On maximal partial spreads of $H(2n+1,q^2)$

D. Luyckx*

October 21, 2005

Abstract

A lower bound for the size of a maximal partial spread of $H(2n+1,q^2)$ is given. For $H(2n+1,q^2)$ in general, and for $H(5,q^2)$ in particular, new upper bounds for this size are also obtained. In [1], maximal partial spreads of $H(3,q^2)$ and $H(5,q^2)$ have been constructed from spreads of $W_3(q)$ and $W_5(q)$ respectively; the construction for $H(5,q^2)$ will be generalized to $H(4n+1,q^2)$, $n \geq 1$, thus yielding examples of maximal partial spreads of $H(4n+1,q^2)$ for all $n \geq 1$.

1 Introduction

In the projective space $\mathsf{PG}(r,q^2)$, a hermitian variety consists of all totally isotropic subspaces of a non-degenerate hermitian polarity, and it is denoted by $H(r,q^2)$. For extensive information on the properties of hermitian varieties, [3] is an excellent source. Here we will focus on odd values of r, so $r = 2n+1$, and study maximal partial spreads of $H(2n+1,q^2)$.

A *partial spread* of $H(2n+1,q^2)$ is a set of pairwise disjoint generators, that is, totally isotropic subspaces of $H(2n+1,q^2)$ of maximal dimension n. A partial spread $\mathcal{S}$ of $H(2n+1,q^2)$ is *maximal* if it is not contained in a larger partial spread, so this occurs if and only if every generator of $H(2n+1,q^2)$ has a non-empty intersection with at least one element of $\mathcal{S}$.

A partial spread that partitions the points of $H(2n+1,q^2)$ is called a *spread*, but it is known that $H(2n+1,q^2)$ does not admit spreads, see Thas [7]. Therefore, the emphasis lies on studying maximal partial spreads of $H(2n+1,q^2)$.

2 On lower bounds

For the hermitian variety $H(3,q^2)$, a lower bound on the size of a maximal partial spread is known: in [2], Ebert and Hirschfeld show that it has at least $2q+1$ elements if $q < 4$ and at least $2q+2$ elements for all $q \geq 4$. To the knowledge of

*The author is Postdoctoral Fellow of the Fund for Scientific Research – Flanders (Belgium) (F.W.O. – Vlaanderen).

the author, no result is known concerning lower bounds on the size of a maximal partial spread of $H(2n+1, q^2)$ for general $n > 1$. In the sequel, a possible way to show the intuitive lower bound of $q+1$ will be demonstrated, by starting from a weaker condition on $\mathcal{S}$ than being a maximal partial spread of $H(2n+1, q^2)$.

Theorem 2.1 *Let $\mathcal{S}$ be a set of generators of $H(2n+1, q^2)$, such that each generator of $H(2n+1, q^2)$ meets at least one element of $\mathcal{S}$ nontrivially. Then $|\mathcal{S}| \geq q+1$.*

Proof.
If each point of $H(2n+1, q^2)$ is contained in an element of $\mathcal{S}$, then clearly

$$|\mathcal{S}| \cdot \frac{q^{2n+2}-1}{q^2-1} \geq \frac{(q^{2n+1}+1)(q^{2n+2}-1)}{q^2-1},$$

