A note on quasi-Hermitian varieties and singular quasi-quadrics

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Abstract

Quasi-quadrics were introduced by Penttila, De Clerck, O'Keefe and Hamilton in [2]. They are defined as point sets which have the same intersection numbers with respect to hyperplanes as non-singular quadrics. We extend this definition in two ways.

The first extension is to *quasi-Hermitian varieties*, which are point sets which have the same intersection numbers with respect to hyperplanes as non-singular Hermitian varieties.

The second one is to *singular quasi-quadrics*, i.e. point sets $\mathcal{K}$ which have the same intersection numbers with respect to hyperplanes as singular quadrics. Our starting point was to investigate whether every singular quasi-quadric is a cone over a non-singular quasi-quadric. This question is tackled in the case of a point set $\mathcal{K}$ with the same intersection numbers with respect to hyperplanes as a point over an ovoid.

1 Introduction

In [2] quasi-quadrics were introduced, i.e. point sets $\mathcal{K}$ in $\mathrm{PG}(n,q)$ which have the same intersection numbers with respect to hyperplanes as non-singular quadrics. In that paper there is a free construction of these structures, yielding an overwhelming amount of examples. In this paper we define quasi-Hermitian varieties, i.e. the analogous concept of quasi-quadrics for Hermitian varieties and provide similar free constructions of them.

In [4], we proved that if one additionally assumes that $\mathcal{K}$ has the same intersection numbers with respect to spaces of codimension 2 as non-singular

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quasi-quadrics (non-singular Hermitian varieties), then $\mathcal{K}$ is a non-singular quadric (non-singular Hermitian variety).

The goal of this paper is to extend the theory to singular quadrics (Hermitian varieties). We prove similar results in the low-dimensional case.

2 Quasi-Hermitian varieties

Definition 1
A set of points H in $PG(n, q^2)$, is called a *quasi-Hermitian variety* in $PG(n, q^2)$, if its intersection numbers with hyperplanes are the size of a non-singular Hermitian variety $H(n-1, q^2)$, namely

$$\frac{(q^n + (-1)^{n-1})(q^{n-1} - (-1)^{n-1})}{q^2 - 1}$$

or the size of a cone with vertex a point p and base a non-singular Hermitian variety $H(n-2, q^2)$, shortly denoted by $pH(n-2, q^2)$, namely

$$\frac{(q^n + (-1)^{n-1})(q^{n-1} - (-1)^{n-1})}{q^2 - 1} + (-1)^{n-1}q^{n-1}.$$

We will call hyperplanes intersecting H in $|H(n-1, q^2)|$ points *secant*, the other ones *tangent*.

We will show that not all quasi-Hermitian varieties are Hermitian varieties. Our first construction is the Hermitian analogue of a construction method by Penttila, De Clerck, O'Keefe and Hamilton, a method which they call pivoting.

Let $H(n, q^2)$ be a non-singular Hermitian variety. Take a point p on $H(n, q^2)$ and consider the tangent space Π of the Hermitian variety at p. This space intersects the Hermitian variety in a cone with vertex p and base a non-singular Hermitian variety $H(n-2, q^2)$ lying in a $PG(n-2, q^2)$. We replace this non-singular Hermitian variety $H(n-2, q^2)$ by a quasi-Hermitian variety in $PG(n-2, q^2)$, say H'. We call the set of points contained in $(H(n, q^2) - pH(n-2, q^2)) \cup pH'$ a pivoted set of $H(n, q^2)$ with respect to p.

Theorem 2
Every pivoted set of $H(n, q^2)$ with respect to a point p of $H(n, q^2)$ is a quasi-Hermitian variety in $PG(n, q^2)$.

Proof. We have to prove that all hyperplanes intersect the pivoted set in the correct number of points. Since we only replace points in the tangent space

Π through p, we only have to look at the intersection of the hyperplanes α with Π.

1) If α equals Π, then α has the same number of intersection points with the pivoted set as with $H(n, q^2)$.

