A geometric construction of the hyperbolic fibrations associated with a flock, q even

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Abstract

In [2], Baker, Ebert and Penttila show, via algebraic methods, that a regular hyperbolic fibration of $\mathsf{PG}(3, q)$ with constant back half gives rise to a flock of a quadratic cone in $\mathsf{PG}(3, q)$, and conversely. In this paper a geometric construction for q even of the flock from the hyperbolic fibration, and conversely, will be described. A proof will be given that this geometric construction indeed corresponds to the known algebraic one.

1 Introduction and definitions

The essence of this paper is a geometric construction, for q even, of the connection between regular hyperbolic fibrations with constant back half and flocks of a quadratic cone in $\mathsf{PG}(3, q)$. The first section will be concerned with the essential definitions and the algebraic description of the connection between the considered hyperbolic fibrations and flocks. Section 2 provides some lemmas that will be useful in Section 3, which describes the geometric construction itself. In Section 4 it will be explained that the given geometric construction indeed yields the same connection as the known algebraic one.

As first defined in [1], a *hyperbolic fibration* of $\mathsf{PG}(3, q)$ is a collection of $q-1$ hyperbolic quadrics and two lines in $\mathsf{PG}(3, q)$ that partition the points of $\mathsf{PG}(3, q)$. Hyperbolic fibrations are studied because they yield many spreads of $\mathsf{PG}(3, q)$: by selecting one of the ruling families of each quadric in the fibration, a spread of $\mathsf{PG}(3, q)$ is obtained. These spreads in turn give rise to translation planes, which explains the interest for hyperbolic fibrations of $\mathsf{PG}(3, q)$, see [1], [3], [2].

An easy example of a hyperbolic fibration is the so-called *hyperbolic pencil* or *H-pencil*, which is a pencil of quadrics of the appropriate types. Other examples of hyperbolic fibrations can be found in [1] and in [3], but up to now all known hyperbolic fibrations are regular with a constant back half. This means the following. A hyperbolic fibration is called *regular* if the two lines in the fibration form a conjugate (skew) pair with respect to each of the polarities associated with the $q-1$ hyperbolic quadrics of the fibration. Denote the two skew lines of the fibration by L_0 and L_∞, respectively, and suppose without loss of generality that coordinates are chosen such that $L_0 : X_2 = X_3 = 0$ and $L_\infty : X_0 = X_1 = 0$. It is an easy exercise to show that every quadric in a regular hyperbolic fibration will then have an equation of the form $aX_0^2 + bX_0X_1 + cX_1^2 + dX_2^2 + eX_2X_3 + fX_3^2 = 0$, for some $a, b, c, d, e, f \in \mathsf{GF}(q)$ with the property that both $aX^2 + bX + c$ and $dX^2 + eX + f$ are irreducible over $\mathsf{GF}(q)$. Any hyperbolic quadric with an equation of this form will be abbreviated by $V[a, b, c, d, e, f]$. The triple (a, b, c) is sometimes called the *front half* of the quadric $V[a, b, c, d, e, f]$ and likewise (d, e, f) is called its *back half.* In all known hyperbolic fibrations, one can fix either the front half or the back half for the six-tuples representing the hyperbolic quadrics of the fibration. Such a hyperbolic fibration is said to have *constant front half,* respectively *constant back half.* In this paper, we will simply say that such a hyperbolic fibration has a *constant half.* Note that the notion of having a constant half is only meaningful for regular hyperbolic fibrations.

Geometrically, having a constant half implies that all quadrics of the fibration intersect either L_0 (constant front half) or L_∞ (constant back half) in the same pair of conjugate points with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. From now on we will say that a hyperbolic fibration *agrees on* L_0, respectively *agrees on* L_∞, precisely when all quadrics of the fibration intersect L_0, respectively L_∞, in the same pair of conjugate points with respect to $\mathsf{GF}(q^2)$. This notion is independent of the choice of the coordinate system and also meaningful for non-regular hyperbolic fibrations.

The H-pencil may be represented as

$$\{V[0, 0, 0, a, b, c]\} \cup \{V[a, b, c, at, bt, ct] \mid t \in \mathsf{GF}(q)\},$$

with $aX^2 + bX + c$ irreducible over $\mathsf{GF}(q)$. Note that the variety $V[0, 0, 0, a, b, c]$ is nothing but L_0 and similarly $V[a, b, c, 0, 0, 0]$ corresponds to L_∞. One sees that the H-pencil agrees on both L_0 and L_∞.

