Trialities and 1-systems of $Q^+(7, q)$

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Abstract

The image of a 1-system of $Q^+(7, q)$ under a triality of the D_4-geometry, attached to $Q^+(7, q)$, will be investigated. Attention will mainly be paid to the case of a locally hermitian, semiclassical 1-system of a $Q(6, q)$, embedded in $Q^+(7, q)$. It is found that its image under a triality is always locally hermitian and semiclassical as well. Moreover, it is a proper 1-system of $Q^+(7, q)$ whenever the original 1-system of $Q(6, q)$ is not a spread of some generalized hexagon $\mathsf{H}(q)$ on $Q(6, q)$. Finally, some results concerning isomorphisms will be obtained.

1 Definitions

1.1 Trialities, T-correspondences and generalized hexagons

Consider the projectively unique hyperbolic quadric $Q^+(7, q)$ in $\mathsf{PG}(7, q)$. It is well known, see for instance [4, 22.4], that the generators of $Q^+(7, q)$, which are 3-dimensional subspaces, can be subdivided in two subsets, often called *families*, in the following way. Two distinct generators are said to belong to the same family if and only if their intersection is empty or a line. Hence generators of different families intersect in a point or a plane and furthermore, every plane of $Q^+(7, q)$ is contained in exactly two generators, namely one from each family.
If we denote the two families of generators of $Q^+(7, q)$ by $\mathcal{F}_1$ and $\mathcal{F}_2$, the set of lines on $Q^+(7, q)$ by $\mathcal{L}$ and the set of points of $Q^+(7, q)$ by $\mathcal{P}$, then a D_4-geometry Ω can be attached to $Q^+(7, q)$, in the following way. The *0-points* of Ω are the points of $Q^+(7, q)$; the *lines* are just the lines of $Q^+(7, q)$; the *1-points* are the elements of one family of generators, say $\mathcal{F}_1$; and the *2-points* are the elements of the other family of generators. Incidence is symmetrized containment for i-points and lines, $i = 0, 1, 2$, and also for 0-points and j-points, $j = 1, 2$. A 1-point $G_1 \in \mathcal{F}_1$ is said to be incident with a 2-point $G_2 \in \mathcal{F}_2$ if and only if the intersection $G_1 \cap G_2$ is a plane of $Q^+(7, q)$. An important property of the geometry Ω is the fact that every permutation of the set $\{\mathcal{P}, \mathcal{F}_1, \mathcal{F}_2\}$ defines a

geometry which is isomorphic to Ω.

Definitions

A *triality* of the geometry Ω, attached to $Q^+(7,q)$ as above, is a map τ:

$$\tau : \mathcal{L} \to \mathcal{L}, \mathcal{P} \to \mathcal{F}_1, \mathcal{F}_1 \to \mathcal{F}_2, \mathcal{F}_2 \to \mathcal{P}$$

preserving the incidence in Ω and such that τ^3 is the identity.
A point $p \in \mathcal{P}$ is called an *absolute* point of τ if p is incident with $p^\tau \in \mathcal{F}_1$. A 1- or 2-point, that is, a generator $G_i \in \mathcal{F}_i$, $i = 1, 2$, is said to be *absolute* for τ if it is incident with its image G_i^τ. A line $L \in \mathcal{L}$ is called *absolute* for τ if $L = L^\tau$.

Although the condition "τ^3 is the identity" is an explicit part of the definition of a triality, most of the results that will be obtained in this paper, hold under weaker conditions too. Often it is not necessary to require that τ^3 is the identity. Therefore we introduce another less common definition, which is due to Tits [13].

Definition

A *T-correspondence* of the geometry Ω is a map θ:

$$\theta : \mathcal{L} \to \mathcal{L}, \mathcal{P} \to \mathcal{F}_1, \mathcal{F}_1 \to \mathcal{F}_2, \mathcal{F}_2 \to \mathcal{P}$$

which preserves incidence in Ω.

The definitions of absolute points and lines for a T-correspondence θ remain the same as for a triality τ.

Remark that the assignment of the names $\mathcal{F}_1$ and $\mathcal{F}_2$ to the families of generators of $Q^+(7,q)$ is arbitrary, and hence the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$ may be interchanged in the above definitions. As such, $\tau^{-1} = \tau^2$ is also a triality and both θ^{-1} and θ^2 are T-correspondences. For convenience, we agree on the convention that a triality, respectively a T-correspondence, is always a map as in the definitions above, unless otherwise mentioned, for some choice of $\mathcal{F}_1$ and $\mathcal{F}_2$ which we consider to be fixed throughout this paper.

Trialities are very interesting for the study of generalized hexagons, as some trialities produce generalized hexagons. This was shown by Tits in his celebrated paper "Sur la trialité et certains groupes qui s'en déduisent" [13]. Generalized hexagons are defined as follows.

Definitions

A *generalized hexagon* is an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines such that the following three axioms are satisfied:

GH1 Γ contains no ordinary k-gon for $2 \leq k \leq 5$.

GH2 Any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary hexagon in Γ.

GH3 There exists an ordinary sevengon in Γ.

If $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ only has to satisfy conditions (i) and (ii), then it is called a *weak generalized hexagon*. A *thick line* of a weak generalized hexagon Γ is a line of Γ which contains at least three points of Γ.

The following theorem states the relation between trialities and generalized hexagons.

Theorem 1.1 (Tits [13]) *Let τ be a triality of the geometry Ω. Suppose that one of the following equivalent hypotheses is satisfied:*

(i) there exists at least one absolute i-point, for some $i \in \{0, 1, 2\}$, and every absolute i-point is incident with at least two absolute lines;

(ii) there exists a sequence of absolute lines $(L_1, L_2, \ldots, L_d)$, $d > 2$, such that L_i is concurrent with L_{i+1}, indices modulo d.

Then for every $i \in \{0, 1, 2\}$, the geometry $\Gamma^{(i)}$ with point set $\mathcal{P}^{(i)}_{\text{abs}}$ the set of absolute i-points, with line set $\mathcal{L}_{\text{abs}}$ the set of absolute lines and with the natural incidence, is a weak generalized hexagon with thick lines.

Over the finite field $\mathsf{GF}(q)$, there exist essentially two examples of generalized hexagons arising from a triality. The first one is the so-called *split Cayley hexagon*, denoted by $\mathsf{H}(q)$, which exists for every q. This hexagon has order (q, q) and an interesting fact is that all points and lines of $\mathsf{H}(q)$ are contained in a hyperplane $\mathsf{PG}(6, q)$ of $\mathsf{PG}(7, q)$. Also, two points of $\mathsf{H}(q)$ are opposite (that is, at distance 6) in $\mathsf{H}(q)$ if and only if they are not joined by a line of $Q^+(7, q)$ and moreover, the $q + 1$ lines through a point of $\mathsf{H}(q)$ form a flat pencil of lines in a totally singular plane of $Q^+(7, q)$.
The other example is the *twisted triality hexagon* $\mathsf{T}(q'^3, q')$ and it exists only if $q = q'^3$ for some prime power q', but this example will not be important for the sequel.
For more information about trialities and their relation to generalized hexagons, the reader is referred to the original paper of Tits [13], or to Chapter 2 of the monograph *"Generalized Polygons"* [14] by Van Maldeghem.

To finish this subsection, we give two easy properties of T-correspondences.

