The uniqueness of the 1-system of $Q^{-}(7,q)$, q even

D. Luyckx*
J. A. Thas
Ghent University, Dept. of Pure Mathematics and Computer Algebra,
Galglaan 2, B-9000 Ghent, Belgium
dluyckx@cage.rug.ac.be, jat@cage.rug.ac.be

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Abstract

For q odd, it was shown in [3] that the elliptic quadric $Q^{-}(7,q)$ possesses a unique 1-system, the so-called classical 1-system. Here, the same result will be obtained for even q.

1 Basic properties of 1-systems of $Q^{-}(7,q)$

A *1-system* $\mathcal{M}$ of the elliptic quadric $Q^{-}(7,q)$ is a set $\{L_0, L_1, \ldots, L_{q^4}\}$ of q^4+1 lines of $Q^{-}(7,q)$ with the property that every plane of $Q^{-}(7,q)$ containing a line L_i of $\mathcal{M}$ has an empty intersection with $(L_0 \cup L_1 \cup \ldots \cup L_{q^4}) \setminus L_i$. We denote the union of all elements of $\mathcal{M}$ by $\widetilde{\mathcal{M}}$. Concerning the generators of $Q^{-}(7,q)$, which are planes, the following result is shown by Shult and Thas in [5] in a more general context; here it is stated for 1-systems of $Q^{-}(7,q)$ in particular.

Theorem 1.1 (Shult and Thas [5]) *If $\mathcal{M}$ is a 1-system of the elliptic quadric $Q^{-}(7,q)$, then every generator of $Q^{-}(7,q)$ contains exactly $q+1$ points of $\widetilde{\mathcal{M}}$.*

In [2] a similar result for lines of $Q^{-}(7,q)$ is obtained, as it is shown that every line of $Q^{-}(7,q)$ has 0, 1, 2 or $q+1$ points in common with $\widetilde{\mathcal{M}}$, where the latter occurs if and only if the line belongs to $\mathcal{M}$. In combination with Theorem 1.1, this implies that every totally singular plane of $Q^{-}(7,q)$ either contains a line of $\mathcal{M}$, or a $(q+1)$-arc of points of $\widetilde{\mathcal{M}}$.

Let $\mathcal{M}$ be an arbitrary 1-system of $Q^{-}(7,q)$. If L_1, L_2 and L_3 are arbitrary lines of $\mathcal{M}$, then $\langle L_1, L_2\rangle$ is 3-dimensional and it intersects $Q^{-}(7,q)$ in a hyperbolic quadric $Q^{+}(3,q)$. This hyperbolic quadric $Q^{+}(3,q)$ contains no points

*The first author is Research Assistant of the Fund for Scientific Research – Flanders (Belgium) (F.W.O.).

of $\widetilde{\mathcal{M}}$, apart from the points on L_1 and L_2, for otherwise there would exist a line of $Q^+(3,q)$ meeting L_1 and L_2 and containing at least three points of $\widetilde{\mathcal{M}}$, a contradiction since such a line cannot be a line of $\mathcal{M}$. Hence L_3 has no point in common with $\langle L_1, L_2\rangle$ and it follows that $\langle L_1, L_2, L_3\rangle$ is 5-dimensional for every three distinct lines L_1, L_2 and L_3 of $\mathcal{M}$.

Both for q even and for q odd, there is exactly one 1-system of $Q^-(7,q)$ known, which is the unique 1-system of $Q^-(7,q)$ if q is odd, see [3]. It is called the *classical* 1-system of $Q^-(7,q)$ and can be constructed as follows. In the extension $\mathsf{PG}(7,q^2)$ of $\mathsf{PG}(7,q) = \langle Q^-(7,q)\rangle$, there exist two disjoint 3-spaces γ and $\overline{\gamma}$, which are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, polar with respect to the polarity defined by the extension $Q^+(7,q^2)$ of $Q^-(7,q)$, and such that $\gamma \cap Q^+(7,q^2)$ is an elliptic quadric $Q^-(3,q^2)$. The classical 1-system $\mathcal{M}$ of $Q^-(7,q)$ then consists of all lines $x\overline{x} \cap \mathsf{PG}(7,q)$, where x varies on $Q^-(3,q^2)$ and $\overline{x}$ is its conjugate, so is a point of $\overline{\gamma} \cap Q^+(7,q^2)$. In Shult and Thas [5], one can find a proof that the set of lines $\mathcal{M} = \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in Q^-(3,q^2)\}$ is indeed a 1-system of $Q^-(7,q)$.