so $|\mathcal{S}| \geq q^{2n+1}+1 > q+1$. If there exists a point p of $H(2n+1, q^2)$ that does not belong to any member of $\mathcal{S}$, we will proceed by induction on n. By a simple counting argument, the theorem is fulfilled for $n = 1$. So assume that the result is true for $H(2n-1, q^2)$, $n \geq 2$. Every element X of $\mathcal{S}$ meets $p^\perp \cap H(2n+1, q^2) = pH(2n-1, q^2)$ in an $(n-1)$-dimensional subspace X_p. Set $\pi(X) := \langle p, X_p \rangle$, so $\pi(X)$ is the unique generator of $H(2n+1, q^2)$ that passes through p and meets $X \in \mathcal{S}$ in an $(n-1)$-dimensional subspace. In the residue $H(2n-1, q^2)$ of p, the set $\pi(\mathcal{S}) := \{\pi(X)/p \mid X \in \mathcal{S}\}$ meets each generator of $H(2n-1, q^2)$ nontrivially. Indeed, if $X \in \mathcal{S}$ and Y is a generator of $H(2n+1, q^2)$ on p, then $\dim(X \cap Y) = \dim(\pi(X) \cap Y) - 1$. Let $X' \in \mathcal{S}$ be such that Y and X' have at least a point in common, then $\dim(X' \cap Y) \geq 0$, so $\dim(\pi(X') \cap Y) \geq 1$ and in the residue of p, $\pi(X')/p$ and Y/p meet nontrivially. By the induction hypothesis we have $|\pi(\mathcal{S})| \geq q+1$. As $|\mathcal{S}| \geq |\pi(\mathcal{S})|$, the result follows. $\square$

If the elements of $\mathcal{S}$ need not be pairwise disjoint, this bound is sharp: the set of $q+1$ generators of $H(2n+1, q^2)$ that pass through a given totally isotropic subspace of dimension $n-1$, meets all generators of $H(2n+1, q^2)$ nontrivially. If we ask in addition that the elements of $\mathcal{S}$ must be pairwise disjoint, we obtain a lower bound for maximal partial spreads.

Theorem 2.2 *If $\mathcal{S}$ is a maximal partial spread of $H(2n+1, q^2)$, then $|\mathcal{S}| \geq q+1$.*

3 On upper bounds

A general upper bound on the number of elements of a partial spread of $H(2n+1, q^2)$ is not known. Concerning upper bounds on the number of elements of a partial spread $\mathcal{S}$ of $H(5, q^2)$, it has been shown in [6] that $|\mathcal{S}| \leq q^2(q^2+q-1)$. It is the aim to improve upon this upper bound, and we will start with a general theorem on partial spreads of $H(2n+1, q^2)$.

Theorem 3.1 *Let $\mathcal{S}$ be a partial spread of $H(2n+1,q^2)$, and define*

$$k = \begin{cases} \frac{n}{2}+1 & n \text{ even,} \\ \frac{n+1}{2}+1 & n \text{ odd.} \end{cases}$$

Then $|\mathcal{S}| \leq q^{k^2}+1-\varepsilon(n,q)$, where

$$\varepsilon(n,q) = \begin{cases} 0 & n \text{ even,} \\ \frac{q^{2k}-1}{q+1} & n \text{ odd.} \end{cases}$$

Proof.
Let $A \in \mathcal{S}$ and let $B \subset A$ be a subspace of dimension $n-k$. For every $X \in \mathcal{S}_A := \mathcal{S} \setminus \{A\}$, let $\pi(X)$ be the unique geverator of $H(2n+1,q^2)$ containing B and such that $\dim(\pi(X) \cap X) = k-1$. If X and Y are distinct members of $\mathcal{S}_A$, then $\dim(X \cap Y) = -1$. On the other hand, suppose that $\dim(\pi(X) \cap \pi(Y)) = n-h$ for some $h \leq k$. Then $X \cup Y \subseteq X \cup \pi(X) \cup \pi(Y) \cup Y$ and hence we find that $\dim(X \cup Y) \leq 3n+h-2k+2$. It follows that $\dim(X \cap Y) \geq 2k-n-h-2$. Since $\dim(X \cap Y) = -1$, this yields $-1 \geq 2k-n-h-2$, or $1 \leq -2k+n+h+2$. For n odd this implies that $h \geq 2$, while n even leads to $h \geq 1$. In both cases $\dim(\pi(X) \cap \pi(Y)) = n-h \leq n-1$, so that π is injective on $\mathcal{S}_A$, and $|\mathcal{S}_A| = |\pi(\mathcal{S}_A)|$, with $\pi(\mathcal{S}_A) := \{\pi(X) \mid X \in \mathcal{S}_A\}$. As the elements of $\mathcal{S}_A$ are disjoint from A, the elements of $\pi(\mathcal{S}_A)$ meet A exactly in B. If considered in the residue $H(2k-1,q^2)$ of B, $\pi(X)/B$ is disjoint from A/B, for each $X \in \mathcal{S}_A$. Hence $|\pi(\mathcal{S}_A)| \leq q^{k^2}$, which is well known to be the number of generators of $H(2k-1,q^2)$ that are disjoint from a given generator. It follows that $|\mathcal{S}| \leq q^{k^2}+1$.

