Noname manuscript No.
(will be inserted by the editor)

A Higman inequality for regular near polygons

Frédéric Vanhove

Received: date / Accepted: date

Abstract The inequality of Higman for generalized quadrangles of order (s,t) with $s > 1$ states that $t \leq s^2$. We will generalize this by proving that the intersection number c_i of a regular near $2d$-gon of order (s,t) with $s > 1$ satisfies the tight bound $c_i \leq (s^{2i} - 1)/(s^2 - 1)$, and we give properties in case of equality. It is known that hemisystems in generalized quadrangles meeting the Higman bound induce strongly regular subgraphs. We will also generalize this by proving that a similar subconstituent in regular near $2d$-gons meeting the bounds would induce a distance-regular graph with classical parameters $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d + 1)/2)$ with q an odd prime power.

Keywords Distance-regular graphs · Regular near polygons · Dual polar graphs · Hemisystems · Classical parameters

1 Introduction

We refer the reader to Section 2 for the definitions of for instance (finite) generalized polygons, near polygons and polar spaces.

Feit and Higman [16] showed that (finite) generalized n-gons of order $(s,t) \neq (1,1)$ with $n \geq 3$ can only exist if $n \in \{3,4,6,8,12\}$; if $n = 12$ then $s = 1$ or $t = 1$. If $s > 1$, then the following inequalities must hold: if $n = 4$ then $t \leq s^2$ ([18]), if $n = 6$ then $t \leq s^3$ ([17]), and if $n = 8$ then $t \leq s^2$ ([18]). Bose and Shrikhande [4] also proved that if $n = 4$ and $t = s^2$, then for any triple of non-adjacent vertices the number of vertices adjacent to all three is independent of the chosen triple, namely $s+1$. This property actually characterizes generalized quadrangles of order (s, s^2) with $s > 1$ (see also Section 1.2 in [25]).

Near polygons were introduced by Shult and Yanushka in [27] and include the generalized polygons. Restrictions on the parameters of regular near polygons were

This research is supported by the Research Foundation Flanders-Belgium (FWO-Vlaanderen).

Department of Mathematics, Ghent University,
Krijgslaan 281-S22, 9000 Ghent, Belgium
E-mail: fvanhove@cage.ugent.be

obtained in for instance [6], [23], [21], [20] and [19]. In particular, Hiraki and Koolen proved in [20] that if Γ is a regular near $2d$-gon of order (s,t) with $s > 1$, then $t < s^{4d/r-1}$ for a certain integer $r \geq 1$.

We will generalize the inequality on the parameters of generalized quadrangles to regular near $2d$-gons, and give a similar property in case of equality. The necessary tools will be introduced in Section 3 and our main result will be given in Theorems 1 and 2 in Section 4.

Segre proved in [26] for the unique classical generalized quadrangle of order (q,q^2) with q a prime power, that if each singular line meets a non-trivial subset of points S in exactly m points, then $m = (q+1)/2$. Such sets of points in any generalized quadrangle of order (s,s^2) are known as *hemisystems*. We will generalize this result in Section 5.

It was also proved in [32] (in the classical case) and in [8] (for all generalized quadrangles of order (s,s^2)) that hemisystems induce a strongly regular subgraph. We will generalize this result in Section 6 by proving that a similar subset of points in the regular near $2d$-gon arising from the polar space $H(2d-1,q^2)$ would induce a distance-regular subgraph of diameter d with classical parameters $(d,b,\alpha,\beta) = (d,-q,-(q+1)/2,-((-q)^d+1)/2)$. The existence of such graphs remains an open problem for $d \geq 3$.

2 Preliminaries

2.1 Distance-regular graphs

All graphs will be assumed to be finite, undirected, connected and without loops or multiple edges. In any graph Γ, we will write $d(x,y)$ for the distance between any two vertices x and y, and $\Gamma_i(x)$ will denote the set of vertices at distance i from a given vertex x. The *diameter* of Γ is the maximum distance between its vertices. A *clique* in a graph is a set of mutually adjacent vertices, and a clique is *maximal* if it is not a proper subset of another clique. A *triangle* is a clique of size three. A subset of vertices with no two elements adjacent is a *coclique*. A graph is *regular* with valency k if every vertex has exactly k neighbours.

A graph Γ is *distance-regular* if there are natural numbers b_i with $i \in \{0,\ldots,d-1\}$ and c_i with $i \in \{1,\ldots,d\}$, known as *intersection numbers*, such that $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i$ for any two vertices x and y at distance $i \in \{1,\ldots,d\}$, and $|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i$ for any two vertices x and y at distance $i \in \{0,\ldots,d-1\}$. A distance-regular graph of diameter d has *classical parameters* (d,b,α,β) if:

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right),$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right),$$

with $\begin{bmatrix} i \\ 1 \end{bmatrix}_b = i$ if $b = 1$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}_b = (b^i - 1)/(b-1)$ if $b \neq 1$ (see [5] for more information).

We say a graph is *strongly regular* and write $srg(v,k,\lambda,\mu)$ if it is regular with valency k and every two distinct vertices have exactly λ or μ neighbours in common, depending on whether or not these two vertices are adjacent. The strongly regular graphs $srg(v,k,\lambda,\mu)$ with $k < v-1$ and $\mu > 0$ are precisely the distance-regular graphs of diameter two.

2.2 Near polygons in general

We will only introduce near n-gons for even n. A more general discussion can be found in [5] or [12].

A graph Γ of diameter $d \geq 2$ is a *near* $2d$*-gon* if the following two axioms are satisfied:

1. a vertex not in a triangle C and adjacent to two vertices in C is adjacent to the third as well,
2. for every vertex x and every maximal clique ℓ with $x \notin \ell$, there is a unique vertex in ℓ at minimal distance from x.

Note that the first axiom implies that through any two adjacent vertices, there is a unique maximal clique, and we will refer to them as the *singular lines*. We will also refer to the vertices of a near $2d$-gon as *points*.

A *regular near* $2d$*-gon* is a distance-regular near $2d$-gon. The intersection numbers b_i and c_i of such a regular near $2d$-gon with valency k satisfy $k = b_i + sc_i$ for every $i \in \{1, \dots, d-1\}$, and $k = sc_d$, for a certain fixed parameter s (see for instance Theorem 6.4.1 in [5]). Every singular line has size $s + 1$ in this case, and every point is on c_d singular lines. A regular near $2d$-gon is said to be of order (s, t) if the singular line size is $s + 1$ and every point is on $t + 1$ singular lines. If x and y are two points at distance i with $i \in \{1, \dots, d\}$, then there are precisely c_i singular lines through y at distance $i - 1$ from x. Similarly, if x and y are two points at distance i with $i \in \{0, \dots, d - 1\}$, then there are exactly b_i/s singular lines through y at distance i from x. We will also let t_i denote $c_i - 1$ for every $i \in \{1, \dots, d\}$.