Property 1.2 *Let θ be a T-correspondence of Ω. If L and M are two distinct concurrent lines, contained in a totally singular plane π on $Q^+(7, q)$, then the same holds for L^θ and M^θ.*

Proof.
If $L \cap M$ is the point x and $G_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{F}_2$ are the two generators of $Q^+(7,q)$ through π, then L^θ and M^θ are lines in the intersection of the two generators $x^\theta \in \mathcal{F}_1$ and $G_1^\theta \in \mathcal{F}_2$, and they contain the point G_2^θ. □

Property 1.3 *Let θ be a T-correspondence of Ω and let π be a totally singular plane on $Q^+(7,q)$. Then the set of lines in π is mapped by θ onto the set of lines through a point in a generator of $Q^+(7,q)$.*

Proof.
Let again $G_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{F}_2$ denote the two generators through the plane π. Then all lines of π are mapped by θ onto all lines in the generator $G_1^\theta \in \mathcal{F}_2$ through the point G_2^θ. □

1.2 1-Systems

Consider a quadric $\mathcal{Q} \in \{Q^-(5,q), Q(6,q), Q^+(7,q)\}$. A *1-system* of $\mathcal{Q}$ is a set $\mathcal{M}$ of q^3+1 lines $L_0, L_1, \ldots, L_{q^3}$ with the property that every generator of $\mathcal{Q}$ which contains a line L_i of $\mathcal{M}$, is disjoint from all lines $L_j \in \mathcal{M}$, $j \neq i$. Thus, if $\mathcal{Q} = Q^-(5,q)$, a 1-system of $\mathcal{Q}$ is nothing but a spread of $Q^-(5,q)$. The set of all points on the lines of $\mathcal{M}$ will be denoted by $\tilde{\mathcal{M}}$, so $\tilde{\mathcal{M}}$ is the union of all elements of $\mathcal{M}$.

As an example, we give the definition of a hermitian spread of $Q^-(5,q)$. Let $Q^+(5,q^2)$ be the extension to $\mathsf{GF}(q^2)$ of $Q^-(5,q)$. Then there exist two disjoint and conjugate planes π and $\overline{\pi}$ on $Q^+(5,q^2)$, which contain no point of $Q^-(5,q)$. The set of lines of $Q^-(5,q)$, the extensions to $\mathsf{GF}(q^2)$ of which have a point in common with both π and $\overline{\pi}$, forms a spread $\mathcal{S}$ of $Q^-(5,q)$; see for instance [11]. Since the common points of π and the extensions to $\mathsf{GF}(q^2)$ of the lines of $\mathcal{S}$ form a hermitian curve $H(2,q^2)$ in π, this spread is called a *hermitian* spread of $Q^-(5,q)$.

Clearly, every 1-system of a $Q^-(5,q)$ or a $Q(6,q)$ which is embedded in $Q^+(7,q)$, is also a 1-system of $Q^+(7,q)$. If a 1-system of $Q^+(7,q)$ is not contained in a hyperplane of the ambient space $\mathsf{PG}(7,q)$ of $Q^+(7,q)$, then it is said to be a *proper* 1-system of $Q^+(7,q)$. Similarly, a 1-system of $Q(6,q) \subseteq \mathsf{PG}(6,q)$ is *proper* if and only if it is not contained in a hyperplane of $\mathsf{PG}(6,q)$.

Let $\mathcal{M}$ be a 1-system of $\mathcal{Q}$, with again $\mathcal{Q} \in \{Q^-(5,q), Q(6,q), Q^+(7,q)\}$. Then $\mathcal{M}$ is *locally hermitian* at a line $L \in \mathcal{M}$ if and only if for every line $M \in \mathcal{M} \setminus \{L\}$, the regulus of lines containing L of the hyperbolic quadric $\langle L, M\rangle \cap \mathcal{Q} = Q^+(3,q)$, is completely contained in $\mathcal{M}$. Hence a locally hermitian 1-system of $\mathcal{Q}$ consists of q^2 reguli through a special line L, and these reguli will often be denoted by $R_1, R_2, \ldots, R_{q^2}$. If $\mathcal{Q}$ is either a $Q(6,q)$ or a $Q^+(7,q)$, then it holds for every locally hermitian 1-system of $\mathcal{Q}$ that $\langle R_i, R_j\rangle$ is 5-dimensional if $i \neq j$, and moreover, $\langle R_i, R_j\rangle \cap \mathcal{Q}$ is an elliptic quadric $Q^-_{ij}(5,q)$, for the following reason. If $\mathcal{Q}_{ij} := \langle R_i, R_j\rangle \cap \mathcal{Q}$ is not elliptic, then it contains totally singular planes. If $M \neq L$ is a line of R_i, then there exists in particular a totally singular plane α of $\mathcal{Q}_{ij}$ through the line M. But $\alpha \cap \langle R_j \rangle$ is at least a point, which implies that α meets at least one line of R_j in a point. This contradicts the definition of a 1-system of $\mathcal{Q}$ and we conclude that $\mathcal{Q}_{ij}$ is indeed an elliptic quadric $Q^-_{ij}(5, q)$.

Suppose next that $\mathcal{M}$ is a 1-system of $\mathcal{Q}$, with $\mathcal{Q} \in \{Q(6, q), Q^+(7, q)\}$, which is locally hermitian at the line $L \in \mathcal{M}$, and let x be an arbitrary point on L. Consider a 5-dimensional subspace γ of the tangent hyperplane $T_x(\mathcal{Q})$ of $\mathcal{Q}$ at x, with the property that $x \notin \gamma$, so that $\gamma \cap \mathcal{Q} := \mathcal{Q}'$ is a parabolic quadric $Q(4, q)$, respectively a hyperbolic quadric $Q^+(5, q)$. Every regulus R_i of $\mathcal{M}$ through L has a unique transversal through the point x, which meets $\mathcal{Q}'$ in a point n_i. Together with the point $l := L \cap \gamma$, we thus obtain $q^2 + 1$ points on $\mathcal{Q}'$. If two points l and n_i, respectively n_i and n_j with $i \neq j$, were collinear on $\mathcal{Q}'$, then the totally singular plane $\langle x, l, n_i \rangle$, respectively $\langle x, n_i, n_j \rangle$, would contain at least $2q + 1$ points of $\tilde{\mathcal{M}}$. This is a contradiction to the elementary properties of 1-systems, see Shult and Thas [10, Theorem 8]. It follows that the $q^2 + 1$ points $l, n_1, n_2, \ldots, n_{q^2}$ form an ovoid of $\mathcal{Q}'$. This ovoid is called the *projection along reguli of $\mathcal{M}$ from x onto γ* and denoted by $\mathcal{O}_x$. If the ovoid $\mathcal{O}_x$ is the classical ovoid $Q^-(3, q)$ of $\mathcal{Q}'$ for all points x on L, then $\mathcal{M}$ is said to be *semiclassical*.

Finally, let $\mathcal{M}$ be a 1-system of $Q(6, q)$ and suppose that $\mathcal{M}$ is locally hermitian at the line $L \in \mathcal{M}$. Denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$ and consider two distinct reguli R_i, R_j of $\mathcal{M}$ through L. Then $\langle R_i, R_j \rangle \cap Q(6, q)$ is an elliptic quadric $Q^-_{ij}(5, q)$, as has been explained above. This quadric $Q^-_{ij}(5, q)$ defines a generalized quadrangle which is the point-line dual of the generalized quadrangle $H(3, q^2)$ arising from a non-singular hermitian variety in $\mathsf{PG}(3, q^2)$, see Payne and Thas [9]. With R_i and R_j correspond point sets W_i and W_j on $H(3, q^2)$, which are Baer sublines of lines in $\mathsf{PG}(3, q^2)$ and have exactly one point of $H(3, q^2)$ in common. So W_i and W_j are contained in exactly one common plane which intersects $H(3, q^2)$ in a hermitian curve $H(2, q^2)$ of $H(3, q^2)$. This implies that the reguli R_i and R_j uniquely define a hermitian spread $\mathcal{S}_{ij}$ of $Q^-_{ij}(5, q)$. In $\mathsf{PG}(5, q) = \langle Q^-_{ij}(5, q) \rangle$, consider a 3-dimensional subspace $\mathsf{PG}(3, q)$ skew to L. If $\mathcal{S}_{ij} = \{L, M_1, M_2, \ldots, M_{q^3}\}$ and $\tau := T_L(Q^-_{ij}(5, q))$ is the tangent space of $Q^-_{ij}(5, q)$ at L, then the lines $\langle L, M_i \rangle \cap \mathsf{PG}(3, q)$, $i = 1, 2, \ldots, q^3$, together with $\tau \cap \mathsf{PG}(3, q) := L'$ form a regular spread S of $\mathsf{PG}(3, q)$; see Bloemen, Thas and Van Maldeghem [1]. Let x be an arbitrary point on L and denote the q^2 reguli of $\mathcal{S}_{ij}$ through L by $R'_1, R'_2, \ldots, R'_{q^2}$. Then each of these q^2 reguli has a unique transversal through x, which must be a generating line of the cone $T_x(Q^-_{ij}(5, q)) \cap Q^-_{ij}(5, q) := xQ^-_x(3, q)$. Together with the line L, this yields all $q^2 + 1$ lines containing x of the cone $xQ^-_x(3, q)$. For every point $z \in Q^-_x(3, q)$, $z \notin L$, the plane $\langle L, z \rangle$ intersects $\mathsf{PG}(3, q)$ in a point. In this way, we obtain q^2 points of $\mathsf{PG}(3, q)$ which form a plane π_x together with the line L'. This plane π_x is in fact the projection from L onto $\mathsf{PG}(3, q)$ of $\langle xQ^-_x(3, q) \rangle \setminus L$. Each conic of $Q^-_x(3, q)$ through $l' := L \cap Q^-_x(3, q)$ corresponds to a line of π_x, different from L'; this line belongs to the opposite regulus of a regulus of S through L'. Conversely, any such line of π_x corresponds to a conic of $Q^-_x(3,q)$ on l'. Hence each regulus of S containing L' defines a conic of $Q^-_x(3,q)$ through l', and conversely. The set of q reguli of $\mathcal{S}_{ij}$ defined by a regulus of S containing L' will be called an $\mathcal{R}$-*conic* of $Q(6,q)$. If $x' \in L \setminus \{x\}$, then the plane $\pi_{x'}$ defines the same $\mathcal{R}$-conics as the plane π_x. In particular, for every point y on L, the q transversals through y of the elements of an $\mathcal{R}$-conic, together with the line L, form a cone with vertex y and a non-singular conic $\mathcal{C}_y \subseteq Q^-_y(3,q)$ through the point $L \cap Q^-_y(3,q)$ as base.

