Hermitian Veronesean Caps

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Abstract

In [4], a characterization theorem for Veronesean caps in $\mathsf{PG}(N, \mathbb{K})$, with $\mathbb{K}$ a skewfield, is proved. This result extends the theorem for the finite case proved in [6]. In this paper, we prove analogous results for Hermitian caps of index 2, extending the results obtained in the finite case in [1] in a non-trivial way.

1 Introduction

In [4], we showed that a cap endowed with the structure of ovals is a projective space using Veblen's axiom and as Corollary we obtained a characterization for quadric Veronesean varieties as a representation of a projective space in another projective space where lines of the former are ovals. In this paper we consider Hermitian Veronesean varieties. Here, a characterization as unions of ovoids with an additional assumption on the tangent planes can be proved in much the same spirit as for quadric Veroneseans, except that, after using Veblen's axiom, one needs an argument to show that all ovoids are isomorphic elliptic quadrics (in the finite case there is only one isomorphism class of elliptic quadrics in projective 3-space over a fixed finite field). There remains to show a characterization of Hermitian Veroneseans as a representation of a projective space in another one where the lines are ovoids. In the finite case, the key lemma is Lemma 3.2 of [7], which is proved with a typical finiteness argument, using counting and dividing. Moreover, later in the proof, in order to arrive at the André representation of an affine plane, one heavily uses the fact that the number of points is already correct. These two facts seemed, up to now, too heavy obstacles for the general case. However, in the present paper we use entirely different ideas to generalize Lemma 3.2 of [7]. The proof we provide is more geometric and also holds in the finite case, and it also provides enough insight in the matter to analyze the smallest case, which is an exception to the theorem, see Remark 4.2. We will describe this case in detail.

We end the paper with an easy application, providing a simple alternative geometric definition of the Hermitian Veronesean. This application generalizes a result of Lunardon [3].

The paper is organized as follows. In Section 2, we introduce the necessary notions: we review the Veblen-Young theorem [8], which is crucial in our arguments, define Hermitian Veroneseans, and state our main results.

2 Notation and main results

2.1 Axiomatization of projective spaces

A good exposition on the foundations of projective and polar spaces can be found on Peter Cameron's website, and the paragraph below is based on these lecture notes. At the end of the 19th century a lot of work was done on the axiomatization of projective spaces, starting with Pasch. This work culminated in 1910 when Veblen and Young provided a beautiful characterization of projective spaces [8] based on the following axiom.

Veblen's axiom

If a line intersects two sides of a triangle but does not contain their intersection then it also intersects the third side.

Theorem 2.1 (Veblen-Young theorem) *Let $(X, \mathcal{L})$ be a thick linear space satisfying Veblen's axiom. Then one of the following holds:*

(1) $X = \mathcal{L} = \emptyset$.

(2) $|X| = 1$, $\mathcal{L} = \emptyset$.

(3) $\mathcal{L} = \{X\}, |X| \geq 3$.

(4) $(X, \mathcal{L})$ *is a projective plane.*

(5) $(X, \mathcal{L})$ *is a projective space over a skew field, not necessarily of finite dimension.*

2.2 Quadric Veronesean caps

In [4] we proved the following

Theorem 2.2 *Let X be a Veronesean cap in* $\mathrm{PG}(N, \mathbb{K})$*. Then K is a field and there exists a natural number $n \geq 2$ (called the index of X), a projective space* $\Pi' := \mathsf{PG}(n(n+3)/2, \mathbb{K})$ *containing* Π*, a subspace R of* Π' *skew to* Π*, and a quadric Veronesean* $\mathcal{V}_n$ *of index n in* Π'*, with* $R \cap \mathcal{V}_n = \emptyset$*, such that X is the (bijective) projection of* $\mathcal{V}_n$ *from R onto* Π*. The subspace R can be empty, in which case X is projectively equivalent to* $\mathcal{V}_n$*.*

As an application, we showed the following characterization, which basically replaces Condition (V2) with a dimension restriction, and (V3) with the condition that the geometry of points and ovals is a projective space.

