André embeddings of affine planes

J. A Thas H. Van Maldeghem

Abstract

An André embedding is a representation of a point-line geometry $\mathcal{S}$ with approximately s^2 points on a line in a planar space with approximately s points per line, but such that the lines of $\mathcal{S}$ are contained in planes of the planar space. An example is the André representation (also sometimes called the Bose-Bruck representation) of an affine translation plane of order q^2 (with kernel of order at least q) in 4-dimensional affine space $\mathsf{AG}(4, q)$, using a line spread at infinity. In this paper, we classify all André embeddings of affine planes of order q^2 in $\mathsf{PG}(4, q)$, $q > 2$, and obtain, besides the natural extension to $\mathsf{PG}(4, q)$ of the above example, two other related constructions. We also consider André embeddings of affine planes of order q^2 in $\mathsf{PG}(d, q)$, with $d > 4$ and $q > 2$.

1 Introduction

In the theory of translation planes, the so-called Bruck-Bose representation [2] of a translation plane plays a central and prominent role. In fact, this representation already appeared in the work of André [1] much earlier, and so we shall call it in this paper the André representation. In essence, this representation shows the equivalence of a translation plane over a quasifield which is at most n-dimensional over its kernel $\mathbb{K}$ with a spread of $(n-1)$-dimensional subspaces in the $(2n-1)$-dimensional projective space $\mathsf{PG}(2n-1, \mathbb{K})$. Then one embeds $\mathsf{PG}(2n-1, \mathbb{K})$ in a $2n$-dimensional space $\mathsf{PG}(2n, \mathbb{K})$ and the points of the (affine) translation plane are the points of $\mathsf{PG}(2n, \mathbb{K}) \setminus \mathsf{PG}(2n-1, \mathbb{K})$, whereas the lines are the n-dimensional subspaces of $\mathsf{PG}(2n, \mathbb{K})$ intersecting $\mathsf{PG}(2n-1, \mathbb{K})$ in precisely a spread element, with natural incidence.

The lowest dimensional nontrivial case appears in projective 4-space, for $n = 2$. In this case, the points of any line of the affine translation plane $\mathcal{A}$ are contained in a projective plane. One can see this as an embedding of $\mathcal{A}$ in $\mathsf{PG}(4, \mathbb{K})$ where points of $\mathcal{A}$

are represented by points of $\mathsf{PG}(4,\mathbb{K})$ and lines of $\mathcal{A}$ by planes of $\mathsf{PG}(4,\mathbb{K})$. A similar situation occurs with the Veronesean embedding of a Pappian projective plane $\mathsf{PG}(2,\mathbb{K})$: there, points of $\mathsf{PG}(2,\mathbb{K})$ are represented by points of $\mathsf{PG}(5,\mathbb{K})$ and lines of $\mathsf{PG}(2,\mathbb{K})$ by planes of $\mathsf{PG}(5,\mathbb{K})$. Characterizations of the latter embedding are given in [5, 6, 7]. Note that Veronesean embeddings were recently used to construct authentications codes, and hence it is worthwhile to try to find other varieties which can relate to codes. In the present paper, we characterize the André representation. Our motivation is not only to do this because of the similarity with (quadric) Veroneseans and to find a way to axiomatically distinguish these embeddings, but also for practical reasons: such characterization is necessary in order to classify the Hermitian Veronesean embeddings of projective spaces, which recently gained interest because of their connection with triality and ovoids of the triality quadric, see [3], and also because their similarity with quadric Veroneseans suggests a possible application to authentication codes.

However, exactly because of the existence of the Veronesean embeddings of projective planes, the additional axioms to distinguish these from the André representations for the moment only work in the finite case. Indeed, the point is that Veronesean embeddings occur in spaces containing planes of the same size as the embedded plane whereas in the André setting the embedded plane has, so to speak, a dimensional double in size compared to the planes of the ambient projective space. In the infinite case, this seems hard to capture in axioms, whereas in the finite case, a simple condition on the size of the planes suffices.

