A characterization of the finite Veronesean by intersection properties

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Abstract. A combinatorial characterization of the Veronese variety of all quadrics in $\mathbf{PG}(n,q)$ by means of its intersection properties with respect to subspaces is obtained. The result relies on a similar combinatorial result on the Veronesean of all conics in the plane $\mathbf{PG}(2,q)$ by Ferri [2], Hirschfeld and Thas [4], and Thas and Van Maldeghem [9], and a structural characterization of the quadric Veronesean by Thas and Van Maldeghem [8].

Keywords: quadric Veronesean, intersection numbers.

1. Introduction

An important branch in combinatorics is the characterization of algebraically defined objects in a combinatorial way. In several situations, it might occur that one has information about the intersection numbers with subspaces of a certain point set $\mathcal{K}$, but no substantial structural information. In such cases, characterization results classifying the possible structures having these properties can be very useful. In this paper, we characterize the finite Veronese variety by means of such intersection properties and some structural information. For the smallest Veronesean, the conic, this was already done (in the odd case) by Segre, in his celebrated characterization of conics ("every set of $q+1$ points in $\mathbf{PG}(2,q)$, q odd, no three of which are collinear, is a conic") [5]. This was in fact the starting point of this kind of results. For the Veronese surface of all conics in $\mathbf{PG}(2,q)$, it was already done by Ferri [2], Hirschfeld and Thas [4], and Thas and Van Maldeghem [9].

Definition 1.1. *The* Veronese variety $\mathcal{V}_n^{2^n}$ *of all quadrics of* $\mathbf{PG}(n,q)$, $n \geq 1$ *is the variety*

$$\mathcal{V}_n^{2^n} = \{p(x_0^2, x_1^2, \cdots, x_n^2, x_0x_1, x_0x_2, \cdots, x_{n-1}x_n) \mid\mid (x_0, \cdots, x_n) \text{ is a point of } \mathbf{PG}(n,q)\}$$

of $\mathbf{PG}(\frac{n(n+3)}{2}, q)$*; this variety has dimension n and order 2^n. The natural number n is called the* index *of* $\mathcal{V}_n^{2^n}$.

For the basic properties of Veroneseans we refer to [4].

The image of an arbitrary hyperplane of $\mathbf{PG}(n,q)$ under the Veronesean map is a quadric Veronesean $\mathcal{V}_{n-1}^{2^{n-1}}$, and the subspace generated by it has dimension $N_{n-1} = \frac{(n-1)(n+2)}{2}$. Such a subspace is called

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a $\mathcal{V}_{n-1}$*-subspace*. In particular for $n = 2$, the $\mathcal{V}_1$-subspaces are called *conic planes*. The image of a line of $\mathbf{PG}(n,q)$ is a plane conic, and if q is even, then the set of nuclei of all such conics is the Grassmanian of the lines of $\mathbf{PG}(n,q)$ and hence generates a subspace of dimension $\frac{(n-1)(n+2)}{2}$, which we call the *nucleus subspace* of $\mathcal{V}_n^{2^n}$, see [8].

One can also consider the Veronesean from a matrix point of view.

Theorem 1.2. *The quadric Veronesean $\mathcal{V}_n^{2^n}$ of $\mathbf{PG}(n,q)$ consists of all points $p(y_{0,0},\cdots,y_{n,n},y_{0,1},\cdots,y_{n-1,n})$ of $\mathbf{PG}(\frac{n(n+3)}{2},q)$ for which $[y_{ij}]$, with $y_{i,j} = y_{j,i}$ for $i \neq j$, is a symmetric matrix of rank 1.*

For $\mathcal{V}_n^{2^n}$, one can define a so-called *tangent space* at each point, and these tangent spaces have nice intersection properties. These can be used to characterize them.

Definition 1.3. *The tangent space of $\mathcal{V}_n^{2^n}$ at $p \in \mathcal{V}_n^{2^n}$ is the union of the tangent lines at p of the conics on $\mathcal{V}_n^{2^n}$ containing p (for $q = 2$ one considers the conics which are the images of the lines of $\mathbf{PG}(n,2)$).*

The set of tangent spaces can also be described algebraically, as shown in the following example for q odd.

Example 1.4. *Starting with $\mathbf{PG}(2,q)$, the mapping $\zeta : \mathbf{PG}(2,q) \to \mathbf{PG}(5,q)$ with*

$$\zeta(x_0,x_1,x_2) = (x_0^2,x_1^2,x_2^2,x_0x_1,x_0x_2,x_1x_2)$$

defines the quadric Veronesean $\mathcal{V}_2^4$.

If $p = (a,b,c)$, one can define a plane $D(p)$ which has the following representation

$$D(p) = \{(ax_0,bx_1,cx_2,ax_1+bx_0,ax_2+cx_0,bx_2+cx_1) \mid\mid x_0,x_1,x_2 \in \mathbb{F}_q\}\ .$$

This set $\mathcal{F}$ of q^2+q+1 planes in $\mathbf{PG}(5,q)$ has the following properties:

(P1) Each two of these planes intersect in a point.
(P2) Each three of these planes have an empty intersection.

If q is odd, then $D(p)$ is the tangent plane to $\mathcal{V}_2^4$ at p.

In 1958, Tallini [6] (see also [4]) showed that every set of q^2+q+1 planes in $\mathbf{PG}(5,q)$, q odd, for which (P1) and (P2) hold, must be isomorphic to the set $\mathcal{F}$ of Example 1.4. Furthermore, we have the following theorem.

Theorem 1.5. *If q is odd, then $\mathbf{PG}(5,q)$ admits a polarity which maps the set of all conic planes of $\mathcal{V}_2^4$ onto the set of all tangent planes of $\mathcal{V}_2^4$.*

This allows to state a dual version of Tallini's result.

Theorem 1.6. *If $\mathcal{L}$ is a set of q^2+q+1 planes of $\mathbf{PG}(5,q)$, q odd, with the following properties*

(i) There is no point belonging to all elements of $\mathcal{L}$.
(ii) Any two distinct elements of $\mathcal{L}$ have exactly one point in common.
(iii) Any three distinct elements of $\mathcal{L}$ generate $\mathbf{PG}(5,q)$.

Then $\mathcal{L}$ is the set of all conic planes of a Veronesean $\mathcal{V}_2^4$.

This result was generalized in [8] to the following characterization of the set of $\mathcal{V}_{n-1}$-subspaces of a finite quadric Veronesean $\mathcal{V}_n^{2^n}$.

Theorem 1.7. *([8]) Let $\mathcal{F}$ be a set of $\frac{q^{n+1}-1}{q-1}$ subspaces of dimension $\frac{(n-1)(n+2)}{2}$ in $\mathbf{PG}(N = \frac{n(n+3)}{2}, q), n \geq 2$, with the following properties:*

(VS1) Each two members of $\mathcal{F}$ generate a hyperplane of $\mathbf{PG}(N, q)$.
(VS2) Each three elements of $\mathcal{F}$ generate $\mathbf{PG}(N, q)$.
(VS3) No point is contained in every member of $\mathcal{F}$.
(VS4) The intersection of any nonempty collection of members of $\mathcal{F}$ is a subspace of dimension $N_i = \frac{i(i+3)}{2}$ for some $i \in \{-1, 0, 1, \cdots, n-1\}$.
(VS5) There exist 3 members Ω_1, Ω_2, Ω_3 of $\mathcal{F}$ with $\Omega_1 \cap \Omega_2 = \Omega_2 \cap \Omega_3 = \Omega_3 \cap \Omega_1$.