If n is odd, $\dim(\pi(X) \cap \pi(Y)) \leq n-2$ for all $X, Y \in \mathcal{S}_A$, $X \neq Y$. For a fixed $A' \in \mathcal{S}_A$, the generators X on B that meet $\pi(A')$ in a subspace of dimension $n-1$ cannot be members of $\pi(\mathcal{S}_A)$, which leads to the improved upper bound. □

By applying the above theorem if $n=2$, w already obtain that $|\mathcal{S}| \leq q^4+1$ for a (maximal) partial spread of $H(5,q^2)$. This bound can be pushed down to q^4, by the following theorem.

Theorem 3.2 *If $\mathcal{S}$ is a partial spread of $H(5,q^2)$, then $|\mathcal{S}| \leq q^4$.*

Proof.
Since every partial spread of $H(5,q^2)$ can be completed to a maximal one, we may assume $\mathcal{S}$ to be maximal. Then every plane of $H(5,q^2)$ has a non-empty intersection with at least one element of $\mathcal{S}$. In particular, there exist planes that meet exactly one element of $\mathcal{S}$ in a line and, possibly, planes that meet no element of $\mathcal{S}$ in a line but some elements of $\mathcal{S}$ in a point. Let π be an arbitrary element of $\mathcal{S}$ and p a point of π. We now count the number of planes of $H(5,q^2)$ on p, not in $\mathcal{S}$, which meet no element of $\mathcal{S}$ in a line. Since there are $(q+1)(q^3+1)$

planes of $H(5,q^2)$ on p, $(q^2+1)q$ of which meet π in a line while $|\mathcal{S}|-1$ of them intersect some element of $\mathcal{S}\setminus\{\pi\}$ in a line, there exist

$$(q+1)(q^3+1)-(q^2+1)q-1-(|\mathcal{S}|-1)=q^4+1-|\mathcal{S}| \tag{1}$$

such planes. As this number cannot be negative, one obtains the upper bound from Theorem 3.1: $|\mathcal{S}|\leq q^4+1$.

If $|\mathcal{S}|=q^4+1$, then (1) equals zero and since $\pi\in\mathcal{S}$ and the point p were chosen arbitrarily, it follows that all planes of $H(5,q^2)$ either belong to $\mathcal{S}$ or meet an element of $\mathcal{S}$ in a line. The number of planes of $H(5,q^2)$ that intersect some element of $\mathcal{S}$ in a line equals $|\mathcal{S}|\cdot(q^4+q^2+1)q$, so that

$$|\mathcal{S}|+|\mathcal{S}|(q^4+q^2+1)q=(q^4+1)(q^5+q^3+q+1)$$

must be the total number of generators of $H(5,q^2)$. This is a contradiction, because the total number of generators is given by $(q+1)(q^3+1)(q^5+1)>(q^4+1)(q^5+q^3+q+1)$. It follows that $|\mathcal{S}|\leq q^4$. $\square$

So far, there are no examples known of maximal partial spreads of $H(5,q^2)$ of sizes close to the upper bound from Theorem 3.2, nor of sizes close to the theoretical lower bound of $q+1$. In the next section, we will describe the only known example of a maximal partial spread of $H(5,q^2)$, which has size q^3+1, and show that it generalizes to examples of maximal partial spreads of $H(4n+1,q^2)$ for all $n\geq 1$.