The ordinary $2d$-gons are precisely the regular near $2d$-gons of order $(1, 1)$. In the following subsections, we will discuss two important families of (regular) near polygons: generalized polygons and dual polar graphs.

2.3 Generalized polygons

Generalized $2d$-gons were introduced by Tits in [33] and are near $2d$-gons with $|\Gamma_1(x) \cap \Gamma_{i-1}(y)| = 1$ for any two points x and y at distance i with $1 \leq i \leq d - 1$. Note that we consider generalized $2d$-gons as collinearity graphs, instead of as point-line geometries. A generalized $2d$-gon has order (s, t) if it is a regular near $2d$-gon of order (s, t). Some of the known conditions on its parameters were already given in Section 1.

The generalized 4-gons or *generalized quadrangles* are precisely the near 4-gons. The dual polar graphs of diameter $d = 2$ from the next subsection will all be examples of generalized quadrangles.

Generalized 6-gons or *generalized hexagons* of order $(1, q), (q, 1), (q, q), (q^3, q)$ and (q, q^3) exist for every prime power q. Generalized 8-gons or *generalized octagons* of order (q, q^2) and (q^2, q) exist with q any odd power of 2, and of order $(1, q)$ or $(q, 1)$ for any prime power q. Finally, generalized 12-gons or *generalized dodecagons* of order $(1, q)$ and $(q, 1)$ exist for any prime power q. In all three cases, no examples of other orders $(s, t) \neq (1, 1)$ are known (see 6.5 in [5] for more information).

The graph with the singular lines of a generalized $2d$-gon of order (s, t) as vertices, with two adjacent when having exactly one point in common, is a generalized $2d$-gon of order (t, s), and is referred to as the *dual*.

2.4 Dual polar graphs

A classical finite *polar space* is an incidence structure, consisting of the totally isotropic subspaces of a finite-dimensional vector space V over a finite field, with respect to a certain non-denegerate sesquilinear or quadratic form f. The *rank* of the polar space is the dimension d of the maximal totally isotropic subspaces or simply *maximals*. Two totally isotropic subspaces of different dimension are said to be *incident* if one is included in the other. We now list all classical finite polar spaces of rank d:

- the hyperbolic quadric $Q^+(2d-1,q)$, with $V = V(2d,q)$ and f a nondegenerate quadratic form of maximal Witt index d,
- the Hermitian variety $H(2d-1,q^2)$, with $V = V(2d,q^2)$ and f a nondegenerate Hermitian form,
- the parabolic quadric $Q(2d,q)$, with $V = V(2d+1,q)$ and f a nondegenerate quadratic form,
- the symplectic space $W(2d-1,q)$, with $V = V(2d,q)$ and f a nondegenerate alternating form,
- the Hermitian variety $H(2d,q^2)$, with $V = V(2d+1,q^2)$ and f a nondegenerate Hermitian form,
- the elliptic quadric $Q^-(2d+1,q)$, with $V = V(2d+2,q)$ and f a nondegenerate quadratic form of Witt index d.

The *dual polar graph* corresponding with a classical polar space is the graph Γ on its maximals, with two vertices adjacent if they intersect in a subspace of codimension one. This graph is a regular near $2d$-gon, and two vertices are at distance i if and only if they intersect in a subspace of codimension i (see Section 9.4 in [5]). In particular, they are at maximum distance d if and only if their intersection is a trivial subspace. Table 1 provides the singular line size $s+1$ and the parameter $t_2 = c_2 - 1$ for the dual polar graph corresponding with all classical finite polar spaces of rank d. (The notation for the dual polar graph in the first column is based on the embedding in a projective space, the notation in the second is the one related to Chevalley groups.)

		(s, t_2)
$Q^+(2d-1,q)$	$D_d(q)$	$(1,q)$
$H(2d-1,q^2)$	${}^2A_{2d-1}(q)$	(q,q^2)
$Q(2d,q)$	$B_d(q)$	(q,q)
$W(2d-1,q)$	$C_d(q)$	(q,q)
$H(2d,q^2)$	${}^2A_{2d}(q)$	(q^3,q^2)
$Q^-(2d+1,q)$	${}^2D_{d+1}(q)$	(q^2,q)

Table 1 The dual polar graphs from classical finite polar spaces

The parameter c_i is then equal to $(t_2^i - 1)/(t_2 - 1)$ if $1 \leq i \leq d$. In particular, the number of singular lines through each vertex is given by $c_d = t+1 = (t_2^d - 1)/(t_2 - 1)$. The number of vertices is $\prod_{i=1}^{d}(s\ t_2^{i-1} + 1)$.

The dual polar graphs from $W(3,q)$ and $Q(4,q)$ are dual to each other, and so are those from $H(3,q^2)$ and $Q^-(5,q)$ (see for instance 3.2.1 and 3.2.3 in [25]).

3 Algebraic techniques

3.1 Association schemes and Bose-Mesner algebras

As each distance-regular graph induces an association scheme, we first describe these combinatorial structures. Bose and Shimamoto [3] introduced the notion of a d-class *association scheme* on a finite set Ω as a pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ a set of symmetric (non-empty) relations on Ω, such that the following axioms hold: (i) R_0 is the identity relation, (ii) $\mathcal{R}$ is a partition of Ω^2, (iii) there are constants p_{ij}^k, known as *intersection numbers*, such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k. An immediate consequence is that each relation R_i is regular, and we will denote its valency by k_i.

If Γ is a graph with vertex set Ω and diameter d, and if we denote the i-distance relation by R_i, then $(\Omega, \{R_0, \ldots, R_d\})$ is an association scheme if and only if Γ is distance-regular (see for instance 4.1.A in [5]).

If $(\Omega, \{R_0, \ldots, R_d\})$ is an association scheme we will always write A_i for the symmetric $(0, 1)$-matrix, the rows and columns of which are indexed by the elements of Ω, with $(A_i)_{x,y} = 1$ if $(x, y) \in R_i$ and $(A_i)_{x,y} = 0$ if $(x, y) \notin R_i$. Axiom (iii) can be algebraically expressed as $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$, and hence the vector space spanned by $\{A_0, \ldots, A_d\}$ is a commutative $(d + 1)$-dimensional algebra of symmetric matrices, known as the *Bose-Mesner algebra*. It can be shown (see for instance 2.2 in [5]) that the Bose-Mesner algebra has a unique basis of minimal idempotents $\{E_0, \ldots, E_d\}$, with $E_i E_j = \delta_{ij} E_i$, $E_0 + \ldots + E_d = I$ and $E_0 = J/|\Omega|$ where J denotes the all-one matrix. As these minimal idempotents are symmetric, they define orthogonal projections and hence they are positive semidefinite.