Then either $\mathcal{F}$ is the set of $\mathcal{V}_{n-1}$-subspaces of a quadric Veronesean $\mathcal{V}_n^{2^n}$ in $\mathbf{PG}(n, q)$ or q is even, there are two members $\Omega_1, \Omega_2 \in \mathcal{F}$ with the property that no other member of $\mathcal{F}$ contains $\Omega_1 \cap \Omega_2$, and there is a unique subspace Ω of dimension $\frac{(n-1)(n+2)}{2}$ such that $\mathcal{F} \cup \{\Omega\}$ is the set of $\mathcal{V}_{n-1}$-subspaces together with the nucleus subspace of a quadric Veronesean $\mathcal{V}_n^{2^n}$. In particular, if $n = 2$, then the statement holds under the weaker hypothesis of $\mathcal{F}$ satisfying (**VS1**), (**VS2**), (**VS3**) *and* (**VS5**).

For $n = 2$ one can classify all examples that do not satisfy (**VS5**) by a result of [1], and the only possibilities are $q = 2$ and $q = 4$. This classification remains open for $n \geq 3$, although an infinite class of examples is known for $q = 2$, see [8].

In particular for $n = 2$, this result generalizes Theorem 25.2.14 of [4] to q even, and allows to generalize Theorem 25.3.14 of [4] to q even, and so one obtains

Theorem 1.8. *([7]) If $\mathcal{K}$ is a set of k points of $\mathbf{PG}(5, q)$, $q \neq 2, 4$, which satisfies the following conditions*

(i) *$|\Pi_4 \cap \mathcal{K}| = 1,\ q+1,\ 2q+1$ for every hyperplane Π_4 of $\mathbf{PG}(5, q)$ and there exists a hyperplane Π_4 for which $|\Pi_4 \cap \mathcal{K}| = 2q + 1$.*
(ii) *Any plane of $\mathbf{PG}(5, q)$ with four points in $\mathcal{K}$ has at least $q + 1$ points in $\mathcal{K}$.*

Then $\mathcal{K}$ is the point set of a Veronesean $\mathcal{V}_2^4$.

A theorem by Zanella [10] gives an upper bound for the intersection of k-dimensional subspaces with the quadric Veronese variety, so for the intersections $\Pi_k \cap \mathcal{V}_n$.

Theorem 1.9. *([10]) Consider the Veronese variety defined by the mapping*

$$\zeta : \mathbf{PG}(n, q) \to \mathbf{PG}(\frac{n(n+3)}{2}, q),$$

$$(x_0, x_1, \cdots, x_n) \to (x_0^2, x_1^2, \cdots, x_{n-1}x_n).$$

If k, a are natural numbers such that $k + 1 \leq \frac{(a+3)(a+2)}{2}$, then the intersections $\Pi_k \cap \mathcal{V}_n$ contain at most

$$\frac{q^{a+1} - 1}{q - 1} + q^{k - \frac{(a+2)(a+1)}{2}}$$

points.

Applying this for small dimensions yields the upper bounds $q + 1$, $q + 2$, $2q + 1$ and $q^2 + q + 1$ for $k = 2$, $k = 3$, $k = 4$ and $k = 5$ respectively.

A result of the second and third author [9] of this paper characterizes Veronese varieties in terms of ovals.

Theorem 1.10. *Let X be a set of points in $\Pi := \mathbf{PG}(M,q)$, $M > 2$, spanning Π, and let $\mathcal{P}$ be a collection of planes such that for any $\pi \in \mathcal{P}$, the intersection $X \cap \pi$ is an oval in π. For $\pi \in \mathcal{P}$ and $x \in X \cap \pi$, we denote by $T_x(\pi)$ the tangent line to $X \cap \pi$ at x in π. We assume the following three properties.*

(U) *Any two points $x, y \in X$ lie in a unique member of $\mathcal{P}$ which we denote by $[x,y]$.*
(NE) *If $\pi_1, \pi_2 \in \mathcal{P}$ and $\pi_1 \cap \pi_2$ is non-empty then $\pi_1 \cap \pi_2 \subset X$.*
(TP) *If $x \in X$ and $\pi \in \mathcal{P}$ with $x \notin \pi$, then each of the lines $T_x([x,y])$, $y \in X \cap \pi$, is contained in a plane of Π, denoted by $T(x,\pi)$.*

Then there exists a natural number $n \geq 2$ (called the index of X), a projective space $\Pi' := \mathbf{PG}(\frac{n(n+3)}{2}, q)$ containing Π, a subspace R of Π' skew to Π, and a quadric Veronesean $\mathcal{V}_n$ of index n in Π', with $R \cap \mathcal{V}_n = \emptyset$, such that X is the (bijective) projection of $\mathcal{V}_n$ from R onto Π. The subspace R can be empty, in which case X is projectively equivalent to $\mathcal{V}_n$.

To conclude this introduction, we define k-arcs in $\mathbf{PG}(3,q)$.

Definition 1.11. *A k-arc of $\mathbf{PG}(3,q)$, $k \geq 4$ is a set of k points, no 4 of which are coplanar.*

By Theorem 21.2.4 and Theorem 21.3.8 of [3], we have the following

Theorem 1.12. *If $q \geq 4$, then $k \leq q+1$.*

2. First characterization

We want to use the following set of conditions to characterize the quadric Veronesean. Consider a set $\mathcal{K}$ of $\frac{q^{n+1}-1}{q-1}$ points spanning $\mathbf{PG}(\frac{n(n+3)}{2}, q)$, with $n \geq 2$, such that the following conditions are satisfied.

(P) If a plane intersects $\mathcal{K}$ in more than three points then it contains exactly $q+1$ points of $\mathcal{K}$. Furthermore, any two points p_1, p_2 of $\mathcal{K}$ are contained in a plane containing $q+1$ points of $\mathcal{K}$.
(S) If a 3-space Π_3 intersects $\mathcal{K}$ in more than 4 points then there are four points of $\mathcal{K}$ contained in a plane of Π_3. In particular, by (P), this implies that if $|\Pi_3 \cap \mathcal{K}| > 4$, then $|\Pi_3 \cap \mathcal{K}| \geq q+1$.
(V) If a 5-space Π_5 intersects $\mathcal{K}$ in more than $2q+2$ points then it intersects $\mathcal{K}$ in exactly q^2+q+1 points.

Definition 2.1. *Planes intersecting $\mathcal{K}$ in $q+1$ points and 5-spaces intersecting $\mathcal{K}$ in q^2+q+1 points will be called* big planes *and* big 5-spaces *respectively.*

Assume $q \geq 5$ in the following.

We will prove the following main theorem.

Theorem 2.2. *If $q \geq 5$, then the set $\mathcal{K}$ is the point set of the Veronese variety of all quadrics of $\mathbf{PG}(n,q)$.*

Remark.

A counterexample for $q = 2$, $n > 2$, to the previous theorem is given by removing one point of a Veronese variety and replacing it by a point in the projective space which corresponds with a matrix of maximal rank, using the correspondence of Theorem 1.2.

A counterexample for $q = 3$, $n = 2$, is given by the point set formed by the points of an elliptic quadric $\mathcal{E}$ lying in a space $\Pi_3 \subset \mathbf{PG}(5,3)$ and 3 points on a line $L \subset \mathbf{PG}(5,3)$ which does not intersect Π_3.

First of all we have to prove that these conditions are well-chosen, meaning the object we want to characterize satisfies them.

Theorem 2.3. *The Conditions (P), (S) and (V) above hold for the Veronesean $\mathcal{V}_n^{2^n}$.*

Proof. For Condition (P), we cannot use Lemma 25.3.1 of [4] directly, since we don't know a priori that every plane containing more than three points of $\mathcal{V}_n^{2^n}$ is contained in a 5-space intersecting $\mathcal{K}$ in a $\mathcal{V}_2^4$ but a slight adaptation of the argument works. Suppose that the plane π contains at least four distinct points q_1, q_2, q_3, q_4 of $\mathcal{V}_n^{2^n}$. By Corollary 1 of Theorem 25.1.9 of [4], the points q_i, q_j, with $i \neq j$, are contained in a unique conic of $\mathcal{V}_n^{2^n}$. Let C', in the plane π', be the conic defined by q_1 and q_2, and let C'', in the plane π'', be the conic defined by q_2 and q_3. Suppose that $C' \neq C''$. By Theorem 1.9 the conic planes π' and π'' generate a 4-space Π_4 such that $|\Pi_4 \cap \mathcal{K}| \leq 2q + 1$. But besides the $2q+1$ points in $C' \cup C''$, the point q_4 would also be contained in this 4-space, a contradiction. Hence $|\pi \cap \mathcal{K}| \geq q + 1$ and by Theorem 1.9, $|\pi \cap \mathcal{K}| = q + 1$. Conditions (S) and (V) can be proved using a coordinatization and checking the different possibilities for the position of the inverse images of the points in $\mathbf{PG}(n, q)$.