4 A maximal partial spread of $H(4n+1,q^2)$, $n\geq 1$

Before we can describe the only known example of a maximal partial spread of $H(5,q^2)$, which can be found in [1], we need to recall some facts concerning commuting polarities of $\mathsf{PG}(2n+1,q^2)$. Let $H(2n+1,q^2)$ be a hermitian variety of $\mathsf{PG}(2n+1,q^2)$ with associated polarity τ. Consider a symplectic polarity σ of $\mathsf{PG}(2n+1,q^2)$ which commutes with τ and set $\nu:=\sigma\tau=\tau\sigma$. From the work of B. Segre, see [4, pp. 128 – 132] it is known that ν is an involutory collineation of $\mathsf{PG}(2n+1,q^2)$ fixing $\frac{q^{2n+2}-1}{q-1}$ points on $H(2n+1,q^2)$, but fixing no point off $H(2n+1,q^2)$. The structure of all subspaces fixed by ν, both those which are totally isotropic for τ and those which are not, is isomorphic to a Baer subspace $\mathsf{PG}(2n+1,q):=\Sigma_0$ of $\mathsf{PG}(2n+1,q^2):=\Sigma$. The symplectic polarity σ induces a symplectic polar space $W_{2n+1}(q)$ in Σ_0, all points of which lie on $H(2n+1,q^2)$ and such that all generators of $W_{2n+1}(q)$ extend to generators of $H(2n+1,q^2)$. In particular, all points of $H(2n+1,q^2)$ not belonging to Σ_0 lie on a unique extended totally isotropic line of $W_{2n+1}(q)$, for the following reason. If two extended lines of Σ_0 meet in a point, then this common point must be a point of Σ_0. Hence the points of $H(2n+1,q^2)\setminus\Sigma_0$ lie on at most one extended totally isotropic line of $W_{2n+1}(q)$. Now $W_{2n+1}(q)$ has $\frac{(q^{2n}-1)(q^{2n+2}-1)}{(q-1)(q^2-1)}$ totally isotropic lines, each of them containing $q^2 - q$ points of $\Sigma \setminus \Sigma_0$. By adding the points of $W_{2n+1}(q)$, one obtains

$$q\frac{(q^{2n}-1)(q^{2n+2}-1)}{(q^2-1)} + \frac{q^{2n+2}-1}{q-1} = \frac{(q^{2n+2}-1)(q^{2n+1}+1)}{(q^2-1)}$$

points, which is exactly the total number of points of $H(2n+1, q^2)$. One concludes that indeed all points of $H(2n+1, q^2)$ not belonging to Σ_0 lie on a unique extended totally isotropic line of $W_{2n+1}(q)$.

In [1], commuting polarities of $\mathsf{PG}(5, q^2)$ are exploited in order to construct a maximal partial spread of $H(5, q^2)$ of size $q^3 + 1$, as follows. If $\mathcal{S}$ is a spread of $W_5(q)$, then the extensions to $\mathsf{PG}(5, q^2)$ of the elements of $\mathcal{S}$ form a set of pairwise disjoint generators of $H(5, q^2)$ and it is shown in [1] that the extended elements of $\mathcal{S}$ in fact form a maximal partial spread of $H(5, q^2)$.

In the sequel, we will show that this construction of a maximal partial spread of $H(5, q^2)$ can be generalized to $H(4n+1, q^2)$ for all $n \geq 1$. The proof that will be given uses induction on n and also provides an alternative proof for the case $H(5, q^2)$.

In the proof of the following lemma, projections will be of major importance. Therefore, we draw the attention to the following well-known situation. Let $\mathsf{PG}(2, q^2)$ be a projective plane containing a Baer subplane $\mathsf{PG}(2, q)$. Consider a point p of $\mathsf{PG}(2, q^2) \setminus \mathsf{PG}(2, q)$ and a line L of $\mathsf{PG}(2, q^2)$ not through p. As p lies on a unique (extended) line of $\mathsf{PG}(2, q)$, the line L contains a unique point p' that is the projection from p of $q + 1$ points of $\mathsf{PG}(2, q)$. The q^2 other lines of $\mathsf{PG}(2, q^2)$ through p contain a unique point of $\mathsf{PG}(2, q)$, so that the q^2 points of $L \setminus \{p'\}$ are the projections from p onto L of exactly one point of $\mathsf{PG}(2, q)$. In particular, all $q^2 + 1$ points of L are reached by the projection of $\mathsf{PG}(2, q)$ from p onto L.