If Γ is a distance-regular graph with diameter d, then the corresponding adjacency matrix A_1 has exactly $d + 1$ distinct eigenvalues. Every minimal idempotent E_j corresponds with such an eigenvalue λ_j such that $A_1 E_j = \lambda_j E_j$, and the column span of E_j is precisely the (right) eigenspace of A_1 for λ_j. Conversely, if any non-zero element C of the Bose-Mesner algebra satisfies $A_1 C = \lambda C$, then λ must be one of the eigenvalues λ_j of A_1 and C must be a scalar multiple of E_j. We refer to 4.1.B and 4.1.C in [5] for proofs and much more information. Every non-zero vector in the column span of any minimal idempotent is also an eigenvector for all A_i. If for some minimal idempotent E the corresponding eigenvalue of A_i is given by λ_i, then E can also be written, up to a positive scalar, as:

$$\sum_{i=0}^{d} \frac{\lambda_i}{k_i} A_i.$$

The latter follows from the orthogonality relations between the eigenvalues of an association scheme (see for instance Lemma 2.2.1(iv) in [5]).

For any set Ω, we will denote by $\mathbb{R}\Omega$ the real vector space with an orthonormal basis indexed by the elements of Ω. Note that the elements of the Bose-Mesner algebra of any association scheme on Ω define endomorphisms of $\mathbb{R}\Omega$.

For any subset $S \subseteq \Omega$, the *characteristic vector* of S is the column vector χ_S with entry 1 in the positions corresponding with elements of S, and zero in all others. For any two subsets S_1 and S_2, the product $(\chi_{S_1})^T \chi_{S_2}$ is equal to $|S_1 \cap S_2|$. More generally, if $(\Omega, \{R_0, \ldots, R_d\})$ is an association scheme, then for any two subsets $S_1, S_2 \subseteq \Omega$, the number $(\chi_{S_1})^T A_i \chi_{S_2} = (\chi_{S_2})^T A_i \chi_{S_1}$ is equal to $|(S_1 \times S_2) \cap R_i|$.

3.2 A particular minimal idempotent for regular near $2d$-gons

We will now consider a specific minimal idempotent. The following result is in fact already implicitly given in many proofs. We will follow that of Theorem 3.1.4 in [12].

Lemma 1 *Let Γ be a regular near $2d$-gon of order (s,t). The element $M = \sum_{i=0}^{d} 1/(-s)^i A_i$ of the Bose-Mesner algebra is a minimal idempotent up to a positive scalar, and its column space is precisely the eigenspace of the eigenvalue $-(t+1)$ of A_1. The corresponding eigenvalue λ_i of A_i is given by $k_i/(-s)^i$.*

Proof Let b_i and c_i be the intersection numbers of Γ and set $b_{-1} = b_d = c_0 = c_{d+1} = 0$. We also define A_{-1} and A_{d+1} as zero matrices. This allows us to algebraically express the property of intersection numbers:

$$A_1 A_i = b_{i-1} A_{i-1} + (k - b_i - c_i) A_i + c_{i+1} A_{i+1}, \forall i \in \{0, \dots, d\}.$$

We can now write:

$$\begin{aligned}
A_1 M &= A_1 \left(\sum_{i=0}^{d} \frac{(-1)^i}{s^i} A_i \right) \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i} (A_1 A_i) \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i} (b_{i-1} A_{i-1} + (k - b_i - c_i) A_i + c_{i+1} A_{i+1}) \\
&= \sum_{i=0}^{d} \frac{(-1)^{i+1}}{s^{i+1}} (b_i A_i) + \sum_{i=0}^{d} \frac{(-1)^i}{s^i} ((k - b_i - c_i) A_i) + \sum_{i=0}^{d} \frac{(-1)^{i-1}}{s^{i-1}} (c_i A_i) \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i} \left(-\frac{b_i}{s} + (k - b_i - c_i) + (-s c_i) \right) A_i \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i} \left(-\frac{k}{s} \right) A_i \\
&= -(t+1) M.
\end{aligned}$$

where we used the identities $b_i = k - s c_i$ and $k = s(t+1)$ in the last two steps. Hence $-(t+1)$ must be an eigenvalue of A_1 and M must be a scalar multiple of the corresponding minimal idempotent E. As both $trace(M) = trace(A_0)$ and $trace(E)$ are positive, this scalar must be positive.

Finally, as E can also be written as $\sum_{i=0}^{d} (\lambda_i / k_i) A_i$ up to a positive scalar and $\lambda_0 = k_0 = 1$, this proves the last part of the lemma. □

From now on, we will always let M denote the element $M = \sum_{i=0}^{d} (-\frac{1}{s})^i A_i$ of the Bose-Mesner algebra, corresponding with a regular near $2d$-gon of order (s,t).

4 Upper bound on the intersection number c_j

We now come to the main result of this paper. It was inspired by and generalizes Lemma 5.1 in [2]. The proof will make implicit use of Delsarte's linear programming bound (see for instance formula 4.3 in [15]).

Theorem 1 *Let Γ be a regular near 2d-gon of order (s,t) with $s>1$. Consider two vertices a and b at distance j with $1 \le j \le d$. Define $v = \alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T$ with $\alpha,\beta,\gamma \in \mathbb{R}$ and $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$. Then:*

$$c_j \le \frac{s^{2j}-1}{s^2-1}.$$

Moreover, $Mv = 0$ if and only if both $c_j = (s^{2j}-1)/(s^2-1)$ and (α,β,γ) is a scalar multiple of

$$\left(s\frac{s^{2j-2}-1}{s^2-1}, (-1)^j s^{j-1}, 1\right).$$

Proof Given a and b at distance j with $1 \le j \le d$, there are exactly c_j points on a common singular line with a and at distance $j-1$ from b. Hence T has size c_j, and no two points in T are on the same such singular line.

We will now consider $v^T A_i v$ for every $i \in \{0,\dots,d\}$. Note that for any two subsets of points S_1 and S_2, the value of $(\chi_{S_1})^T A_i \chi_{S_2} = (\chi_{S_2})^T A_i \chi_{S_1}$ is given by the number of ordered pairs $(\omega_1,\omega_2) \in (S_1 \times S_2)$ with $d(\omega_1,\omega_2) = i$. Our assumptions immediately yield:
$(\chi_{\{a\}})^T A_0 \chi_{\{a\}} = (\chi_{\{b\}})^T A_0 \chi_{\{b\}} = 1, (\chi_{\{a\}})^T A_i \chi_{\{a\}} = (\chi_{\{b\}})^T A_i \chi_{\{b\}} = 0$ if $1 \le i \le d$,
$(\chi_{\{a\}})^T A_i \chi_{\{b\}} = 0$ if $i \ne j$, and $(\chi_{\{a\}})^T A_j \chi_{\{b\}} = 1$,
$(\chi_{\{a\}})^T A_i \chi_T = 0$ if $i \ne 1$, and $(\chi_{\{a\}})^T A_1 \chi_T = |T| = c_j$,
$(\chi_{\{b\}})^T A_i \chi_T = 0$ if $i \ne j-1$, and $(\chi_{\{b\}})^T A_{j-1} \chi_T = |T| = c_j$.

