Antidesigns and regularity of partial spreads in dual polar graphs

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Abstract

We give several examples of designs and antidesigns in classical finite polar spaces. These types of subsets of maximal totally isotropic subspaces generalize the dualization of the concepts of m-ovoids and tight sets of points in generalized quadrangles. We also consider regularity of partial spreads and spreads. The techniques that we apply were developed by Delsarte. In some polar spaces of small rank, some of these subsets turn out to be completely regular codes.

Keywords: dual polar graph, antidesigns, (partial) spreads, completely regular codes.

1 Introduction

Classical finite polar spaces are incidence structures, consisting of the totally isotropic subspaces in a vector space with respect to a non-degenerate sesquilinear or quadratic form (see Subsection 2.3 for an explicit description of all types). The vertices of the associated *dual polar graph* that we will consider, are the *maximal totally isotropic subspaces* or simply *maximals*, with two vertices adjacent if they meet in a subspace of codimension one. The *rank* of the polar space is the dimension of its maximals.

It is our aim to study certain types of subsets of vertices in the dual polar graph. The terminology and techniques that we will use were developed by Delsarte, and will be explained in Section 2. In [8], he introduced powerful algebraic techniques to study subsets in association schemes (see Subsection 2.1 for the definition). He also developed a general theory of regular semilattices in [10], providing a generalizing notion of t-designs in several association schemes and an algebraic characterization of them. Roos [16] then introduced the notion of t-*antidesigns* in these schemes, which behave in a nice way with respect to t-designs.

We will consider *partial spreads* in Section 4. These are sets of maximals, all at maximum distance in the dual polar graph. For a particular Hermitian polar space of rank three, a tight upper bound for the size of partial spreads was given in [6], together with

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regularity properties of those meeting the bound. A generalizing tight upper bound for arbitrary odd rank d was given in [24], and we will use algebraic techniques in Subsection 4.2 to prove that partial spreads meeting that bound in these polar spaces are $(d-1)$-antidesigns and possess similar regularity properties. A partial spread is a *spread* if every 1-dimensional totally isotropic subspace is incident with exactly one of its elements. We will prove in Subsection 4.3 that in parabolic quadrics and symplectic spaces of odd rank d, spreads are also $(d-1)$-antidesigns and therefore exhibit a "higher regularity" than in other polar spaces. This will generalize a result for rank three by Thas in [20].

We will conclude by giving more examples of antidesigns in Sections 5 and 6.

2 Background

2.1 Association schemes

Bose and Shimamoto [4] introduced the notion of a d-class *association scheme* on a finite set Ω as a pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ a set of symmetric (non-empty) relations on Ω, such that the following axioms hold:

(i) R_0 is the identity relation,

(ii) $\mathcal{R}$ is a partition of Ω^2,

(iii) there are *intersection numbers* p_{ij}^k such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

The relations $R_i (i > 0)$ are all symmetric regular relations with valency p_{ii}^0, and hence define regular graphs on Ω.

Consider the real vector space $\mathbb{R}\Omega$, with an orthonormal basis indexed by the elements of Ω. It can be shown (see for instance [2]) that $\mathbb{R}\Omega$ has an orthogonal decomposition into $d+1$ subspaces V_j called *strata*, all of them eigenspaces (or subspaces of eigenspaces) of the relations R_i of the association scheme. We will write $V^\perp$ for the orthogonal complement in $\mathbb{R}\Omega$ of a subspace V. The $(d+1) \times (d+1)$-matrix P, where P_{ji} is the eigenvalue of the relation R_i for the stratum V_j, is called the *matrix of eigenvalues* of the association scheme. Let Δ_m be the diagonal matrix with $(\Delta_m)_{jj}$ the dimension of the eigenspace V_j, and let Δ_n be the diagonal matrix with $(\Delta_n)_{ii}$ the valency of the relation R_i, then P satisfies the orthogonality relation $P^T \Delta_m P = |\Omega| \Delta_n$ (see for instance Lemma 2.2.1 (iv) in [5]).

The *characteristic vector* of a subset S in Ω is the vector χ_S in $\mathbb{R}\Omega$ with $(\chi_S)_x = 1$ if $x \in S$, and $(\chi_S)_x = 0$ if $x \notin S$. In [8], the *outer distribution* and *inner distribution* of a non-empty subset S of Ω is introduced. The outer distribution of S is the $|\Omega| \times (d+1)$-matrix B, with $B_{x,i} = |\{x' \in S | (x, x') \in R_i\}|$. The inner distribution $\mathbf{a} := (\mathbf{a}_0, \ldots, \mathbf{a}_d)$ of S is defined as follows:

$$\mathbf{a}_i = \frac{1}{|S|} |\{(S \times S) \cap R_i\}|, \quad \text{for all } i \in \{0, \ldots, d\}.$$

Hence, the i-th entry of $\mathbf{a}$ equals the average number of elements $x' \in S$, such that $(x, x') \in R_i$ for some $x \in S$. It follows immediately from the definitions that $\mathbf{a}_0 = 1$, and that the sum of all of its entries must equal $|S|$. The following theorem is due to Delsarte, and by use of the orthogonality relation, can be expressed in the following form (see for instance Proposition 2.5.2 in [5]).