With the same notation as in the beginning of this section, we can now prove the following result.

Lemma 4.1 *Every generator of $H(4n+1, q^2)$ has at least one point in common with Σ_0.*

Proof.
The proof will proceed by induction. For $n = 0$, it follows from [4] that the point sets of $H(1, q^2)$ and $W_1(q)$ coincide. As the generators of $H(1, q^2)$ are exactly its points, the claim is thus trivially fulfilled. So assume that $n \geq 1$ and that the claim holds for $H(4n - 3, q^2)$.

Let p be an arbitrary point of $H(4n + 1, q^2)$, not belonging to Σ_0. We will show that every generator of $H(4n + 1, q^2)$ through p has at least one point in common with Σ_0. By construction, p is a point of some extended generator of $W_{4n+1}(q)$, so that p lies on a unique extended totally isotropic line L of $W_{4n+1}(q)$. Now p^τ is a $4n$-dimensional subspace of Σ and so is its conjugate $\overline{p^\tau}$ with respect to $\mathsf{GF}(q^2)$. As p is a point of $\Sigma \setminus \Sigma_0$ by assumption and $\overline{p^\tau} = \overline{p}^\tau$, it holds that $p^\tau \neq \overline{p^\tau}$; as a consequence $p^\tau \cap \overline{p^\tau}$ is $(4n-1)$-dimensional, which implies that $p^\tau \cap \Sigma_0$ is a $\mathsf{PG}(4n-1, q)$. The extended totally isotropic line L of $W_{4n+1}(q)$ on which p is known to lie must be a line of $p^\tau \cap \Sigma_0$, and hence σ must induce a cone $LW_{4n-3}(q)$ in $p^\tau \cap \Sigma_0$.

Now let g be an arbitrary generator of $H(4n+1, q^2)$ through p. If g contains L, then it has a non-empty intersection with Σ_0 since L is an extended line of $W_{4n+1}(q)$. Hence we assume that g does not contain L. The generator g of $H(4n+1, q^2)$ has a point in common with Σ_0 if and only if it meets $LW_{4n-3}(q)$ non-trivially. Consider a point r of Σ_0 on L and a $(4n-1)$-dimensional subspace $\mathsf{PG}(4n-1, q^2)$ of p^τ containing neither p nor r and such that $\mathsf{PG}(4n-1, q^2) \cap \Sigma_0$ is a subspace $\mathsf{PG}(4n-2, q)$ of Σ_0. Then $\mathsf{PG}(4n-1, q^2) \cap H(4n+1, q^2)$ is a hermitian variety $H(4n-1, q^2)$ and σ induces a cone $rW_{4n-3}(q)$ in $\mathsf{PG}(4n-2, q)$. Let g' be the projection of g from p onto $H(4n-1, q^2)$. Then g meets Σ_0 non-trivially if and only if g' contains at least one point of the projection of $LW_{4n-3}(q)$ from p onto $H(4n-1, q^2)$. By the remark concerning projections preceding this lemma, this projection consists of all points on the extended totally isotropic lines through r of $rW_{4n-3}(q)$. As g' is a generator of $H(4n-1, q^2)$ not through r, it meets the extension of $\mathsf{PG}(4n-2, q)$ in a $(2n-2)$-dimensional totally isotropic subspace h of $rH(4n-3, q^2) = r^\tau \cap H(4n-1, q^2)$. Let π be a hyperplane of $r^\tau \cap \mathsf{PG}(4n-1, q^2)$ not through r but containing h, then $\pi \cap H(4n+1, q^2)$ is a hermitian variety $H(4n-3, q^2)$. Moreover, the extended totally isotropic lines through r of $rW_{4n-3}(q)$ meet π in the point set of a $W_{4n-3}(q)$ and $H(4n-3, q^2)$ can be obtained by extending the generators of $W_{4n-3}(q)$. By the induction hypothesis, all generators of $H(4n-3, q^2)$ contain at least one point of $W_{4n-3}(q)$, and it follows that this also applies to h. But as $h \subseteq g'$, this implies that g' contains at least one point of the projection of $LW_{4n-3}(q)$ from p onto $H(4n-1, q^2)$ and consequently g meets Σ_0 non-trivially. Since this holds for all generators through p and p was chosen arbitrarily, the lemma follows. □