In [6] and [7], it is shown that the locally hermitian, semiclassical 1-systems of $Q(6,q)$ have a special property, which is formulated in the following lemma.

Lemma 1.4 ([6], [7]) *Let $\mathcal{M}$ be a proper 1-system of $Q(6,q)$, which is locally hermitian at some line $L \in \mathcal{M}$ and in addition semiclassical. Denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$. Then for every $i \neq j$, the elliptic quadric $Q^-_{ij}(5,q) = \langle R_i, R_j \rangle \cap Q(6,q)$ contains exactly q reguli of $\mathcal{M}$ through L and these reguli form an $\mathcal{R}$-conic.*

To conclude this section, a construction is given for locally hermitian spreads of the elliptic quadric $Q^-(5,q)$.

Let $\mathcal{S}$ be a spread of $Q^-(5,q)$, which is locally hermitian at some line $L \in \mathcal{S}$. Consider the tangent space $\gamma := T_L(Q^-(5,q))$ of $Q^-(5,q)$ at the line L and denote the q^2 reguli of $\mathcal{S}$ through L by $R_1, R_2, \ldots, R_{q^2}$. Then for every regulus R_i, it holds that $\langle R_i \rangle^\perp$, with $\perp$ the polarity of $Q^-(5,q)$, is a line L_i in γ, and the q^2+1 lines $L, L_1, L_2, \ldots, L_{q^2}$ form a line spread S in γ. Conversely, one can reconstruct a locally hermitian spread of $Q^-(5,q)$ from every line spread S' in γ which contains the line L, just by reversing the above construction. If a locally hermitian spread of $Q^-(5,q)$ is obtained in this way from a line spread S of γ through L, it will be denoted by $\mathcal{S}(S)$. It can be shown, see [2], that $\mathcal{S}(S)$ is hermitian if and only if the spread S is the regular line spread in γ, which is determined by two conjugate lines T and $\overline{T}$ with respect to $\mathsf{GF}(q^2)$, such that $\langle L, T \rangle$ and $\langle L, \overline{T} \rangle$ are the generators containing L of the extension $Q^+(5,q^2)$ of $Q^-(5,q)$ to $\mathsf{GF}(q^2)$.

2 T-correspondences and locally hermitian 1-systems of $Q(6,q)$

Before atention is paid to the image under a T-correspondence (and hence also a triality) of a locally hermitian 1-system of an induced $Q(6,q) \subseteq Q^+(7,q)$, it is first shown that T-correspondences map 1-systems of $Q^+(7,q)$ onto 1-systems, in the following easy, but basic, theorem.

Theorem 2.1 *Let θ be a T-correspondence of Ω and consider an arbitrary 1-system $\mathcal{M}$ of $Q^+(7,q)$. Then $\mathcal{M}^\theta$ is also a 1-system of $Q^+(7,q)$.*

Proof.
Suppose that the claim is false and $\mathcal{M}^\theta$ is not a 1-system of $Q^+(7,q)$. This implies that there exist a line $M \in \mathcal{M}^\theta$ and a generator G of $Q^+(7,q)$ on M such that G meets some line N of $\mathcal{M}^\theta \setminus \{M\}$ at least in a point x. In that case $\langle M, x\rangle$ is a totally singular plane on $Q^+(7,q)$ and as such contained in exactly two generators G_1 and G_2 of $Q^+(7,q)$; assume without loss of generality that $G = G_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{F}_2$. It then holds that $N^{\theta^{-1}}$ and $M^{\theta^{-1}}$ are lines of $\mathcal{M}$, the generator $x^{\theta^{-1}} \in \mathcal{F}_2$ contains $N^{\theta^{-1}}$, $G_1^{\theta^{-1}}$ is a point of $M^{\theta^{-1}}$ and as $x \in G_1$, it follows that the point $G_1^{\theta^{-1}}$ is contained in $x^{\theta^{-1}}$. In other words, $x^{\theta^{-1}}$ is a generator of $Q^+(7,q)$ through the line $N^{\theta^{-1}} \in \mathcal{M}$, which has the point $G_1^{\theta^{-1}}$ in common with the line $M^{\theta^{-1}} \in \mathcal{M} \setminus \{N^{\theta^{-1}}\}$. This is a contradiction to the fact that $\mathcal{M}$ is a 1-system of $Q^+(7,q)$, which proves the theorem. $\square$

Since locally hermitian 1-systems are characterized by the fact that they consist of a number of reguli through some common line, it is important to understand the action of a T-correspondence on a regulus.

Lemma 2.2 *Let θ be a T-correspondence of Ω and consider a 3-dimensional subspace γ of $\mathsf{PG}(7,q)$ such that $\gamma \cap Q^+(7,q) = Q^+(3,q)$. Then θ maps each regulus of lines of $Q^+(3,q)$ onto a regulus of lines of some $Q^+(3,q)' = \gamma' \cap Q^+(7,q)$, with γ' also a 3-dimensional subspace of $\mathsf{PG}(7,q)$.*

Proof.
Denote the two reguli of the hyperbolic quadric $Q^+(3,q)$ by $\{K_0, K_1, \ldots, K_q\}$ and $\{T_0, T_1, \ldots, T_q\}$. Since for $i \neq j$, any two lines K_i and K_j, respectively T_i and T_j, are disjoint and not contained in a generator of $Q^+(7,q)$, the same holds for their images K_i^θ and K_j^θ, respectively T_i^θ and T_j^θ. On the other hand, every K_i meets every T_j in a point, which implies that K_i^θ and T_j^θ are contained in a generator of $Q^+(7,q)$, for $i, j \in \{0, 1, \ldots, q\}$. If $K_i^\theta \cap T_j^\theta$ were a point y, then K_i and T_j would be contained in the generator $y^{\theta^{-1}}$ of $Q^+(7,q)$, clearly a contradiction. So K_i^θ is disjoint from T_j^θ, for all $i, j \in \{0, 1, \ldots, q\}$ and the situation is such that every T_j^θ is contained in $(K_i^\theta)^\perp$, for all $i = 0, 1, \ldots, q$. Consequently, the $q+1$ lines T_j^θ, $j = 0, 1, \ldots, q$, must form a regulus of $Q^+(3,q)'' := \langle K_0^\theta, K_1^\theta\rangle^\perp \cap Q^+(7,q)$ and this in turn implies that the lines $K_0^\theta, K_1^\theta, \ldots, K_q^\theta$ form a regulus of $\langle Q^+(3,q)''\rangle^\perp \cap Q^+(7,q) := Q^+(3,q)'$, where $Q^+(3,q)' = \langle K_0^\theta, K_1^\theta\rangle \cap Q^+(7,q)$. This proves the lemma. $\square$

This lemma ensures that every T-correspondence θ maps a 1-system $\mathcal{M}$ of $Q^+(7,q)$ that is locally hermitian at some line L, onto a 1-system $\mathcal{M}^\theta$ that is locally hermitian at the line L^θ.
The next theorem we will prove, tells something about the dimension of $\langle \mathcal{M}^\theta\rangle$, but first we need another lemma.