$\square$

We prove some upper bounds on the number of points of $\mathcal{K}$ contained in low-dimensional spaces.

Lemma 2.4. *If $n > 2$, every 4-space contains at most $2q + 2$ points of $\mathcal{K}$.*

Proof. Let Π be a 4-space. By Condition (V), it follows directly that $|\Pi \cap \mathcal{K}| \leq q^2 + q + 1$ and clearly $|\Pi \cap \mathcal{K}| = q^2 + q + 1$ also yields a contradiction.

Suppose that $2q+2 < |\Pi \cap \mathcal{K}| < q^2+q+1$. Again by Condition (V), every 5-space through Π contains exactly $q^2 + q + 1$ points of $\mathcal{K}$. The number of 5-spaces through a fixed 4-space in $\mathbf{PG}(\frac{n(n+3)}{2}, q)$ is equal to $\frac{q^{\frac{n(n+3)}{2}-4}-1}{q-1}$. Hence, we get at least

$$\frac{q^{\frac{n(n+3)}{2}-4} - 1}{q - 1} + 2q + 2 > \frac{q^{n+1} - 1}{q - 1}$$

points in $\mathcal{K}$, a contradiction for $n > 2$.

$\square$

Lemma 2.5. *Any line l meets $\mathcal{K}$ in at most 2 points. Hence, a plane π with $|\pi \cap \mathcal{K}| = q + 1$ intersects $\mathcal{K}$ in an oval.*

Proof. First suppose that $|l \cap \mathcal{K}| = 3$. If $n > 2$, then consider 3 planes π_1, π_2, π_3 through l containing more than 3 points of $\mathcal{K}$ and hence by Condition (P) $q + 1$ points of $\mathcal{K}$. Then $\dim\langle\pi_1, \pi_2, \pi_3\rangle \leq 4$. For $q > 5$, this yields a contradiction by Lemma 2.4. If $q = 5$, then consider a 3-space Π_3 through l containing at least 9 points of $\mathcal{K}$ inside a big 5-space Π_5. But then considering all 4-spaces through Π_3 inside Π_5, by Lemma 2.4, we get at most $6 \cdot 3 + 9 = 27$ points in $\Pi_5 \cap \mathcal{K}$, a contradiction.

If $n = 2$ then we get the following equation for the number α of planes through l which contain exactly $q + 1$ points of $\mathcal{K}$:

$$\alpha(q - 2) + 3 = q^2 + q + 1.$$

This yields a contradiction if $q \geq 5$. Next, suppose that $|l \cap \mathcal{K}| = x$, with $3 < x < q + 1$. Consider all planes through l. Then clearly, we get too many points for our set $\mathcal{K}$, a contradiction. Finally, if $|l \cap \mathcal{K}| = q + 1$, we also get a contradiction as planes can contain at most $q + 1$ points of $\mathcal{K}$.

$\square$

The previous lemma allows us for $n = 2$ to prove the same upper bound as in Lemma 2.4.

Lemma 2.6. *Every 4-space intersects $\mathcal{K}$ in at most $2q+2$ points. Hence, every 3-space contained in a big 5-space intersects $\mathcal{K}$ in at most $q+3$ points.*

Proof. For $n > 2$ this is Lemma 2.4. Next let $n = 2$.

Suppose there exists a 3-space Π_3 which contains two planes π_1 and π_2 which intersect $\mathcal{K}$ in ovals $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively which have two points p_1, p_2 of $\mathcal{K}$ in common. Consider two points r_1 and r_2, different from p_1 and p_2, which lie on $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Then there are at most 4 planes through the line $\langle r_1, r_2\rangle$ which are not $(q+1)$-planes, namely the planes containing either the point p_1 or p_2 or those which intersect π_i in a tangent line to $\mathcal{O}_i$ at r_i for $i = 1$ or $i = 2$.

Hence, we get at least

$$2 + (q-3)(q-1) + 4 = q^2 - 4q + 9$$

points in $\Pi_3 \cap \mathcal{K}$.

The bound above is already sufficient for the remainder of the proof if $q > 5$. But since we now know there is a point p in $\Pi_3 \cap \mathcal{K}$ not contained in $\mathcal{O}_1 \cup \mathcal{O}_2$ we can consider all planes through the line $\langle p, p_1\rangle$ inside Π_3. In this case, we get at most three exceptions, namely the plane containing p_2 and those which intersect π_i in a tangent line to $\mathcal{O}_i$ at p_1. Hence we get at least

$$2 + 3 + (q-2)(q-1) = q^2 - 3q + 7$$

points in $\Pi_3 \cap \mathcal{K}$.

If one would carry out this argument a bit more carefully one can get up to q^2+1 points in $\Pi_3 \cap \mathcal{K}$, and hence this intersection is an ovoid. However this does not shorten the reasonings made in the remainder of this proof.

Hence if there are three such 3-spaces we distinguish the following cases.

Case (i): Any two of them only intersect in a line. Then the union of the 3-spaces contains at least $3(q^2 - 3q + 7) - 3 \cdot 2$ points of $\mathcal{K}$, a contradiction since $q \geq 5$.

Case (ii): There are two of them which intersect in a plane. Then we get a 4-space Π_4 containing at least $2(q^2 - 3q + 7) - (q+1) = 2q^2 - 7q + 13$ points of $\mathcal{K}$. Consider a point p in $\mathcal{K}$ not contained in Π_4. Through p and any point r in $\Pi_4 \cap \mathcal{K}$ there passes an oval of $\mathcal{K}$ by Condition (P). If none of these ovals have two points of $\mathcal{K}$ in common, we get too many points, a contradiction. If two of these ovals have two points of $\mathcal{K}$ in common then the 3-space spanned by these two ovals contains at least $q^2 - 3q + 7$ points of $\mathcal{K}$. Hence, we get at least

$$2q^2 - 7q + 13 + q^2 - 3q + 7 - (q+1)$$

points in $\mathcal{K}$, a contradiction since $q \geq 5$.

If there are exactly one or two such 3-spaces we consider a 4-space Π_4 containing such a 3-space Π_3 and a point p in $\mathcal{K}$ not contained in Π_4. Through p and each point r in $\Pi_4 \cap \mathcal{K}$ there passes an oval by Condition (P). For each such point r we choose exactly one such oval. If we have two ovals of $\mathcal{K}$ through p and a point r of $\mathcal{K}$ in Π_4, then these two ovals define a 3-space Π_3' containing at least $q^2 - 3q + 7$ points of $\mathcal{K}$, and then the line rp lies in at most $q+1$ planes of the solid containing an oval of $\Pi_3' \cap \mathcal{K}$. If there are more than $q+1$ ovals through p sharing two points of $\mathcal{K}$, then there would be another 3-space Π_3'' through p sharing at least $q^2 - 3q + 7$ points with $\mathcal{K}$. Now Π_3' and Π_3'' are different from the solid Π_3 in Π_4 sharing at least $q^2 - 3q + 7$ points of $\mathcal{K}$. But this contradicts the assumption that there are no three such solids. Hence we clearly get too many points in $\mathcal{K}$, a contradiction.