Theorem 2.3 *Let X be a set of points in the projective space $\mathsf{PG}(d,\mathbb{K})$, with $\mathbb{K}$ any skew field of order at least 3. Suppose that*

(V1*) *for any pair of points $x, y \in X$, there is a unique plane denoted $[x,y]$ such that $[x,y] \cap X$ is an oval, denoted $X([x,y])$;*

(V2*) *the set X endowed with all subsets $X([x,y])$, has the structure of the point-line geometry of a projective space $\mathsf{PG}(n,\mathbb{F})$, for some skew field $\mathbb{F}$, $n \geq 3$, or of any projective plane Π (and we put $n = 2$ in this case);*

(V3*) $d \geq \frac{1}{2}n(n+3)$.

Then $d = \frac{1}{2}n(n+3)$ and X is the point set of a quadric Veronesean of index n. In particular, $\mathbb{F} \equiv \mathbb{K}$ if $n \geq 3$, and Π is isomorphic to $\mathsf{PG}(2,\mathbb{K})$ if $n = 2$.

2.3 Hermitian Veronesean caps

An *ovoid* O in a 3-dimensional projective space Σ is a set of points of Σ such that no line of Σ intersects O in at least 3 points, and for every point $x \in O$, there is a unique plane π through x intersecting O in only x and containing al lines through x that meet O in only x. The plane π is called the *tangent plane* at x to O and denoted $T_x(O)$.

Let X be a spanning point set of PG$(N,\mathbb{K})$, with $\mathbb{K}$ any skew field, and let Ξ be a collection of 3-dimensional projective subspaces of PG$(N,\mathbb{K})$, called the elliptic spaces of X, such that, for any $\xi \in \Xi$, the intersection $\xi \cap X$ is an ovoid $X(\xi)$ in ξ (and then, for $x \in X(\xi)$, we sometimes denote $T_x(X(\xi))$ simply by $T_x(\xi)$. We call X a Hermitian cap if the following properties hold :

(H1) Any two points x and y lie in a unique element of Ξ, denoted by $[x,y]$.

(H2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X$.

(H3) If $x \in X$ and $\xi \in \Xi$, with $x \notin \pi$, then each of the planes $T_x([x,y])$, $y \in \xi \cap X$, is contained in a fixed 4-dimensional subspace of PG$(N,\mathbb{K})$, denoted by $T(x,\xi)$.

In [1], it was shown that the following are examples of Hermitian Veronesean caps.

Hermitian Veronesean varieties

Let n be a positive integer, let L be a quadratic extension of a field K, and consider the projective spaces $\mathsf{PG}(n,\mathbb{L})$ and $\mathsf{PG}(N,\mathbb{K})$ with $N = n(n+2)$. Let $r \in \mathbb{L} \setminus \mathbb{K}$ be arbitrary. Then the *Hermitian Veronesean variety of index n* is the image of the map $\pi : \mathsf{PG}(n,\mathbb{L}) \to \mathsf{PG}(N,\mathbb{K})$

$$\pi(\langle(x_0, x_1, \ldots, x_n)\rangle) = \langle(y_{i,j})_{0 \leq i,j \leq n}\rangle,$$

with $y_{i,i} = x_i\bar{x}_i$, $y_{i,j} = x_i\bar{x}_j + \bar{x}_i x_j$ (for $i < j$), and $y_{i,j} = r x_i\bar{x}_j + \bar{r}\bar{x}_i x_j$ (for $i > j$).

Moreover, in the finite case, it is proved in [1] that every Hermitian Veronesean cap is a suitable projection of some Hermitian Veronesean variety. Below we generalize this to the infinite finite dimensional case.

The following is our second main result.

Theorem 2.4 *Let X is a Hermitian cap in $\Pi = \mathsf{PG}(N, \mathbb{K})$, $N > 3$, with corresponding set Ξ of elliptic spaces. Then $\mathbb{K}$ is commutative. Also, X endowed with all $X(\xi)$, for $\xi \in \Xi$, is the point-line structure of a projective space $\mathsf{PG}(n, \mathbb{L})$, with $\mathbb{L}$ a quadratic extension of $\mathbb{K}$, and X is projectively equivalent to a quotient of a Hermitian Veronesean variety of index n. If $n = 2, 3$, then $N = n(n+2)$ and X is projectively equivalent to a Hermitian Veronesean variety of index n.*

As an application, we also show the following elegant characterization, which basically replaces Condition (H2) with a dimension restriction, and (H3) with the condition that the geometry of points and ovoids is a projective space.