This lemma easily implies the following result.

Theorem 4.2 *Let $\mathcal{S}$ be a spread of a symplectic polar space $W_{4n+1}(q)$ and let $H(4n+1, q^2)$ be the hermitian variety obtained by extending the generators of $W_{4n+1}(q)$. Then the extended elements of $\mathcal{S}$ form a maximal partial spread of $H(4n+1, q^2)$ of size $q^{2n+1} + 1$.*

Proof.
Since $\mathcal{S}$ is a spread of $W_{4n+1}(q)$, the extended elements of $\mathcal{S}$ are pairwise disjoint. By Lemma 4.1, every generator of $H(4n+1, q^2)$ meets at least one extended element of $\mathcal{S}$ non-trivially, which proves the theorem. □

For $n \geq 1$, there is only one example of a spread of $W_{4n+1}(q)$ known, see for instance [5]. This spread is a symplectic $2n$-spread of the projective space

$\mathsf{PG}(4n+1,q)$ which is regular. By Theorem 4.2, it is now known that the extensions of the elements of this spread form a maximal partial spread of $H(4n+1,q^2)$.

Remark

By using commuting polarities, one sees that a spread $\mathcal{S}$ of $W_{4n+3}(q)$ also yields a partial spread of $H(4n+3,q^2)$, but it is not immediately clear whether this partial spread is maximal or not. The equivalent of Lemma 4.1 is not valid in this case. By a similar induction argument as was used in the proof of Lemma 4.1, one can show that the number of generators of $H(4n+3,q^2)$, disjoint from Σ_0, equals

$$\prod_{j=0}^{n} q^{4j+2}(q^{4j+2}-1).$$

In this case it is hence necessary to show that the points of the *extensions* of the elements of $\mathcal{S}$ block all generators of $H(4n+3,q^2)$.

Acknowledgement

The author would like to thank the referees for their very useful remarks and suggestions.

References

[1] A. Aguglia, A. Cossidente, and G. L. Ebert. Complete spans on Hermitian varieties. In *Proceedings of the Conference on Finite Geometries (Oberwolfach, 2001)*, volume 29, pages 7–15, 2003.

[2] G. L. Ebert and J. W. P. Hirschfeld. Complete systems of lines on a Hermitian surface over a finite field. *Des. Codes Cryptogr.*, 17(1-3):253–268, 1999.

[3] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[4] B. Segre. Forme e geometrie hermitiane, con particolare riguardo al caso finito. *Ann. Mat. Pura Appl.*, 70:1–201, 1965.

[5] J. A. Thas. Two infinite classes of perfect codes in metrically regular graphs. *J. Combinatorial Theory Ser. B*, 23(2-3):236–238, 1977.

[6] J. A. Thas. A note on spreads and partial spreads of Hermitian varieties. *Simon Stevin*, 63(2):101–105, 1989.

[7] J. A. Thas. Old and new results on spreads and ovoids of finite classical polar spaces. In *Combinatorics '90 (Gaeta, 1990)*, volume 52 of *Ann. Discrete Math.*, pages 529–544. North-Holland, Amsterdam, 1992.

Author's address

Deirdre Luyckx
Ghent University
Dept. of Pure Mathematics and Computer Algebra
Krijgslaan 281, S22
BE-9000 Ghent, Belgium
dluyckx@cage.ugent.be