Finally, as every two distinct points in T are on distinct singular lines through a, they cannot be collinear, and hence they are at distance two. This yields: $(\chi_T)^T A_0 \chi_T = |T| = c_j, (\chi_T)^T A_2 \chi_T = |T|(|T|-1) = c_j(c_j-1)$ and $(\chi_T)^T A_i \chi_T = 0$ if $i \notin \{0,2\}$.

We will now work out the following:

$$s^j(v^T M v) = \sum_{i=0}^{d} (-1)^i s^{j-i} (v^T A_i v) = \sum_{i=0}^{d} (-1)^i s^{j-i} (\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T)^T A_i (\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T).$$

For any $j \ge 1$ this is equal to

$$s^j(\alpha^2+\beta^2+\gamma^2 c_j) - s^{j-1}(2\alpha\gamma)c_j + s^{j-2}\gamma^2 c_j(c_j-1) + (-1)^{j-1}s(2\beta\gamma)c_j + (-1)^j(2\alpha\beta).$$

We can rewrite this as $(\alpha,\beta,\gamma)F(\alpha,\beta,\gamma)^T$ with:

$$F = \begin{pmatrix} s^j & (-1)^j & -s^{j-1}c_j \\ (-1)^j & s^j & (-1)^{j-1}sc_j \\ -s^{j-1}c_j & (-1)^{j-1}sc_j & c_j s^{j-2}(s^2+c_j-1) \end{pmatrix}.$$

We compute the determinant of F:

$$\begin{aligned} \mathrm{Det}(F) &= (-1)^j c_j \mathrm{Det} \begin{pmatrix} s^j & (-1)^j & -s^{j-1}c_j \\ 1 & (-1)^j s^j & -sc_j \\ -s^{j-1} & (-1)^{j-1}s & s^{j-2}(s^2+c_j-1) \end{pmatrix} \\ &= (-1)^j c_j \mathrm{Det} \begin{pmatrix} 0 & -(-1)^j(s^{2j}-1) & c_j s^{j-1}(s^2-1) \\ 1 & (-1)^j s^j & -sc_j \\ 0 & (-1)^j s(s^{2j-2}-1) & -(c_j-1)s^{j-2}(s^2-1) \end{pmatrix} \end{aligned}$$

$$= -c_j s^{j-2}(s^2-1)\mathrm{Det}\begin{pmatrix} -(s^{2j}-1) & c_j s \\ s(s^{2j-2}-1) & -(c_j-1)\end{pmatrix}$$
$$= c_j s^{j-2}(s^2-1)((s^{2j}-1) - c_j(s^2-1)).$$

We know from Lemma 1 that M is a minimal idempotent up to a positive scalar and thus positive semidefinite. Hence $v^T M v \geq 0$ for all $\alpha, \beta, \gamma \in \mathbb{R}$. Thus F is positive semidefinite, and hence its determinant must be non-negative, and it is positive definite if and only if this determinant is positive. We find that $c_j \leq \frac{s^{2j}-1}{s^2-1}$ since $s > 1$. We can also write:

$$Mv = 0 \Longleftrightarrow v^T M v = 0 \Longleftrightarrow (\alpha, \beta, \gamma) F (\alpha, \beta, \gamma)^T = 0.$$

As F is positive semidefinite, the latter will hold if and only if both F is not positive definite and $F(\alpha, \beta, \gamma)^T = 0$. This is possible if and only if both $c_j = \frac{s^{2j}-1}{s^2-1}$ and (α, β, γ) is a scalar multiple of $(s\frac{s^{2j-2}-1}{s^2-1}, (-1)^j s^{j-1}, 1)$. □

We now give a property of those regular near $2d$-gons attaining one of the bounds from the previous theorem. It is in fact based on properties of outer distributions of subsets in association schemes (see Theorem 3.3 in [13]).

Theorem 2 *Let Γ be a regular near $2d$-gon of order (s,t) with $s > 1$. Suppose $c_j = (s^{2j}-1)/(s^2-1)$ for some $j \in \{1, \ldots, d\}$, and consider three vertices a, b, c with $d(a,b) = j, d(a,c) = d, d(b,c) = k$. The set $\Gamma_1(a) \cap \Gamma_{j-1}(b) \cap \Gamma_{d-1}(c)$ has size:*

$$\frac{s^{2j-1} + (-1)^{j+k+d} s^{d-k+j} - (-1)^{j+k+d} s^{d-k+j-1} - 1}{s^2-1}.$$

Proof Let T be $\Gamma_1(a) \cap \Gamma_{j-1}(b)$. We know from Theorem 1 that $v = s\frac{s^{2j-2}-1}{s^2-1}\chi_{\{a\}} + (-1)^j s^{j-1}\chi_{\{b\}} + \chi_T$ satisfies $Mv = 0$. Hence we have in particular:

$$0 = (\chi_{\{c\}})^T M v = (\chi_{\{c\}})^T \left(\sum_{i=0}^{d} \frac{(-1)^i}{s^i} A_i\right)\left(s\frac{s^{2j-2}-1}{s^2-1}\chi_{\{a\}} + (-1)^j s^{j-1}\chi_{\{b\}} + \chi_T\right).$$

As $d(a,c) = d$ all elements of T are at distance at least $d-1$ from c. Hence if x denotes $|T \cap \Gamma_{d-1}(c)|$, then $|T \cap \Gamma_d(c)| = |T| - x = c_j - x$. The assumptions now imply: $(\chi_{\{c\}})^T A_d \chi_{\{a\}} = (\chi_{\{c\}})^T A_k \chi_{\{b\}} = 1$, $(\chi_{\{c\}})^T A_i \chi_{\{a\}} = 0$ if $i \neq d$, $(\chi_{\{c\}})^T A_i \chi_{\{b\}} = 0$ if $i \neq k$, $(\chi_{\{c\}})^T A_{d-1}\chi_T = x$, $(\chi_{\{c\}})^T A_d \chi_T = c_j - x$ and $(\chi_{\{c\}})^T A_i \chi_T = 0$ if $i \notin \{d-1, d\}$. Hence we obtain:

$$\frac{(-1)^d}{s^d} s\frac{s^{2j-2}-1}{s^2-1} + \frac{(-1)^k}{s^k}(-1)^j s^{j-1} + \frac{(-1)^{d-1}}{s^{d-1}}x + \frac{(-1)^d}{s^d}(c_j - x) = 0.$$

Since we assume $c_j = \frac{s^{2j}-1}{s^2-1}$, we can rewrite:

$$s\frac{s^{2j-2}-1}{s^2-1} + (-1)^{d+k+j} s^{d-k+j-1} + \frac{s^{2j}-1}{s^2-1} = (s+1)x.$$

This yields $|(\Gamma_1(a) \cap \Gamma_{j-1}(b)) \cap \Gamma_{d-1}(c)| = |T \cap \Gamma_{d-1}(c)| = x =$

$$\frac{s^{2j-1} + (-1)^{j+k+d} s^{d-k+j} - (-1)^{j+k+d} s^{d-k+j-1} - 1}{s^2-1}.$$

□

The following corollary generalizes the Higman inequality between the parameters (s,t) of generalized quadrangles, and also gives a property in case of equality.