2) Next suppose that α intersects Π in an $(n-2)$-dimensional space. If α contains p then there are two cases to consider. The first possibility is that α intersects $H(n-2, q^2)$ and H' in the same number of points, in which case the total intersection number of α and the pivoted set is the same as the intersection number of α with $H(n, q^2)$. The second possibility is that α has different intersection numbers with $H(n-2, q^2)$ and H', but then this difference is equal to,

$$|H(n-3, q^2)| - |pH(n-4, q^2)| = (-1)^n q^{n-3},$$

and so the total difference for the intersection size is $q^2(-1)^n q^{n-3} = (-1)^n q^{n-1}$ which equals

$$|H(n-1, q^2)| - |pH(n-2, q^2)|,$$

hence we get a valid intersection number.

If α does not contain p, then α intersects the intersection of the pivoted set and Π in a set of size $|H'| = |H(n-2, q^2)|$, hence the number of intersection points is unchanged. $\square$

The second construction of a quasi-Hermitian variety only works in odd dimension since in even dimension the generators are too small for this construction to work. It is the Hermitian analogue of a theorem of Delanote [3].

Theorem 3
Let Π be an $(n-1)$-dimensional space lying on $H(2n+1, q^2)$. Consider the $q+1$ generators G_i, $1 \leq i \leq q+1$ on $H(2n+1, q^2)$ through Π. Consider also spaces Π_i, $1 \leq i \leq q+1$ through Π inside the tangent space Π^* of $H(2n+1, q^2)$ at Π which intersect the Hermitian variety exactly in Π. Consider

$$H' = (H(2n+1, q^2) \setminus \cup_i G_i) \cup (\cup_i \Pi_i)$$

This set H' is a quasi-Hermitian variety in $PG(2n+1, q^2)$.

Proof. Again we only have to look at the intersection of the hyperplanes α with the $(n+1)$-space Π^* since only there we replace points. If α contains Π^* then α has the same number of intersection points with H' as with $H(2n+1, q^2)$. So suppose that α intersects Π^* in an n-dimensional subspace.

1) If α intersects Π^* in one of the generators G_i, then it is a tangent hyperplane, hence we get the following number of points in $\alpha \cap H'$.

$$|pH(2n-1,q^2)| - |G_i \backslash \Pi| = |H(2n,q^2)|.$$

2) If $\alpha \cap \Pi^*$ is one of the Π_i then α is a secant hyperplane, hence we get the following number of points in $\alpha \cap H'$.

$$|\Pi_i \backslash \Pi| + |H(2n,q^2)| = |pH(2n-1,q^2)|.$$

3) If $\alpha \cap \Pi^*$ is an n-dimensional space containing Π different from the generators G_i and the spaces Π_j, then clearly we get a correct number of intersection points in $\alpha \cap H'$.

4) The last possibility is that α intersects each of the $q+1$ n-dimensional spaces G_i in an $(n-1)$-dimensional space P_i with $P_i \cap \Pi = \alpha \cap \Pi = Y$ an $(n-2)$-dimensional space.

Let $\Pi_j \cap \alpha = P_{q+1+j}$, with $j = 1, 2, ..., q+1$. We have replaced $(\cup_{j=1}^{q+1} P_j) \backslash Y$ by $(\cup_{j=q+2}^{2(q+1)} P_j) \backslash Y$. This clearly yields a correct number of intersection points in $\alpha \cap H'$.

4) $\square$

Next we prove the Hermitian analogue of a remark in the Ph.D. thesis of Delanote [3]. Again we give a construction of a quasi-Hermitian variety, one which only works in odd dimension for $q = 2$.

Theorem 4
Consider $H' = H(2n+1,q^2) \backslash G$ where G is a generator of $H(2n+1,q^2)$. The complement of H' in $PG(2n+1,q^2)$ is a quasi-Hermitian variety in $PG(2n+1,q^2)$ if and only if $q = 2$.

Proof. A hyperplane α either contains G or intersects G in an $(n-1)$-dimensional space. If α contains G we know α is a tangent hyperplane. So the possible intersections of H' with hyperplanes are

$$|pH(2n-1,q^2)| - |G| = \frac{q^{4n+1} - q^{2n+1}}{q^2 - 1},$$

$$|H(2n,q^2)| - |PG(n-1,q^2)| = \frac{q^{4n+1} - q^{2n+1}}{q^2 - 1},$$

$$|pH(2n-1,q^2)| - |PG(n-1,q^2)| = \frac{q^{4n+1} - q^{2n+1}}{q^2 - 1} + q^{2n}.$$

So we get a two-character set. When looking at the complement of H' in $PG(2n+1, q^2)$ we get the following two intersection numbers with hyperplanes.

$$h_1 = \frac{q^{4n+2} - q^{4n+1} + q^{2n+1} - 1}{q^2 - 1}$$

$$h_2 = \frac{q^{4n+2} - q^{4n+1} + q^{2n+1} - q^{2n+2} + q^{2n} - 1}{q^2 - 1}$$

Hence we get the right intersection numbers if and only if $q = 2$. $\square$

3 Singular quasi-quadrics

First we recall the theorem of Bose and Burton.