Theorem 2.1 *Let S be a non-empty subset in an association scheme $(\Omega, \{R_0, \ldots, R_d\})$. If for some stratum V_j, the eigenvalue of the relation R_i is given by λ_i, while the valency of R_i is k_i, then:*

$$\sum_{i=0}^{d} \frac{\lambda_i}{k_i} \boldsymbol{a}_i \geq 0,$$

with equality if and only if $\chi_S \in V_j^{\perp}$. In that case, the outer distribution B of S satisfies the equation $\sum \frac{\lambda_i}{k_i} B_{x,i} = 0$ for every $x \in \Omega$.

2.2 Distance-regular graphs

Let Γ be a connected undirected graph with diameter d on a set of vertices Ω. We let $d(x, y)$ denote the distance between two vertices x and y in the graph. For every i in $\{0, \ldots, d\}$, we let Γ_i denote the graph on the same set Ω, with two vertices adjacent if and only if they are at distance i in Γ, and we write R_i for the corresponding symmetric relation on Ω. The graph Γ is said to be *distance-regular* if $(\Omega, \{R_0, R_1, \ldots, R_d\})$ is an association scheme. It can be shown (see [5]) that this is equivalent with the existence of parameters b_i and c_i, such that for every $(v, v_i) \in R_i$, there are c_i neighbours v_{i-1} of v_i with $(v, v_{i-1}) \in R_{i-1}$ if $i \in \{1, \ldots, d\}$, and b_i neighbours v_{i+1} of v_i with $(v, v_{i+1}) \in R_{i+1}$ if $i \in \{0, \ldots, d-1\}$. These parameters b_i and c_i are known as the *intersection numbers* of the distance-regular graph Γ.

A *code* in a distance-regular graph is a non-empty subset of the set of vertices. The *distance* of a vertex x to a code C, denoted by $d(x, C)$, is $\min\{d(x, y) | y \in C\}$. The *covering* radius of C, denoted by $t(C)$, is $\max\{d(x, C) | x \in \Omega\}$. If the sets $\{y | d(x, y) \leq t(C)\}$ with $x \in C$ partition the set of vertices, then the code C is *perfect*. The *minimum distance* $\delta(C)$ of a code C with $|C| > 1$ is $\min\{d(x, y) | x, y \in C, x \neq y\}$.

A code C is called *s-regular* if for every $x \in \Omega$ with $d(x, C) = l \leq s$, the entry $B_{x,i}$ of its outer distribution B only depends on l and i. If C is $t(C)$-regular, or hence if $B_{x,i}$ only depends on $d(x, C)$ and i, the code C is *completely regular*. Every perfect code is a completely regular code (see for instance Theorem 11.1.1 in [5]).

2.3 Polar spaces and the dual polar graph

A classical finite *polar space* is an incidence structure, consisting of the totally isotropic subspaces of a finite-dimensional vector space V over a finite field, with respect to a certain non-denegerate sesquilinear or quadratic form f. The *rank* of the polar space is the dimension of the maximal totally isotropic subspaces or simply *maximals*. Two totally isotropic subspaces of different dimension are said to be *incident* if one is included in the other. The classical finite polar spaces of rank two are the classical finite *generalized quadrangles*. We will refer to a-dimensional subspaces simply as *a-spaces*. A polar space will be said to have parameters (q, q^e) if each totally isotropic 2-space (or *line*) is incident with exactly $q+1$ totally isotropic 1-spaces (or *points*), and every totally isotropic $(d-1)$-space is incident with exactly $q^e + 1$ maximals. We now explicitly list all different types of classical finite polar spaces of rank d, together with their parameters. For the sake of clarity, we give the notation related to Chevalley groups, as well as the more geometric notation, based on the embedding of the polar space in a projective space.

- the hyperbolic quadric $D_d(q)$ or $Q^+(2d-1, q)$, with $V = \mathrm{GF(q)}^{2\mathrm{d}}$ and f a non-degenerate quadratic form of Witt index d, with parameters $(q, 1)$,
- the Hermitian variety ${}^2A_{2d-1}(q)$ or $H(2d-1, q^2)$, with $V = \mathrm{GF(q^2)}^{2\mathrm{d}}$ and f a non-degenerate Hermitian form, with parameters (q^2, q),
- the parabolic quadric $B_d(q)$ or $Q(2d, q)$, with $V = \mathrm{GF(q)}^{2\mathrm{d}+1}$ and f a non-degenerate quadratic form, with parameters (q, q),
- the symplectic space $C_d(q)$ or $W(2d-1, q)$, with $V = \mathrm{GF(q)}^{2\mathrm{d}}$ and f a non-degenerate symplectic form, with parameters (q, q),
- the Hermitian variety ${}^2A_{2d}(q)$ or $H(2d, q^2)$, with $V = \mathrm{GF(q^2)}^{2\mathrm{d}+1}$ and f a non-degenerate Hermitian form, with parameters (q^2, q^3),
- the elliptic quadric ${}^2D_{d+1}(q)$ or $Q^-(2d+1, q)$, with $V = \mathrm{GF(q)}^{2\mathrm{d}+2}$ and f a non-degenerate quadratic form of Witt index d, with parameters (q, q^2).

We will often use the Gaussian coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q$, which gives the number of subspaces of dimension b in a vector space of dimension a over GF(q) with $0 \le b \le a$:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^{b} \frac{q^{a+1-i} - 1}{q^i - 1}.$$

If $b < 0$ or $b > a$, the coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is defined to be zero. More generally, the number of b-spaces in a vector space V of dimension a over GF(q) and through a fixed c-space in V, is given by: $\begin{bmatrix} a-c \\ b-c \end{bmatrix}_q$.