From now on it is assumed that q is even and it will be shown that also in this case, the classical 1-system is the unique 1-system of $Q^-(7,q)$.

2 A 1-system of $Q^-(7,q)$ is an egg of $\mathsf{PG}(7,q)$

In this section, it will be shown that every 1-system of $Q^-(7,q)$, q even, is an egg of the ambient space $\mathsf{PG}(7,q)$ of $Q^-(7,q)$, so we start with the general definition of an egg $O(n,2n,q)$.

An *egg* $O(n,2n,q)$ of $\mathsf{PG}(4n-1,q)$ is a set of $q^{2n}+1$ $(n-1)$-dimensional subspaces $\pi_0, \pi_1, \ldots, \pi_{q^{2n}}$ of $\mathsf{PG}(4n-1,q)$, every three of which generate a $\mathsf{PG}(3n-1,q)$ and such that each element π_i of $O(n,2n,q)$ is contained in a $\mathsf{PG}^{(i)}(3n-1,q)$, having no point in common with $(\pi_0 \cup \pi_1 \cup \cdots \cup \pi_{q^{2n}}) \setminus \pi_i$. The space $\mathsf{PG}^{(i)}(3n-1,q)$ is called the *tangent space* of $O(n,2n,q)$ at π_i. An egg $O(n,2n,q)$ is called *good* at an element π_i if for all distinct π_j, π_k, $j \neq i \neq k$, the space generated by π_i, π_j and π_k contains exactly q^n+1 elements of $O(n,2n,q)$. An egg $O(n,2n,q)$ is called *regular* if it is constructed in the following way.

Consider the algebraic extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$ and the corresponding extension $\mathsf{PG}(4n-1,q^n)$ of $\mathsf{PG}(4n-1,q)$. Consider n 3-dimensional subspaces $\mathsf{PG}^{(1)}(3,q^n), \mathsf{PG}^{(2)}(3,q^n), \ldots, \mathsf{PG}^{(n)}(3,q^n)$ of $\mathsf{PG}(4n-1,q^n)$, which generate the space $\mathsf{PG}(4n-1,q^n)$ and constitute a conjugate n-tuple with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. Let $\mathcal{O}$ be an ovoid of $\mathsf{PG}^{(1)}(3,q^n)$. With every point $p^{(1)}$ of $\mathcal{O}$, there correspond $n-1$ points $p^{(2)}, p^{(3)}, \ldots, p^{(n)}$ such that the points $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$ constitute a conjugate n-tuple with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. The points $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$ define an $(n-1)$-dimensional subspace of $\mathsf{PG}(4n-1,q)$. If we let $p^{(1)}$ vary in $\mathcal{O}$, we obtain $q^{2n}+1$ such $(n-1)$-dimensional subspaces of $\mathsf{PG}(4n-1,q)$, which form a regular egg $O(n,2n,q)$. Regular eggs are good at each of their elements. In [6], Thas shows that the converse also holds.

Theorem 2.1 (Thas [6]) *Every egg $O(n, 2n, q)$ of* $\mathsf{PG}(4n-1, q)$ *which is good at each of its elements, is regular.*

In the special case where the ovoid $\mathcal{O}$ of $\mathsf{PG}^{(1)}(3, q^n)$ is an elliptic quadric $Q^-(3, q^n)$, the egg is said to be *classical*.

Concerning 1-systems of $Q^-(7, q)$, with q even, a first result is obtained in the following lemma.

Lemma 2.2 *Let π be a totally singular plane of $Q^-(7, q)$, containing a $(q+1)$-arc $\mathcal{O}$ of points of $\widetilde{\mathcal{M}}$. If $x \in \mathcal{O}$ is arbitrary and $M \in \mathcal{M}$ is the line of $\mathcal{M}$ containing x, then for any point $z \in \pi \setminus \mathcal{O}$, the plane $\langle z, M\rangle$ is totally singular if and only if z lies on the tangent line of $\mathcal{O}$ at x.*

Proof.
If the plane $\langle z, M\rangle$ is totally singular and xz is a secant line of $\mathcal{O}$, then the generator $\langle z, M\rangle$ of $Q^-(7, q)$ contains the line M of $\mathcal{M}$ and a point of $\widetilde{\mathcal{M}}$ not on M, namely the point $(\mathcal{O} \cap xz) \setminus \{x\}$. This contradicts the definition of a 1-system of $Q^-(7, q)$ and it follows that xz must be the tangent line of $\mathcal{O}$ at x.