In general, one could define an André embedding of a point-line geometry $\mathcal{S}$ with approximately s^2 points on a line as a representation of $\mathcal{S}$ in a planar space $\mathcal{S}'$ with approximately s points per line, such that points of $\mathcal{S}$ correspond to points of $\mathcal{S}'$, and such that the lines of $\mathcal{S}$ are contained in planes of the planar space $\mathcal{S}'$. More exactly, we define an *André embedding of an affine plane $\mathcal{A}$ of order q^2* in the projective space $\mathsf{PG}(d,q)$, $d \geq 4$, as a representation of $\mathcal{A}$ in $\mathsf{PG}(d,q)$ where the points of $\mathcal{A}$ are points of $\mathsf{PG}(d,q)$ which generate $\mathsf{PG}(d,q)$, and where the induced lines of $\mathcal{A}$ are contained in planes of $\mathsf{PG}(d,q)$. The same definition holds for projective planes of order q^2 in projective spaces $\mathsf{PG}(d,q)$, $d \geq 4$. We briefly say that the plane is *André embedded* in $\mathsf{PG}(d,q)$.

In the present paper, we classify the André embeddings of affine and projective planes of order q^2. This yields a characterization of the André representation of affine translation planes in the case $n = 2$ mentioned above. In the next section, we state our Main Result in detail, after we define the relevant examples.

As for notation, we will denote an affine and projective plane with a triple mentioning the point set, the line set and the incidence relation.

Finally, we would like to remark that all our results exclude the case $q = 2$, where our methods completely fail. However, it should not be too difficult to settle the case $q = 2$ separately. We did not try to do this, since it would take too much space, and this would overemphasize this small case.

2 Three constructions and statement of the Main Result

Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \in)$ be an affine translation plane of order q^2 and kernel containing $\mathsf{GF}(q)$. Then, by [1], there is a line spread $\mathcal{S}$ of $\mathsf{PG}(3, q)$ such that $\mathcal{A}$ can be represented as follows. Embed $\mathsf{PG}(3, q)$ in $\mathsf{PG}(4, q)$. Then the points of $\mathcal{A}$ are all points of $\mathsf{PG}(4, q) \setminus \mathsf{PG}(3, q)$ and the lines of $\mathcal{A}$ are the planes of $\mathsf{PG}(4, q)$ meeting $\mathsf{PG}(3, q)$ in a member of $\mathcal{S}$; incidence is the natural one. This clearly defines an André embedding of $\mathcal{A}$ in $\mathsf{PG}(4, q)$, which we will call the *standard André embedding* of $\mathcal{A}$.

Next, consider the projective completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ (denote the line at infinity by A_∞). We can embed $\widehat{\mathcal{A}}$ in $\mathsf{PG}(4, q)$ by first letting A_∞ correspond to an arbitrarily chosen plane π_∞ in $\mathsf{PG}(3, q)$, by secondly noting that π_∞ contains a unique member S of $\mathcal{S}$, by thirdly choosing an arbitrary point x on S, and by finally letting the points of A_∞ correspond to all points of the set $(\pi_\infty \setminus S) \cup \{x\}$. This way, we obtain an André embedding of $\widehat{\mathcal{A}}$ in $\mathsf{PG}(4, q)$, which we call a *standard André embedding* of $\widehat{\mathcal{A}}$.

Now, choose an arbitrary line L in $\widehat{\mathcal{A}}$, with $L \neq A_\infty$, and denote the associated affine plane by $\mathcal{A}_L$. If we delete in the standard André embedding of $\widehat{\mathcal{A}}$ all points corresponding to L, then we obtain an André embedding of $\mathcal{A}_L$ in $\mathsf{PG}(4, q)$, which we call a *nonstandard André embedding* of $\mathcal{A}_L$. If the plane of $\mathsf{PG}(4, q)$ corresponding to L contains the spread line S, then we say that the corresponding nonstandard André embedding of $\mathcal{A}_L$ is of Type I, otherwise it is of Type II.

Main Result. *If the affine plane $\mathcal{A}$ of order q^2, $q > 2$, is André embedded in $\mathsf{PG}(d, q)$, with $d \geq 4$, then $d = 4$, the projective completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ is a projective translation plane, and the embedding is either the standard André embedding of the (translation) affine plane $\mathcal{A}$, or a nonstandard André embedding of $\mathcal{A}$ (and $\widehat{\mathcal{A}}$ has a translation line belonging to $\mathcal{A}$).*

An immediate corollary is the following.