Lemma 2.3 *Suppose that θ is a T-correspondence of Ω and let $\mathcal{A}$ be the line set of a generalized hexagon $\mathsf{H}(q)$ on $Q^+(7,q)$. Then the set $\mathcal{A}^\theta$ is the set of lines of a generalized hexagon $\mathsf{H}(q)'$ on $Q^+(7,q)$.*

Proof.
The hexagon $\mathsf{H}(q)$ consists of the absolute points and lines of some triality τ of the D_4-geometry Ω, attached to $Q^+(7,q)$, which satisfies the hypotheses of Theorem 1.1. Let L be a line of $\mathsf{H}(q)$, that is, L is absolute for τ. Then the line L^θ satisfies

$$(L^\theta)^{\theta^{-1}\tau\theta} = L^{\tau\theta} = L^\theta,$$

so L^θ coincides with its image under the map $\theta^{-1}\tau\theta$. It is obvious that $\theta^{-1}\tau\theta$ is also a triality of Ω, but to prove the claim completely, we must ensure that $\theta^{-1}\tau\theta$ satisfies one of the hypotheses of Theorem 1.1. We will check the second of the two equivalent conditions from that theorem.
On $\mathsf{H}(q)$ there certainly exists an ordinary hexagon, consisting of the sequence of lines $(L_1, L_2, L_3, L_4, L_5, L_6)$, where L_i is concurrent with L_{i+1}, indices modulo 6. Then $(L_1^\theta, L_2^\theta, L_3^\theta, L_4^\theta, L_5^\theta, L_6^\theta)$ is a sequence of absolute lines for $\theta^{-1}\tau\theta$. Since L_i is concurrent with L_{i+1} on $\mathsf{H}(q)$ for $i = 1, 2, \ldots, 6$ and indices taken modulo 6, $\langle L_i, L_{i+1}\rangle$ is a totally singular plane of $Q^+(7,q)$. Now by Property 1.2 this implies that L_i^θ and L_{i+1}^θ have an absolute point for $\theta^{-1}\tau\theta$ in common, for $i = 1, 2, \ldots, 6$ and indices taken modulo 6. Hence the triality $\theta^{-1}\tau\theta$ satisfies (ii) of Theorem 1.1 and so its absolute points and lines are the points and lines of a generalized hexagon.
Finally, the generalized hexagon of the absolute points and lines of $\theta^{-1}\tau\theta$ must be isomorphic to the one defined by τ, because we have that $M^{\theta^{-1}\tau\theta} = M$ if and only if $(M^{\theta^{-1}})^\tau = M^{\theta^{-1}}$. So a line M of $Q^+(7,q)$ is absolute for $\theta^{-1}\tau\theta$ if and only its inverse image $M^{\theta^{-1}}$ is absolute for τ. Hence both hexagons have the same order (q,q).
This proves the lemma completely. $\square$

Using this lemma, we can prove the following interesting theorem.

Theorem 2.4 *Let $\mathcal{A}$ be a set of lines on a $Q(6,q) \subseteq Q^+(7,q)$ and consider a T-correspondence θ of Ω. Then the lines of $\mathcal{A}^\theta$ lie in an induced $Q(6,q)' \subseteq Q^+(7,q)$ if and only if the lines of $\mathcal{A}$ are lines of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$.*

Proof.
Denote by $\mathcal{B}$ the set of all lines of $Q(6,q)$ and suppose that $\mathcal{A}^\theta \subseteq Q(6,q)'$ for some parabolic quadric $Q(6,q)' \subseteq Q^+(7,q)$. We investigate which lines of $\mathcal{B}^\theta$ are lines of this $Q(6,q)'$. If p is an arbitrary point of $Q(6,q)'$, then $p^{\theta^{-1}}$ is a generator of $Q^+(7,q)$. So $p^{\theta^{-1}}$ intersects $Q(6,q)$ in a totally singular plane, containing q^2+q+1 lines of $\mathcal{B}$. By Property 1.3, this implies that the q^2+q+1 lines of $\mathcal{B}^\theta$ through the point p are exactly the lines on p contained in some generator G of $Q^+(7,q)$ through p. But G meets $Q(6,q)'$ in a totally singular plane α of $Q(6,q)'$,

and hence the common lines of $\mathcal{B}^\theta$ and $Q(6,q)'$ through the point p are exactly the $q+1$ lines on p in the totally singular plane α. In other words, the common lines of $\mathcal{B}^\theta$ and $Q(6,q)'$ through an arbitrary point of $Q(6,q)'$ form a pencil of lines in some totally singular plane of $Q(6,q)'$ containing this point. Now, by Theorem 3.7 of [12], this implies that the common lines of $\mathcal{B}^\theta$ and $Q(6,q)'$ either form the set of all lines in the q^3+1 planes of a spread of $Q(6,q)'$, or they form the line set of a generalized hexagon $\mathsf{H}(q)'$ on $Q(6,q)'$. The first case cannot occur in the current situation, because in that case, $(\mathcal{B}^\theta)^{\theta^{-1}} = \mathcal{B}$ would contain q^2+q+1 lines through a point of $Q^+(7,q)$ in a generator of $Q^+(7,q)$. This is of course a contradiction, as the rank of $Q(6,q)$ is 3. So the lines of $\mathcal{B}^\theta$ on $Q(6,q)'$ form the set of lines of a generalized hexagon $\mathsf{H}(q)'$ on $Q(6,q)'$. Since $\mathcal{A}^\theta \subseteq \mathcal{B}^\theta$ and the lines of $\mathcal{A}^\theta$ lie on $Q(6,q)'$ by assumption, all lines of $\mathcal{A}^\theta$ must be lines of $\mathsf{H}(q)'$. If we now apply Lemma 2.3 to the T-correspondence θ^{-1}, we find that all lines of $\mathcal{A}$ are lines of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$.

As the converse has already been shown in Lemma 2.3, the proof of the theorem is complete. □

Suppose as a special case that $\mathcal{M}$ is a locally hermitian 1-system of $Q(6,q) \subseteq Q^+(7,q)$, not isomorphic to a spread of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$. Then by the above results, the image $\mathcal{M}^\theta$ of $\mathcal{M}$ under a T-correspondence θ is a locally hermitian 1-system of $Q^+(7,q)$ that is not contained in a non-tangent hyperplane of $\mathsf{PG}(7,q)$. If it were contained in a tangent hyperplane of $Q^+(7,q)$, this would imply the existence of a set of q^3+1 lines on $Q^+(5,q)$ with the property that every plane of $Q^+(5,q)$ which contains one of these lines, is disjoint from the other q^3 lines. From [10, Theorem 4], it follows that this is a contradiction. We conclude that $\mathcal{M}^\theta$ is a proper locally hermitian 1-system of $Q^+(7,q)$.
In the next section we will have a closer look at $\mathcal{M}^\theta$ if the 1-system $\mathcal{M}$ of $Q(6,q)$ is not only locally hermitian, but also semiclassical.

3 T-correspondences and locally hermitian, semiclassical 1-systems of $Q(6,q)$

Suppose again that θ is a T-correspondence of Ω. We now examine the special case where $\mathcal{M}$ is a locally hermitian 1-system of a $Q(6,q) \subseteq Q^+(7,q)$, which is in addition semiclassical. Let $L \in \mathcal{M}$ be a line at which $\mathcal{M}$ is locally hermitian and denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$, as usually. We already know that $\mathcal{M}^\theta$ is locally hermitian at the line L^θ, but one would also like to know whether $\mathcal{M}^\theta$ is semiclassical or not.