Definition 5
A *blocking set with respect to t-spaces* in $PG(n, q)$ is a set B of points such that every t-dimensional subspace of $PG(n, q)$ meets B in at least one point.

The following result by Bose and Burton gives a nice characterization of the smallest ones [1].

Theorem 6
If B is a blocking set with respect to t-spaces in $PG(n, q)$ then $|B| \geq |PG(n-t, q)|$ and equality holds if and only if B is an $(n-t)$-dimensional subspace.

Next we introduce the concept of singular quasi-quadric

Definition 7
A set $\mathcal{K}$ in $PG(n, q)$ having the same number of points as a singular quadric Q and for which each intersection number with respect to hyperplanes is also an intersection number of Q with respect to hyperplanes, is called a *singular quasi-quadric*.

A natural question is whether each singular quasi-quadric is formed by the point set of a vertex over a quasi-quadric. The smallest non-trivial case to investigate is the case of a point over an oval, this was solved in [5]. In this paper we investigate the case of a point over an ovoid. It seems to be hard to generalize this result to either greater vertex, because of the growing number of hyperplane intersection possibilities or to greater base, because the dimension of a generator becomes small compared to the dimension of the ambient space.

Consider a set $\mathcal{K}$ of $q^3 + q + 1$ points in $PG(4, q)$ such that every hyperplane intersects $\mathcal{K}$ in $q + 1$, $q^2 + 1$ or $q^2 + q + 1$ points. A solid intersecting $\mathcal{K}$ in i points will be called an *i-solid*.

Theorem 8
Every $(q + 1)$-solids contains a line which intersects $\mathcal{K}$ in at least q points. If there are at least three $(q + 1)$-solids which intersect $\mathcal{K}$ in a full line, then the set $\mathcal{K}$ is a cone with vertex a point p and base an ovoid.

If we do not assume there are at least three $(q+1)$-solids which intersect $\mathcal{K}$ in a line, we have the following counterexamples:

Example 9
Let $q = 2$ and let $\mathcal{O}$ be an ovoid in a hyperplane Γ of $\mathrm{PG}(4, q)$. Let π be a tangent plane at $\mathcal{O}$ in Γ, say at the point x of $\mathcal{O}$. Let $p_1 \neq x$ and $p_2 \neq x$ be two different points in π and consider two disjoint lines L_1 and L_2, through p_1 and p_2 respectively, which are not contained in Γ. Then the point set $\mathcal{K} = \mathcal{O} \cup L_1 \cup L_2$ satisfies all the desired intersection properties.

Remark 10
Placing the lines L_1 and L_2 in different positions yields other examples for the case $q = 2$.

Example 11
Let $\mathcal{O}$ be an ovoid in a hyperplane Γ of $\mathrm{PG}(4, q)$, let p be a point not in Γ and consider the cone $\mathcal{K} := p\mathcal{O}$. Let π be a tangent plane at $\mathcal{O}$ in Γ, say at the point x of $\mathcal{O}$, and let L be a line in π through x. Then the set $\mathcal{K}' := \mathcal{K} \setminus px \cup L$ satisfies all the desired intersection properties.

We will prove Theorem 8 in several steps, which are described below.

Lemma 12
The number of $(q + 1)$-solids is $q^2 + 1$.