The number of points in the polar space is given by $\begin{bmatrix} d \\ 1 \end{bmatrix}_q (q^{d+e-1} + 1)$, and the number of maximals by $\prod_{i=1}^{d} (q^{i+e-1} + 1)$ (see for instance Lemma 9.4.1 in [5]).

Consider a classical finite polar space of rank d with parameters (q, q^e). The *dual polar graph* has as vertices the maximals, and two vertices are defined to be adjacent when their intersection has dimension $d-1$. We refer to Lemma 9.4.2 and Theorem 9.4.3 in [5] for the following results.

- Γ is distance-regular with diameter d,
- two vertices x and y are at distance i if and only if $x \cap y$ has dimension $d - i$,

- two pairs of vertices (x_1, y_1) and (x_2, y_2) are in the same orbit of the automorphism group of Γ, if and only if $d(x_1, y_1) = d(x_2, y_2)$,
- the intersection numbers are given by $b_i = q^{i+e} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q$ for every $i \in \{0, \dots, d-1\}$ and $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$ for every $i \in \{1, \dots, d\}$,
- every vertex is at distance i from exactly $q^{i(i-1)/2} q^{ie} \begin{bmatrix} d \\ i \end{bmatrix}_q$ vertices.

In the remainder of this paper, the full set of maximals will be denoted by Ω, and the ordering relations of the association scheme $(\Omega, \{R_0, \dots, R_d\})$ induced by the dual polar graph will be such that R_i corresponds with the i-distance relation.

2.4 The regular semilattice associated with the polar space

In [10], Delsarte developed a general theory of semiregular lattices, which gives a meaning to the strata in many well-known association schemes. See [19] for a treatment of the case of the dual polar graph. We will only give the results we need from [10] with respect to the dual polar graph.

Consider a classical finite polar space of rank d with parameters (q, q^e). We will denote the column span of any matrix M by $\mathrm{Im}(M)$, and its transpose by M^T. Let C_i denote the incidence matrix between the totally isotropic subspaces of dimension i and the maximals. This means that the rows are indexed by the i-spaces of the polar space, and the columns by the vertices of the dual polar graph, with $(C_i)_{yx} = 1$ if $y \subseteq x$ and $(C_i)_{yx} = 0$ if $y \nsubseteq x$. Now define V_0 as $\mathrm{Im}(C_0^T) = \langle \chi_\Omega \rangle$, and V_i as $\mathrm{Im}(C_i^T) \cap \ker(C_{i-1})$ if $1 \leq i \leq d$. Now $\mathrm{Im}(C_i^T) = V_0 \perp \ldots \perp V_i$ if $0 \leq i \leq d$, and the subspaces V_i are precisely the strata of the association scheme induced by the dual polar graph. The eigenvalue of the dual polar graph for the subspace V_i is $q^e \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q$ for every $i \in \{0, \dots, d\}$ (see for instance Theorem 4.23 in [13]). The *dual degree set* of a set of vertices S of the dual polar graph, is the set of non-zero indices i such that $\chi_S \notin V_i^\perp$. If the size r of the dual degree set of S with $|S| > 1$ satisfies $\delta(S) \geq 2r - 1$, then S is a completely regular code (see for instance Theorem 11.1.1(iv) in [5]). Note that the latter is certainly the case if $r = 1$.

If the dual degree set of S contains no index from $\{1, \dots, t\}$ with $1 \leq t \leq d$, we say that S is a *t-design*. It follows from the theory of semiregular lattices that this is the case, if and only if there is a constant λ such that every totally isotropic t-space is incident with exactly λ elements of S. Similarly, we say that S is a *t-antidesign* (with $1 \leq t \leq d$) if the dual degree set of S is a subset of $\{1, \dots, t\}$. Antidesigns were introduced by Roos because of the following important property, which we will present here in the particular case of the dual polar graph.

Theorem 2.2 ([9]) *Let X and Y be two subsets of vertices in the dual polar graph associated with a classical finite polar space. If their dual degree sets are disjoint, then $|X \cap Y| = \frac{|X||Y|}{|\Omega|}$. Conversely, if $|X \cap (Y^g)|$ is independent of the element g of the automorphism group of the dual polar graph, then the dual degree sets of X and Y are disjoint.*

A similar result (for distance-regular graphs in general) that does not rely on a group action is proved in [7]. In particular, we have the following special case.

Corollary 2.3 *Every t-design X and t-antidesign Y in a dual polar graph have exactly $\frac{|X||Y|}{|\Omega|}$ elements in common.*

The following result gives the most straightforward way to construct t-antidesigns.

Theorem 2.4 *Let π_t be a totally isotropic t-space in a classical finite polar space of rank d with $1 \leq t \leq d$. If S is the set of maximals incident with π_t, then S is a t-antidesign.*

Proof We can write $\chi_S = C_t^T \chi_{\pi_t}$. As $\mathrm{Im}(C_t^T) = V_0 \perp \ldots \perp V_t$, this yields the desired result. □

Tight sets in generalized quadrangles were introduced by Payne in [14]. These are subsets of points such that the number of pairs of collinear points in it reaches a certain upper bound. Thas defined an m-ovoid in a generalized quadrangle in [21] as a subset of points such that every line intersects it in m points. It can be shown (see for instance Theorem 2.1 in [12]) that 1-designs and 1-antidesigns in classical finite polar spaces of rank two (hence in classical finite generalized quadrangles), are precisely the dual concepts of m-ovoids and tight sets of points, respectively.