Consider a regular hyperbolic fibration $\mathcal{H}$ that agrees on L_∞. Then $\mathcal{H}$ may be represented by

$$\begin{aligned} \mathcal{H} = \{V[d, e, f, 0, 0, 0], &V[0, 0, 0, d, e, f]\} \\ &\cup \{V[a_i, b_i, c_i, d, e, f] \mid i = 1, \ldots, q-1\}, \end{aligned} \tag{1}$$

with $dX^2 + eX + f$ irreducible over $\mathsf{GF}(q)$. Note that also $a_iX^2 + b_iX + c_i$ and $(a_i - a_j)X^2 + (b_i - b_j)X + (c_i - c_j)$ must be irreducible over $\mathsf{GF}(q)$ for all $i, j \in \{1, 2, \ldots, q-1\}$, $i \neq j$.

A *flock* (see for instance [4]) of a quadratic cone $\mathcal{K}$ with vertex p in $\mathsf{PG}(3, q)$ is a partition of the points of $\mathcal{K} \setminus \{p\}$ into q disjoint irreducible conics. It is customary to work with the set $\mathcal{F}$ of q planes whose intersections with $\mathcal{K}$ yield the flock. If $\mathcal{K}$ has equation $X_0X_2 = X_1^2$, then the planes of $\mathcal{F}$ have equations of the form $aX_0 + bX_1 + cX_2 + X_3 = 0$. Any such plane will be represented by $\pi[a, b, c, 1]$. In [2], the following connection between regular hyperbolic fibrations with a constant back half and flocks of a quadratic cone was first observed.

Theorem 1.1 (Baker, Ebert, Penttila [2]) *With the above notation (1), $\mathcal{H}$ is a hyperbolic fibration of $\mathsf{PG}(3, q)$ if and only if $\mathcal{F} := \{\pi[a_i, b_i, c_i, 1] \mid i = 1, 2, \ldots, q-1\} \cup \{\pi[0, 0, 0, 1]\}$ is a flock of the quadratic cone $\mathcal{K}$ in $\mathsf{PG}(3, q)$ with equation $X_0X_2 = X_1^2$.*

Flocks of a quadratic cone are not only related to hyperbolic fibrations, but also to a plethora of other interesting objects, like ovoids of $Q^+(5, q)$, spreads of $\mathsf{PG}(3, q)$ and translation planes (Walker [10] and Thas independently), generalized quadrangles (Knarr [7], Thas [8], [9]), q-clans and herds of ovals if q is even (see Johnson and Payne [6] for an overview). Of these connections, only the one between flocks and hyperbolic fibrations has so far been described algebraically but not geometrically. It is our aim to fill this gap for q even by providing a geometric explanation of Theorem 1.1.

Note that the flock corresponding to a hyperbolic fibration as in Theorem 1.1 always contains the plane $\pi_0 := \pi[0, 0, 0, 1]$ with equation $X_3 = 0$. Hence to a given flock there might correspond inequivalent regular hyperbolic fibrations, according to which plane is chosen as π_0. This matter was sorted out in [2], as follows.

Theorem 1.2 (Baker, Ebert, Penttila [2]) *The number of mutually inequivalent regular hyperbolic fibrations with constant back half obtained from a given flock $\mathcal{F}$ of a quadratic cone is the number of orbits of $\mathrm{Aut}(\mathcal{F})$ on its conics (planes).*

2 Preliminary results

From now on we assume that q is even. In this section, two easy preliminary lemmas are given that will be of use in the construction of Section 3.

Lemma 2.1 *For q even, there exists a unique irreducible conic in $\mathsf{PG}(2, q)$ with a given nucleus and containing three distinct given points.*

Proof.
First of all, it is assumed that the three given points and the nucleus form a set of 4 points, no three of which are collinear, for otherwise there cannot exist an irreducible conic satisfying the conditions of the lemma. The group $\mathsf{PGL}(3, q)$ of all projectivities of $\mathsf{PG}(2, q)$ acts sharply transitively on the skeletons ([5, p. 32]), so that we may assume without loss of generality that the given nucleus is the point $n = (0, 1, 0)$, while the other three points are $(1, 0, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$.