Proof. Call the number of $(q+1)$-solids, (q^2+1)-solids and (q^2+q+1)-solids a, b and c respectively. Counting the total number of solids in a 4-space, the incident pairs (p, α) where p is a point of $\mathcal{K}$ and α a solid, and the number of ordered triples (p, r, α) where p and r are distinct points of $\mathcal{K}$ lying in the solid α respectively, yields the following equations

$$\begin{aligned} a+b+c &= \frac{q^5-1}{q-1}, \\ a(q+1)+b(q^2+1)+c(q^2+q+1) &= (q^3+q+1)\frac{q^4-1}{q-1}, \\ a(q+1)q+b(q^2+1)q^2+c(q^2+q+1)(q^2+q) &= (q^3+q+1)(q^3+q)\frac{q^3-1}{q-1}. \end{aligned}$$

Solving these equations completes the proof. □

Lemma 13

(i) *Every plane which does not meet $\mathcal{K}$ is contained in exactly two $(q+1)$-solids.*

(ii) *Two $(q+1)$-solids intersect in at most one point of $\mathcal{K}$.*

(iii) *Any plane contains at most $2q+1$ points of the set $\mathcal{K}$.*

(iv) *All $(q+1)$-solids which intersect $\mathcal{K}$ in a line have a point of $\mathcal{K}$ in common.*

Proof. Consider a plane π and suppose that $|\pi \cap \mathcal{K}| = x$. Consider all solids through π in $PG(4,q)$ and denote the number of them which are $(q+1)$-solids, (q^2+1)-solids and (q^2+q+1)-solids by a, b and c respectively. This yields the following equation:

$$x + a(q+1-x) + b(q^2+1-x) + c(q^2+q+1-x) = q^3+q+1.$$

After simplifying we get $a+c-x = (a-1)q$. This proves (i), (ii) and (iii) immediately. Hence, all $(q+1)$-solids which intersect $\mathcal{K}$ in a line have a point in common, otherwise we get a plane intersecting $\mathcal{K}$ in at least $3q$ points. □

Lemma 14

Every $(q+1)$-solid intersects $\mathcal{K}$ contains a line which intersects $\mathcal{K}$ in at least q points.

Proof.

Let Σ be a $(q+1)$-solid, and let L be any line of Σ having non-trivial intersection with $\mathcal{K}$. Suppose that L intersects $\mathcal{K}$ in $1+k$ points. We calculate a lower bound for the number of exterior planes (i.e. not intersecting $\mathcal{K}$) of Σ. One easily sees there are at least $(q-k)(q^2+k)$ exterior lines intersecting L. Furthermore, on each such line there are at least k exterior planes. Since every exterior plane intersects L, such plane contains exactly $q+1$ exterior lines intersecting L. It follows there are at least

$$E = \frac{k(q-k)(q^2+k)}{q+1}$$

exterior planes in Σ. By (i) of Lemma 13 this implies there at least $E+1$ $(q+1)$-solids in $\mathrm{PG}(4,q)$. Hence, by Lemma 12, $E \leq q^2$ must hold. We obtain that $k \in \{0,1,q-1,q\}$ or $(k,q)=(2,4)$. We will deal with $(k,q)=(2,4)$ at the end of the proof. So suppose that $k \in \{0,1,q-1,q\}$, and that $\Sigma \cap \mathcal{K}$ would be an arc in Σ. Let π be a plane of Σ intersecting $\mathcal{K}$ in $l+1$ points. We may assume without loss of generality that $l \geq 2$. In π there are exactly $q^2+q+1-(q+1-l)-(l+1)l/2$ lines exterior to $\mathcal{K}$, and through each of these lines there pass at least l planes of Σ exterior to $\mathcal{K}$. As the total number of exterior planes in Σ can be at most q^2 it follows that $q^2 \leq l^2/2$, a contradiction.

We now deal with the case $(k,q)=(2,4)$. There is a line L containing exactly 3 points of $\Sigma \cap \mathcal{K}$. Let M be the line spanned by the two remaining points in $\Sigma \cap \mathcal{K}$.

(a) If $L \cap M = \emptyset$ then in Σ there are 18 planes exterior to $\mathcal{K}$. By (i) of Lemma 13 each of them is contained in two $(q+1)$-solids. This yields a contradiction since there are only 17 $(q+1)$-solids by Lemma 12.

(b) If $L \cap M$ is a point $p \notin \mathcal{K}$ then inside $\pi = \langle L,\, m\rangle$ there are 5 lines exterior to $\mathcal{K}$. Hence in Σ there are 20 planes exterior to $\mathcal{K}$. By (i) of Lemma 13 each of them is contained in two $(q+1)$-solids. This yields a contradiction since there are only 17 $(q+1)$-solids by Lemma 12.