It should be noted that the definitions of tight sets of points and m-ovoids have also been generalized for polar spaces of arbitrary rank in [11] and [18], respectively. See [3] for a discussion of both types of sets of points.

In the hyperbolic quadric $D_d(q)$, the dual polar graph is bipartite, and each totally isotropic $(d-1)$-space is incident with exactly one element of both parts. Hence each half is a $(d-1)$-design. Apart from this example, no non-trivial t-designs with $t \geq 2$ in dual polar graphs are known to the author. However, many 1-designs are known, among which the spreads that we will discuss in Section 4. We will also encounter several examples of 1-antidesigns and $(d-1)$-antidesigns in classical finite polar spaces of rank d in Sections 4, 5 and 6.

3 A criterion for $(d-1)$-antidesigns

Consider a dual polar graph Γ with diameter d, and its corresponding orthogonal decomposition into strata: $V_0 \perp \ldots \perp V_d$, using the same ordering as in Subsection 2.4. As the association scheme induced by the dual polar graph is related to a regular semilattice, we could use Theorem 9 from [10] to calculate the entire matrix of eigenvalues. However, the subspace V_d is of particular interest to us, and we will only consider the corresponding eigenvalues of each relation of the association scheme. In [24], a general but not so elegant technique from the theory of distance-regular graphs was used for this. We will now use yet another method, which only works for V_d.

Lemma 3.1 *Let Γ denote the dual polar graph associated with a classical finite polar space of rank d with parameters (q, q^e). If λ_i denotes the eigenvalue of the i-distance* *relation in Γ corresponding with the subspace V_d, and if k_i is the valency of the i-distance relation, then $\lambda_i/k_i = \left(-\frac{1}{q^e}\right)^i$.*

Proof For every $i \in \{0, \ldots, d\}$, let A_i be the symmetric matrix, the rows and columns of which indexed by the vertices of the dual polar graph, with $(A_i)_{xy} = 1$ if $d(x, y) = i$ and $(A_i)_{xy} = 0$ in all other cases. Let Ω_{d-1} denote the set of totally isotropic $(d-1)$-spaces in the polar space. For each $i \in \{0, \ldots, d-1\}$, we also let W^i denote the matrix, the columns of which are indexed by the totally isotropic $(d-1)$-spaces and the rows by the vertices of Γ, with $(W^i)_{xy} = 1$ if $\dim(x \cap y) = (d-1) - i$ and $(W^i)_{xy} = 0$ in all other cases.

We now consider $W^i C_{d-1}$. For every two maximals x_1 and x_2, we have:

$$(W^i C_{d-1})_{x_1 x_2} = \sum_{y \in \Omega_{d-1}} (W^i)_{x_1 y} (C_{d-1})_{y x_2} = |\{y \in \Omega_{d-1} |\, \dim(x_1 \cap y) = (d-1) - i, y \subset x_2\}|.$$

If a maximal x_2 contains a totally isotropic $(d-1)$-space y, meeting a maximal x_1 in a $(d-1-i)$-space, then $x_1 \cap x_2$ can only have dimension $d-i$ or $d-i-1$. In the first case, there are exactly $\begin{bmatrix} d \\ d-1 \end{bmatrix}_q - \begin{bmatrix} d-(d-i) \\ (d-1)-(d-i) \end{bmatrix}_q = q^i \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q$ possibilities for a $(d-1)$-space y in x_2 intersecting x_1 in a $(d-i-1)$-space, and in the second case, there are exactly $\begin{bmatrix} d-(d-i-1) \\ (d-1)-(d-i-1) \end{bmatrix}_q = \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q$ possibilities for such a subspace y. Hence we can write for all $i \in \{0, \ldots, d-1\}$:

$$W^i C_{d-1} = q^i \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q A_i + \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q A_{i+1}.$$

We know that $V_d = (\mathrm{Im}(C_{d-1}^T))^\perp = \ker(C_{d-1})$. Let v be any non-zero vector in V_d. As $W^i C_{d-1} v = 0$, $A_i v = \lambda_i v$ and $A_{i+1} v = \lambda_{i+1} v$, it follows from the above that $0 = q^i \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q \lambda_i + \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q \lambda_{i+1}$.

On the other hand, the valencies k_i and k_{i+1} are linked by the intersection numbers: $b_i k_i = c_{i+1} k_{i+1}$, for all $i \in \{0, \ldots, d-1\}$, or hence: $q^{i+e} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q k_i = \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q k_{i+1}$. We can hence conclude that for every $i \in \{0, \ldots, d-1\}$, the following holds: $\frac{\lambda_{i+1}}{k_{i+1}} = -\frac{1}{q^e} \frac{\lambda_i}{k_i}$. As $k_0 = \lambda_0 = 1$, the desired result now follows immediately by induction. □

The simplicity of the ratio λ_i/k_i obtained in the last lemma, allows us to formulate a fairly simple criterion for $(d-1)$-antidesigns in a dual polar graph with diameter d.