Theorem 2.5 *Let X be a set of points in the projective space $\mathsf{PG}(d, \mathbb{K})$, with $\mathbb{K}$ any skew field of order at least 3. Suppose that*

- (H1*) *for any pair of points $x, y \in X$, there is a unique 3-dimensional subspace denoted $[x, y]$ such that $[x, y] \cap X$ is an ovoid, denoted $X([x, y])$;*
- (H2*) *the set X endowed with all subsets $X([x, y])$, has the structure of the point-line geometry of a projective space $\mathsf{PG}(n, \mathbb{L})$, for some skew field $\mathbb{L}$, $n \geq 3$, or of any projective plane Π (and we put $n = 2$ in this case);*
- (H3*) $d \geq n(n+2)$.

Then $d = n(n+2)$ and X is the point set of a Hermitian Veronesean variety of index n. In particular, $\mathbb{L}$ is a quadratic extension of $\mathbb{K}$ if $n \geq 3$, and Π is isomorphic to $\mathsf{PG}(2, \mathbb{F})$ if $n = 2$, where $\mathbb{F}$ is a quadratic extension of $\mathbb{K}$.

3 Hermitian Veronesean caps

3.1 The projective space associated with the cap

Let $\mathcal{H} = (X, \Theta)$ be a Hermitian cap, where X is a set of points in $\mathsf{PG}(N, \mathbb{K})$, for some skew field $\mathbb{K}$, and Θ is a set of elliptic spaces satisfying (H1), (H2) and (H3) introduced before.

Associated with $\mathcal{H}$ we can consider the geometry $\mathcal{P}$ having point set X and line set the set Ξ, endowed with the natural incidence.

Lemma 3.1 *$\mathcal{P}$ is a projective space.*

Proof First of all $\mathcal{P}$ is a linear space by (H1).

Let x_{12}, x_{23} and x_{13} be three points of X and denote $O_1 = [x_{12}, x_{13}]$, $O_2 = [x_{12}, x_{23}]$ and $O_3 = [x_{13}, x_{23}]$. Let O_4 be an oval intersecting O_1 in a point x_{14} and O_2 in a point x_{24}, both different from x_{12}. Our purpose is to show that the Veblen's axiom holds, which means that we have to show that O_4 intersects O_3. Of course, we may assume that $O_3 \neq O_4$ and that O_4 does not contain x_{13} nor x_{23}. Clearly $6 \leq \dim V \leq 8$ and we claim that $V := \langle O_1, O_2, O_3 \rangle$ contains O_4.

Indeed, let us first show that V contains O_4. Since both $T_{x_{13}}(O_3)$ and $T_{x_{13}}(O_1)$ belong to $\langle O_1, O_3 \rangle \subseteq V$, also $T_{x_{13}}([x_{13}, x_{24}])$ does by applying (H3) with as point x_{13} and as ovoid O_2 , and hence $\langle [x_{13}, x_{24}] \rangle = \langle T_{x_{13}}([x_{13}, x_{24}]), x_{24} \rangle$ is contained in V. Likewise, applying (H3) to x_{24} and O_1 and reasoning as above it follows that O_4 is contained in V.

If V were 6-dimensional, then O_4 and O_3 would meet, and Veblen's axiom would follow automatically.

Next, suppose that $\dim V = 8$. Now we project $V \setminus \langle O_2 \rangle$ from O_2 onto a four-dimensional space Π of V disjoint from $\langle O_2 \rangle$. The ovoids O_3 and O_4 together with their tangent planes at their intersection point with O_2 are mapped onto two full planes of Π, say Π_3 and Π_4, respectively. Note that Π_3 and Π_4 generate Π, as one has similarly as above that O_1 is contained in $\langle O_2, O_3, O_4 \rangle$, and so the latter is 8-dimensonal. Let x be the unique intersection point of Π_3 and Π_4. There are basically four different possibilities.