Now consider a 4-space Π_4 which intersects $\mathcal{K}$ in x points. Consider a point p of $\mathcal{K}$ not in Π_4. By Condition (P) through every 2 points of $\mathcal{K}$ there passes an oval of $\mathcal{K}$. Consider all ovals through p and a point r of $\Pi_4 \cap \mathcal{K}$. Any two of these ovals can intersect in at most one point. Since any oval through p intersects $\Pi_4 \cap \mathcal{K}$ in at most 2 points, we get at least $\frac{x}{2}$ ovals, which all contain $q-2$ points in $\Pi_5 \cap \mathcal{K}$ different from p and the x points in $\Pi_4 \cap \mathcal{K}$. Hence we get the following equation,

$$\frac{x}{2}(q-2) + x + 1 \leq q^2 + q + 1.$$

This yields $x \leq 2q + 2$.

Consider a 3-space Π_3 in a big 5-space Π_5 which intersects $\mathcal{K}$ in $q + 3 + y$ points. Since all 4-spaces through Π_3 inside Π_5 intersect $\mathcal{K}$ in at most $2q + 2$ points we get the following inequality

$$(q+1)(q-1-y) + q + 3 + y \geq q^2 + q + 1.$$

This implies $y \leq 0$.

□

Now we are able to lower the bound of Lemma 2.6.

Lemma 2.7. (i) *Every 4-space Π_4 intersecting $\mathcal{K}$ in more than $q + 1$ points contains a plane which intersects $\mathcal{K}$ in an oval.*

(ii) *Every 4-space Π_4 contains at most $2q + 1$ points of $\mathcal{K}$.*

Proof. (i) Suppose that $|\Pi_4 \cap \mathcal{K}| > q + 1$. Since $q \geq 5$, by a result on arcs, namely Theorem 27.6.3 of [4], there are 5 points which are contained in a 3-space. By Condition (S), it follows that there are 4 of them which are contained in a plane π. Hence, by Lemma 2.5, π intersects $\mathcal{K}$ in an oval.

(ii) Suppose that $|\Pi_4 \cap \mathcal{K}| = 2q + 2$. Consider a plane π in Π_4 intersecting $\mathcal{K}$ in an oval $\mathcal{O}$. Such a plane always exists by (i).

Consider 2 points a and b in $\Pi_4 \cap \mathcal{K}$, but not in π, such that $\langle a, b\rangle \cap \pi = \emptyset$. Note that this is always possible, since at most $q + 3$ points of $\mathcal{K}$ are contained in a 3-space by Lemma 2.6.

Consider a third point c in $(\Pi_4 \cap \mathcal{K})\backslash\pi$, and let p be the intersection point of π and $\pi' = \langle a,\ b,\ c\rangle$.

We distinguish the following cases.

Case (i): $p \in \mathcal{O}$.

Since π' contains at least 4 points of $\mathcal{K}$ it contains at least $q + 1$ points of $\mathcal{K}$ by Condition (P). The planes π and π' both intersect $\mathcal{K}$ in an oval, $\mathcal{O}$ and $\mathcal{O}'$. Denote the remaining point in $\Pi_4 \cap \mathcal{K}$ by p'. Consider a plane π'' spanned by p' and two points a' and b' belonging to $\mathcal{O}\backslash\{p\}$. The planes π' and π'' intersect in a point r. If r belongs to $\mathcal{K}$ then π'' contains at least 4 and hence, by Condition (P), $q + 1$ points of $\mathcal{K}$. If r does not belong to $\mathcal{K}$ we may assume it is not the nucleus of $\mathcal{O}'$, otherwise we can restart the reasoning with two other points of $\mathcal{O}$. Then the 3-space spanned by π'' and a bisecant to $\mathcal{O}'$ through r, but not through p, contains at least 5 and hence by Condition (S) at least $q + 1$ points. Since $q \geq 5$, in both cases we get more than $2q+2$ points in $\Pi_4 \cap \mathcal{K}$, a contradiction by Lemma 2.6.

Case (ii): $p \notin \mathcal{O}$.

(ii.A) First of all, we assume that not all points in $\Pi_4 \cap \mathcal{K}$ are contained in $\pi \cup \pi'$. Since not all points in $\Pi_4 \cap \mathcal{K}$ are contained in $\pi \cup \pi'$, we may assume that p is not the nucleus of $\mathcal{O}$. Indeed, if p would be the nucleus of $\mathcal{O}$ we consider a point c' of $\Pi_4 \cap \mathcal{K}$ not in $\pi' \cup \pi$ and the plane $\pi'' = \langle a,\ b,\ c'\rangle$ which then intersects π in a point p', with p' not the nucleus of $\mathcal{O}$. So in that case we continue the reasonings with π'' instead of π'. Consider two secants of $\mathcal{O}$ through p, say l and l'. The 3-spaces $\langle \pi', l\rangle$ and $\langle \pi', l'\rangle$ both contain at least 5 points of $\mathcal{K}$, hence they both contain a plane intersecting $\mathcal{K}$ in an oval. These planes have to coincide, since also π intersects $\mathcal{K}$ in an oval, otherwise we get too many points in $\Pi_4 \cap \mathcal{K}$. Hence, the plane π' intersects $\mathcal{K}$ in an oval $\mathcal{O}'$. This yields a contradiction with the assumption at the beginning of this paragraph. Note that as a byproduct we proved that if a 4-space contains at least $q+5$ points of $\mathcal{K}$, it contains two planes which intersect $\mathcal{K}$ in an oval, hence $|\Pi_4 \cap \mathcal{K}| \geq 2q+1$. Indeed we only used 4 points a, b, c and c' in $\mathcal{K}$ but not in π to find the second oval $\mathcal{O}'$. Furthermore, these ovals can have at most one point in common, otherwise they only span a 3-space, but a 3-space inside Π_5 intersects $\mathcal{K}$ in at most $q+3$ points by Lemma 2.6.

(ii.B) Next, we may suppose that $\Pi_4 \cap \mathcal{K}$ is a union of two ovals $\mathcal{O}$ and $\mathcal{O}'$ contained in planes π and π' which intersect in a point p.

(ii.B.1) If p is the nucleus of neither $\mathcal{O}$ nor of $\mathcal{O}'$, then consider a secant t of $\mathcal{O}$ and a secant t' of $\mathcal{O}'$ through p. The plane spanned by t and t' contains 4 and hence $q+1$ points of $\mathcal{K}$, and so $|\Pi_4 \cap \mathcal{K}| > 2q+2$, a contradiction.

(ii.B.2) If p is the nucleus of $\mathcal{O}$, but not the nucleus of $\mathcal{O}'$, then consider a 5-space Π_5 containing Π_4 which intersects $\mathcal{K}$ in q^2+q+1 elements. Since p is not the nucleus of $\mathcal{O}'$, there is a secant l' of $\mathcal{O}'$ through p. Hence, by Lemma 2.6, the 3-space Π_3 spanned by $\mathcal{O}$ and l' contains exactly $q+3$ elements of $\mathcal{K}$. It follows that exactly one of the 4-spaces containing Π_3 in Π_5 intersects $\mathcal{K}$ in $2q+1$ points, while all the other 4-spaces through Π_3 in Π_5 contain $2q+2$ points of $\mathcal{K}$.

By the foregoing there is a 4-space $\Pi_4' \neq \Pi_4$ containing Π_3 inside Π_5 which intersects $\mathcal{K}$ in $2q+2$ elements on two ovals $\mathcal{O}$ and $\mathcal{O}''$, where the planes of $\mathcal{O}'$ and $\mathcal{O}''$ intersect in l'. Hence the 3-space Π_3' spanned by $\mathcal{O}'$ and $\mathcal{O}''$ contains at least $2q$ elements of $\mathcal{K}$. Consider all 4-spaces through Π_3' inside Π_5. By Lemma 2.6 we get at most $2q+2(q+1) = 4q+2$ points in $\Pi_5 \cap \mathcal{K}$, a contradiction since $q \geq 5$.