To prove the converse, consider an arbitrary point y of $M \setminus \{x\}$. Then $y^\perp$ intersects π in some line K through x. If K were a secant of $\mathcal{O}$, then the totally singular plane $\langle y, K\rangle$ would again contain the line $M \in \mathcal{M}$, and some other point of $\widetilde{\mathcal{M}}$, not on M, a contradiction. Hence K must be the unique tangent of $\mathcal{O}$ at x and one concludes that the plane $\langle z, M\rangle$ is totally singular for every point z of $K \setminus \{x\}$. □

In Shult and Thas [5] it is proved that a 1-system of $Q^-(7, q)$ has two intersection numbers with respect to hyperplanes. In particular, a hyperplane H of $\mathsf{PG}(7, q)$ contains exactly one line of $\mathcal{M}$ if and only if it is the tangent hyperplane of $Q^-(7, q)$ at some point of $\widetilde{\mathcal{M}}$. Otherwise, H contains exactly q^2+1 lines of $\mathcal{M}$. In the next lemma, it is stated that whenever H is the tangent hyperplane of $Q^-(7, q)$ at a point $p \notin \widetilde{\mathcal{M}}$, then the q^2+1 lines of $\mathcal{M}$ in H lie in fact in a 5-dimensional subspace of $\mathsf{PG}(7, q)$.

Lemma 2.3 *For every point p of $Q^-(7, q) \setminus \widetilde{\mathcal{M}}$, the q^2+1 lines of $\mathcal{M}$ in $p^\perp$ are contained in a 5-dimensional subspace of* $\mathsf{PG}(7, q)$.

Proof.
Consider a totally singular plane π of $Q^-(7, q)$ through the point p, and such that π meets $\widetilde{\mathcal{M}}$ in a $(q+1)$-arc $\mathcal{O}$. As q is even, the $(q+1)$-arc $\mathcal{O}$ has a nucleus n, which is the intersection of all tangent lines of $\mathcal{O}$. If $p = n$, then p lies on all $q+1$ tangents of $\mathcal{O}$ and by the previous lemma, it follows that $\langle p, M\rangle$ is a totally singular plane for all lines $M \in \mathcal{M}$ containing a point of $\mathcal{O}$. Hence the $q+1$ lines

of $\mathcal{M}$ which intersect π in a point, are contained in $p^\perp$. If p is not the nucleus n of $\mathcal{O}$, then p lies on exactly one tangent line of $\mathcal{O}$, say at the point $x \in \mathcal{O}$. Again by the previous lemma, the line $M \in \mathcal{M}$ through x is the unique line of $\mathcal{M}$ intersecting π with the property that $\langle p, M\rangle$ is a totally singular plane. It follows that this line M is the only line of $\mathcal{M}$ containing a point of $\mathcal{O}$, which is completely contained in $p^\perp$.