Let x be a point on L^θ. Then the q^2 transversals of the reguli of $\mathcal{M}^\theta$ through L^θ are the images under θ of q^2 lines in the generator $x^{\theta^{-1}} \in \mathcal{F}_2$ through L. From the proof of Lemma 2.2, it follows that if T is a transversal on x of a regulus R_i^θ, with R_i^θ the image under θ of some regulus R_i of $\mathcal{M}$ containing L, then $T^{\theta^{-1}}$ is a line of $\langle R_i \rangle^\perp$, with $\perp$ the polarity of $Q^+(7,q)$. As x is a point of T, the line $T^{\theta^{-1}}$

is thus the unique line in the intersection of $x^{\theta^{-1}}$ and $\langle Q_i^+(3,q)\rangle = \langle R_i\rangle^\perp$. So the transversals through x of the reguli of $\mathcal{M}^\theta$ containing L^θ, are the images under θ of the q^2 lines $\langle R_i\rangle^\perp \cap x^{\theta^{-1}}$, $i = 1, 2, \ldots, q^2$. These q^2 lines in $x^{\theta^{-1}}$ are pairwise disjoint, since no two transversals through x of the reguli of $\mathcal{M}^\theta$ containing L^θ, are contained in a totally singular plane of $Q^+(7,q)$. For the same reason, all of them are also disjoint from L, so that $\{L\} \cup \{\langle R_i\rangle^\perp \cap x^{\theta^{-1}} \mid i = 1, 2, \ldots, q^2\}$ is a spread of the 3-dimensional projective space $x^{\theta^{-1}}$. If $x^{\theta^{-1}} := G$, then we denote this spread by $\mathcal{S}_G$. Apparently, the structure of $\mathcal{S}_G$ determines the configuration of the transversals on x of the reguli of $\mathcal{M}^\theta$ through L^θ.

Lemma 3.1 *The 1-system $\mathcal{M}^\theta$ is semiclassical if and only if for every generator $G \in \mathcal{F}_2$ containing L, the spread $\mathcal{S}_G$ is regular.*

Proof.
Let x on L^θ be fixed and consider in $x^{\theta^{-1}} := G$ the spread $\mathcal{S}_G$. Suppose that $\{K_0, K_1, \ldots, K_q\}$ is a regulus of lines of $\mathcal{S}_G$ and denote its opposite regulus by $\{T_0, T_1, \ldots, T_q\}$. Then $K_0^\theta, K_1^\theta, \ldots, K_q^\theta$ and $T_0^\theta, T_1^\theta, \ldots, T_q^\theta$ are lines through the point x. As K_i and K_j are disjoint for $i \neq j$, the lines K_i^θ and K_j^θ are not contained in a totally singular plane of $Q^+(7,q)$. On the other hand, $K_i \cap T_j$ is a point for all $i, j \in \{0, 1, \ldots, q\}$, so that $\langle K_i^\theta, T_j^\theta\rangle$ must be a totally singular plane of $Q^+(7,q)$, by Property 1.2. Hence the only possible configuration is that $\langle K_0^\theta, K_1^\theta, \ldots, K_q^\theta\rangle \cap Q^+(7,q)$ is a quadratic cone $x\mathcal{C}$ with generating lines $K_0^\theta, K_1^\theta, \ldots, K_q^\theta$ and similarly $\langle T_0^\theta, T_1^\theta, \ldots, T_q^\theta\rangle \cap Q^+(7,q)$ is a quadratic cone $x\mathcal{C}'$, but such that $\langle x\mathcal{C}\rangle^\perp \cap Q^+(7,q) = x\mathcal{C}'$. One immediately sees that the converse also holds: if $K_0^\theta, K_1^\theta, \ldots, K_q^\theta$, respectively $T_0^\theta, T_1^\theta, \ldots, T_q^\theta$, are the generators of a quadratic cone $x\mathcal{C}$, respectively $x\mathcal{C}'$, with $\langle x\mathcal{C}\rangle^\perp \cap Q^+(7,q) = x\mathcal{C}'$ as an additional property, then $\{K_0, K_1, \ldots, K_q\}$ is a regulus in G with opposite regulus $\{T_0, T_1, \ldots, T_q\}$.
It now easily follows that $\mathcal{M}^\theta$ is semiclassical if and only if the spread $\mathcal{S}_G$ is a regular spread of G, for all generators $G \in \mathcal{F}_2$ through the line L. $\square$

Thanks to this lemma, we are able to show that $\mathcal{M}^\theta$ is semiclassical whenever $\mathcal{M}$ itself is semiclassical, by relying on the properties of locally hermitian, semiclassical 1-systems of $Q(6,q)$. We start with the case of a proper 1-system of $Q(6,q)$.

Theorem 3.2 *Let $\mathcal{M}$ be a proper 1-system of a $Q(6,q) \subseteq Q^+(7,q)$, which is locally hermitian at the line $L \in \mathcal{M}$, and consider a T-correspondence θ of Ω. If $\mathcal{M}$ is semiclassical, then $\mathcal{M}^\theta$ is semiclassical too.*

Proof.
Suppose that $\mathcal{M}$ is locally hermitian at the line L and furthermore semiclassical, and let the q^2 reguli of $\mathcal{M}$ through L be denoted by $R_1, R_2, \ldots, R_{q^2}$, as always. Consider two arbitrary reguli R_i and R_j of $\mathcal{M}$ through L. Then we know from

Lemma 1.4 that the elliptic quadric $\langle R_i, R_j \rangle \cap Q^+(7,q) = Q^-_{ij}(5,q)$ contains exactly q reguli of $\mathcal{M}$ through L. To make the notation easier, let $i = 1$ and $j = 2$ and denote the q reguli of $\mathcal{M}$ through L in $Q^-_{12}(5,q)$ by $R_1, R_2, \ldots, R_q$. Let $G \in \mathcal{F}_2$ be an arbitrary generator of $Q^+(7,q)$ through the line L. We shall show that the lines $\langle R_i \rangle^\perp \cap G$, $i = 1, 2, \ldots, q$, together with L, form a regulus of lines of $\mathcal{S}_G$. Of course, the proof holds for all R_i, R_j, $i, j \in \{1, 2, \ldots, q^2\}$ with $i \neq j$, and not only for R_1 and R_2.

As the reguli $R_1, R_2, \ldots, R_q$ form an $\mathcal{R}$-conic of $Q^-_{12}(5,q)$ by Lemma 1.4, the transversals of $R_1, R_2, \ldots, R_q$ through x, together with the line L, form a quadratic cone $x\mathcal{C}$ on $Q^-_{12}(5,q)$ for every point x on L. Hence $\langle x\mathcal{C} \rangle^\perp \cap Q^+(7,q)$ is a quadratic cone $x\mathcal{C}'$ through x and contained in $L^\perp$, but not containing L. The 3-space $\langle x\mathcal{C}' \rangle$ meets every generator through L in a line, which is a generating line of $x\mathcal{C}'$. Denote the line $x\mathcal{C}' \cap G$ by A_x.
We also know that $\langle x\mathcal{C} \rangle$ meets each $\langle R_i \rangle$, $i = 1, 2, \ldots, q$, in a plane through L. As a consequence, $\langle x\mathcal{C}' \rangle \cap \langle Q^+_i(3,q) \rangle$ is a plane as well, for all $i = 1, 2, \ldots, q$, where $Q^+_i(3,q) := \langle R_i \rangle^\perp \cap Q^+(7,q)$. Since every $Q^+_i(3,q)$ is disjoint from L and every line of $x\mathcal{C}'$ meets L in the point x, the quadratic cone $x\mathcal{C}'$ and $Q^+_i(3,q)$ have a non-singular conic $\mathcal{C}_i$ in common. But $\mathcal{C}_i$ shares exactly one point with the line A_x, which shows that A_x has a unique point in common with $Q^+_i(3,q)$, for $i = 1, 2, \ldots, q$. In other words: A_x meets L and the q lines $\langle R_i \rangle^\perp \cap G$, $i = 1, 2, \ldots, q$, in a point.
Suppose that two such lines A_x and A_y, for distinct points x and y on L, share a point u, say $u \in Q^+_1(3,q)$. Since $x \neq y$, $\langle A_x, A_y \rangle$ must then be a plane that contains L and the $q-1$ lines $\langle R_i \rangle^\perp \cap G$, $i = 2, 3, \ldots, q$. But this means that L and $\langle R_i \rangle^\perp$ are not disjoint for $i = 2, 3, \ldots, q$, a contradiction. So every two lines A_x and A_y, for distinct $x, y \in L$, are disjoint.
By the above, it follows that $\{L\} \cup \{\langle R_i \rangle^\perp \cap G \mid i = 1, 2, \ldots, q\}$ is a regulus of lines of $\mathcal{S}_G$, with opposite regulus $\{A_x \mid x \in L\}$. This property does not only hold for the q reguli $R_1, R_2, \ldots, R_q$, but for any q reguli of $\mathcal{M}$ through L which are contained in an elliptic quadric $Q^-_{ij}(5,q)$. As such, we have found that for any two lines $\langle R_i \rangle^\perp \cap G$ and $\langle R_j \rangle^\perp \cap G$, $i \neq j$, the regulus defined by these two lines and L completely consists of lines of $\mathcal{S}_G$. Consequently, there are $q(q+1)$ reguli through L in $\mathcal{S}_G$ and by Theorem 3.1 of Gevaert, Johnson and Thas [3], it follows that $\mathcal{S}_G$ is regular.
Finally, since the generator $G \in \mathcal{F}_2$ through L was chosen arbitrarily, the desired result follows from Lemma 3.1. □