Now there exists a unique irreducible conic $\mathcal{C}$ with nucleus n and containing these three points, namely $\mathcal{C} : X_0X_2 = X_1^2$. This proves the lemma. $\square$

Lemma 2.2 *Let $\mathcal{C}$ be an irreducible conic in some plane π of $\mathsf{PG}(3, q)$, q even, with nucleus n in π, and let L_0 be a line of π, disjoint from $\mathcal{C}$. Consider a line L_∞ of $\mathsf{PG}(3, q)$ containing n but not contained in π, and a pair of conjugate points $\{p, \overline{p}\}$ with respect to $\mathsf{GF}(q^2)$ on L_∞. Then there exists a unique hyperbolic quadric $Q^+(3, q)$ containing $\mathcal{C}$, having L_0 and L_∞ as conjugate lines with respect to its polarity and such that its extension $Q^+(3, q^2)$ to $\mathsf{GF}(q^2)$ meets L_∞ in the points p and $\overline{p}$.*

Proof.
First note that any hyperbolic quadric $Q^+(3, q)$ having L_0 and L_∞ as conjugate lines with respect to its polarity and such that $Q^+(3, q^2)$ meets L_∞ in p and $\overline{p}$, intersects π in some conic $\mathcal{C}'$ which is disjoint from L_0 and has nucleus $n = L_\infty \cap \pi$.

One can count that the number of hyperbolic quadrics $Q^+(3, q)$ having L_0 and L_∞ as a conjugate pair equals $\frac{1}{4}q^2(q-1)^3$. Each one of them contains a pair of conjugate points (with respect to $\mathsf{GF}(q^2)$) of L_∞, and conversely each such pair is contained in the same number of hyperbolic quadrics (having L_0 and L_∞ as conjugate pair). As there are $\frac{1}{2}q(q-1)$ pairs of conjugate points on L_∞, this yields $\frac{1}{2}q(q-1)^2$ hyperbolic quadrics which in addition meet L_∞ in p and $\overline{p}$, if considered over $\mathsf{GF}(q^2)$.

On the other hand, one similarly counts the number of irreducible conics in π having nucleus n and disjoint from the line L_0: this number also equals $\frac{1}{2}q(q-1)^2$. Hence the lemma follows. $\square$

3 The construction geometrically

Let $\mathcal{K}$ be a quadratic cone in $\mathsf{PG}(3, q)$, q even, and consider a flock $\mathcal{F} = \{\pi_0, \pi_1, \ldots, \pi_{q-1}\}$ of $\mathcal{K}$. Denote the conic which is the intersection of π_i with $\mathcal{K}$ by $\mathcal{C}_i$. As q is even by assumption, the cone $\mathcal{K}$ has a nucleus line N through the vertex v of $\mathcal{K}$. This nucleus line intersects π_i in the nucleus n_i of $\mathcal{C}_i$, for all $i = 0, 1, \ldots, q-1$, and these nuclei are all distinct.

Set $n_0 := n$. Since n is a point of the nucleus line of $\mathcal{K}$, every line of $\mathsf{PG}(3, q)$ through n is tangent to $\mathcal{K}$. Consider a plane π of $\mathsf{PG}(3, q)$ through v but not containing n. We will project the conics of the flock $\mathcal{F}$ from n onto π. As n is a point of the plane π_0, the $q + 1$ points of $\mathcal{C}_0$ are projected onto the $q + 1$ points of some line L_0 of π. On the other hand, n is not contained in any of the planes π_i, $i = 1, 2, \ldots, q-1$, so that each conic $\mathcal{C}_i$, $i = 1, 2, \ldots, q-1$, is projected onto a conic $\mathcal{C}_i'$ of π. The vertex v of $\mathcal{K}$ is projected onto itself. We thus obtain a set of $q-1$ conics $\mathcal{C}_i'$, $i = 1, 2, \ldots, q-1$, one line L_0 and a point v in π which together partition the points of π. Moreover, the point v is the nucleus of each conic $\mathcal{C}_i'$, since $vn = N$ is the nucleus line of $\mathcal{K}$. Hence every plane of $\mathsf{PG}(3, q)$ through N meets $\mathcal{K}$ in a generator of $\mathcal{K}$ containing exactly one point of each element of the flock $\mathcal{F}$. After projection onto π, this means that every line of π through v is tangent to every conic $\mathcal{C}'_i$.