Corollary 1 *Let Γ be a regular near $2d$-gon of order (s,t) with $s>1$. Then:*

$$t+1 \leq \frac{s^{2d}-1}{s^2-1},$$

and if equality holds, then for any triple of points a,b and c mutually at distance d, the set $\Gamma_1(a)\cap\Gamma_{d-1}(b)\cap\Gamma_{d-1}(c)$ has size:

$$\frac{(s^d-(-1)^d)(s^{d-1}+(-1)^d)}{s^2-1}.$$

Proof This follows immediately from Theorems 1 and 2 with $j=d$ and $k=d$. □

Suppose the regular near $2d$-gon is the dual polar graph arising from a classical finite polar space of rank d. Then for any two vertices a and c at distance d, there is a bijective correspondence between the one-dimensional subspaces or 1-*spaces* of c and the elements of $\Gamma_1(a)\cap\Gamma_{d-1}(c)$, as each such 1-space p in c is in a unique neighbour ω of a in the dual polar graph, which will intersect c in precisely p.

For dual polar graphs from classical finite polar spaces of diameter d and order (s,t) with $s>1$, the bound from Corollary 1 is attained if and only if $t_2=s^2$. It follows from Table 1 that this is the case if and only if Γ is the dual polar graph on the maximals of $H(2d-1,q^2)$, when Γ is of order $(s,t)=(q,(q^{2d}-1)/(q^2-1)-1)$. Corollary 1 then yields that for any three maximals a,b and c mutually at maximum distance d, the size of $\Gamma_1(a)\cap\Gamma_{d-1}(b)\cap\Gamma_{d-1}(c)$ is given by $(q^d-(-1)^d)(q^{d-1}+(-1)^d)/(q^2-1)$. As these vertices are all in $\Gamma_1(a)\cap\Gamma_{d-1}(c)$, they correspond with a set of 1-spaces in c. Thas already described this set of 1-spaces (instead of just determining its size) for this particular graph in [31]. For the sake of completeness, we mention the result in a somewhat different from.

Lemma 2 *Let a,b and c be maximals in the polar space $H(2d-1,q^2)$, pairwise intersecting trivially. The set of 1-dimensional subspaces p of c, such that the unique neighbour in the corresponding dual polar graph of a through p also intersects b in a 1-space, is precisely the set of $(q^d-(-1)^d)(q^{d-1}+(-1)^d)/(q^2-1)$ isotropic 1-spaces of an induced polar space $H(d-1,q^2)$ in c.*

Finally, we would like to remark that the property from Corollary 1 does not characterize the regular near $2d$-gons of order (s,t) meeting the bound on t. The dual polar graph arising from the polar space $W(2d-1,q)$, which is of order $(s,t)=(q,(q^d-1)/(q-1)-1)$, provides a counterexample if d is odd, as was worked out in Theorem 21 of [22], although $t_2=s\neq s^2$ in this case. We again state the result in an adapted form.

Lemma 3 *Let a,b and c be maximals in the polar space $W(2d-1,q)$ with d odd, pairwise intersecting trivially. The set of 1-dimensional subspaces p of c, such that the unique neighbour in the corresponding dual polar graph of a through p also intersects b in a 1-space, is precisely the set of $(q^{d-1}-1)/(q-1)$ isotropic 1-spaces of a hyperplane (if q is even) or of an induced polar space $Q(d-1,q)$ (if q is odd) in c.*

5 On m-ovoids in regular near $2d$-gons meeting the bound

An *ovoid* of a regular near $2d$-gon Γ is a set S of points, such that each singular line contains a unique point of S. If Γ is of order (s,t) with set of vertices Ω, then the ovoids are precisely the cocliques of size $|\Omega|/(s+1)$.

More generally, we will say that a subset of points S in a regular near $2d$-gon is an *m-ovoid* if every singular line contains exactly m points of S. Thas introduced this concept for generalized quadrangles in [30]. We first prove a fundamental algebraic property of m-ovoids.

Lemma 4 *If S is an m-ovoid of a regular near $2d$-gon Γ of order (s,t) with set of vertices Ω, then its characteristic vector χ_S can be written as $(m/(s+1))\chi_\Omega + Mw$ for some vector w.*

Proof Let Ω and $\mathcal{L}$ denote the sets of points and of singular lines, respectively. Let C be the incidence matrix between points and singular lines, the columns of which are indexed by the points of Γ, and the rows by the singular lines, with $C_{\ell,a} = 1$ if $a \in \ell$ and $C_{\ell,a} = 0$ if $a \notin \ell$. As each singular line contains $s+1$ points and exactly m elements of S, we can write $C\chi_\Omega = (s+1)\chi_\mathcal{L}$ and $C\chi_S = m\chi_\mathcal{L}$. We also know that two points can only be in a common singular line if they are either equal (when they are on $t+1$ common singular lines) or at distance one (when they are on a unique common singular line). This can be expressed algebraically as: $C^TC = A_1 + (t+1)A_0$, which implies:

$$(A_1+(t+1)A_0)(\chi_S-\frac{m}{s+1}\chi_\Omega) = (C^TC)(\chi_S-\frac{m}{s+1}\chi_\Omega) = C^T\left((m\chi_\mathcal{L})-\frac{m}{s+1}((s+1)\chi_\mathcal{L})\right) = 0.$$

Hence $\chi_S - \frac{m}{s+1}\chi_\Omega$ is an eigenvector with eigenvalue $-(t+1)$ of A_1, and so it follows from Lemma 1 that it is in the column span of M, which is the corresponding minimal idempotent up to a positive scalar. □

Lemma 5 *If S is an m-ovoid in a regular near $2d$-gon Γ of order (s,t), then for every point $a \in S$ and every $i \in \{0,\dots,d\}$:*

$$|\Gamma_i(a) \cap S| = k_i\left(\frac{m}{s+1} + \left(-\frac{1}{s}\right)^i\left(1-\frac{m}{s+1}\right)\right).$$