(1) *There is a point x_i of $O_i \setminus O_2$ projected onto x from $\langle O_2 \rangle$, for $i \in \{3, 4\}$, and $x_3 \neq x_4$.*

In this case, since the space $\langle x_3, x_4, O_2 \rangle = \langle x, O_2 \rangle$ is 4-dimensional, the line $\langle x_3, x_4 \rangle$ meets the elliptic space $\langle O_2 \rangle$ in a point y. This implies that the elliptic space of the ovoid $[x_3, x_4]$ intersects $\langle O_2 \rangle$ in y, implying $y \in X$, contradicting $[x_3, x_4]$ being an ovoid.

(2) *There is a point x_3 of $O_3 \setminus O_2$ projected onto x from $\langle O_2 \rangle$, and the tangent plane $T_{x_{24}}(O_4)$ to O_4 at x_{24} projects from $\langle O_2 \rangle$ onto a line L_4 through x.*

In this case, clearly $\langle T_{x_{24}}(O_4) \rangle$ is contained in $\langle O_2, L_4 \rangle$, which also contains $T_{x_{24}}(O_2)$. Hence, by our axioms, the 5-space $\langle O_2, L_4 \rangle$ also contains $T_{x_{24}}([x_{13}, x_{24}])$ (since the ovoids O_2, O_4 and $[x_{13}, x_{24}]$ all intersect O_1). Similarly, since the ovoids $[x_{13}, x_{24}], O_2$ and $[x_3, x_{24}]$ all meet the ovoid O_3, the plane $T_{x_{24}}([x_3, x_{24}])$ belongs to $\langle O_2, L_4 \rangle$, which implies that $[x_3, x_{24}]$ belongs to the 5-space $\langle O_2, L_4 \rangle$ and so $\langle [x_3, x_{24}] \rangle$ meets $\langle O_2 \rangle$ in a line, contradicting our axioms.

(3) *The tangent plane $T_{x_{24}}(O_i) =: L_i$ to O_i at x_{24} projects onto lines L_3 and L_4 through x from $\langle O_2\rangle$, for all $i \in \{3,4\}$.*

In this case, as above, the 5-space $\langle O_2, L_4\rangle$ contains $T_{x_{24}}([x_{13}, x_{24}])$. It follows that the 7-space $U := \langle O_2, L_3, L_4, , x_{13}\rangle$ contains $[x_{13}, x_{24}], O_2$ and O_3. But, as above, using (H3) with x_{13} and O_2 one easily deduces that U also contains O_1, and so U coincides with V, a contradiction.

(4) The only remaining possibility is that *there is a point z of $(O_3 \cap O_4) \setminus O_2$ projected onto x from $\langle O_2\rangle$.* But then $O_3 \cap O_4$ is nonempty, and that is exactly what we had to prove.

Finally, suppose that $\dim V = 7$. Now we project $V \setminus \langle O_2\rangle$ from O_2 onto a three-dimensional space Σ of V disjoint from $\langle O_2\rangle$. The ovoids O_3 and O_4 together with their tangent planes at their intersection point with O_2 are mapped onto two full planes of Σ, say Π_3 and Π_4, respectively. Let L be the intersection line of Π_3 and Π_4. We distinguish three cases.

(1) *Both tangent planes $T_{x_{23}}(O_3)$ and $T_{x_{24}}(O_4)$ are projected from O_2 onto lines L_3 and L_4 which are distinct from L.* Then we can pick a point y on L not contained in L_3 nor L_4 and continue as in (1) of the case $\dim V = 8$.

(2) *The tangent plane $T_{x_{24}}(O_4)$ is projected onto L from O_2 and $T_{x_{23}}(O_3)$ onto a line $L_3 \neq L$.* Then choose a point x_3 of O_3 that is projected onto a point of L and reason as in (2) of the case $\dim V = 8$ to conclude that $[x_3, x_{24}]$ belongs to the 5-space $\langle O_2, L\rangle$ and so $\langle [x_3, x_{24}]\rangle$ meets $\langle O_2\rangle$ in a line, contradicting our axioms.