Lemma 3.2 *If S is a non-empty subset in the dual polar graph associated with a classical finite polar space of rank d with parameters (q, q^e), then its inner distribution $\boldsymbol{a}$ satisfies*

$$\sum_{i=0}^{d} \left(-\frac{1}{q^e}\right)^i \boldsymbol{a}_i \geq 0,$$

with equality if and only if S is a $(d-1)$-antidesign. Moreover, in that case the outer distribution B of S also satisfies the equation $\sum_{i=0}^{d} \left(-\frac{1}{q^e}\right)^i B_{x,i} = 0$ for every maximal x.

Proof This follows immediately from Theorem 2.1 and Lemma 3.1. □

4 Partial spreads as antidesigns

4.1 Partial spreads in general

A *partial spread* in a classical finite polar space is a set of maximals, all intersecting trivially. In other words, a partial spread is a set of vertices in the corresponding dual polar graph, all at maximal distance. In a classical finite polar space of rank d with parameters (q, q^e), the number of points incident with a fixed maximal is $\begin{bmatrix} d \\ 1 \end{bmatrix}_q$ and the total number of points in the polar space is $\begin{bmatrix} d \\ 1 \end{bmatrix}_q (q^{d+e-1}+1)$. Hence $q^{d+e-1}+1$ is certainly an upper bound for the size of a partial spread S, and it is reached if and only if every point is incident with exactly one element of S, or hence if and only if S is a 1-design. In that case, we say that the partial spread S is a *spread*.

Lemma 4.1 *Let S be a (non-empty) partial spread in a classical finite polar space of rank d with parameters (q, q^e). If d is even, then S cannot be a $(d-1)$-antidesign. If d is odd, then $|S| \leq q^{de}+1$, and this bound is reached if and only if S is a $(d-1)$-antidesign.*

Proof As all distinct elements of a partial spread S are at distance d, the inner distribution $\mathbf{a}$ of S is simply: $(1, 0, \ldots, 0, |S|-1)$. Lemma 3.2 yields that $1 + \frac{(-1)^d}{(q^e)^d}(|S|-1) \geq 0$, with equality if and only if S is a $(d-1)$-antidesign. This yields the desired result. $\square$

Comparing the bound $q^{de}+1$ with the upper bound $q^{d+e-1}+1$ from the above, we see that (non-empty) partial spreads cannot be $(d-1)$-antidesigns if $e > 1$. If $e = 0$, or hence if the polar space is the hyperbolic quadric $D_d(q)$, then Lemma 4.1 yields the trivial upper bound 2 for odd d. We will consider the cases $e = \frac{1}{2}$ and $e = 1$ in Subsections 4.2 and 4.3, respectively.

4.2 Partial spreads in Hermitian varieties

In this Subsection, we will focus on the Hermitian varieties ${}^2A_{2d-1}(q)$. Non-existence of spreads of ${}^2A_{2d-1}(q)$ was already obtained by Thas in [22]. A construction for partial spreads of size q^d+1 in ${}^2A_{2d-1}(q)$ was given in [1]. The inequality in Lemma 4.1 was used in [24] to prove that this size is in fact the maximum size if d is odd. We will now give properties of partial spreads meeting that bound in this polar space.

Theorem 4.2 *If S is a (non-empty) partial spread in ${}^2A_{2d-1}(q)$ with d odd, then $|S| \leq q^d+1$ and S is a $(d-1)$-antidesign if and only if its size reaches this bound. If this is the case, then:*

- *S is a 1-regular code and for every maximal x the outer distribution B of S satisfies the equation $\sum_{i=0}^{d} \left(-\frac{1}{q}\right)^i B_{x,i} = 0$,*
- *every maximal intersecting an element of S in a $(d-1)$-space, intersects precisely q^{d-1} elements of S in a point and intersects the other $q^d - q^{d-1}$ elements of S trivially,*
- *every maximal at distance $d-1$ from S (hence not meeting any element of S in more than a point) meets exactly $\frac{q^d+1}{q+1}$ elements of S in a point.*

Finally, a partial spread of size q^3+1 in ${}^2A_5(q)$ is completely regular.

Proof In this case, the polar space has parameters (q^2, q), and hence the desired bound and the condition for S being a $(d-1)$-antidesign follow immediately from Lemma 4.1. Suppose from now on that $|S|$ is indeed q^d+1. Lemma 3.2 also implies that the outer distribution B of S satisfies the stated equality.

If x is a maximal at distance 1 from S, then x can only be at distance d or $d-1$ from any other element of S, and hence $B_{x,0}=0, B_{x,1}=1$ and $B_{x,i}=0$ if $2\leq i\leq d-2$. Hence the remaining unknown entries $B_{x,d-1}$ and $B_{x,d}$ satisfy the equation $-\frac{1}{q}+\frac{B_{x,d-1}}{q^{d-1}}-\frac{B_{x,d}}{q^d}=0$. As $1+B_{x,d-1}+B_{x,d}=|S|=q^d+1$ must also hold, one easily obtains that $B_{x,d-1}=q^{d-1}$ and $B_{x,d}=q^d-q^{d-1}$. This establishes 1-regularity of S.