Proof We know from Lemma 4 that χ_S can be written as $(m/(s+1))\chi_\Omega + Mw$. Note that $A_i\chi_\Omega = k_i\chi_\Omega$ and $A_i(Mw) = \lambda_i(Mw)$, where λ_i denotes the eigenvalue of A_i corresponding with the column span of M (see Lemma 1). We can now write:

$$\begin{aligned}
|\Gamma_i(a) \cap S| &= (\chi_{\{a\}})^T A_i \chi_S \\
&= (\chi_{\{a\}})^T A_i\left(\frac{m}{s+1}\chi_\Omega + Mw\right) \\
&= \frac{m}{s+1}k_i((\chi_{\{a\}})^T\chi_\Omega) + (\chi_{\{a\}})^T\lambda_i(Mw) \\
&= \frac{m}{s+1}k_i + \lambda_i(\chi_{\{a\}})^T\left(\chi_S - \frac{m}{s+1}\chi_\Omega\right) \\
&= \frac{m}{s+1}k_i + \lambda_i\left(1-\frac{m}{s+1}\right).
\end{aligned}$$

Applying the formula $\lambda_i/k_i = (-1)^i/s^i$ from Lemma 1 now completes the proof. □

The technique used in the following proof is based on the concept of design-orthogonal pairs of vectors (see for instance Theorem 6.7 in [15]).

Lemma 6 *If S is an m-ovoid in a regular near $2d$-gon of order (s,t) with $s>1$ and with $c_j = (s^{2j}-1)/(s^2-1)$ for some $j \in \{1,\dots,d\}$, and a and b are two elements of S at distance j in Γ, then*

$$|S \cap \Gamma_1(a) \cap \Gamma_{j-1}(b)| = m\frac{(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s^2-1} - s\frac{(s^j-(-1)^j)(s^{j-2}+(-1)^j)}{s^2-1}.$$

Proof Let T denote the subset $\Gamma_1(a) \cap \Gamma_{j-1}(b)$ and take $\alpha = s\frac{s^{2j-2}-1}{s^2-1}$ and $\beta = (-1)^j s^{j-1}$. We know from Theorem 1 that $v = \alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \chi_T$ satisfies $Mv = 0$. We now consider $(\chi_S)^T v$:

$$(\chi_S)^T v = (\chi_S)^T(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \chi_T) = \alpha + \beta + |S \cap T|.$$

On the other hand, Lemma 4 implies that χ_S can be written as $(m/(s+1))\chi_\Omega + Mw$. Hence:

$$\begin{aligned}(\chi_S)^T v &= \left(\frac{m}{s+1}\chi_\Omega + Mw\right)^T v\\ &= \frac{m}{s+1}(\chi_\Omega)^T v + w^t(Mv)\\ &= \frac{m}{s+1}(\chi_\Omega)^T(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \chi_T)\\ &= \frac{m}{s+1}(\alpha+\beta+|T|) = \frac{m}{s+1}(\alpha+\beta+c_j).\end{aligned}$$

Hence we obtain:

$$|S \cap (\Gamma_1(a) \cap \Gamma_{j-1}(b))| = |S \cap T| = \frac{m}{s+1}(\alpha+\beta+c_j) - (\alpha+\beta),$$

which yields the desired result after substituting for α, β and c_j. □

We can now severely restrict the size of m-ovoids in a regular near $2d$-gon if at least one of the non-trivial bounds from Theorem 1 is met.

Theorem 3 *If Γ is a regular near $2d$-gon of order (s,t) with $s>1$ and $c_j = (s^{2j}-1)/(s^2-1)$ for some $j \in \{2,\dots,d\}$, then m-ovoids with $0<m<s+1$ can only exist for $m=(s+1)/2$.*

Proof Suppose S is an m-ovoid with $0<m<s+1$. Consider any point b in S. We will count the number N of pairs (p,a) of adjacent points in $(\Gamma_{j-1}(b)\cap S) \times (\Gamma_j(b) \cap S)$ in two ways. The size of $\Gamma_{j-1}(b) \cap S$ is given by Lemma 5. For each point p in $\Gamma_{j-1}(b)\cap S$, there are b_{j-1}/s singular lines through p such that the distance from b to this line is $d(p,b) = j-1$. The other points on those singular lines are precisely the neighbours of p at distance j from b. Each such singular line contains exactly $m-1$ points in $S\backslash\{p\}$, all at distance j from b. Hence:

$$N = k_{j-1}\left(\frac{m}{s+1} + \left(1-\frac{m}{s+1}\right)\left(-\frac{1}{s}\right)^{j-1}\right)\frac{b_{j-1}}{s}(m-1).$$

We also know the size of $\Gamma_j(b) \cap S$ from Lemma 5, and for each point a in that subset, the number of its neighbours in S at distance $j-1$ from b is given by Lemma 6. We find:

$$N = k_j \left(\frac{m}{s+1} + \left(1 - \frac{m}{s+1}\right)\left(-\frac{1}{s}\right)^j \right) \times \\ \left(m\frac{(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s^2-1} - s\frac{(s^j-(-1)^j)(s^{j-2}+(-1)^j)}{s^2-1} \right).$$

When setting $m = x(s+1)$ and using the identity $k_{j-1}b_{j-1} = k_j c_j$ and the assumption $c_j = (s^{2j}-1)/(s^2-1)$, we see that x must be a root of the following polynomial in x:

$$\left(x + (1-x)\left(-\frac{1}{s}\right)^{j-1} \right) \frac{s^{2j}-1}{s(s^2-1)}(x(s+1)-1) - \\ \left(x + (1-x)\left(-\frac{1}{s}\right)^j \right)\left(x\frac{(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s-1} - s\frac{(s^j-(-1)^j)(s^{j-2}+(-1)^j)}{s^2-1} \right),$$

which can be rewritten as:

$$\frac{(-1)^j(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s^j(s-1)}(x-1)(2x-1).$$

Since we assumed that $j \geq 2$ and $0 < m < s+1$, we see that $m/(s+1) = x = 1/2$. $\square$

The $((s+1)/2)$-ovoids of generalized quadrangles of order (s,s^2) (or thus with $c_2 = (s^4-1)/(s^2-1)$) are known as *hemisystems*. For the dual polar graph Γ arising from the polar space $H(3,q^2)$, which is a generalized quadrangle of order (q,q^2), Theorem 3 was already obtained for odd q by Segre in [26] and for even q by Bruen and Hirschfeld in [7]. Segre also proved that there is a unique hemisystem (up to equivalence) if $q = 3$. A construction for hemisystems in the dual polar graph from $H(3,q^2)$ for every odd prime power q was given in [10]. The restriction on m was obtained for all generalized quadrangles of order (s,s^2) in [30]. A hemisystem in a non-classical generalized quadrangle of order $(5,5^2)$ was constructed in [1]. Very recently, it was proved in [2] that hemisystems exist in all flock generalized quadrangles (see [29] for more information on the latter).

