intersects H_2 in an ideal non-thick subhexagon with that intersection, and every member of Ω that intersect H_2 in a configuration S_7^3 with the unique line that contains three points of the intersection. □

Now Observation 3 also follows from the previous theorem, if we use the first argument of the above proof.

5. Near-octagon of order $(2,4)$

5.1. **Two useful lemmas.** Using the results of the previous section, we now show two lemmas that will be very useful for the proof of Theorem 13 below.

Lemma 11. *Let H and H' be two elements of Ω, and let p be a point of H and p' a point of H'. If p and p' are collinear in $\mathsf{H}(4)$, then the line L joining them belongs to both of H and H'.*

Proof. Suppose by way of contradiction that L does not belong to either H or H'. First suppose that $\mathsf{H} \cap \mathsf{H}'$ is a subhexagon Δ of order $(1, 2)$ in both H and H'. It is well known (and easy to see with an elementary count) that every point of H and of H' not on a line of Δ belongs to a point regulus $\mathcal{R}(x, y)$, with x, y points of Δ. It follows that both p and p' are at distance 4 of all points of a line of any of the ideal twin planes of Δ (noting that this is obvious if p or p' belongs to a line of Δ). Since lines in these ideal twin planes intersect non-trivially, we deduce that there is a point z of Δ at distance 4 from both p and p'. This gives rise to a pentagon in $\mathsf{H}(4)$ unless L is at distance 3 from z, in which case L is on the shortest paths from z to p and p' and hence L belongs to both H and H'.

Suppose now secondly that $\mathsf{H} \cap \mathsf{H}'$ contains three collinear points (say, on a common line M) and all lines of H through these points. There is at least one point z on M of $\mathsf{H} \cap \mathsf{H}'$ at distance at most 4 from p. Considering the line $\mathrm{proj}_z p$, which belongs to both H and H', we see that the shortest path joining p' and $\mathrm{proj}_z p$ contains L; hence L belongs to H'. Symmetrically, L belongs to H. □

Lemma 12. *Two opposite lines L, M of $\mathsf{H}(4)$ are contained in at most 4 members of Ω; if they are contained in at least one, then they are contained in exactly 4 members, and there are unique points $x \in L$ and $y \in M$ at distance 4 such that L, M, x, y are not contained in any member of Ω.*

Proof. Let H_0 be a member of Ω containing L, M, and let x_i, $i = 0, 1, 2, 3, 4$, denote the points on L, and y_j, $j = 0, 1, 2, 3, 4$, the points on M, with x_i at distance 4 from y_i, $i \in \{0, 1, 2, 3, 4\}$. Without loss, we may assume that H_0 contains x_1, x_2, x_3 and consequently also y_1, y_2, y_3. By construction of Ω, there is a unique element H_k, $k \in \{1, 2\}$, of Ω intersecting H_0 in the ideal non-thick subhexagon $\Delta(x_k, y_3)$ of H_0. Clearly $\mathsf{H}_1 \cap \mathsf{H}_2$ contains L, M, x_3 and y_3 and hence must contain a further element on L, since it must clearly be of type S_{21}^{14}. Say x_0 and y_0 also belong to H_1 and H_2. The same argument repeated for H_1 in stead of H_0 shows that also x_0, x_1, x_2 and M are contained in a member H_3 of Ω. Now, if also x_0, y_0, L and M were contained in a member of Ω, then this member would intersect at least one of $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3$ in L, M and unique points on these lines, contradicting the fact that such intersection must be either configuration S_{21}^{14}, or S_7^3. Putting $x = x_4$ and $y = y_4$, the lemma follows. □

5.2. **The near-octagon.** Finally we are ready to prove that the split Cayley generalized hexagon of order 4 has a full subgeometry isomorphic to the dual of the near-octagon of order $(2,4)$. This provide a geometrical interpretation of $\mathsf{J}_2 : 2$ as a maximal subgroup of $\mathsf{G}_2(4)$. Furthermore, the following theorem immediately implies that the dual split Cayley generalized hexagon of order 2 is a full subgeometry of this near-octagon.