Corollary. *Every André embedding of a projective plane of order q^2, $q > 2$, in $\mathsf{PG}(d, q)$, $d \geq 4$, is the standard André embedding.*

3 Proof of the Main Result

We first prove the assertion for $d = 4$.

We assume throughout that $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \in)$ is an affine plane of order q^2, which is André embedded in $\mathsf{PG}(4, q)$, with $q > 2$. For every $L \in \mathcal{L}$, we denote by π_L the corresponding plane in $\mathsf{PG}(4, q)$. For two points $x, y \in \mathcal{P}$, we denote by xy the unique line of $\mathcal{A}$ incident with both x and y, and by $\langle x, y\rangle$ the unique line of $\mathsf{PG}(4, q)$ containing both x and y.

First we claim that, if $x \in \mathcal{P}$ and x belongs to π_L, for some $L \in \mathcal{L}$, then $x \in L$. Indeed, suppose not. Since $(q + 1)(q - 1) < q^2$, we can choose q points $y_1, \ldots, y_q \in L$ such that $x, y_1, \ldots, y_q$ are collinear in $\mathsf{PG}(4, q)$. The line $\langle x, y_i\rangle$ only contains two points of xy_i; hence all points of $\pi_{xy_i} \setminus \langle x, y_i\rangle$ except two belong to xy_i, for all $i \in \{1, 2, \ldots, q\}$. Since $q > 2$, there is a line A in π_{xy_2} through y_1 all points of which, except y_1, belong to xy_2. Set $A = \{y_1, z_1, \ldots, z_q\}$. At least two lines y_1z_i, say y_1z_1 and y_1z_2, are not parallel to xy_3. Hence the planes $\pi_{y_1z_1}$ and $\pi_{y_1z_2}$ intersect the plane π_{xy_3} in two distinct lines of $\mathsf{PG}(4, q)$ (indeed distinct, because clearly $\pi_{y_1z_1} \neq \pi_{y_1z_2}$), so that each of these lines contains exactly one point of xy_3. It follows that these lines contain in total $2q - 2$ points which do not belong to xy_3 and which neither belong to the line $\langle x, y_3\rangle$ of $\mathsf{PG}(4, q)$. So $2q - 2 \leq 2$, a contradiction.

Our claim is proved.

For ease of speech, we will call any point of $\mathsf{PG}(4, q) \setminus \mathcal{P}$ an *imaginary point*. For convenience, imaginary points will be denoted with letters at the beginning of the alphabet such as a, b, c, up to m, possibly furnished with subscripts. Likewise, lines of $\mathsf{PG}(4, q)$ will be denoted by A, B, C, possibly furnished with subscripts. Points of $\mathcal{A}$ will be denoted with lower case letters at the end of the alphabet, and lines of $\mathcal{A}$ with capital letters ranging from L to Z.

A plane of $\mathsf{PG}(4, q)$ containing all the points of a line of $\mathcal{A}$ will be referred to as an *A-plane*.

Now we first assume that for every pair of nonparallel lines $L, M \in \mathcal{L}$, the intersection $\pi_L \cap \pi_M$ is a point.

Let $N \in \mathcal{L}$ be arbitrary and let $x \in \mathcal{P}$ with $x \notin N$. Let ξ be a solid of $\mathsf{PG}(4, q)$ containing π_N but not x. Then the $q^2 + 1$ planes π_K through x intersect ξ in the lines of a spread $\mathcal{S}$ of ξ. It follows that exactly one line of $\mathcal{S}$ is contained in π_N. The corresponding plane π_R thus meets π_N in a line and hence R and N must be parallel, by our assumption. Our first claim above also implies that $C := \pi_R \cap \pi_N$ consists of imaginary points only, and these are the only imaginary points in the planes π_N and π_R. It now also follows that for all lines R' of $\mathcal{A}$ parallel to N, the plane $\pi_{R'}$ contains C. We will call C a *special line.*