Remark that the previous theorem is not valid for locally hermitian spreads of an elliptic quadric $Q^-(5,q)$, which is embedded in $Q^+(7,q)$. The following result solves the corresponding question in this case: it is possible to show that $\mathcal{M}^\theta$ is semiclassical if and only if the spread $\mathcal{M}$ of $Q^-(5,q)$ arises from a regular line spread of $\mathsf{PG}(3,q)$ by the construction, explained in Section 1.2.

Theorem 3.3 *Let $\mathcal{M}$ be a spread of an elliptic quadric $Q^-(5,q)$, embedded in*

$Q^+(7,q)$, which is locally hermitian at some line $L \in \mathcal{M}$, and consider a T-correspondence θ of Ω. Then the image $\mathcal{M}^\theta$ of $\mathcal{M}$ under θ is semiclassical if and only if $\mathcal{M} = \mathcal{S}(S)$ for a regular line spread S of $T_L(Q^-(5,q))$.

Proof.
Suppose that $\mathcal{M}$ is locally hermitian at the line $L \in \mathcal{M}$ and denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$. By Lemma 3.1, $\mathcal{M}^\theta$ is semiclassical if and only if for every generator $G \in \mathcal{F}_2$ of $Q^+(7,q)$ through L, the spread $\mathcal{S}_G$ is a regular line spread of G. As $\mathcal{M}$ is locally hermitian, we can write $\mathcal{M} = \mathcal{S}(S)$ for some line spread of the tangent space $\beta := T_L(Q^-(5,q))$ of $Q^-(5,q)$ at L. Let T be the line $\langle Q^-(5,q)\rangle^\perp$, with $\perp$ the polarity of $Q^+(7,q)$. Then $\langle \beta, T\rangle$ is the 5-dimensional subspace $L^\perp$, with $\perp$ again the polarity of $Q^+(7,q)$. Also, both β and $\langle L, T\rangle = \beta^\perp$ only intersect $Q^+(7,q)$ in L. Now the spread S in β consists of L and the q^2 lines $\langle R_i\rangle^\perp \cap \beta$, and every subspace $\langle R_i\rangle^\perp$ contains the line T. So the spread S is in fact the projection from T onto β of the spaces $\langle L, T\rangle \setminus T$ and $\langle R_i\rangle^\perp \setminus T$, $i = 1, 2, \ldots, q^2$.
If $G \in \mathcal{F}_2$ is a generator of $Q^+(7,q)$ through L, then $\beta \cap G$ is the line L. It also holds that the spread $\mathcal{S}_G$ consists of L and the q^2 lines $\langle R_i\rangle^\perp \cap G$, so it is the projection from T onto G of $\langle L, T\rangle \setminus T$ and $\langle R_i\rangle^\perp \setminus T$, $i = 1, 2, \ldots, q^2$. Hence it is clear that S is a regular line spread of β if and only if $\mathcal{S}_G$ is a regular line spread in G. This holds for all generators $G \in \mathcal{F}_2$ of $Q^+(7,q)$ containing the line L, so that the theorem follows. □

By now, we have quite extensive knowledge on what $\mathcal{M}^\theta$ looks like if $\mathcal{M} \subseteq Q(6,q)$ is locally hermitian and semiclassical and θ is a T-correspondence of Ω. If $\mathcal{M}$ is a spread of an induced elliptic quadric $Q^-(5,q) \subseteq Q(6,q)$, then Theorem 3.3 provides a criterion to decide whether $\mathcal{M}^\theta$ is semiclassical or not. Also, $\mathcal{M}^\theta$ spans the whole $\mathsf{PG}(7,q)$ whenever $\mathcal{M}$ is not hermitian.
If $\mathcal{M}$ is a proper locally hermitian, semiclassical 1-system of $Q(6,q)$, then $\mathcal{M}^\theta$ is also locally hermitian and semiclassical. Moreover, it is not contained in a hyperplane of $\mathsf{PG}(7,q)$, provided that $\mathcal{M}$ is not a spread of some $\mathsf{H}(q)$ on $Q(6,q)$. In particular, it then follows that $\mathcal{M}$ and $\mathcal{M}^\theta$ are not isomorphic.

In the next section, we will have a look at $\mathcal{M}^\theta$ in view of the results of [8], since a characterization of the proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$ was obtained there.

4 $\mathcal{M}^\theta$ compared to the known locally hermitian 1-systems of $Q^+(7,q)$

It is the aim of this section to compare the 1-systems $\mathcal{M}^\theta$, with $\mathcal{M}$ and θ as in the foregoing section, to the other previously known proper locally hermitian 1-systems of $Q^+(7,q)$, namely the ones that arise from an ovoid $\mathcal{O}$ of a hermitian polar space $H(3,q^2)$. This means the following.

Consider the quadratic extension $Q^+(7,q^2)$ of $Q^+(7,q)$ and let γ and $\overline{\gamma}$ be two disjoint generators of $Q^+(7,q^2)$, which are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$ and contain no point of $\mathsf{PG}(7,q)$. Denote by T the set of lines of $Q^+(7,q)$, the extensions of which to $\mathsf{GF}(q^2)$ have a point in common with both γ and $\overline{\gamma}$. It is known that the set T constitutes a partition of the point set of $Q^+(7,q)$ and that the intersection points of the extensions to $\mathsf{GF}(q^2)$ of the lines of T with γ, form a hermitian variety $H(3,q^2)$ in γ, and similarly for $\overline{\gamma}$. Let $\mathcal{O}$ be an ovoid of $H(3,q^2) \subseteq \gamma$ and let $\overline{\mathcal{O}} \subseteq \overline{\gamma}$ be its conjugate. Then by Shult and Thas [10], the set of lines $\mathcal{M} := \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in \mathcal{O}\}$ is a 1-system of $Q^+(7,q)$.

The next theorem states under which conditions $\mathcal{M}^\theta$, with $\mathcal{M}$ and θ as in the previous section, was previously known.

Theorem 4.1 *Suppose that $\mathcal{M}$ is a locally hermitian, semiclassical 1-system of a $Q(6,q) \subseteq Q^+(7,q)$ and let θ be a T-correspondence of Ω. Then $\mathcal{M}^\theta$ arises from an ovoid of $H(3,q^2)$ if and only if $\mathcal{M}$ is a spread of an induced $Q^-(5,q) \subseteq Q(6,q)$.*

Proof.
For this proof, let $\mathsf{PG}(7,q^2)$ be the extension of $\mathsf{PG}(7,q)$ to $\mathsf{GF}(q^2)$; the quadric $Q^+(7,q)$ extends to a hyperbolic quadric $Q^+(7,q^2)$.