Theorem 13. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be the incidence geometry with $\mathcal{P}$, respectively $\mathcal{L}$, the union of all points, respectively lines, contained in the order 2 subhexagons of Ω, and with induced incidence relation. Then the dual Γ^D of Γ is a near-octagon of order $(2,4)$ with 315 points and 525 lines..*

Proof. We will prove this Theorem in the following 3 steps.

Step 1: Γ has 525 points, 315 lines and order $(4,2)$.
Step 2: Two elements of Γ are at most distance 7 apart.
Step 3: Γ^D satisfies property $(*)$.

Step 1: Γ has order $(4,2)$. Consider an element H of Ω and count the number of elements $\mathsf{H}' \in \Omega \setminus \{\mathsf{H}\}$ on a fixed line L of H. There are exactly 36 elements of Ω which intersect H in 21 lines, while there are 63 elements of Ω that intersect H in 7 lines. Hence a double counting on the couples (H', L), with $\mathsf{H}' \in \Omega \setminus \{\mathsf{H}\}$ and $L \in \mathsf{H} \cap \mathsf{H}'$, together with the fact that $\mathsf{J}_2 : 2$ acts transitively on the elements in Ω yields that every line in $\mathcal{L}$ is contained in 20 subhexagons of Ω. Similarly, a double counting of the pairs (H', p), with $\mathsf{H}' \in \Omega \setminus \{\mathsf{H}\}$ and $p \in \mathsf{H} \cap \mathsf{H}'$ tells us that there are 12 elements of Ω on every point of $\mathcal{P}$. One now readily checks that the cardinalities of $\mathcal{P}$ and $\mathcal{L}$ equal 525 and 315, respectively.

We will now prove that Γ has order $(4,2)$. Consider a point p in $\mathcal{P}$ and suppose H is one of the 12 elements of Ω on p. Obviously, within H there are 3 lines on p, each of which belongs to $\mathcal{L}$. Hence there are at least 3 lines on every point in Γ. On the other hand, a fourth line of $\mathcal{L}$ on p would imply that there exists a subhexagon $\mathsf{H}' \in \Omega$ for which the number of lines on p in $\mathsf{H} \cap \mathsf{H}'$ is at most 2, a contradiction (two elements of Ω always intersect in a substructure having 3 lines on any one of its intersection points). Hence there are exactly 3 lines through every point in Γ and consequently also 5 points on every one of its lines (apply an easy double count to see).

Step 2: a point and a line in Γ are never at distance 9 from each other. Let L be a line at distance 9 from a point x. Lemma 11 together with the fact that all points of $\mathsf{H}(4)$ on L belong to Γ imply that, in $\mathsf{H}(4)$, the distance from x to L is not 3, hence it is 5 and we can consider a point y on L opposite x in $\mathsf{H}(4)$. Since there are exactly 5 lines through each of x and y in $\mathsf{H}(4)$, and exactly $3 > \frac{5}{2}$ of each of them belong to Γ, there are lines L_x and L_y through x and y, respectively, belonging to Γ and at distance 4 from each other. Again, Lemma 11 now implies that the shortest path connecting x and y containing L_x and L_y belongs entirely to Γ and so x and L are at distance 7, a contradiction.

It also clear that the diameter of the incidence graph of Γ is at least 7, as otherwise Γ would be a subhexagon of $\mathsf{H}(4)$, a contradiction.

Step 3: Γ^D satisfies property $(*)$. We will now show that for each point P and every line l in Γ^D, there exists a unique point Q on l nearest to P. Dualizing this situation we consider a point p of $\mathcal{P}$ and a line L of $\mathcal{L}$ and prove that there exists a unique line of Γ incident with p nearest to L in Γ.