Since the q^2+1 special lines lie in the q^2+1 planes π_K through x, we immediately see that the former are mutually disjoint, and hence that their union forms the complete set of q^3+q^2+q+1 imaginary points. Suppose that this set does not constitute a solid. Then we can find a point $z \in \mathcal{P}$ and two imaginary points a, b such that z, a, b are collinear. Let A and B be the respective special lines containing a and b. Letting z play the role of x above, we see that the two planes $\langle z, A\rangle$ and $\langle z, B\rangle$ (which correspond to two intersecting lines of $\mathcal{A}$) meet in a line, a contradiction.

It is now clear that we obtain a standard André embedding of $\mathcal{A}$ (and hence the latter is a translation affine plane).

So, from now on, we may assume that there are intersecting lines $L, M \in \mathcal{L}$ with $\pi_L \cap \pi_M$ a line A of $\mathsf{PG}(4,q)$. Let $L \cap M = \{u\}$. By our claim above, the line A contains exactly q imaginary points. The remaining imaginary point of π_L and of π_M is denoted by ℓ and m, respectively.

Let $L' \in \mathcal{L}$ be parallel to L in $\mathcal{A}$ and suppose by way of contradiction that $\pi_{L'}$ does not contain ℓ. Then it contains exactly one imaginary point a of A. Since L' is not parallel to M, the planes $\pi_{L'}$ and π_M have a point $z \in \mathcal{P}$ in common. But then the line $\langle a, z\rangle$ belongs to both planes, and this line contains at least two members of $\mathcal{P}$, a contradiction. Hence $\pi_L \cap \pi_{L'} = \{\ell\}$. The point ℓ will be called the *special point of* π_L.

Suppose now, by way of contradiction, that for some line T not parallel to L, the plane π_T contains ℓ. If $L \cap T = \{t\}$, then $\pi_T \cap \pi_L = \langle \ell, t\rangle$ and hence T and L would have at least two points in common, a contradiction. Hence the only planes π_X, $X \in \mathcal{L}$, containing ℓ are the q^2 planes corresponding to the lines of $\mathcal{A}$ parallel to L.

Now we claim that all points of the line $\langle \ell, m\rangle$ are imaginary points. Indeed, suppose by way of contradiction that some point $w \in \langle \ell, m\rangle$ belongs to $\mathcal{P}$. Let $L' \in \mathcal{L}$ be parallel to L and incident with w. Then, by the foregoing, $\pi_{L'}$ contains ℓ and hence m. Since m is the special point of π_M, this implies that L' is parallel to M, a contradiction. Our claim follows.

Now let b be an arbitrary imaginary point on A and let B be a line of $\mathsf{PG}(4,q)$ through b. Suppose that B contains at least two points v, w of $\mathcal{A}$, and that it is not contained in $\pi_L \cup \pi_M$. Since π_{vw} contains $b \notin \{\ell, m\}$, the line $vw \in \mathcal{L}$ is not parallel to either L or M. If it met L in a point distinct from u, then vw and L would share at least two points, a contradiction. We conclude that π_{vw} contains u and hence A.

If α_A denotes the number of planes π_X, $X \in \mathcal{L}$, containing A, then the foregoing implies that there are exactly $q\alpha_A$ lines of $\mathsf{PG}(4,q)$ through b containing at least two members of $\mathcal{P}$, and the total number of points of $\mathcal{A}$ covered by such lines is equal to $(q^2-1)\alpha_A$. All other points — and there are $q^4-(q^2-1)\alpha_A$ of these — are responsible for different lines through b. Hence, there are exactly $q^4-(q^2-q-1)\alpha_A$ lines through b containing at least one member of $\mathcal{P}$. Expressing that this number does not exceed q^3+q^2+q+1, we obtain

$$\alpha_A \geq q^2 - \frac{q+1}{q^2-q-1}.$$

Consequently $\alpha_A \geq q^2$ and so $\alpha_A \in \{q^2, q^2+1\}$.

From the previous paragraph, we deduce two cases.