(ii.B.3) Finally, if p is the nucleus of both $\mathcal{O}$ and $\mathcal{O}'$, then consider a 3-space Π_3 spanned by $\mathcal{O}$ and a tangent l to $\mathcal{O}'$ through p. Consider a big 5-space Π_5 through Π_4. Consider all 4-spaces through Π_3 inside Π_5. No 4-space through Π_3 inside Π_5 different from Π_4 can intersect $\mathcal{K}$ in $2q+2$ points as well. Indeed, by Case (i) and the previous subcases of Case (ii) such a 4-space Π_4' again has to intersect $\mathcal{K}$ in two ovals $\mathcal{O}$ and $\mathcal{O}''$, contained in planes π and π'' respectively. The planes π' and π'' have to intersect in the tangent line l, and p again has to be the nucleus of both the ovals $\mathcal{O}$ and $\mathcal{O}''$.

But then the 3-space Π_3'' spanned by $\mathcal{O}'$ and $\mathcal{O}''$ contains at least $2q+1$ points of $\mathcal{K}$. Consider all 4-spaces through Π_3'' inside Π_5. Then by Lemma 2.6 $|\Pi \cap \mathcal{K}| \leq 2q+1+q+1 = 3q+2$, a contradiction since $q \geq 5$.

Consider now all 4-spaces through Π_3 inside Π_5. Exactly one of them intersects $\mathcal{K}$ in $2q+2$ points by the previous and all the others intersect $\mathcal{K}$ in at most $2q+1$ points. Hence, by an easy inspection, there is exactly one 4-space through Π_3 in Π_5 containing exactly $2q$ points of $\mathcal{K}$, but this yields a contradiction by the remark made at the end of Case (ii.A).

□

Remark. A 4-space intersecting $\mathcal{K}$ in $2q+1$ points will be called a *big 4-space.*

Lemma 2.8. (i) *Inside a big 5-space* Π_5 *all 3-spaces contain at most* $q+2$ *points. Furthermore, all 4-spaces inside* Π_5 *through a 3-space intersecting* $\mathcal{K}$ *in* $q+2$ *points are big ones.*
(ii) *A big 4-space* Π_4 *contained in a big 5-space* Π_5 *intersects* $\mathcal{K}$ *in two ovals* $\mathcal{O}_1, \mathcal{O}_2$ *with* $\mathcal{O}_1 \cap \mathcal{O}_2 = \{P\}$, $P \in \mathcal{K}$.

Proof. (i) Suppose a 3-space Π_3 of the big 5-space Π_5 intersects $\mathcal{K}$ in $q+2+x$ points, with $x \geq 0$. Then considering all 4-spaces in Π_5 through Π_3, we get at most $(2q+1-(q+2+x))(q+1)+q+2+x = q^2+q+1-xq$ points in $\Pi_5 \cap \mathcal{K}$ by Lemma 2.7, a contradiction if $x > 0$. The second part follows directly if $x = 0$.

(ii) By (i) of Lemma 2.7 there is a plane π in Π_4 which intersects $\mathcal{K}$ in an oval. We claim we can find a second plane in Π_4 which intersects $\mathcal{K}$ in an oval. Take 3 points contained in $\Pi_4 \cap \mathcal{K}$ not lying in π.

These points span a plane π'. The space $\langle \pi, \pi' \rangle$ is a 4-space, otherwise we get a 3-space intersecting $\mathcal{K}$ in more than $q+2$ points, contradicting (i).

If π' contains exactly 3 points of $\mathcal{K}$ then consider all 3-spaces through π' in Π_4. If none of them contains at least 5 points of $\mathcal{K}$ we get at most $q+1+3 = q+4$ points in $\Pi_4 \cap \mathcal{K}$, a contradiction. So there is a 3-space Π_3 through π' in Π_4 containing more than 4 points of $\mathcal{K}$, hence by Conditions (P) and (S) we find a plane π'' containing $q+1$ points of $\mathcal{K}$ inside Π_3. Clearly π and π'' are different since π and π' span a 4-space.

If the two different planes π and π'' which intersect $\mathcal{K}$ in an oval intersect in a point, then we are done by Lemma 2.7. Suppose that π and π'' intersect in a line. Then the 3-space $\Pi_3' = \langle \pi, \pi'' \rangle$ intersects $\mathcal{K}$ in more than $q+2$ points, contradicting (i).

□

Lemma 2.9. *Every 4-space contained in a big 5-space Π_5 intersects $\mathcal{K}$ in 1, $q+1$ or $2q+1$ points and each such big 5-space contains at least one 4-space intersecting $\mathcal{K}$ in exactly $2q+1$ points. Hence, each big 5-space intersects $\mathcal{K}$ in a $\mathcal{V}_2^4$.*

Proof. Denote the number of points belonging to $\mathcal{K}$ contained in a 4-space $\Pi_i \subset \Pi_5$ by x_i; here Π_5 is a big 5-space. In the following sum and all the others below, i runs over all 4-spaces Π_i contained in Π_5. We have

$$\sum_i (x_i - 1)(x_i - (q+1))(x_i - (2q+1)) = 0. \tag{1}$$

Indeed, by a standard counting technique counting in two different ways respectively the number of pairs (p, Π) in Π_5, where $p \in \mathcal{K}$ and Π is a 4-space in Π_5, the number of triples $(p_1, p_2, \Pi), p_1 \neq p_2 \in \Pi \cap \mathcal{K}$ and Π a 4-space in Π_5, and the quadruples $(p_1, p_2, p_3, \Pi), p_i \in \Pi \cap \mathcal{K}$, where the points p_i are all distinct and Π is a 4-space in Π_5 yields

$$\sum_i x_i = \frac{\left(q^2+q+1\right)\left(q^5-1\right)}{q-1}, \tag{2}$$

$$\sum_i x_i(x_i - 1) = \frac{\left(q^2+q+1\right)\left(q^2+q\right)\left(q^4-1\right)}{q-1}, \tag{3}$$

$$\sum_i x_i(x_i-1)(x_i-2) = \frac{\left(q^2+q+1\right)\left(q^2+q\right)\left(q^2+q-1\right)\left(q^3-1\right)}{q-1}. \tag{4}$$

Now Equations (2), (3) and (4) together lead to Equation (1).

If a 4-space Π_4 inside a big 5-space contains more than $q+1$ points of $\mathcal{K}$, then it is a big one. Indeed, by Lemma 2.7 there is a plane in Π_4 intersecting $\mathcal{K}$ in an oval. Hence we can find a 3-space in Π_4 which intersects $\mathcal{K}$ in $q+2$ points. The claim now follows from (i) of Lemma 2.8.

Suppose $|\Pi_4 \cap \mathcal{K}| = x$, $\Pi_4 \subset \Pi_5$, with $4 \leq x < q+1$. Let Π_3 be a 3-space containing 4 points p_1, p_2, p_3, p_4 in $\Pi_4 \cap \mathcal{K}$. Hence, by Condition (S), $|\Pi_3 \cap \mathcal{K}| = 4$. Consider all 4-spaces through Π_3 inside Π_5. If there are less than 4 big ones among them, we get less than

$$3(2q-3) + (q-2)(q-3) + 4 = q^2 + q + 1$$

points in $\Pi_5 \cap \mathcal{K}$, a contradiction.

By (ii) of Lemma 2.8, in each of the at least 4 big 4-spaces inside Π_5 containing Π_3 the points p_1, p_2, p_3, p_4 are contained in 2 ovals. Hence either there is an oval containing 3 of them, which yields a contradiction, or there is a pair p_i, p_j contained in two different ovals. But the latter yields a contradiction by (i) of Lemma 2.8.

Suppose now that $|\Pi_4 \cap \mathcal{K}| = 3$. Consider a 3-space Π_3 in Π_4 containing the 3 points p_1, p_2, p_3 of $\Pi_4 \cap \mathcal{K}$ and all 4-spaces inside Π_5 containing Π_3. By the previous arguments these intersect $\mathcal{K}$ in 3, $q+1$ or $2q+1$ points. Denote the number of them intersecting $\mathcal{K}$ in $q+1$ and $2q+1$ points by α and β respectively. This yields the following equation

$$\alpha(q-2) + \beta(2q-2) + 3 = q^2 + q + 1.$$

We deduce that α is a multiple of $q-1$. If $\alpha = q-1$, then we get at most $(q-1)(q-2) + 2q + 1$ points in $\Pi_5 \cap \mathcal{K}$, a contradiction.