Similarly, if x is at distance $d-1$ from S, then $B_{x,i}=0$ if $0\leq i\leq d-2$. Here, the remaining unknown entries $B_{x,d-1}$ and $B_{x,d}$ satisfy the equation $\frac{B_{x,d-1}}{q^{d-1}}-\frac{B_{x,d}}{q^d}=0$. As $B_{x,d-1}+B_{x,d}=|S|=q^d+1$ must also hold, one now obtains that $B_{x,d-1}=\frac{q^{d+1}+1}{q+1}$ and $B_{x,d}=q\frac{q^{d+1}+1}{q+1}$.

If $d=3$, a maximal can only be at distance $0,1$ or $2=d-1$ from S, and hence the above implies complete regularity of S. □

The previous theorem generalizes a result from [6], where it was shown that partial spreads in ${}^2A_5(q)$ have size at most q^3+1, and that when this bound is reached, every maximal at distance 2 from the partial spread meets exactly q^2-q+1 of its elements in a point.

Theorem 4.2 motivates us to consider $(d-1)$-designs in ${}^2A_{2d-1}(q)$. We first mention the following famous result by Segre.

Theorem 4.3 ([17]) *If T is a set of lines (different from the empty set and the full set of lines) in the classical generalized quadrangle ${}^2A_3(q)$, such that each point is on precisely λ lines of T, then q is odd and $\lambda=(q+1)/2$.*

Corollary 4.4 *If T is a $(d-1)$-design (different from the empty or full set of maximals) in ${}^2A_{2d-1}(q)$ with $d\geq 2$, such that each totally isotropic $(d-1)$-space is incident with exactly λ elements of T, then q is odd and $\lambda=(q+1)/2$.*
If d is odd, then exactly half of the elements of any partial spread of size q^d+1 are in T.

Proof Let π be any totally isotropic $(d-2)$-space. The residual incidence geometry of π, consisting of the totally isotropic $(d-1)$-spaces and the maximals through π as the points and lines, respectively, is isomorphic to the generalized quadrangle ${}^2A_3(q)$. The elements of T through π will correspond with a set T' of lines in this ${}^2A_3(q)$, such that each point in ${}^2A_3(q)$ is on exactly λ elements of S'. Theorem 4.3 now yields that $\lambda=(q+1)/2$.

Now let Ω_{d-1} denote the set of $(d-1)$-spaces in the polar spaces. As each element of Ω_{d-1} is incident with $q+1$ maximals and $(q+1)/2$ elements of T, we obtain: $\frac{|T|}{|\Omega|}=\frac{|T|\left[{d \atop 1}\right]_q}{|\Omega|\left[{d \atop 1}\right]_q}=\frac{|\Omega_{d-1}|(q+1)/2}{|\Omega_{d-1}|(q+1)}=\frac{1}{2}$. If S is any partial spread of size q^d+1, then Corollary 2.3 and Theorem 4.2 yield that $|S\cap T|=|S||T|/|\Omega|=|S|/2$. □

4.3 Spreads of parabolic quadrics and symplectic spaces

We now move on to spreads of parabolic quadrics and symplectic spaces. In $B_d(q)$ and $C_d(q)$, a partial spread is a spread when it has size q^d+1. The symplectic space $C_d(q)$ has a spread for all $d \geq 2$. If q is even, then the parabolic quadric $B_d(q)$ is isomorphic to $C_d(q)$ and hence has a spread as well. If q is odd, then $B_d(q)$ has no spreads for all even $d \geq 2$. While $B_3(q)$ is known to have spreads for many odd values of q, including all powers of 3, no spreads of $B_d(q)$ with q odd, d odd and $d \geq 5$ are known. We refer to [22] for proofs of these results and much more information on spreads in general.

In any classical finite polar space of rank d with parameters (q, q^e), a spread S is always a 1-regular code in the dual polar graph. Indeed, if a maximal x is adjacent to some element s_0 of S, it is at distance $d-1$ or d from any other element of the set, and hence it can intersect the other elements of S in at most a point. As all $\begin{bmatrix} d \\ 1 \end{bmatrix}_q - \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q = q^{d-1}$ points in x but not in s_0 must be contained in some element of S, the maximal x is at distance $d-1$ from exactly q^{d-1} elements of S, and at distance d from the remaining $q^d - q^{d-1}$ elements of S. However, the next result will improve this regularity in $B_d(q)$ and $C_d(q)$ in case the rank d is odd.

Theorem 4.5 *For any odd $d \geq 3$, a (non-empty) partial spread S of the parabolic quadric $B_d(q)$ or of the symplectic space $C_d(q)$ is a $(d-1)$-antidesign if and only if it is a spread. In that case, it is also a 1-design and a 2-regular code.*
For every spread S of $B_d(q)$ or $C_d(q)$ with d odd, the outer distribution B of S satisfies the equation $\sum_{i=0}^{d} \left(-\frac{1}{q}\right)^i B_{x,i} = 0$ for any maximal x, and in particular:

- *if x intersects an element of S in a $(d-1)$-space, it intersects exactly q^{d-1} elements of S in a point and intersects $q^d - q^{d-1}$ elements trivially,*
- *if x intersects an element of S in a $(d-2)$-space (with $d \geq 5$), then x intersects exactly q^{d-3} elements of S in a line, $q^{d-1} - q^{d-3}$ elements in a point, and intersects the remaining $q^d - q^{d-1}$ elements of S trivially,*
- *if x intersects no element of S in more than a line, then x intersects $\frac{q^{d-1}-1}{q^2-1}$ elements of S in a line, q^{d-1} elements in a point and intersects the remaining $q^d - q^2\frac{q^{d-1}-1}{q^2-1}$ elements of S trivially.*

Spreads in $B_d(q)$ or $C_d(q)$ with d odd have covering radius at most $d-2$, and if $d=3$ or $d=5$, they are completely regular codes.