6 Construction of an induced distance-regular graph

In the regular near $2d$-gons where every intersection number c_j meets the bound from Theorem 1, we can construct another distance-regular graph by use of a (non-trivial) m-ovoid.

In general, a hemisystem in any generalized quadrangle of order (s,s^2) induces a strongly regular graph with parameters $srg((s+1)(s^3+1)/2, (s-1)(s^2+1)/2, (s-3)/2, (s-1)^2/2)$ (this was proved in [8]). For the dual polar graph from $H(3,q^2)$, this result was already obtained by Thas in [32]; for $q = 3$ the induced graph on the unique hemisystem is isomorphic to the Gewirtz graph. The following lemma generalizes these facts to regular near $2d$-gons meeting the bounds from Theorem 1. It only requires assumptions on the parameters, but we we will later see that for $d \geq 3$ they actually force the near polygon to be the dual polar graph from $H(2d-1,q^2)$.

Lemma 7 *Let Γ be a regular near $2d$-gon of order (s,t) with $s>1$ and $c_j=(s^{2j}-1)/(s^2-1)$ for every $j\in\{1,\dots,d\}$. Suppose S is an $((s+1)/2)$-ovoid. Let Γ' be the induced subgraph of Γ on S. The distance between any two vertices in Γ' is the same as in Γ, and Γ' is distance-regular with diameter d and intersection numbers:*

$$b'_j=\frac{s^{2d}-s^{2j}}{2(s+1)}, \forall j\in\{0,\dots,d-1\}; c'_j=\frac{(s^j-(-1)^j)(s^{j-1}-(-1)^j)}{2(s+1)}, \forall j\in\{1,\dots,d\}.$$

Proof Consider any elements $a,b\in S$ at distance j in Γ with $j\in\{1,\dots,d\}$. Lemma 6 yields, after substituting $(s+1)/2$ for m, that:

$$|S\cap(\Gamma_1(a)\cap\Gamma_{j-1}(b))|=\frac{(s^j-(-1)^j)(s^{j-1}-(-1)^j)}{2(s+1)},$$

which is in particular at least one. Induction on j now yields that the distance between a and b in the induced subgraph is also j.

Now consider any two elements a and b of S at distance j in Γ with $0\le j\le d-1$. There are precisely b_j/s singular lines through a at distance j from b. Only on these singular lines through a can points at distance $j+1$ from b and adjacent to a be found, and each such singular line contains exactly $(s-1)/2$ points of $S\backslash\{a\}$. Hence:

$$|S\cap(\Gamma_1(a)\cap\Gamma_{j+1}(b))|=\frac{b_j}{s}\frac{s-1}{2}=\frac{k-sc_j}{s}\frac{s-1}{2}=(c_d-c_j)\frac{s-1}{2}=\frac{s^{2d}-s^{2j}}{2(s+1)},$$

where we let c_0 be zero. Note also that the last value is non-zero if $0\le j\le d-1$ so the diameter of Γ' is precisely d. $\square$

If Γ is the dual polar graph arising from $H(2d-1,q^2)$, then it is a regular near $2d$-gon of order $(s,t)=(q,(q^{2d}-1)/(q^2-1)-1)$ with parameters $c_j=(q^{2j}-1)/(q^2-1)$ for every $j\in\{1,\dots,d\}$ and hence meeting the requirements of Lemma 7. The following lemma characterizes these graphs as the only regular near $2d$-gons meeting the requirements of Lemma 7 for any $d\ge 3$.

Lemma 8 *Suppose Γ is a regular near $2d$-gon of order (s,t) with $d\ge 3$ and $s>1$. If $c_j=(s^{2j}-1)/(s^2-1)$ for all $j\in\{1,\dots,d\}$, then s is a prime power q and Γ is the dual polar graph arising from the polar space $H(2d-1,q^2)$.*

Proof The assumptions imply that $b_j=s(s^{2d}-s^{2j})/(s^2-1)$ for every $j\in\{0,\dots,d-1\}$. Hence Γ has classical parameters $(d,b,\alpha,\beta)=(d,s^2,0,s)$. Theorem 9.4.4 in [5] characterizes the regular near $2d$-gons of order (s,t) with $s>1$, $d\ge 3$ and with classical parameters $(d,b,0,\beta)$ as either a dual polar graph arising from a classical finite polar space or a Hamming graph. However, since Hamming graphs have intersection numbers $c_j=j<(s^{2j}-1)/(s^2-1)$ (see for instance Theorem 9.2.1 in [5]) we can exclude the last possibility. The condition $t_2=c_2-1=s^2$ and Table 1 now yield that Γ can only be the dual polar graph arising from $H(2d-1,q^2)$ with $q=s$. $\square$

Because of Lemma 8 the result from Lemma 7 comes down to the following if the diameter is at least three.

Theorem 4 *Let S be a $((q+1)/2)$-ovoid in the dual polar graph Γ from $H(2d-1,q^2)$ with q odd. The induced subgraph Γ' on S is distance-regular with classical parameters:*

$$(d,b,\alpha,\beta)=\left(d,-q,-\left(\frac{q+1}{2}\right),-\left(\frac{(-q)^d+1}{2}\right)\right).$$

The distance between any two vertices in Γ' is the same as in Γ.

Proof This follows immediately from Lemma 7 and the definition of classical parameters in Subsection 2.1. □

Let Γ be the dual polar graph from $H(2d-1,q^2)$ with q an odd prime power. We first observe that any $((q+1)/2)$-ovoid in Γ would also yield a $((q+1)/2)$-ovoid in the residual graph induced on the set of vertices through a fixed 1-dimensional isotropic subspace of the polar space, which is isomorphic to the dual polar graph arising from $H(2(d-1)-1,q^2)$. The case $d=2$ was already discussed at the end of Section 5, but even for $d=3$ no constructions are known to the author. Theorem 4 here yields that if S is a set of $(q+1)(q^3+1)(q^5+1)/2$ maximals in $H(5,q^2)$ such that every isotropic 2-space is in exactly $(q+1)/2$ elements of S, the induced graph Γ' on S is distance-regular with diameter three and intersection array $\{b'_0,b'_1,b'_2;c'_1,c'_2,c'_3\}$:

$$\{(q^3-1)(q^2-q+1)/2, q^2(q-1)(q^2+1)/2, q^4(q-1)/2; 1, (q-1)^2/2, (q^2-q+1)(q^2+1)/2\}.$$

In general, no distance-regular graphs with the classical parameters found in Theorem 4 of diameter at least three seem to be known. Weng [35] proved that distance-regular graphs with classical parameters (d,b,α,β) with $b<-1, d\geq 4, c_2>1$ and with triangles must either be in one of two known families, or satisfy $(d,b,\alpha,\beta)=(d,-q,-(q+1)/2,-((-q)^d+1)/2)$ for some odd prime power q. For $q=3$ the induced graph on a 2-ovoid in the dual polar graph from $H(2d-1,q^2)$ would be a triangle-free distance-regular graph with classical parameters $(d,b,\alpha,\beta)=(d,-3,-2,-((-3)^d+1)/2)$, the non-existence of which was conjectured for $d\geq 3$ in Conjecture 4.11 in [24].