First assume that $\mathcal{M}$ is a spread of an induced $Q^-(5,q) \subseteq Q(6,q)$. Then the extension of $Q^-(5,q)$ to $\mathsf{GF}(q^2)$ is a hyperbolic quadric $Q^+(5,q^2)$ and it holds that $\langle Q^+(5,q^2)\rangle^\perp \cap Q^+(7,q^2)$ is a secant line meeting $Q^+(7,q^2)$ in two points x and $\overline{x}$, which are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. As every line of $\mathcal{M}$, considered as a line over $\mathsf{GF}(q^2)$, lies in a totally singular plane of $Q^+(7,q^2)$ through x, and in another one through $\overline{x}$, every line of $\mathcal{M}^\theta$, again considered over $\mathsf{GF}(q^2)$, has a point in common with the disjoint generators x^θ and $\overline{x}^\theta$ of $Q^+(7,q^2)$. This implies that $\mathcal{M}^\theta = \{y\overline{y} \cap \mathsf{PG}(7,q) \mid y \in \mathcal{O}\}$ for some ovoid $\mathcal{O}$ of the hermitian variety $H(3,q^2)$ which is defined in x^θ by the lines of $Q^+(7,q)$, the extensions of which meet both x^θ and $\overline{x}^\theta$.

Conversely, assume that there exist two generators G and $\overline{G}$ of $Q^+(7,q^2)$, which are disjoint and conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. Further, suppose that $\mathcal{M}^\theta = \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in \mathcal{O}\}$ for some ovoid $\mathcal{O}$ of the hermitian variety $H(3,q^2)$, defined in G by the lines of $Q^+(7,q)$, the extensions of which to $\mathsf{GF}(q^2)$ meet both G and $\overline{G}$ in a point. There are two possibilities: either G and $\overline{G}$ are elements of the first family of generators of $Q^+(7,q^2)$, or they both belong to the second family of generators.
If G and $\overline{G}$ are elements of $\mathcal{F}_1$, then $G^{\theta^{-1}}$ and $\overline{G}^{\theta^{-1}}$ are points of $Q^+(7,q^2) \setminus Q^+(7,q)$. Moreover, every line of $\mathcal{M}^\theta$ has a point in common with G and $\overline{G}$, and consequently every line of $\mathcal{M}$, if considered over $\mathsf{GF}(q^2)$, lies in a totally singular plane of $Q^+(7,q^2)$ through $G^{\theta^{-1}}$ and in another one containing $\overline{G}^{\theta^{-1}}$. But this means that the line $\langle G^{\theta^{-1}}, \overline{G}^{\theta^{-1}}\rangle \cap \mathsf{PG}(7,q)$ is an external line of $Q^+(7,q)$, such that all lines of $\mathcal{M}$ are contained in the elliptic quadric

$\langle G^{\theta^{-1}}, \overline{G}^{\theta^{-1}} \rangle^{\perp} \cap Q^{+}(7,q) = Q^{-}(5,q)$.
In the case where $G, \overline{G} \in \mathcal{F}_2$, the spaces $G^{\theta^{-1}}$ and $\overline{G}^{\theta^{-1}}$ are generators of $Q^{+}(7,q^2)$ of the first family. Consider an arbitrary line $M \in \mathcal{M}^{\theta}$ and denote the common point of the extension of M to $\mathsf{GF}(q^2)$ and G by x. Then $M^{\theta^{-1}}$ is a line in the generator $x^{\theta^{-1}} \in \mathcal{F}_2$. But x is a point of G, which means that $x^{\theta^{-1}} \cap G^{\theta^{-1}}$ is a plane, so that the extension to $\mathsf{GF}(q^2)$ of $M^{\theta^{-1}}$ has a point y in common with $G^{\theta^{-1}}$. Similarly, the extension to $\mathsf{GF}(q^2)$ of $M^{\theta^{-1}}$ also has the point $\overline{y}$ in common with $\overline{G}^{\theta^{-1}}$. As $M \in \mathcal{M}^{\theta}$ was chosen arbitrarily, this implies that $\mathcal{M}$ too arises from an ovoid $\mathcal{O}'$ of the hermitian polar space $H(3,q^2)'$ in $G^{\theta^{-1}}$, by the usual construction. On the other hand, $\mathcal{M}$ is by assumption contained in a hyperplane of $\mathsf{PG}(7,q)$, so $\langle \mathcal{O}' \rangle$ cannot coincide with the 3-dimensional subspace $G^{\theta^{-1}}$. Thus $\langle \mathcal{O}' \rangle$ must be a plane in $G^{\theta^{-1}}$ and it follows that $\mathcal{M}$ is contained in the 5-dimensional subspace $\langle \mathcal{O}', \overline{\mathcal{O}'} \rangle$, considered over $\mathsf{GF}(q)$. We conclude that $\mathcal{M}$ is a spread of the elliptic quadric $\langle \mathcal{O}', \overline{\mathcal{O}'} \rangle \cap Q^{+}(7,q) = Q^{-}(5,q)$. □

By the information of this theorem, it is clear that $\mathcal{M}^{\theta}$ is not isomorphic to an example of a locally hermitian 1-system of $Q^{+}(7,q)$ which was known before, provided that $\langle \mathcal{M} \rangle = \mathsf{PG}(6,q)$ and $\mathcal{M}$ is not a spread of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$. Thus we have obtained an alternative description to the one in [8], of a subclass in the class of all proper locally hermitian, semiclassical 1-systems of $Q^{+}(7,q)$, namely the subclass of all such 1-systems of the form $\mathcal{M}^{\theta}$, with $\mathcal{M}$ and θ as above.

5 Isomorphism results for $\mathcal{M}^{\theta}$

Another question that appears naturally, is the following. In [6] and [7], it has been shown that there exist several orbits under the action of $\mathsf{P\Gamma O}(7,q)$ in the set of the proper locally hermitian, semiclassical 1-systems of $Q(6,q)$. It seems obvious in this context to wonder whether two non-isomorphic 1-systems $\mathcal{M}_1$ and $\mathcal{M}_2$ of $Q(6,q)$ yield non-isomorphic 1-systems $\mathcal{M}_1^{\theta}$ and $\mathcal{M}_2^{\theta}$ of $Q^{+}(7,q)$, with θ a T-correspondence as is usual in this chapter. It will turn out that this is indeed the case.

We start with two easy cases; the first one is a corollary of Theorem 4.1.

Corollary 5.1 *Let θ be a T-correspondence of Ω. A spread $\mathcal{M}$ of an induced $Q^{-}(5,q) \subseteq Q^{+}(7,q)$ is a hermitian spread of $Q^{-}(5,q)$ if and only if $\mathcal{M}^{\theta}$ is a hermitian spread of some $Q^{-}(5,q)' \subseteq Q^{+}(7,q)$.*

Proof.
$\mathcal{M}$ is a hermitian spread of a $Q^{-}(5,q)$ on $Q^{+}(7,q)$ if and only if there hold two things: firstly $\mathcal{M}$ must be a spread of a $Q^{-}(5,q)$ and secondly $\mathcal{M}$ must arise from an ovoid $\mathcal{O}$ of $H(3,q^2)$, which is in this case the classical ovoid $\mathcal{O} \cong H(2,q^2)$. By applying Theorem 4.1 to θ, it follows from the first property that $\mathcal{M}^{\theta}$ arises from some ovoid $\mathcal{O}'$ of a hermitian variety $H(3,q^2)'$. On the other hand, $\mathcal{M} =$

$(\mathcal{M}^\theta)^{\theta^{-1}}$ arises from an ovoid of $H(3,q^2)$, so that, again by Theorem 4.1 which we now apply to the T-correspondence θ^{-1}, $\mathcal{M}^\theta$ is a 1-system of a $Q^-(5,q)' \subseteq Q^+(7,q)$. By combining both results, the statement of the corollary follows. □

Corollary 5.2 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be locally hermitian, semiclassical spreads of the hexagon $\mathsf{H}(q)$, embedded in a $Q(6,q) \subseteq Q^+(7,q)$. Then $\mathcal{M}_1 \cong \mathcal{M}_2$ under the action of $\mathsf{P\Gamma O}^+(8,q)$ if and only if $\mathcal{M}_1^\theta \cong \mathcal{M}_2^\theta$ under the action of $\mathsf{P\Gamma O}^+(8,q)$, with θ a T-correspondence of Ω.*

Proof.
This is an immediate consequence of Lemma 2.3, Corollary 5.1 and the fact that both for q odd and for q even, there exist at most two non-isomorphic locally hermitian, semiclassical spreads of $\mathsf{H}(q)$: the hermitian spread of a $Q^-(5,q)$ and $\mathcal{S}_{[9]}$ if q is odd and $q \equiv 1 \bmod 3$, see [1], respectively $\mathcal{S}_{[\delta]}$ if q is even and $q = 2^{2e}$, see [5]. □

The next theorem handles the general case.