Proof In this case, the polar space has parameters (q, q), and so it follows from Lemma 4.1 that S is a $(d-1)$-antidesign if and only if $|S| = q^d + 1$, or hence if and only if S is a spread. Suppose from now on that this is the case. As each point is on exactly one element of S, it is also a 1-design. Lemma 3.2 also implies that the outer distribution B of S satisfies the stated equality.

For every maximal x, the sum $B_{x,0} + \ldots + B_{x,d}$ must equal $|S|$. As each of the $\begin{bmatrix} d \\ 1 \end{bmatrix}_q$ points in x must be on a unique element of S, we also have that $\sum_{i=0}^{d} B_{x,i} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q = \begin{bmatrix} d \\ 1 \end{bmatrix}_q$.

If x is at distance 1 from S, then $B_{x,0} = 0, B_{x,1} = 1$ and $B_{x,i} = 0$ if $2 \leq i \leq d-2$. If x is at distance 2 from S and $d \geq 5$, then $B_{x,0} = 0, B_{x,1} = 0, B_{x,2} = 1$ and $B_{x,i} = 0$ if $2 \leq i \leq d-3$. Finally, if x is at distance at least $d-2$ from S, then $B_{x,i} = 0$ if $0 \leq i \leq d-3$. In all cases, the three equations given above, allow explicit computation of the remaining entries $B_{x,d-2}, B_{x,d-1}$ and $B_{x,d}$ of the outer distribution. In particular, we see that for no maximal x it is possible that $B_{x,0} = \ldots B_{x,d-2} = 0$, and hence the covering radius of S is at most $d-2$. This establishes 2-regularity of S (if $d = 3$, then every maximal is in S or at distance $d-2 = 1$ from S).

If $d = 3$ or $d = 5$, a maximal can only be at distance $0, 1, 2$ or $d-2$ from S, and hence the above implies complete regularity of S. □

For $C_3(q)$, the previous result was already noted by Thas in [20], where the spreads are not only completely regular, but even perfect codes.

5 Embeddings of dual polar graphs as antidesigns

Consider a classical finite polar space of rank d, consisting of the totally isotropic subspaces with respect to a non-degenerate sesquilinear or quadratic form on a vector space V. By choosing a hyperplane of V, the restriction of this form to which is non-degenerate, one can obtain the following embeddings:

- the hyperbolic quadric $D_d(q)$ in the parabolic quadric $C_d(q)$,
- the Hermitian variety ${}^2A_{2d-1}(q)$ in the Hermitian variety ${}^2A_{2d}(q)$,
- the parabolic quadric $B_d(q)$ in the elliptic quadric ${}^2D_{d+1}(q)$.

Theorem 5.1 *Consider one of the three embeddings constructed in the above, and let Ω' and Ω denote the sets of maximals of the smaller and the bigger polar space, respectively. Now Ω' is a 1-antidesign and a completely regular code in the bigger dual polar graph.*

Proof

Let P' and P denote the sets of points in the smaller and bigger polar space, respectively, and suppose the first has parameters (q, q^e). Each maximal in Ω is incident with exactly $\begin{bmatrix} d \\ 1 \end{bmatrix}_q$ points in P. A maximal is also incident with exactly $\begin{bmatrix} d \\ 1 \end{bmatrix}_q$ or $\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q$ points of P' if it is in Ω' or $\Omega \backslash \Omega'$, respectively. We can rewrite this algebraically, using the incidence matrix between points and maximals as defined in Subsection 2.4:

$$\begin{aligned} C_1^T \chi_{P'} &= \begin{bmatrix} d \\ 1 \end{bmatrix}_q \chi_{\Omega'} + \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q (\chi_\Omega - \chi_{\Omega'}) = q^{d-1} \chi_{\Omega'} + \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q \chi_\Omega, \\ C_1^T \chi_P &= \begin{bmatrix} d \\ 1 \end{bmatrix}_q \chi_\Omega. \end{aligned}$$

We now see that $\chi_{\Omega'} \in \mathrm{Im}(C_1^T) = V_0 \perp V_1$, and hence Ω' is a 1-antidesign in the bigger dual polar graph. As the dual degree set of Ω' has size one, this is certainly a completely regular code. □

We now give a short algebraic proof of a result that can also be obtained by counting arguments (see for instance [18] for the result on spreads).

Corollary 5.2 *Consider one of the three embeddings given above, and let Ω' and Ω denote the sets of maximals of the smaller and the bigger polar space, respectively. Suppose the rank of both polar spaces is d, and that the smallest has parameters (q, q^e). Now every 1-design S in the bigger dual polar graph has exactly $(q^e+1)|S|/(q^{d+e}+1)$ elements in common with Ω'.*
In particular, if S is a spread of the bigger polar space, then $|S \cap \Omega'| = q^e + 1$.