(c) If $L \cap M$ is a point $p \in \mathcal{K}$ then inside $\pi = \langle L,\, M\rangle$ there are 4 lines exterior to $\mathcal{K}$. Hence in Σ there are 16 planes exterior to $\mathcal{K}$. By (i) of Lemma 13 each of them is contained in two $(q+1)$-solids. Assume that at least one $(q+1)$-solid intersects $\mathcal{K}$ in a line. This yields a contradiction since there are only 17 $(q+1)$-solids by Lemma 12.

So we may suppose there are no $(q+1)$-solids intersecting $\mathcal{K}$ in a full line. Hence every $(q+1)$-solids contains either 1 line intersecting $\mathcal{K}$ in 4 points (type I) or 2 lines intersecting $\mathcal{K}$ each in 3 points (type II). From the above, the intersection of two $(q+1)$-solids of type II can never contain a point of $\mathcal{K}$ (as there are 16 exterior planes in such $(q+1)$-solids). Hence there must be $(q+1)$-solids of type I, otherwise we get too many points in $\mathcal{K}$. Let Π_1 be such solid, and call h_1 the unique point on L_1 not belonging to $\mathcal{K}$, where L_1 is the unique line of Π_1 intersecting $\mathcal{K}$ in 4 points. We immediately see that h_1 is contained in at least 13 $(q+1)$-solids. Furthermore, as all exterior planes of Π_1 pas through h_1, the point h_1 is contained in all solids of type

II. Now define Π_2 and h_2 analogously to Π_1 and h_1 for a second $(q+1)$-solid Π_2 of type I. Assume that $h_1 \neq h_2$. Since also h_2 is contained in at least 13 $(q+1)$-solids it follows that the line h_1h_2 is contained in at least 9 $(q+1)$-solids. This implies the existence of a plane through h_1h_2 containing at least three $(q+1)$-solids, a contradiction. Hence $h_1 = h_2$, and the point h_1 is contained in all $(q+1)$-solids. Now consider any $(q+1)$-solid Σ of type II. Then $h_1 \in \Sigma$, and furthermore every exterior plane of Σ must contain h_1. This is clearly impossible, hence there are no solids of type II.

□

Lemma 15
Suppose there are at least three different $(q+1)$-solids which intersect $\mathcal{K}$ in a line. Then all $(q+1)$-solids intersect $\mathcal{K}$ in a line.

Proof. Suppose there is a $(q+1)$-solid Π_3 which does not contain the common intersection point p of all $(q+1)$-solids which intersect $\mathcal{K}$ in a line L_i. The space Π_3 intersects each of the lines L_i in a point of $\mathcal{K}$. Since there are at least three such lines, we get, using Lemma 14, a plane containing more than $2q+1$ points of $\mathcal{K}$, a contradiction.

Consequently every plane π not intersecting $\mathcal{K}$ is contained in at most one $(q+1)$-solid, namely $\langle p, \pi\rangle$, a contradicting (i) of Lemma 13. Hence every $(q+1)$-solid is blocked by $\mathcal{K}$, so Theorem 6 implies that all $(q+1)$-solids intersect $\mathcal{K}$ in a line. □

Now we can complete the proof of Theorem 8.

Proof. By Lemma 15 all q^2+1 $(q+1)$-solids intersect $\mathcal{K}$ in a line, and these lines have a point p in common. Since $1 + q(q^2+1) = q^3+q+1$, it follows that p is collinear with all other points of the set $\mathcal{K}$.

Let M be a line not through p containing at least three points of $\mathcal{K}$, say r, s and t. Then t is contained in the plane π spanned by the lines $\langle p, r\rangle$ and $\langle p, s\rangle$. Hence π intersects $\mathcal{K}$ in at least $2q+2$ points, a contradiction by (iii) of Lemma 13. Hence, all lines not through p intersect $\mathcal{K}$ in at most 2 points. Consider a solid Π not through p. If $q > 2$ then by the above, Π intersects $\mathcal{K}$ in an ovoid. If $q = 2$, then an ovoid is a set of 5 points in $PG(3,2)$ no four of which are coplanar. Let π be an arbitrary plane in Π. If $|\pi \cap \mathcal{K}| \geq 4$ then the solid $\langle p, \pi\rangle$ would contain more than 7 points, a contradiction. This completes the proof. □

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