Hence, we find that $\alpha = 0$ and $\beta = \frac{q+2}{2}$. This already yields a contradiction if q is odd. As $q > 2$, the points p_1, p_2, p_3 are contained in 2 ovals in each of the at least 3 different big 4-spaces of Π_5 containing Π_3 by (ii) of Lemma 2.8. This yields a contradiction.

Finally, suppose that $|\Pi_4 \cap \mathcal{K}| = 2$. Consider a 3-space Π_3 in Π_4 containing the 2 points p_1, p_2 of $\Pi_4 \cap \mathcal{K}$ and all 4-spaces inside Π_5 containing Π_3. By the previous these all intersect $\mathcal{K}$ in 2, $q+1$ or $2q+1$ points. Denote the number of them intersecting $\mathcal{K}$ in $q+1$ and $2q+1$ points by α and β respectively. This yields the following equation,

$$\alpha(q-1) + \beta(2q-1) + 2 = q^2 + q + 1.$$

This yields that $\beta - 1$ is a multiple of $q-1$. If $\beta = 1$, we get at most $(q-1)(q-1) + 2q + 1 = q^2 + 2$ points of $\mathcal{K}$ in Π_5, a contradiction. If $\beta = q$, we get exactly $2q^2 - q + 2$ points in $\Pi_5 \cap \mathcal{K}$, also a contradiction.

In the previous paragraphs we proved that if a 4-space contains at least 2 points of $\mathcal{K}$, then it contains at least $q+1$ points of $\mathcal{K}$. By (i) of Lemma 2.8 and (ii) of Lemma 2.7 the only possibilities in this case are $q+1$ and $2q+1$. By Equation (1), this implies that there are no 4-spaces which have an empty intersection with $\mathcal{K}$.

Hence, every 4-space contained in Π_5 intersects $\mathcal{K}$ in 1, $q+1$ or $2q+1$ points.

We prove there is a 4-space contained in Π_5 which intersects $\mathcal{K}$ in $2q+1$ points. If this is not the case, then consider a 3-space in Π_5 containing $x > 1$ points of $\mathcal{K}$. We get the following equality:

$$(q+1)(q+1-x) + x = q^2 + q + 1,$$

hence $x = 1$, a contradiction.

Hence by Theorem 1.8, Π_5 intersects $\mathcal{K}$ in a Veronese variety $\mathcal{V}_2^4$.

□

Theorem 2.10. *The set $\mathcal{K}$ is a Veronese variety $\mathcal{V}_n^{2^n}$.*

Proof. We check the conditions of Theorem 1.10. The set $\mathcal{P}$ consists of all planes intersecting $\mathcal{K}$ in an oval.

Any two points of $\mathcal{K}$ are contained in at least one oval of $\mathcal{K}$ by Condition (P) and Lemma 2.5. If two points p_1, p_2 are contained in two ovals, namely $\mathcal{O}_1$ in the plane π_1 and $\mathcal{O}_2$ in the plane π_2, then these ovals span a 3-space Π_3 containing too many points of $\mathcal{K}$, a contradiction. Indeed, consider a point r on the intersection line l of π_1 and π_2 which is not the nucleus of $\mathcal{O}_1$ neither of $\mathcal{O}_2$ and two bisecants l_1 and l_2 through r to $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Then the plane spanned by l_1 and l_2 contains at least 4 points of $\mathcal{K}$ and hence by Condition (P) $q+1$ points of $\mathcal{K}$. In this way we get at least $2q + q - 3 = 3q - 3 \geq 2q + 2$ (since $q \geq 5$) points in $\Pi_3 \cap \mathcal{K}$, a contradiction by Lemma 2.6. Hence, Property (U) is proved.

To prove Property (NE), consider two planes π_1 and π_2 which intersect $\mathcal{K}$ in an oval. If $\pi_1 \cap \pi_2$ is a point then the property follows directly from Lemma 2.7. If $\pi_1 \cap \pi_2$ is a line then we get a 3-space Π_3

containing at least and so exactly $2q+1$ points. But then there are 4-spaces through Π_3 containing more than $2q+1$ points of $\mathcal{K}$, a contradiction.

For Property (TP), take a point p not contained in a plane π which intersects $\mathcal{K}$ in an oval $\mathcal{O}_1$. Consider two points r and s on $\mathcal{O}_1$ and the ovals $\mathcal{O}_2$ and $\mathcal{O}_3$ which are uniquely determined by p and r, and p and s respectively. The point set of the ovals $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are contained in a 5-space Π_5 intersecting $\mathcal{K}$ in more than $2q+2$ points. Hence, by Lemma 2.9 $\Pi_5 \cap \mathcal{K}$ is a Veronesean $\mathcal{V}_2^4$. Take an arbitrary point t on $\mathcal{O}_1$ and consider the oval determined by p and t. Since $\Pi_5 \cap \mathcal{K}$ is a $\mathcal{V}_2^4$, this oval is contained in Π_5. For each of these ovals there is a tangent at p to these ovals. By Lemma 25.4.2 of [4] the union of these tangents forms a plane.

$\square$

3. Second characterization

In this section, we show that for $n > 2$, we can replace the set of conditions of Section 2 by the following set of conditions. Furthermore, we provide a counterexample for the case $n = 2$.

Consider a set $\mathcal{K}$ of $\frac{q^{n+1}-1}{q-1}$ points spanning $\mathbf{PG}(\frac{n(n+3)}{2}, q)$, with $n > 2$, such that

(P') If π is a plane then the intersection $\pi \cap \mathcal{K}$ contains at most $q+1$ points of $\mathcal{K}$.
(S') If a 3-space Π_3 intersects $\mathcal{K}$ in more than 4 points, then $|\Pi_3 \cap \mathcal{K}| \geq q+1$ and $\Pi_3 \cap \mathcal{K}$ is not a $(q+1)$-arc.
(V') If a 5-space Π_5 intersects $\mathcal{K}$ in more than $2q+2$ points then it intersects $\mathcal{K}$ in exactly q^2+q+1 points. Furthermore, any two points p_1, p_2 of $\mathcal{K}$ are contained in a 5-space containing q^2+q+1 points of $\mathcal{K}$.

Lemma 3.1. *Every 4-space contains at most $2q+2$ points of $\mathcal{K}$. Hence, a 3-space contained in a big 5-space contains at most $q+3$ points of $\mathcal{K}$.*

Proof. Exactly the same as the proof of Lemma 2.6 using Condition (V'), since we only used there that part of Condition (V).

$\square$

Lemma 3.2. *For $n > 2$, $q > 7$, if a plane π contains at least 4 points of $\mathcal{K}$, then it contains exactly $q+1$ points of $\mathcal{K}$.*

Proof. First suppose that $4 < |\pi \cap \mathcal{K}| < q+1$. Then all 3-spaces through π contain at least $q+1$ points. This yields at least $\frac{q^{\frac{n(n+3)}{2}-2}-1}{q-1}$ points for the set $\mathcal{K}$, a contradiction since $n > 2$.

Next, suppose that $|\pi \cap \mathcal{K}| = 4$. Consider points a, b and c of $\mathcal{K}$ such that $\langle \pi, a\rangle$, $\langle \pi, b\rangle$ and $\langle \pi, c\rangle$ are three different 3-spaces. By Condition (S') each of these three 3-spaces intersects $\mathcal{K}$ in at least $q+1$ points. Then the space $\langle \pi,\ a,\ b,\ c\rangle$ contains at least $3(q-3)+4$ points of $\mathcal{K}$. Hence, since $q > 7$, by Lemma 3.1 and Condition (V') it is a big 5-space Π_5.