Let $\gamma \subseteq p^\perp$ be a 5-dimensional subspace not through p, so that $\gamma \cap Q^-(7,q)$ is an elliptic quadric $Q^-(5,q)$. We project the q^2+1 lines of $\mathcal{M}$ in $p^\perp$ from p onto γ; this yields a set $\mathcal{M}_p$ of q^2+1 mutually disjoint lines of $Q^-(5,q)$. Starting from this set $\mathcal{M}_p$, one can define a substructure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of the generalized quadrangle $Q^-(5,q)$ as follows. The point set $\mathcal{P}$ is the set of all points on the lines of $\mathcal{M}_p$, the line set $\mathcal{B}$ consists of all lines of $Q^-(5,q)$ consisting entirely of points of $\mathcal{P}$, and incidence I is the incidence of $Q^-(5,q)$. We will show that $\mathcal{S}$ is a subquadrangle of $Q^-(5,q)$. To that end, consider an arbitrary line L of $Q^-(5,q)$, which does not belong to $\mathcal{M}_p$. Then $\langle p, L\rangle$ is a totally singular plane of $Q^-(7,q)$ containing a $(q+1)$-arc $\mathcal{O}$ of points of $\widetilde{\mathcal{M}}$. By the previous paragraph it follows that either exactly one, or all $q+1$ points of L lie on a line of $\mathcal{M}_p$, so either L is a line of $\mathcal{B}$, or it meets $\mathcal{P}$ in a unique point. Since this holds for all lines L of $Q^-(5,q)$ not in $\mathcal{M}_p$, it follows that every line of $Q^-(5,q)$ containing at least two distinct points of $\mathcal{P}$ belongs to $\mathcal{B}$. This implies that the substructure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ satisfies the conditions of [4, Theorem 2.3.1] and since the lines of $\mathcal{B}$ are pairwise disjoint, one concludes from [4, Theorem 2.3.1] that $\mathcal{S}$ is a subquadrangle of $Q^-(5,q)$. Moreover, the lines of $\mathcal{M}_p$ partition the point set of this subquadrangle and hence $\mathcal{M}_p$ is a spread of $\mathcal{S}$, so that $|\mathcal{P}| = (q+1)(q^2+1)$. Consequently, $\mathcal{S}$ has order (q,q) and thus it is a generalized quadrangle $Q(4,q)$. It readily follows that the q^2+1 lines of $\mathcal{M}$ in $p^\perp$ all lie in a 5-dimensional subspace ε, where $\varepsilon \cap Q^-(7,q)$ is a cone $pQ(4,q)$ with p as vertex and a parabolic quadric $Q(4,q)$ as base. □

We are now ready to show the main result of this section.

Theorem 2.4 *If q is even, then every 1-system of $Q^-(7,q)$ is an egg $O(2,4,q)$ of $\mathsf{PG}(7,q)$, which is good at each of its elements.*

Proof.
By definition, all q^4+1 lines of a 1-system $\mathcal{M}$ of $Q^-(7,q)$ are pairwise disjoint and in Section 1 it has been explained that every three distinct elements of $\mathcal{M}$ span a $\mathsf{PG}(5,q)$. Moreover, the quadric $Q^-(7,q)$ has a unique 5-dimensional tangent space at each line L of $\mathcal{M}$, and this tangent space is disjoint from all lines of $\mathcal{M} \setminus \{L\}$ by definition. Hence the lines of $\mathcal{M}$ form an egg $O(2,4,q)$ of $\mathsf{PG}(7,q)$ and it only remains to show that $\mathcal{M}$, considered as an egg $O(2,4,q)$, is good at each of its elements.

On $Q^-(7,q)$, there are $(q^4+1)q^2$ points p which do not belong to $\widetilde{\mathcal{M}}$. By the previous lemma, it holds for every such point p that the q^2+1 lines of $\mathcal{M}$ in $p^\perp$

span a 5-dimensional subspace of $\mathsf{PG}(7,q)$, which intersects $Q^-(7,q)$ in a cone $pQ(4,q)$; also, for any three such lines L_i, L_j, L_k, we have that $\langle L_i, L_j, L_k\rangle^\perp \cap Q^-(7,q) = \{p\}$, and so p is uniquely defined by $\{L_i, L_j, L_k\}$. It follows that there exist

$$\frac{q^2(q^4+1)\cdot(q^2+1)q^2(q^2-1)}{3\cdot 2\cdot 1} = \frac{1}{6}(q^4+1)q^4(q^4-1)$$

triples (L_i, L_j, L_k) of distinct lines of $\mathcal{M}$ with the property that $\langle L_i, L_j, L_k\rangle \cap Q^-(7,q)$ is some cone $pQ(4,q)$ containing exactly q^2+1 lines of $\mathcal{M}$. On the other hand, the fact that $|\mathcal{M}| = q^4+1$ implies that $\frac{1}{6}(q^4+1)q^4(q^4-1)$ is also the total number of triples (L_i, L_j, L_k) consisting of three distinct lines of $\mathcal{M}$. Hence it holds for every triple of distinct lines L_i, L_j, L_k of $\mathcal{M}$ that $\langle L_i, L_j, L_k\rangle$ is a 5-dimensional subspace of $\mathsf{PG}(7,q)$ which contains exactly q^2+1 lines of $\mathcal{M}$. So the 1-system $\mathcal{M}$, considered as an egg $O(2,4,q)$ of $\mathsf{PG}(7,q)$, is good at each of its elements. $\square$

As has been mentioned in Theorem 2.1, every egg which is good at each of its elements, is regular. This implies that there exist two disjoint and conjugate 3-dimensional subspaces ρ and $\overline{\rho}$ in the extension $\mathsf{PG}(7,q^2)$ of $\mathsf{PG}(7,q)$ to $\mathsf{GF}(q^2)$, and ovoids $\mathcal{O} \subseteq \rho$, respectively $\overline{\mathcal{O}} \subseteq \overline{\rho}$, such that $\mathcal{M} = \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in \mathcal{O}\}$. In the next section, it will be deduced from this property that $\mathcal{M}$ is classical.