It is our aim to construct from $\mathcal{F}$ a regular hyperbolic fibration of $\mathsf{PG}(3,q)$ which agrees on one of its two lines. Hence we set $L_\infty := N$, and the fibration we will construct will agree on L_∞. Choose an arbitrary pair $\{p,\overline{p}\}$ of conjugate points of L_∞ with respect to $\mathsf{GF}(q^2)$. Next, we denote by $Q_i^+(3,q)$, $i = 1,2,\ldots,q-1$, the unique (non-degenerate) hyperbolic quadric of $\mathsf{PG}(3,q)$ determined by

- $L_0^\perp = L_\infty$ with respect to the polarity of $Q_i^+(3,q)$;
- $\pi \cap Q_i^+(3,q) = \mathcal{C}'_i$; and
- the extension $Q_i^+(3,q^2)$ of $Q_i^+(3,q)$ meets L_∞ in the conjugate pair $\{p,\overline{p}\}$.

By Lemma 2.2, the hyperbolic quadric $Q_i^+(3,q)$ exists and is unique. We will now show that the $q-1$ quadrics $Q_i^+(3,q)$, $i = 1,2,\ldots,q-1$, together with L_0 and L_∞, form a regular hyperbolic fibration in $\mathsf{PG}(3,q)$ which agrees on L_∞.

Theorem 3.1 *With the above notation,* $\mathcal{H} := \{Q_i^+(3,q) \mid i = 1,2,\ldots,q-1\} \cup \{L_0, L_\infty\}$ *is a regular hyperbolic fibration of* $\mathsf{PG}(3,q)$ *which agrees on* L_∞*.*

Proof.
By construction, $L_0^\perp = L_\infty$ with respect to the polarity of $Q_i^+(3,q)$, for all $i = 1,2,\ldots,q-1$, and the extenstion $Q_i^+(3,q^2)$ of each $Q_i^+(3,q)$ meets L_∞ in the conjugate pair $\{p,\overline{p}\}$. So if $\mathcal{H}$ is a hyperbolic fibration, it will be a regular one that agrees on L_∞.

Since every $\mathcal{C}'_i$ is disjoint from L_0 and every $Q_i^+(3,q)$ is disjoint from L_∞ by construction, we must show that $Q_i^+(3,q)$ and $Q_j^+(3,q)$ have no common points for all $i \neq j$, in order to obtain a partition of the points of $\mathsf{PG}(3,q)$. So suppose that $Q_i^+(3,q)$ and $Q_j^+(3,q)$ have a point x in common. Denote the plane $\langle x, L_\infty\rangle$ by π'. Then $Q_i^+(3,q) \cap \pi'$ is a conic $\tilde{\mathcal{C}}_i$ and similarly $Q_j^+(3,q) \cap \pi'$ is a conic $\tilde{\mathcal{C}}_j$. The conics $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_j$ share the point x. By the fact that $L_0^\perp = L_\infty$ with respect to the polarity of each $Q_k^+(3,q) \in \mathcal{H}$, the above two conics have the same nucleus $v' := L_0 \cap \pi'$ and their extensions to $\mathsf{GF}(q^2)$ intersect the line L_∞ in the same pair $\{p,\overline{p}\}$ of conjugate points with respect to $\mathsf{GF}(q^2)$. By Lemma 2.1, the extensions to $\mathsf{GF}(q^2)$ of $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_j$, and hence also the conics $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_j$, must coincide. In particular, every line of π' through the nucleus v' of $\tilde{\mathcal{C}}_i = \tilde{\mathcal{C}}_j$ contains a common point of $Q_i^+(3,q)$ and $Q_j^+(3,q)$ and this also holds for the line $\pi \cap \pi'$. But then the conics $\mathcal{C}'_i$ and $\mathcal{C}'_j$ must also have a point in common, which is a contradiction. It follows that the hyperbolic quadrics $Q_i^+(3,q)$, $i = 1,2,\ldots,q-1$, together with the lines L_0 and L_∞, partition the points of $\mathsf{PG}(3,q)$ and thus they form a hyperbolic fibration. This completes the proof. $\square$

Conversely, let $\mathcal{H} := \{Q_i^+(3,q) \mid i = 1,2,\ldots,q-1\} \cup \{L_0, L_\infty\}$ be a regular hyperbolic fibration of $\mathsf{PG}(3,q)$ which agrees on one of its lines, say L_∞. Let

π be an arbitrary plane of $\mathsf{PG}(3,q)$ through the line L_0 and set $L_\infty \cap \pi := v$. Every hyperbolic quadric $Q_i^+(3,q)$ intersects π in a conic $\mathcal{C}_i'$ and these conics are pairwise disjoint on the one hand and disjoint from L_0 on the other hand. So together with v, these $q-1$ conics and L_0 partition the points of π. Moreover, as $\mathcal{H}$ is regular by assumption, the point v is the nucleus of each conic $\mathcal{C}_i'$.