By Lemma 3.1, a 3-space Π_3 in Π_5 contains at most $q+3$ points of $\mathcal{K}$.

From the previous paragraph it follows that we get the following inequality for the number x of 3-spaces Π_3^i through π inside the big 5-space Π_5 containing at least $q+1$ points of $\mathcal{K}$.

$$x(q-1)+4 \geq q^2+q+1.$$

Hence we get $x \geq q+2-\frac{1}{q-1}$, this implies $x \geq q+2$ if $q > 2$.

Now consider a 3-space Π_3'' containing π and at least $q+1$ points of $\mathcal{K}$ which is not contained in Π_5 and consider also the 6-space $\Pi_6 = \langle \Pi_5, \Pi_3'' \rangle$. Take one fixed 3-space Π_3^1 and consider the 5-spaces $\langle \Pi_3^1, \Pi_3'', \Pi_3^i \rangle$ with $i \neq 1$. Each of these 5-spaces intersects $\mathcal{K}$ in more than $2q+2$ points since $q > 7$ and hence is a big 5-space. It follows that $\Pi_6 \cap \mathcal{K}$ contains at least $(q+1)(q^2-q-1)+2q+2 = q^3+1$ points.

Repeating this reasoning yields inductively the following recursion formula for the number of points ϕ_{k+1} in $\Pi_{k+1} \cap \mathcal{K}$ where $\tilde{\Pi}_3$ is a 3-space containing π and at least $q+1$ points of $\mathcal{K}$ which is not contained in Π_k and where $\Pi_{k+1} = \langle \Pi_k, \tilde{\Pi}_3 \rangle$, where $\phi_5 = q^2+q+1$.

$$\phi_{k+1} \geq \left(\frac{\phi_k - 4}{q-1} - 1\right)(q^2-q-1) + 2q + 2. \tag{5}$$

We will adapt the recursion formula to a recursion formula for numbers ψ_k such that $\psi_k \leq \phi_k$ for all $k \geq 5$.

First we rewrite the recursion formula for ϕ_k as follows.

$$\phi_{k+1} = (\phi_k - q - 3)\frac{q^2-q-1}{q-1} + 2q + 2.$$

Since $\frac{q^2-q-1}{q-1} > q-1$ if $q > 2$ we get after a little calculation

$$\phi_{k+1} > (q-1)\phi_k - q^2 + 5.$$

Since $\phi_5 = q^2+q+1$ we can even write for all integers $k \geq 5$

$$\phi_{k+1} > (q-2)\phi_k.$$

Now we set $\psi_5 = \phi_5$ and $\psi_{k+1} = (q-2)\psi_k$. Hence we get $\psi_N = (q-2)^{N-5}(q^2+q+1)$ for all $N \geq 5$. This yields the following inequality

$$(q-2)^{\frac{n(n+3)}{2}-5}(q^2+q+1) \leq \frac{q^{n+1}-1}{q-1}.$$

This is an equality if $n = 2$ and the left hand side increases faster than the right hand side if n increases, hence this yields a contradiction for $n > 2$.

$\square$

The remaining cases are $q = 5$ and $q = 7$. First we prove a lemma for $q = 5$.

Lemma 3.3. *Let $q = 5$, $n > 2$, and consider a plane π which intersects $\mathcal{K}$ in 4 points. If inside a big 5-space Π_5 there is a 4-space Π_4 through π intersecting $\mathcal{K}$ in 12 points then there are no 4-spaces through π inside Π_5 intersecting $\mathcal{K}$ in 11 or 10 points.*

Proof. Suppose the contrary and consider a 6-space Π_6 containing Π_5 which intersects $\mathcal{K}$ in more than 31 points. Such a 6-space always exists otherwise we don't get enough points for the set $\mathcal{K}$.

If Π_5 contains a 4-space Π_4' through π intersecting $\mathcal{K}$ in 11 points, then consider all 5-spaces through Π_4 and Π_4' inside Π_6. By Conditions (S') and (V') the only ones which yield extra points are big 5-spaces. For take a 5-space with an extra point p, then we have at least two extra points. Namely, the 3-space $\langle \pi, p \rangle$ contains more than 4 points of $\mathcal{K}$ and hence by Condition (S') at least 6 points of $\mathcal{K}$.

Denote the number of big 5-spaces inside Π_6 through Π_4 and Π_4' by α and β respectively. We get the following equation

$$19\alpha + 12 = 20\beta + 11$$

If we rewrite this as $1+19(\alpha-\beta)=\beta$, then clearly the only solution with $1 \leq \alpha, \beta \leq 6$ is $\alpha=\beta=1$, a contradiction since Π_6 intersects $\mathcal{K}$ in more than 31 points.

If there is a 4-space Π_4' in Π_5 through π intersecting $\mathcal{K}$ in 10 points then 5-spaces through Π_4' inside Π_6 which yield extra points intersect $\mathcal{K}$ either in 12 or in 31 points by Condition (S') and Condition (V'). Denote the number of big 5-spaces through Π_4 inside Π_6 by x, the number of 5-spaces of Π_6 through Π_4' intersecting $\mathcal{K}$ in 12 points by y and the number of big 5-spaces through Π_4' inside Π_6 by z. Then the following equation is obtained

$$19x+12=2y+21z+10, \text{ with } x \geq 1 \text{ and } x,\ y,\ z \leq 6.$$

The only solution is $x=z=1$ and $y=0$, a contradiction since Π_6 intersects $\mathcal{K}$ in more than 31 points.

□

Lemma 3.4. *For $q=5$ or $q=7$ and $n>2$, a plane π intersecting $\mathcal{K}$ in exactly 4 points is never contained in a big 5-space Π_5.*

Proof. Case (a) $q=5$:

Assume π is contained in a big 5-space Π_5. First of all, a 3-space in Π_5 contains at most 8 points of $\mathcal{K}$ by Lemma 3.1.

Project Π_5 from π onto a plane π' which is skew to π in Π_5. For the 3-spaces through π which contain 6, 7 or 8 points of $\mathcal{K}$, the projection in π' is given weight 2, 3 or 4 respectively.

First suppose there is a 3-space Π_3 in Π_5 which contains π and which intersects $\mathcal{K}$ in 8 points. Then five 4-spaces through Π_3 in Π_5 intersect $\mathcal{K}$ in 12 points and one 4-space through Π_3 in Π_5 intersects $\mathcal{K}$ in 11 points. But this yields a contradiction by Lemma 3.3.

Hence, from now on, we may assume that each 3-space through π inside a big 5-space which contains more than 4 points of $\mathcal{K}$ contains 6 or 7 points of $\mathcal{K}$. Hence if we denote the number of 3-spaces through π inside Π_5 which intersect $\mathcal{K}$ in 6 points by α and those which intersect $\mathcal{K}$ in 7 points by β, we get the following equation,

$$4+2\alpha+3\beta=31.$$

The rest of the proof is case-by-case analysis.

(A) $\beta \geq 7$:

In this case we have a set $\mathcal{P}$ of at least 7 points with weight 3 in π'. Since an oval in $\mathbf{PG}(2,5)$ contains at most 6 points, three points of $\mathcal{P}$ will be collinear. But this implies that the 4-space spanned by the line L containing them and π intersects $\mathcal{K}$ in more than 12 points, a contradiction by Lemma 3.1.