(1) *The first case is that for each line C of $\mathsf{PG}(4,q)$, which is the intersection of two planes $\pi_{L'}, \pi_{M'}$, with $L', M' \in \mathcal{L}$ not parallel, we have $\alpha_C = q^2$.*

Let $\{X_1, X_2, \ldots, X_{q^2}\}$ be the collection of lines of $\mathcal{A}$ whose corresponding planes in $\mathsf{PG}(4,q)$ contain A. All these lines are incident with the point u. Let $U \in \mathcal{L}$ be the remaining line through u. Let $u' \in U$, $u' \neq u$. We claim that u' is contained in two intersecting lines $L', M' \in \mathcal{L}$ such that $\pi_{L'} \cap \pi_{M'}$ is a line A' of $\mathsf{PG}(4,q)$. Indeed, assume on the contrary that the planes corresponding to lines of $\mathcal{A}$ through u' pairwise meet in u'; then one of them, say π_Y, $Y \in \mathcal{L}$, meets π_L in a line distinct from A, implying $|L \cap Y| > 1$, a contradiction. Our claim follows. By assumption, there are now q^2 lines $X'_1, X'_2, \ldots, X'_{q^2}$ of $\mathcal{A}$ such that their corresponding planes contain A'.

We now claim that $A \cap A'$ is empty. Indeed, if not, then A and A' intersect in an imaginary point c. We can choose two lines X_i and X'_j, $i, j \in \{1, 2, \ldots, q^2\}$ such that X_i is parallel to X'_j, and then c must be the special point of π_{X_i}, contradicting the fact that the special point of that plane does not belong to A. Our claim is proved.

Projecting π_U and all π_{X_i}, $i = 1, 2, \ldots, q^2$, from A onto a plane of $\mathsf{PG}(4,q)$ skew to A, we obtain a line (corresponding to π_U) and a set of q^2 points, not any of them on that line; hence $(\pi_{X_1} \cup \pi_{X_2} \cup \cdots \cup \pi_{X_{q^2}}) \setminus A$ is an affine space $\mathsf{AG}(4,q)$.

Let a_i be the special point of π_{X_i}, $i = 1, 2, \ldots, q^2$. For distinct $i, j \in \{1, 2, \ldots, q^2\}$, the line $\langle a_i, a_j \rangle$ only contains imaginary points, but it also contains q points of $\mathsf{AG}(4,q)$; hence these must all be special points of certain planes π_{X_k}, with $k \in \{1, 2, \ldots, q^2\}$. It follows that $\{a_1, a_2, \ldots, a_{q^2}\}$ is the point set of an affine plane $\mathsf{AG}(2,q)$ contained in $\mathsf{AG}(4,q)$. We denote the line at infinity of $\mathsf{AG}(2,q)$ by A_∞. It is contained in the solid $\mathsf{PG}(3,q)$ spanned by A and π_U.

Since A' contains a point u' of $\mathcal{A}$, it is clear that it cannot contain a special point a_i, $i = 1, 2, \ldots, q^2$, and so A' is contained in $\mathsf{PG}(3,q)$ (otherwise q^2 lines of $\mathcal{A}$ through u'

would be parallel). At least q^2-1 A-planes through A' are parallel to some A-plane through A, and hence at least q^2-1 special points of A-planes through A' are contained in $\mathsf{AG}(2,q)$. It follows that all of them are contained in $\mathsf{AG}(2,q)$. Consequently π_U does not contain A'. This also implies that $\mathsf{PG}(3,q)=\langle U,A'\rangle$ and consequently that the affine 4-space defined by all A-planes through A' coincides with $\mathsf{AG}(4,q)$.

Varying u' over U, there arise q^2 mutually disjoint lines A' in $\mathsf{PG}(3,q)$; they all play the same role. So we see that they are all disjoint from A_∞. Consequently, the lines A' together with A_∞ define a spread of $\mathsf{PG}(3,q)$. Notice also that, since no line A' is contained in π_U, we must necessarily have that A_∞ is contained in π_U.