Theorem 5.3 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two proper locally hermitian, semiclassical 1-systems of a $Q(6,q) \subseteq Q^+(7,q)$ and let θ be a T-correspondence of Ω. Then $\mathcal{M}_1^\theta$ and $\mathcal{M}_2^\theta$ are isomorphic under the action of $\mathsf{P\Gamma O}^+(8,q)$ if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic under the action of $\mathsf{P\Gamma O}^+(8,q)$.*

Proof.
In the proof, we will need the element $\mu_p \in \mathsf{PGO}^+(8,q)$, see Hirschfeld and Thas [4, Lemma 22.6.3], with p the point $\langle Q(6,q)\rangle^\perp$ and $\perp$ the polarity of $Q^+(7,q)$, where μ_p stabilizes the points of $Q(6,q)$ and maps an arbitrary point r of $Q^+(7,q) \setminus Q(6,q)$ onto the unique second common point of the line pr and the quadric $Q^+(7,q)$. It can be shown that this map can be extended to an element of $\mathsf{PGO}^+(8,q)$, which is also denoted by μ_p. Let $G_1 \in \mathcal{F}_1$ be a generator of $Q^+(7,q)$. Then $G_1 \cap Q(6,q)$ is a totally singular plane, say π. As π is fixed by μ_p, the image of G_1 under μ_p must be a generator, different from G_1, through π. This generator is uniquely defined as the generator G_2 through π which belongs to $\mathcal{F}_2$. So $G_1^{\mu_p} = G_2$ and similarly $G_2^{\mu_p} = G_1$ and thus μ_p interchanges the families of generators $\mathcal{F}_1$ and $\mathcal{F}_2$.

First assume that $\mathcal{M}_1 \cong \mathcal{M}_2$ under the action of $\mathsf{P\Gamma O}^+(8,q)$, so there exists an element $\alpha \in \mathsf{P\Gamma O}^+(8,q)$ such that $\mathcal{M}_1^\alpha = \mathcal{M}_2$. If α interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\alpha\mu_p \in \mathsf{P\Gamma O}^+(8,q)$ fixes both families of generators and it also maps $\mathcal{M}_1$ onto $\mathcal{M}_2$. So we may assume that α preserves the families $\mathcal{F}_1$ and $\mathcal{F}_2$. In that case we have that

$$(\mathcal{M}_1^\theta)^{\theta^{-1}\alpha\theta} = \mathcal{M}_1^{\alpha\theta} = \mathcal{M}_2^\theta,$$

so that $\theta^{-1}\alpha\theta$ maps $\mathcal{M}_1^\theta$ onto $\mathcal{M}_2^\theta$. But, as α preserves the families of generators, the map $\theta^{-1}\alpha\theta$ stabilizes the sets $\mathcal{P}$, $\mathcal{L}$, $\mathcal{F}_1$ and $\mathcal{F}_2$, preserves the incidence between these sets and stabilizes $Q^+(7,q)$. Hence $\theta^{-1}\alpha\theta$ is an element of $\mathsf{P\Gamma O}^+(8,q)$, which shows that $\mathcal{M}_1^\theta$ and $\mathcal{M}_2^\theta$ are also isomorphic under $\mathsf{P\Gamma O}^+(8,q)$.

To prove the converse, assume that $\mathcal{M}_1^\theta$ and $\mathcal{M}_2^\theta$ are isomorphic under the action of $\mathsf{P\Gamma O}^+(8,q)$, with isomorphism $\alpha \in \mathsf{P\Gamma O}^+(8,q)$. Then it holds that $\mathcal{M}_1^{\theta\alpha} = \mathcal{M}_2^\theta$, from which one derives that $\mathcal{M}_1^{\theta\alpha\theta^{-1}} = \mathcal{M}_2$. Here one should distinguish between the case where α preseves $\mathcal{F}_1$ and $\mathcal{F}_2$ and the case where it interchanges both families of generators.
If α preseves the families $\mathcal{F}_1$ and $\mathcal{F}_2$, one easily sees that $\theta\alpha\theta^{-1}$ is itself an element of $\mathsf{P\Gamma L}(8,q)$, stabilizing $Q^+(7,q)$. So it immediately follows that $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$ under $\mathsf{P\Gamma O}^+(8,q)$, with isomorphism $\theta\alpha\theta^{-1}$.
Suppose next that α interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$. It still holds that the map $\theta\alpha\theta^{-1}$ maps $\mathcal{M}_1$ onto $\mathcal{M}_2$, but now $\theta\alpha\theta^{-1}$ is not a collineation. It is easily seen that $\theta\alpha\theta^{-1}$ still maps lines onto lines, but unfortunately it interchanges points and generators of the first family, while preserving the second family of generators. Now the element $\mu_p \in \mathsf{PGO}^+(8,q)$ comes into play, as we consider the map $\theta\alpha\theta^{-1}\mu_p$. Clearly this map acts as follows on $\mathcal{M}_1$:

$$\mathcal{M}_1^{\theta\alpha\theta^{-1}\mu_p} = \mathcal{M}_2^{\mu_p} = \mathcal{M}_2,$$

so $\theta\alpha\theta^{-1}\mu_p$ also maps $\mathcal{M}_1$ onto $\mathcal{M}_2$. But by the fact that μ_p interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$, $\theta\alpha\theta^{-1}\mu_p$ maps points onto elements of $\mathcal{F}_2$, elements of $\mathcal{F}_2$ onto elements of $\mathcal{F}_1$ and elements of $\mathcal{F}_1$ onto points. As it also preserves the incidence of Ω, $\theta\alpha\theta^{-1}\mu_p$ is a T-correspondence of Ω. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are 1-systems of a $Q(6,q) \subseteq Q^+(7,q)$, it now follows from Theorem 2.4 that $\mathcal{M}_1$ and $\mathcal{M}_2$ are spreads of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$. Consequently Corollary 5.2 yields that $\mathcal{M}_1 \cong \mathcal{M}_2$ under $\mathsf{P\Gamma O}^+(8,q)$.
This proves the theorem. □

Theorem 5.3 learns how many orbits under the action of $\mathsf{P\Gamma O}^+(8,q)$ there are in the set of all 1-systems of $Q^+(7,q)$ of the form $\mathcal{M}^\theta$, with $\mathcal{M}$ a proper locally hermitian, semiclassical 1-system of $Q(6,q) \subseteq Q^+(7,q)$ and θ a fixed T-correspondence or triality of the D_4-geometry Ω, attached to $Q^+(7,q)$. This number of orbits equals the number of orbits under the action of $\mathsf{P\Gamma O}(7,q)$ in the set of all proper locally hermitian, semiclassical 1-systems of $Q(6,q)$. This number is known: for q odd, it equals the number of orbits of $\mathrm{Aut}(\mathsf{GF}(q))$ in the set of squares of $\mathsf{GF}(q) \setminus \{0,1\}$; see the remark following Theorem 7.2 of [6]. If q is even, it equals the number of orbits of $\mathrm{Aut}(\mathsf{GF}(q))$ in the set of all elements of $\mathsf{GF}(q) \setminus \{0\}$ with trace zero; see the remark following Theorem 6.2 in [7].

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