In a classical finite polar space of rank d, *t-designs* are defined as subsets of maximals, such that each isotropic t-space of the polar space is included in exactly m elements of S for some m. Hence m-ovoids in dual polar graphs are precisely the $(d-1)$-designs. Algebraic characterizations for t-designs in this context (as well as in many other association schemes) are given in [28], based on Delsarte's theory of regular semilattices from [14]. Moreover, 1-designs in the dual polar graph arising from $H(5,q^2)$ with size exactly half the number of all maximals were constructed for every odd prime power q in [11]. Any *partial spread* in the polar space $H(2d-1,q^2)$ with d odd, i.e. a subset of pairwise trivially intersecting maximals, of (maximum) size q^d+1 should also intersect any $((q+1)/2)$-ovoid in exactly half its elements (see Corollary 4.4 in [34]).

Finally, it is worth noting that in any dual polar graph, an m-ovoid S_1 and its complement S_2 yield a *regular* or *equitable partition* $\{S_1,S_2\}$: for each point p, the number of neighbours in both parts only depends on whether $p\in S_1$ or $p\in S_2$. Regular partitions of dual polar spaces were discussed in detail with many examples in [9].

Acknowledgements The author is grateful to John Bamberg for pointing his attention to the appendix of [2], to Bart De Bruyn for many helpful discussions on regular near polygons, and to Frank De Clerck for carefully reviewing this manuscript.

References

1. J. Bamberg, F. De Clerck, and N. Durante. A hemisystem of a nonclassical generalised quadrangle. *Des. Codes Cryptogr.*, 51(2):157–165, (2009)
2. J. Bamberg, M. Giudici, and G. Royle. Every flock generalised quadrangle has a hemisystem. Preprint, `arXiv:0912.2574v1` (2009)

3. R. C. Bose and T. Shimamoto. Classification and analysis of partially balanced incomplete block designs with two associate classes. *J. Amer. Statist. Assoc.*, 47:151–184, (1952)
4. R. C. Bose and S. S. Shrikhande. Geometric and pseudo-geometric graphs $(q^2+1,\, q+1,\, 1)$. *J. Geometry*, 2:75–94, (1972)
5. A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-regular graphs.* Springer-Verlag, Berlin (1989)
6. A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata*, 14(2):145–176 (1983)
7. A. A. Bruen and J. W. P. Hirschfeld. Applications of line geometry over finite fields. II. The Hermitian surface. *Geom. Dedicata*, 7(3):333–353 (1978)
8. P. J. Cameron, P. Delsarte, and J.-M. Goethals. Hemisystems, orthogonal configurations, and dissipative conference matrices. *Philips J. Res.*, 34(3-4):147–162 (1979)
9. I. Cardinali and B. De Bruyn. Regular partitions of dual polar spaces. *Linear Algebra Appl.*, 432(2-3):744–769 (2010)
10. A. Cossidente and T. Penttila. Hemisystems on the Hermitian surface. *J. London Math. Soc. (2)*, 72(3):731–741 (2005)
11. A. Cossidente and T. Penttila. On m-regular systems on $H(5, q^2)$. *J. Algebraic Combin.*, 29(4):437–445 (2009)
12. B. De Bruyn. *Near polygons.* Birkhäuser Verlag, Basel (2006)
13. P. Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, (10):vi+97 (1973)
14. P. Delsarte. Association schemes and t-designs in regular semilattices. *J. Combinatorial Theory Ser. A*, 20(2):230–243 (1976)
15. P. Delsarte. Pairs of vectors in the space of an association scheme. *Philips Res. Rep.*, 32(5-6):373–411 (1977)
16. W. Feit and G. Higman. The nonexistence of certain generalized polygons. *J. Algebra*, 1:114–131 (1964)
17. W. Haemers and C. Roos. An inequality for generalized hexagons. *Geom. Dedicata*, 10(1-4):219–222 (1981)
18. D. G. Higman. Invariant relations, coherent configurations and generalized polygons. In *Combinatorics (Proc. Advanced Study Inst., Breukelen, 1974), Part 3: Combinatorial group theory*, pages 27–43. Math. Centre Tracts, No. 57. Math. Centrum, Amsterdam (1974)
19. A. Hiraki and J. Koolen. A generalization of an inequality of Brouwer-Wilbrink. *J. Combin. Theory Ser. A*, 109(1):181–188 (2005)
20. A. Hiraki and J. Koolen. A Higman-Haemers inequality for thick regular near polygons. *J. Algebraic Combin.*, 20(2):213–218 (2004)
21. A. Hiraki and J. Koolen. A note on regular near polygons. *Graphs Combin.*, 20(4):485–497 (2004)
22. A. Klein, K. Metsch, and L. Storme. Small maximal partial spreads in classical finite polar spaces. (to appear) *Advances in geometry.*
23. A. Neumaier. Krein conditions and near polygons. *J. Combin. Theory Ser. A*, 54(2):201–209 (1990)
24. Y. Pan, M. Lu and C. Weng. Triangle-free distance-regular graphs. *J. Algebraic Combin.*, 27(1):23–34 (2008)
25. S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* European Mathematical Society (EMS), Zürich (2009)
26. B. Segre. Forme e geometrie hermitiane, con particolare riguardo al caso finito. *Ann. Mat. Pura Appl. (4)*, 70:1–201 (1965)
27. E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata*, 9(1):1–72 (1980)
28. D. Stanton. t-designs in classical association schemes. *Graphs Combin.*, 2(3):283–286 (1986)
29. J. A. Thas. Generalized quadrangles and flocks of cones. *European J. Combin.*, 8(4):441–452 (1987)
30. J. A. Thas. Interesting pointsets in generalized quadrangles and partial geometries. *Linear Algebra Appl.*, 114/115:103–131 (1989)
31. J. A. Thas. Old and new results on spreads and ovoids of finite classical polar spaces. In *Combinatorics '90 (Gaeta, 1990)*, volume 52 of *Ann. Discrete Math.*, pages 529–544 (1992)
32. J. A. Thas. Ovoids and spreads of finite classical polar spaces. *Geom. Dedicata*, 10(1-4):135–143 (1981)

33. J. Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Études Sci. Publ. Math.*, (2):13–60 (1959)
34. F. Vanhove Antidesigns and regularity of partial spreads in dual polar graphs. Preprint, `http://cage.ugent.be/geometry/Files/364/antidesignsdualpolar.pdf` (2010)
35. C. Weng. Classical distance-regular graphs of negative type. *J. Combin. Theory Ser. B*, 76 (1): 93–116 (1999)