3 The uniqueness result

Using the same notation as above, there remain two more properties to show. Firstly, it must hold that the ovoid $\mathcal{O}$ is the elliptic quadric $Q^-(3,q^2) = \rho \cap Q^+(7,q^2)$, and similarly for its conjugate $\overline{\mathcal{O}}$. Secondly, one must prove that $\overline{\rho} = \rho^\perp$, so the conjugate $\overline{\rho}$ of ρ must coincide with the image $\rho^\perp$ of ρ under the polarity $\perp$ of $Q^+(7,q^2)$. These properties will be shown in the following theorem.

Theorem 3.1 *If q is even, the elliptic quadric $Q^-(7,q)$ has a unique 1-system up to a projectivity.*

Proof.
Let $\mathcal{M}$ be a 1-system of $Q^-(7,q)$, q even. Then by Theorem 2.4, $\mathcal{M}$ is a regular egg of the ambient space $\mathsf{PG}(7,q)$ of $Q^-(7,q)$. Denote the extensions to $\mathsf{GF}(q^2)$ of $\mathsf{PG}(7,q)$ and $Q^-(7,q)$ by $\mathsf{PG}(7,q^2)$, respectively $Q^+(7,q^2)$. In $\mathsf{PG}(7,q^2)$ there exist two disjoint and conjugate 3-spaces ρ and $\overline{\rho}$ such that $\mathcal{M} = \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in \mathcal{O}\}$, for some ovoid $\mathcal{O} \subseteq \rho$.

Since all extensions of lines of $\mathcal{M}$ to $\mathsf{GF}(q^2)$ are lines of $Q^+(7,q^2)$, the ovoid $\mathcal{O}$ must be contained in $\rho \cap Q^+(7,q^2)$. Moreover $\mathcal{O}$ is an ovoid of ρ, which implies that a totally singular line, respectively plane, of $\rho \cap Q^+(7,q^2)$ may contain at most 2, respectively $q+2$ points of $\mathcal{O}$. These observations exclude the possibilities that $\rho \cap Q^+(7,q^2)$ is a hyperbolic quadric $Q^+(3,q^2)$, or a cone $xQ(2,q^2)$, or a cone $LQ^+(1,q^2)$ for some point x or some line L of $Q^+(7,q^2)$; also $\rho \cap Q^+(7,q^2)$ cannot be of type $LQ^-(3,q^2)$ nor can be a unique plane. Furthermore, one easily counts that the number of generators of $Q^+(7,q^2)$ which intersect $Q^-(7,q)$ in a plane equals $2(q^2+1)(q^3+1)(q^4+1)$, that every generator of $Q^+(7,q^2)$ which contains a line of $Q^-(7,q)$ also contains a plane of $Q^-(7,q)$, and that there exist $2(q^2+1)(q^4+1)q^3(q^3-1)$ generators of $Q^+(7,q^2)$ having exactly one point in common with $Q^-(7,q)$. As $2(q^2+1)(q^3+1)(q^4+1)+2(q^2+1)(q^4+1)q^3(q^3-1)$ equals the total number of generators of $Q^+(7,q^2)$, it follows that ρ must contain at least one point of $Q^-(7,q)$ if it is a generator of $Q^+(7,q^2)$. But this contradicts the fact that $\rho \cap \overline{\rho} = \emptyset$ and consequently $\rho \cap Q^+(7,q^2)$ must be an elliptic quadric $Q^-(3,q^2)$, which thus coincides with the ovoid $\mathcal{O}$.