Proof
First note that in all three embeddings under consideration, the bigger polar space has parameters (q, q^{e+1}). We know from Theorem 5.1 that Ω' is a 1-antidesign, and so we can use Corollary 2.3 to see that:

$$|S \cap \Omega'| = \frac{|S||\Omega'|}{|\Omega|} = \frac{\prod_{i=1}^{d}(q^{i+e-1}+1)}{\prod_{i=1}^{d}(q^{i+(e+1)-1}+1)}|S| = \frac{q^e+1}{q^{d+e}+1}|S|.$$

We have seen in Subsection 4.1 that spreads of the bigger polar space are 1-designs of size $q^{d+e}+1$. Hence every spread has exactly q^e+1 elements in common with Ω'. □

The following theorem shows that even when the number of points on one line in the smaller polar space is also different, an embedding can still give us a $(d-1)$-antidesign for polar spaces of rank d.

Theorem 5.3 *Let Γ' and Γ be dual polar graphs, associated with classical finite polar spaces of the same rank d with the first embedded in the last, such that the distance between vertices of Γ' is the same in both graphs. If the smaller polar space has parameters (q, q^{e_1}) and the bigger polar space has parameters $(q^a, (q^a)^{e_2})$, then the vertices of Γ' form a $(d-1)$-antidesign in Γ if and only if $ae_2 - e_1$ is an integer with $0 \le ae_2 - e_1 \le d-1$.*

Proof The inner distribution $\mathbf{a}$ of the set of vertices of Γ' consists of the valencies of the relations between maximals in the smaller polar space: $\mathbf{a}_i = q^{i(i-1)/2} q^{ie_1} \begin{bmatrix} d \\ i \end{bmatrix}_q$, for any $i \in \{0, \ldots, d\}$. The quotient λ_i/k_i from Theorem 2.1 is in this case: $(-1)^i q^{-aie_2}$. Lemma 3.2 now implies that the set of vertices of Γ' is a $(d-1)$-antidesign if and only if:

$$\sum_{i=0}^{d} (-1)^i q^{-aie_2} q^{i(i-1)/2} q^{ie_1} \begin{bmatrix} d \\ i \end{bmatrix}_q = 0.$$

The q-binomial theorem gives us that for any indeterminate z:

$$\sum_{i=0}^{d} q^{i(i-1)/2} z^i \begin{bmatrix} d \\ i \end{bmatrix}_q = \prod_{j=1}^{d}(1 + q^{j-1}z).$$

If we let z be equal to $-q^{e_1 - ae_2}$, we see that the set of vertices of Γ' is a $(d-1)$-antidesign in Γ if and only if $\prod_{j=1}^{d}(1 - q^{j-1+e_1-ae_2}) = 0$, which holds if and only if $ae_2 - e_1$ is an integer with $0 \le ae_2 - e_1 \le d-1$. □

For rank two, the previous theorem yields that the set of lines of a classical generalized quadrangle with parameters $(s', t') = (q, q^{e_1})$, embedded in a classical generalized quadrangle with parameters $(s, t) = (q^a, (q^a)^{e_2})$, is a 1-antidesign (or hence a tight set of lines) if and only if $t = t'$ or $t = s't'$. This is precisely the dual of II.6 in [14] when restricted to classical generalized quadrangles.

6 The hexagon planes from the embedding of $H(q)$ in $B_3(q)$

The Split Cayley hexagon $H(q)$ is a particular incidence structure, namely a generalized hexagon. It can be embedded in the parabolic quadric $B_3(q)$, which leads to an interesting set of maximals or *planes* in this polar space. We refer to [23] for more information and proofs. In this embedding, the points of $H(q)$ are simply the points of $B_3(q)$, and the lines are a particular subset of lines of $B_3(q)$, known as the *hexagon lines*. The collinearity graph of $H(q)$, the vertices of which are points of $H(q)$ and with two points adjacent if they are on a (necessarily unique) common line of $H(q)$, is distance-regular with diameter three. Each point is collinear with $q(q+1)$ points, at distance two from $q^3(q+1)$ points, and at distance three from q^5 points. The number of common adjacent vertices to two vertices in this graph is $q-1$, 1 or 0, depending on the vertices being at distance one, two or three, respectively. Each point p is on a plane p^α of $B_3(q)$, such that another point is collinear with p if and only if it is in p^α. The planes of form p^α are the *hexagon planes*. No two distinct points can yield the same hexagon plane. It follows from the above that the planes p_1^α and p_2^α meet in a line, a point or trivially, depending on p_1 and p_2 being at distance one, two or three, respectively.

Theorem 6.1 *If S is the set of hexagon planes of an embedded $H(q)$ in the parabolic quadric $B_3(q)$, then S is a 1-design, a 2-antidesign and a completely regular code.*

Proof The polar space $B_3(q)$ has parameters (q, q). Let π be any element of S. Suppose $\pi = p^\alpha$. We know from the above that π intersects precisely $q(q+1)$ hexagon planes in a line, $q^2(q+1)$ in a point, and intersects q^5 hexagon planes trivially. Hence the inner distribution of S is given by: $(1, q(q+1), q^3(q+1), q^5)$. Lemma 3.2 implies that S is a 2-antidesign.

On the other hand, a point p_1 is in p_2^α, if and only if p_2 is one of the points in p_1^α. This means that every point is on exactly q^2+q+1 hexagon planes, and thus S is also a 1-design.

Finally, as the dual degree set of S is just $\{2\}$, this code is also completely regular. $\square$

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