Now let $R\neq U$ be a line of $\mathcal{A}$ parallel to U. We claim that π_R contains A_∞ and no further points of $\mathsf{PG}(3,q)$. Clearly, all of the non-imaginary points of π_R are contained in $\mathsf{AG}(4,q)$ since these points must belong to an A-plane through A. If π_R contained a point of A, then it would contain a line of each A-plane through A, contradicting the fact that lines of $\mathcal{A}$ meet in at most one point. Since all A' play the same role, π_R does not meet any of these. Hence it must contain A_∞ and our claim is proved.

If we now remove the line $U=\pi_U\setminus A_\infty$ and all its points, and add the plane $\mathsf{AG}(2,q)$ together with all its points, then we obtain a standard André embedding of some translation affine plane. It now easily follows that the embedding of $\mathcal{A}$ is a nonstandard André embedding of Type I.

We now treat the second case.

(2) *There is a line A of $\mathsf{PG}(4,q)$, which is the intersection of two planes π_L,π_M, with $L,M\in\mathcal{L}$ not parallel, and with $\alpha_A=q^2+1$.*

Let π_{X_i}, $i=0,1,2,\ldots,q^2$, be the A-planes containing A. Let a_i be the special point of π_{X_i}, $i=0,1,2,\ldots,q^2$.

We first claim that the union of q^2 A-planes through A, minus the line A, is an affine 4-space $\mathsf{AG}(4,q)$. Indeed, let $y\in\mathcal{P}$ belong to X_0, with $y\neq u$ (as before $u\in\mathcal{P}$ is the point common to all X_i, $i=0,1,2,\ldots,q^2$). There are two possibilities. Either all A-planes through y intersect mutually in only y, or there exist two such planes intersecting in a line C.

Suppose first that there are two A-planes through y meeting in a line.

At least one of these planes does not contain u, say π_K, $K\in\mathcal{L}$ (in fact, both do not, but we do not need this). Note that one easily sees that A and π_K are not contained in a solid. Let k be the special point of π_K, and let B be the line in π_K containing q imaginary points. Then the A-plane containing A and corresponding to a line of $\mathcal{A}$ parallel to K, intersects π_K in k. Also, q^2-1 other A-planes through A meet π_K in points off B. Hence, these q^2 A-planes, minus their common line A, are contained in the affine space obtained by joining A with $\pi_K \setminus B$, and then deleting A. The claim follows.

Suppose now that all A-planes through y intersect mutually in only y.

In this case, we consider any A-plane π_N not containing u nor y. Then one of the A-planes, say π_K, $K \in \mathcal{L}$, through y intersects π_N in a line. If K and N are not parallel, then the claim follows similarly as in the first case. If they are parallel, then it is easy to see that the claim follows by considering the q^2 A-planes through A whose corresponding line in $\mathcal{A}$ is not parallel to K (again use the easy observation that A and π_K are not contained in a solid).

Hence the claim is proved.

So we may assume that $(\pi_{X_1} \cup \pi_{X_2} \cup \cdots \cup \pi_{X_{q^2}}) \setminus A$ is an affine space $\mathsf{AG}(4,q)$, which does not contain π_{X_0}. We denote by $\mathsf{PG}(3,q)$ its solid at infinity.

As before, it follows that the special points $a_1, a_2, \ldots, a_{q^2}$ form the point set of an affine plane $\mathsf{AG}(2,q)$, and that the corresponding line A_∞ at infinity consists of imaginary points only, among which a_0.

Take an arbitrary point u' of X_0, $u' \neq u$. We claim that there are two A-planes through u' intersecting in a line. Indeed, suppose all A-planes through u' intersect mutually in u'. Then one of them, say π_V, intersects the projective completion $\mathsf{PG}(2,q)$ of $\mathsf{AG}(2,q)$ in a line; this line necessarily coincides with A_∞, as otherwise V is parallel to at least q different lines through u', a contradiction. It follows that π_V is parallel to and hence coincides with π_{X_0}. But π_{X_0} cannot contain A_∞, as it would otherwise contain too many imaginary points (the ones on A and the ones on A_∞). The claim follows.