First we show that every path of length 6 bounded by two lines is contained in exactly three members of Ω. Let $M_0 \mathtt{I} p_0 \mathtt{I} M_1 \mathtt{I} p_1 \mathtt{I} M_2 \mathtt{I} p_2 \mathtt{I} M_3$ be such path γ. Since M_0 and M_3 are clearly opposite in $\mathsf{H}(4)$, we deduce from the proof of Lemma 12 that there are at most three members of Ω containing γ. Since there are precisely $315 \cdot 5 \cdot 2 \cdot 4 \cdot 2 \cdot 4 \cdot 2 = 201600$ such paths, there are at most 604800 pairs (γ, H_2), with γ such a path contained in a member H_2 of Ω. But since $|\Omega| = 100$, and each member of Ω contains $63 \cdot 3 \cdot 2^5 = 6048$ such paths, we see that equality holds above and every such path is contained in exactly three members of Ω. Note that, due to Lemma 12, there are unique points x_0 and x_4 on M_0 and M_4, respectively, such that x_0 and x_4 are not opposite and not contained together with M_0 and M_4 in a member of Ω. We now also deduce that the unique shortest path between x_0 and x_4 cannot be contained in Γ as there would otherwise be three members of Ω containing this path and M_0, M_4, a contradiction to what we just deduced from Lemma 12. Using Lemma 12, it now follows easily that every apartment of $\mathsf{H}(4)$ all of whose members are contained in Γ, is contained in precisely two members of Ω.

Now suppose, by way of contradiction, that there are two paths $p \mathtt{I} L_0 \mathtt{I} x_0 \mathtt{I} L_1 \mathtt{I} x_1 \mathtt{I} L_2 \mathtt{I} x_2 \mathtt{I} L$ and $p \mathtt{I} L_0' \mathtt{I} x_0' \mathtt{I} L_1' \mathtt{I} x_1' \mathtt{I} L_2' \mathtt{I} x_2' \mathtt{I} L$, with $L_0 \neq L_0'$. If $x_2 = x_2'$, then any member of Ω containing p and x_2 contains L (since there are only 3 lines of Γ incident with x_2), and so L is at distance 5 from p, a contradiction.

So $x_2 \neq x_2'$. Note that the shortest path between p and $p' = \mathrm{proj}_L p$ is not contained in Γ, as otherwise the distance between p and L would be 5. Since x_2' is opposite p, we deduce $x_2' \neq p'$. Hence, Lemma 12 implies that the unique path between x_2' and $y_2 = \mathrm{proj}_{L_0} x_2'$ is contained in Γ, and so there is a member H_2 of Ω containing the apartment determined by p, x_0', x_3', x_2' and y_2. But H_2 contains all lines of Γ through x_2', so H_2 contains both the opposite lines L_0 and L, which implies it contains p and p', contradicting Lemma 12 and the definition of p'.

The theorem is proved. □

Now the proof of our Main Result is complete, noting that there is a unique near-octagon of order $(2, 4)$ by [1].

Acknowledgment. The first author is Research Assistant of the Fund for Scientific Research - Flanders (Belgium) (F.W.O.) and the second author is partly supported by the Fund for Scientific Research - Flanders (Belgium) (F.W.O.).

References

[1] A. M. Cohen and J. Tits, On generalized hexagons and a near-octagon whose lines have three points, *European J. Combin.* **6** (1985), 13–27.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[3] A. De Wispelaere and H. Van Maldeghem, Codes from generalized hexagons, *Designs, Codes and Crypt.* **37** (3) (2005), 435–448.

[4] M. A. Ronan, A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227–262.

[5] J. Tits, Sur la trialité et certains groupes qui s'en déduisent. Publ. Math. Inst. Hautes Étud. Sci. **2** (1959), 13–60.

[6] H. Van Maldeghem, *Generalized Polygons*, Birkäuser Verlag, Basel, 1998.

[7] H. Van Maldeghem, Ovoids and spreads arising from involutions, **in** Groups and Geometries (ed. A. Pasini et al.), Birkhäuser Verlag, Basel, *Trends in Mathematics* (1998), 231–236.

Department of Pure Mathematics and Computer Algebra,, Ghent University,, Krijgslaan 281-S22,, B-9000 Ghent,, Belgium, adw@cage.UGent.be, hvm@cage.UGent.be