Next, let n be any point of $L_\infty \setminus \{v\}$ and consider the plane $\pi_0 := \langle n, L_0\rangle$. In π_0, consider a non-degenerate conic $\mathcal{C}_0$ with nucleus n and let $\mathcal{K}$ be the quadratic cone of $\mathsf{PG}(3,q)$ with vertex v and base conic $\mathcal{C}_0$. Then vn is the nucleus line of $\mathcal{K}$. For $i = 1, 2, \ldots, q-1$, we also consider the quadratic cone $\mathcal{K}_i$ with vertex n and base conic $\mathcal{C}_i'$. These cones $\mathcal{K}_i$ have a common vertex, but apart from that they are disjoint, because the conics $\mathcal{C}_i'$, $i = 1, 2, \ldots, q-1$, are pairwise disjoint. Now we have a look at the intersection of $\mathcal{K}$ with $\mathcal{K}_i$, for $i = 1, 2, \ldots, q-1$. Every line through the nucleus n of $\mathcal{C}_0$, so in particular also every generator of $\mathcal{K}_i$, meets $\mathcal{K}$ in a unique point. As a consequence, the cones $\mathcal{K}$ and $\mathcal{K}_i$ have exactly $q+1$ points in common.

Lemma 3.2 *For each $i \in \{1, 2, \ldots, q-1\}$, the $q+1$ common points of $\mathcal{K}$ and $\mathcal{K}_i$ lie in a plane π_i.*

Proof.
Consider three distinct points x, y and z of $\mathcal{K} \cap \mathcal{K}_i$. Then $\langle x, y, z\rangle$ is a plane intersecting $\mathcal{K}$ in some conic $\tilde{\mathcal{C}}$ and $\mathcal{K}_i$ in a conic $\tilde{\mathcal{C}}_i$. By construction of the cones $\mathcal{K}$ and $\mathcal{K}_i$, the line vn is the nucleus line of both of them and hence $vn \cap \langle x, y, z\rangle$ is the nucleus of both $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}_i$. Now Lemma 2.1 implies that $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}_i$ must coincide, so that the $q+1$ common points of $\mathcal{K}$ and $\mathcal{K}_i$ must be exactly those of $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}_i$. This proves the lemma, with $\pi_i = \langle x, y, z\rangle = \langle \tilde{\mathcal{C}}_i\rangle$. $\square$

Lemma 3.3 *For each $i \in \{1, 2, \ldots, q-1\}$, it holds that $\pi_i \cap \pi_0 \cap \mathcal{K} = \emptyset$.*

Proof.
With the same notation as in the previous lemma, $\pi_i \cap \mathcal{K} = \tilde{\mathcal{C}}_i$, $i = 1, 2, \ldots, q-1$. Now $\pi_i \cap \pi_0 \cap \mathcal{K}$ is non-empty if and only if $\tilde{\mathcal{C}}_i$ and $\mathcal{C}_0$ have a point in common. But this implies that their projections from n onto π also share a point. This is obviously not the case, since v, L_0 and the $q-1$ conics $\mathcal{C}_i$ partition the points of π. $\square$

As $\mathcal{K}_i$ and $\mathcal{K}_j$, $i \neq j$, share no points except for their common vertex, one concludes by the two previous lemmas that $\{\pi_i \mid i = 0, 1, \ldots, q-1\}$ is a flock of $\mathcal{K}$. Hence we have constructed a flock from a hyperbolic fibration of $\mathsf{PG}(3,q)$ which is regular and agrees on one of its lines. If a regular hyperbolic fibration agrees on one of its lines, coordinates can be chosen such that it is of the form (1), which implies that it has a constant (back) half.