So we obtain q^2-1 lines $B_1, B_2, \ldots, B_{q^2-1}$ of $\mathsf{PG}(4,q)$ which each are the intersection of two A-planes whose corresponding lines of $\mathcal{A}$ are not parallel. Since clearly none of these lines can contain a special point a_i, $i \in \{0,1,2,\ldots,q^2\}$ (as otherwise two nonparallel lines of $\mathcal{A}$ are parallel to a common one), we see that $B_1, B_2, \ldots, B_{q^2-1}$ are contained in $\mathsf{PG}(3,q)$. Similarly as in (1), the lines $A, B_1, B_2, \ldots, B_{q^2-1}$ are mutually skew. Also, none of A, B_i, $i = 1,2,\ldots,q^2-1$, contains a point of A_∞ as otherwise an A-plane through such a line which corresponds to a line of $\mathcal{A}$ that is parallel to some line X_j, $j \in \{1,2,\ldots,q^2\}$, contains at least q special points of the planes π_{X_i}, $i = 1,2,\ldots,q^2$, a contradiction. Hence $\{A_\infty, A, B_1, B_2, \ldots B_{q^2-1}\}$ is a spread $\mathcal{S}$ of $\mathsf{PG}(3,q)$.

Now let W be a line of $\mathcal{A}$ parallel to but distinct from X_0. Let w be a point on W. Then $w \in \mathsf{AG}(4,q)$. Since q^2 A-planes through w mutually meet in w (because they contain $A, B_1, B_2, \ldots, B_{q^2-1}$), the unique missing A-plane through w must also intersect the others in just w and hence must contain A_∞ (and hence contains a_0 and therefore coincides with π_W).

We conclude that all A-planes except π_{X_0} meet $\mathsf{PG}(3,q)$ in an element of the spread $\mathcal{S}$. Since also $\mathsf{PG}(2,q)$ meets $\mathsf{PG}(3,q)$ in A_∞, we obtain an André embedding of a translation plane by removing the line $X_0 = (\pi_{X_0} \setminus (A \cup \{a_0\})) \cup \{u\}$ of $\mathcal{A}$ and all its points, and adding $\mathsf{AG}(2,q)$ and all its points. It easily follows that the embedding of $\mathcal{A}$ is a nonstandard André embedding of Type II.

This completes the proof for the case $d = 4$.

Now we prove the Main Result for $d \geq 6$.

We use the same notation as in the case $d = 4$.

Let $d \geq 6$. The number of points of $\mathsf{PG}(d,q)$ lying in some A-plane is at most $q^4 + (q^4 + q^2)(q+1) = q^5 + 2q^4 + q^3 + q^2 < q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$, hence there exists an imaginary point c not contained in any A-plane. Consequently c is not contained in any line of $\mathsf{PG}(d,q)$ that contains at least two points of $\mathcal{A}$. So we can project from c onto a suitable hyperplane to obtain an André embedding in some $\mathsf{PG}(d-1,q)$. We can do this procedure $d-5$ times to end up with an André embedding in $\mathsf{PG}(5,q)$. Hence we have reduced this case to the next case.

Finally, we prove the assertion for $d = 5$.

We again use the same notation as in the case $d = 4$.

Let $d = 5$. First suppose that for every pair of intersecting lines L, M of $\mathcal{A}$, the planes π_L and π_M meet in a line of $\mathsf{PG}(5,q)$. We treat this case including the possibility of q being equal to 2. Consider such lines L, M and let S be the solid they span. Since $\pi_L \cap \pi_M$ contains at most $q + 1 < q^2$ points of L, there is some point $y \in L$ not contained in π_M (and only if $q = 2$, this point could be unique). Similarly, there is some point $z \in M$ not contained in π_L. Let x in $\mathcal{A}$ be arbitrary, but not in $L \cup M$. If one of the lines xy and xz can be chosen not parallel to either L or M (and this can only fail when $q = 2$), then the corresponding A-plane must meet π_L and π_M in distinct lines, and hence this A-plane is contained in S. Consequently only the point x with xy parallel to M and xz parallel to L (and with $q = 2$) is possibly not contained in S, but this situation cannot occur since it would require that $\pi_L \cap \pi_M$ contains 3 points of L and also 3 points of M, a contradiction.