(B) $\alpha=6,\ \beta=5$:

Consider a point p of weight 3 in π' and all lines $L_1, \cdots, L_6$ through it. On four of these lines, say $L_1, \cdots, L_4$ we have exactly one other point which has weight 3 otherwise we get a 4-space with more than 12 points. If none of $L_1, \cdots, L_4$ contains a point of weight 2 then the six points of weight 2 have to be distributed over the remaining two lines through p, a contradiction since then we get a 4-space intersecting $\mathcal{K}$ in more than 12 points of $\mathcal{K}$, a contradiction by Lemma 3.1. Hence we have already found a 4-space through π inside Π_5 which intersects $\mathcal{K}$ in 12 points. By Lemma 3.3, this implies that no 4-space through π inside Π_5 can intersect $\mathcal{K}$ in 10 or in 11 points. Now consider a point of weight 2 in π' and all lines through it. Then the 5 points of weight 3 are distributed over these lines as 2+2+1, as 2+1+1+1 or as 1+1+1+1+1. The latter two possibilities clearly yield a 4-space intersecting $\mathcal{K}$ in more than 12 points, a contradiction by Lemma 3.1. Namely, in the last case for instance, since no

4-space inside Π_5 is allowed to intersect $\mathcal{K}$ in exactly 11 points, all points of weight two are contained in one line through p in π'. The other case is similar.

But if it is always the 2+2+1 possibility then through each point of weight 2 there passes a line L containing 4 points of weight 2, which yields a contradiction.

(C) $\alpha = 9$, $\beta = 3$:

Consider a point p of weight 3 in π' and all lines through it. On two of these lines, L_1 and L_2, we have exactly one other point which has weight 3 otherwise we get a 4-space with more than 12 points. If there is no 4-space through π in Π_5 which intersects $\mathcal{K}$ in 12 points, then the 9 points of weight 2 have to be distributed over the remaining 4 lines, which again yields a too big 4-space. Hence there is a 4-space inside Π_5 intersecting $\mathcal{K}$ in 12 points, implying no 4-spaces through π inside Π_5 are allowed to intersect $\mathcal{K}$ in 10 or in 11 points by Lemma 3.3. This is impossible.

(D) $\alpha = 12$, $\beta = 1$:

Denote the 3-space through π inside Π_5 which intersects $\mathcal{K}$ in 7 points by Π_3. We have a set $\mathcal{P}$ of 13 points of weight 2 and 3 in π'. We claim that there has to be a line L containing 4 points of $\mathcal{P}$.

Indeed, consider an arbitrary point p contained in $\mathcal{P}$ and all lines through it. If there is no line which intersects $\mathcal{P}$ in 4 points then all lines of π' through p intersect $\mathcal{K}$ in 3 points. Since p was arbitrary this implies that all lines in π' intersect $\mathcal{P}$ in 0 or 3 points. But consider now a point r in π' not contained in $\mathcal{P}$ and all lines through it. Then we get a contradiction, since 3 does not divide 13. So we may assume that there is a line L in π' which intersects $\mathcal{P}$ in 4 points.

Hence, the 4-space spanned by L and π intersects $\mathcal{K}$ in 12 points. It has to be contained in another big 5-space otherwise we don't get enough points in $\mathcal{K}$. There again there has to be at least one 3-space, say Π_3', through π which intersects $\mathcal{K}$ in 7 points.

But now consider a space $\hat{\Pi}$ spanned by Π_3, Π_3' and another 3-space through π which contains at least 6 points of $\mathcal{K}$.

Then $\hat{\Pi}$ is certainly contained in a big 5-space, otherwise we don't have enough points in the set $\mathcal{K}$. But this is a contradiction since in any big 5-space we already excluded all cases with $\beta > 1$.

Case (b) $q = 7$:

Assume the plane π is contained in a big 5-space Π_5. First of all, a 3-space in Π_5 contains at most 10 points of $\mathcal{K}$ by Lemma 3.1.

Project Π_5 from π onto a plane π' which is skew to π in Π_5. For the 3-spaces through π which contain 8, 9 or 10 points of $\mathcal{K}$, the projection in π' is given weight 4, 5 or 6 respectively. Denote this set of points by $\mathcal{P}$.

First suppose there is a 3-space Π_3 in Π_5 which contains π and which intersects $\mathcal{K}$ in 10 points. Then seven 4-spaces through Π_3 in Π_5 intersect $\mathcal{K}$ in 16 points and one 4-space through Π_3 in Π_5 intersects $\mathcal{K}$ in 15 points. Consider all lines through the point p in π' corresponding with Π_3. There is exactly one line through p which contains one point p' of weight 5.

This implies that inside a big 5-space through Π_3 there is exactly one 3-space Π_3', namely the one which corresponds with p', which contains π and which intersects $\mathcal{K}$ in 9 points.

But a 4-space Π_4 through Π_3 intersecting $\mathcal{K}$ in 16 points has to be contained in at least one other big 5-space $\tilde{\Pi}_5$. Inside $\tilde{\Pi}_5$ we also find a 3-space $\tilde{\Pi}_3$ which contains π and which intersects $\mathcal{K}$ in 9 points. But now the big 5-space spanned by Π_3, Π_3' and $\tilde{\Pi}_3$ contains Π_3 and two 3-spaces which contain π and which intersect $\mathcal{K}$ in 9 points, a contradiction by the previous paragraph.

Hence, from now on, we may assume that each 3-space through π inside a big 5-space through π which contains more than 4 points of $\mathcal{K}$ contains 8 or 9 points of $\mathcal{K}$. Hence if we denote the number of 3-spaces through π inside Π_5 which intersect $\mathcal{K}$ in 8 points by α and those which intersect $\mathcal{K}$ in 9 points by β, we get the following equation,

$$4 + 4\alpha + 5\beta = 57.$$

Remark that inside π' each line through a point of weight 5 can contain at most one other point of weight 4 or 5 otherwise we get a 4-space which intersects $\mathcal{K}$ in more than 16 points. Hence $|\mathcal{P}| = \alpha + \beta \leq 1 + 8 \cdot 1 = 9$.

The only solutions of the above equation for (α, β) are the pairs $(12, 1)$, $(7, 5)$ and $(2, 9)$, which yields a contradiction by the previous paragraph.

□

Remark. The method of proof of the above theorem can also be used for the general case. However, Lemma 3.2 directly excludes all planes containing 4 points of $\mathcal{K}$ for $q > 7$.

Lemma 3.5. *Any line l meets $\mathcal{K}$ in at most 2 points. Hence, a plane π with $|\pi \cap \mathcal{K}| = q + 1$ intersects $\mathcal{K}$ in an oval.*

Proof. Similar to the proof of Lemma 2.5; use Lemma 3.1 and Lemma 3.2.

□

Theorem 3.6. *If $q \geq 5$ and $n > 2$, then the set $\mathcal{K}$ is the point set of the Veronese variety of all quadrics of $\mathbf{PG}(n, q)$.*

Proof. We check Conditions (P), (S) and (V) of Theorem 2.2. Conditions (S) and (V) are implied by Condition (S') and Theorem 1.12 and by Condition (V') respectively. The first part of Condition (P) was proved in Lemma 3.2 for $q > 7$.

Furthermore, for $q = 5$ and $q = 7$ we proved the first part of Condition (P) for all planes which are contained in a big 5-space. In fact, we did only use Condition (P) for these planes in our first characterization.

The second part of Condition (P), namely that every 2 points of $\mathcal{K}$ are contained in an oval of $\mathcal{K}$, is never used to obtain Lemma 2.9 if $n > 2$. If $n = 2$, we did use Condition (P) for the proof of Lemma 2.6. Since every two points are contained in a big 5-space Π_5 by Condition (V), and since $\Pi_5 \cap \mathcal{K}$ is a Veronese surface $\mathcal{V}_2^4$ by Lemma 2.9 the second part of Condition (P) is proved. The proof is finished by Theorem 2.2.

□

The counterexample for the case $n = 2$ is the following. Consider in $\mathbf{PG}(5, q)$ a point p on an ovoid $\mathcal{O}$ in $\mathbf{PG}(3, q)$ and a tangent line L to $\mathcal{O}$ at p. Furthermore, consider a second 3-space Π_3' intersecting Π_3 exactly in L and containing an oval $\mathcal{O}'$ which intersects L in p. Then the set $\mathcal{O} \cup \mathcal{O}'$ fulfills Conditions (P'), (S') and (V') but it is not a Veronesean $\mathcal{V}_2^4$.

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