We proceed to show that $\overline{\rho} = \rho^\perp$. First, let x be an arbitrary point of $\mathcal{O}$. Then the image $x^\perp$ of x with respect to the polarity of $Q^+(7,q^2)$ is a 6-dimensional subspace containing $L^\perp$, with L the line $x\overline{x} \cap \mathsf{PG}(7,q)$ of $\mathcal{M}$. But $L^\perp \cap \mathsf{PG}(7,q)$ is the tangent space $T_L(Q^-(7,q))$ of $Q^-(7,q)$ at L, so that $L^\perp \cap Q^-(7,q) = LQ^-(3,q)$. If $x^\perp$ contained a point of $Q^-(7,q) \setminus L^\perp$, then it would contain a 6-dimensional subspace of $\mathsf{PG}(7,q)$, a contradiction. So $x^\perp$ intersects $Q^-(7,q)$ exactly in $LQ^-(3,q)$. If y is an arbitrary point of $\mathcal{O} \setminus \{x\}$, then $x^\perp$ and the line $y\overline{y}$ have a point in common, say p, which is a point of $Q^+(7,q^2) \setminus Q^-(7,q)$ by the above observation. Let $\mathcal{C}$ be any conic on $\mathcal{O}$ containing x and y. Now the number of points on $y\overline{y}$ which do not belong to $\mathsf{PG}(7,q)$ equals q^2-q, while $\mathcal{C} \setminus \{y\}$ has q^2 points. Hence there must exist distinct points u, u' on $\mathcal{C} \setminus \{y\}$ such that $u^\perp \cap y\overline{y} = u'^\perp \cap y\overline{y} = \{w\}$. But then the point w is a point of $u^\perp \cap y^\perp \cap u'^\perp = \langle \mathcal{C} \rangle^\perp$, and so $z^\perp \cap y\overline{y} = \{w\}(=\{p\})$ for any point z on $\mathcal{C} \setminus \{y\}$. Since $\mathcal{C}$ is any conic on $\mathcal{O}$ containing x and y, it follows that p is a point of $\rho^\perp$ and consequently $\rho^\perp$ has a point in common with all lines $y\overline{y}$, $y \in \mathcal{O}$.

Suppose next that $\overline{\rho} \cap \rho^\perp$ is at least a point r. If $\overline{\rho} \neq \rho^\perp$, then these two 3-spaces have at most a plane in common and $\langle \overline{\rho}, \rho^\perp \rangle$ is at most 6-dimensional. But both $\overline{\rho}$ and $\rho^\perp$ contain a point of each line $x\overline{x}$, $x \in \mathcal{O}$, and as $\overline{\rho} \cap \rho^\perp$ is at most a plane, this implies that $\langle \overline{\rho}, \rho^\perp \rangle \cap \mathsf{PG}(7,q)$, so also any hyperplane of $\mathsf{PG}(7,q)$ through $\langle \overline{\rho}, \rho^\perp \rangle \cap \mathsf{PG}(7,q)$, contains at least q^4-q^2 lines of $\mathcal{M}$. This is a contradiction to the fact that a hyperplane of $\mathsf{PG}(7,q)$ contains either 1 or q^2+1 lines of $\mathcal{M}$ and one concludes that $\overline{\rho} \cap \rho^\perp$ is either empty, or $\overline{\rho}$ and $\rho^\perp$ coincide.

Finally, assume that $\overline{\rho} \cap \rho^\perp$ is empty. Then there exist 3 pairwise disjoint 3-dimensional subspaces ρ, $\overline{\rho}$ and $\rho^\perp$, having a point in common with the extensions to $\mathsf{GF}(q^2)$ of all lines of $\mathcal{M}$. By Theorem 25.6.1 of [1], this implies that all lines $x\overline{x}$, $x \in \mathcal{O}$, are elements of a system of maximal spaces of a Segre variety $\mathcal{S}_{1;3}$ in $\mathsf{PG}(7,q^2)$. On the other hand, the lines of $\mathcal{M}$ are lines of $\mathsf{PG}(7,q)$, and the restriction of $\mathcal{S}_{1;3}$ to $\mathsf{PG}(7,q)$ is a Segre variety $\mathcal{S}'_{1;3}$ of $\mathsf{PG}(7,q)$. This is nevertheless impossible because the system of maximal lines of the Segre variety $\mathcal{S}'_{1;3}$ in $\mathsf{PG}(7,q)$ contains q^3+q^2+q+1 lines and hence cannot contain all q^4+1

lines of $\mathcal{M}$. It follows that $\overline{\rho}$ and $\rho^{\perp}$ are not disjoint either. Consequently, $\overline{\rho} = \rho^{\perp}$ and $\mathcal{M}$ is the classical 1-system of $Q^{-}(7, q)$. $\square$

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