Hence we may assume that there are two A-planes π_L and π_M that meet in a point, and for which the corresponding lines L and M are not parallel in $\mathcal{A}$.

First assume that there exists a point x of $\mathcal{A}$ not in $L \cup M$ and contained in the 4-dimensional space $\xi := \langle \pi_L, \pi_M \rangle$.

There are only $q+1$ lines in ξ through x meeting both planes π_L and π_M; hence for at least q^2-2-q lines of $\mathcal{A}$ through x, the corresponding A-plane contains three non-collinear points of ξ and hence is contained in ξ. Let $q>3$. Now, for any point y of $\mathcal{A}$, not contained in ξ, we can find a line N of $\mathcal{A}$ containing y, not through x, and not parallel to any of the q^2-2-q above mentioned lines. Hence π_N meets ξ in at least $q^2-2-q>q+1$ points, and so must be contained in it. Hence $\xi=\mathsf{PG}(5,q)$, a contradiction.

If $q=3$, then the only case in which the above argument fails is when exactly 4 lines through x have a corresponding A-plane in ξ. If R is such a line, then, with the same reasoning, we may assume that every point on R, except $R\cap L$ and $R\cap M$, is incident with exactly 4 lines contained in ξ (including R). In total, this gives us already $7\cdot 3+1+2=24$ lines of $\mathcal{A}$ contained in ξ. Let y be as above, then each line of $\mathcal{A}$ through y meets the union of these 24 lines in at most 4 points; hence this union contains at most 40 points. A double count reveals that the average number of lines of $\mathcal{A}$ contained in ξ through a point of that union is at least 27/5. Consequently there is some point in $\mathcal{A}$ incident with at least 6 lines of $\mathcal{A}$ contained in ξ. Since $6>q+1=4$, the previous argument now works to obtain a contradiction.

Now assume that, with the above notation, $\xi\cap(L\cup M)=L\cup M$*.*

Set $x=L\cap M$. Since $q^2-1>q+1$, we can find a point $y\in L$ such that $\langle x,y\rangle$ contains at least three points of $\mathcal{A}$; let y' be a point of $\mathcal{A}$ on $\langle x,y\rangle$ distinct from x,y. Now choose two points z,z' in M such that the lines $\langle y,z\rangle$ and $\langle y',z'\rangle$ are skew; this is easy as not all points of M are contained in a line of $\mathsf{PG}(5,q)$ through x. Moreover, since $q^2>q+2$, we can choose z' such that the lines yz and $y'z'$ of $\mathcal{A}$ are not parallel. But then the planes π_{yz} and $\pi_{y'z'}$ meet in a point and span a 4-space containing the point x which does not belong to $\pi_{yz}\cup\pi_{y'z'}$, a contradiction as in the previous paragraph.

The Main Result is completely proved.

References

[1] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe, *Math. Z.* **60** (1954), 156–186.

[2] R. H. Bruck & R. C. Bose, The construction of translation planes from projective spaces, *J. Algebra* **1** (1964), 85–102.

[3] A. De Wispelaere, J. Huizinga & H. Van Maldeghem, Veronesean embeddings of Hermitian unitals, to appear in *European J. Combin.*

[4] J. W. P. Hirschfeld & J. A.Thas, *General Galois Geometries*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1991.

[5] J. A. Thas & H. Van Maldeghem, Characterizations of the finite quadric Veroneseans $\mathcal{V}_n^{2^n}$, *Quart. J. Math.* **55** (2004), 99–113.

[6] J. A. Thas & H. Van Maldeghem, Generalized Veronesean embeddings of finite projective spaces, submitted preprint, see also `http://cage.ugent.be/geometry/preprints.php`.

[7] Z. Akça, A. Bayar, S. Ekmekçi, R. Kaya , J. A. Thas & H. Van Maldeghem, Generalized lax Veronesean embeddings of projective spaces, submitted preprint, see also `http://cage.ugent.be/geometry/preprints.php`.

Address of the authors
Department of Pure Mathematics and Computer Algebra, Ghent University,
Krijgslaan 281-S22, B-9000 Ghent, BELGIUM
`{jat,hvm}@cage.ugent.be`