Note that for constructing the flock from the hyperbolic fibration, one does not need to start from a fibration that agrees on one of its lines. With the above construction, flocks can be obtained from all regular hyperbolic fibrations. If the regular hyperbolic fibration in addition agrees on one of its lines, it can be seen, by adding coordinates as in the next section, that all choices of the plane π containing the line L_0 yield the same flock. If the fibration does not agree on any of its lines, different choices of π may lead to non-isomorphic flocks. So far, however, there are no examples known of such hyperbolic fibrations.

4 The construction algebraically

In this section we add coordinates to the construction of the previous section to show that it is indeed the geometric translation of the algebraic correspondence between a regular hyperbolic fibration with constant back half and a flock of a quadratic cone containing a fixed plane, as described by Baker, Ebert and Penttila in [2], see Theorem 1.1.

Consider a hyperbolic fibration $\mathcal{H}$ consisting of the two lines $L_0 : X_2 = X_3 = 0$ and $L_\infty : X_0 = X_1 = 0$ and $q-1$ hyperbolic quadrics $Q_i^+(3,q)$, $i = 1, 2, \ldots, q-1$, given by

$$Q_i^+(3,q) : a_iX_0^2 + b_iX_0X_1 + c_iX_1^2 + dX_2^2 + eX_2X_3 + fX_3^2 = 0,$$

where $dX^2 + eX + f$, $a_iX^2 + b_iX + c_i$ and $(a_i - a_j)X^2 + (b_i - b_j)X + (c_i - c_j)$ are irreducible over $\mathsf{GF}(q)$, for all $i, j \in \{1, 2, \ldots, q-1\}$, $i \neq j$. This hyperbolic fibration is regular and agrees on L_∞. Let π be the plane $X_2 = 0$. The hyperbolic quadric $Q_i^+(3,q)$, $i \in \{1, 2, \ldots, q-1\}$, then intersects π in the conic $\mathcal{C}_i'$ with equation

$$\mathcal{C}_i' : \begin{cases} X_2 = 0 \\ a_iX_0^2 + b_iX_0X_1 + c_iX_1^2 + fX_3^2 = 0. \end{cases}$$

The conics $\mathcal{C}_i'$, $i = 1, 2, \ldots, q-1$, are pairwise disjoint and disjoint from L_0, and they all have nucleus $v := (0,0,0,1)$, which is the point $L_\infty \cap \pi$. Let n be the point $(0,0,1,0)$ on $L_\infty \setminus \{v\}$ and set $\pi_0 := \langle n, L_0\rangle$, so that $\pi_0 : X_3 = 0$. In π_0 we choose a conic $\mathcal{C}_0$ with nucleus n as follows:

$$\mathcal{C}_0 : \begin{cases} X_3 = 0 \\ X_0X_1 = X_2^2. \end{cases}$$

With these choices, the cone $\mathcal{K}'$ with vertex v and base conic $\mathcal{C}_0$ has equation $X_0X_1 = X_2^2$, and for $i = 1, 2, \ldots, q-1$ we consider the cone $\mathcal{K}_i$ with vertex n and base conic $\mathcal{C}_i'$, having equation $a_iX_0^2 + b_iX_0X_1 + c_iX_1^2 + fX_3^2 = 0$. By some basic calculations, one can check that the intersection $\mathcal{K} \cap \mathcal{K}_i$ is the following conic:

$$\tilde{\mathcal{C}}_i : \begin{cases} \sqrt{a_i}X_0 + \sqrt{c_i}X_1 + \sqrt{b_i}X_2 + \sqrt{f}X_3 = 0 \\ \sqrt{c_i}X_1^2 + \sqrt{b_i}X_1X_2 + \sqrt{a_i}X_2^2 + \sqrt{f}X_1X_3 = 0. \end{cases}$$

This is in fact the intersection of $\mathcal{K}'$ with the plane $\pi_i' : \sqrt{a_i}X_0 + \sqrt{c_i}X_1 + \sqrt{b_i}X_2 + \sqrt{f}X_3 = 0$. Applying the collineation $Y_0 = X_0$, $Y_1 = X_2$, $Y_2 = X_1$, $Y_3 = \sqrt{f}X_3$, followed by the automorphic collineation induced by the field automorphism

$x \mapsto x^2$, the plane π'_i is mapped to the plane $\pi_i : a_i Y_0 + b_i Y_1 + c_i Y_2 + Y_3 = 0$, while $\mathcal{K}'$ is transformed into the cone $\mathcal{K} : Y_0 Y_2 = Y_1^2$. Hence we have obtained the same flock as the one given in Theorem 1.1.

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