(3) *Both tangent planes $T_{x_{23}}(O_3)$ and $T_{x_{24}}(O_4)$ are projected onto L from O_2.* Then arguing as in (3) of the case $\dim V = 8$ yields that $V = \langle O_2, L, x_{13}\rangle$, a contradiction since $\langle O_2, L, x_{13}\rangle$ is 6-dimensional.

Hence we have shown that Veblen's axiom holds. □

Note that $\mathcal{P}$ is not necessarily finite-dimensional at this stage. If $\mathcal{P}$ has finite dimension n, then we say that the Hermitian cap has index n.

3.2 The basic step

Assume that our Hermitian cap has index 2. Then we have

Theorem 3.2 *If $\mathcal{H} = (X, \Theta)$ is a Hermitian cap of index 2 in $\mathsf{PG}(N, \mathbb{K})$, then $N = 8$ and X is projectively equivalent to the Hermitian Veronesean of index 2.*

Proof We choose an arbitrary elliptic space ξ and corresponding ovoid $O := X(\xi)$. Assume, by way of contradiction, that there is a 4-dimensional space U containing ξ and two points x, y of $\mathcal{H}$ not on O. Then $[x, y]$ contains the line xy, which intersects ξ in some point z, which must necessarily belong to O in view of Condition (H2). But then the ovoid $X([x, y])$ contains three collinear points x, y, z, a contradiction.

It follows that, if W_4 is a 4-dimensional subspace of $\mathsf{PG}(8, \mathbb{K})$ skew to ξ, then the projection ρ_4 of $X \setminus O$ from ξ onto W_4 is injective. Let p be any point of O. Then $T(p)$ is 4-dimensional and intersects ξ in the plane $T_p(O)$. Hence $T(p)^{\rho_4}$ is a line, which we denote by L_p. For any member $\xi' \in \Theta$ containing p, with $\xi' \neq \xi$, each line in ξ' through p is either contained in $T(p)$ or intersects $X(\xi')$ in a second point. This immediately implies that the projection of $X(\xi') \setminus \{p\}$ is an affine plane $\alpha_{\xi'}$ in W_4 which, completed with L_p, becomes a projective plane $\pi_{\xi'}$. Choose $q \in O$ with $q \neq p$ and let $\xi'' \in \Theta$ be arbitrary but such that $q \in \xi''$ and $\xi'' \neq \xi$. Then all points of $(X \setminus O)^{\rho_4}$ arise in all planes of W that are spanned by L_p and some point of $\alpha_{\xi''}$, and no such point is contained in L_p . It follows that the projection of $X \setminus O$ from ξ coincides with the affine space obtained from W_4 by deleting the 3-space Σ generated by L_p and L_q (this is indeed a 3-space as if the lines L_p and L_q would intersect in a point x, then the projection of the elliptic spaces through an arbitrary point z of $X \setminus O$ and p, respectively q, would intersect in the set $\langle z^{\rho_4}, x\rangle \setminus \{x\}$, a contradiction to the injectivity of ρ_4 and Condition (H2)). It also follows from this argument that, for any point $r \in O$, the line L_r is contained in Σ. Moreover, if s is an arbitrary point of Σ, then we can choose two points s_1, s_2 in $W_4 \setminus \Sigma$ such that s, s_1, s_2 are collinear. Considering the inverse images under ρ_4 of s_1 and s_2, and the unique member ξ^* of Θ containing both these inverse images, we see that, if $\xi^* \cap \xi = \{t\}$, the point s is contained in L_t. We conclude that the lines L_r, for r ranging through O, form a spread $\mathcal{S}$ of Σ, and we obtain an André construction of the projective plane $\mathcal{P}$. Hence each line of $\mathcal{P}$ is a translation line. So $\mathcal{P}$ is a Moufang plane. The inverse image of Σ under ρ_4 is a 7-dimensional space which we shall denote by $T(\xi)$ or $T(O)$ and refer to as the tangent space to X at ξ or at O. Clearly, it intersects X in O.

Since $T(O)$ contains $T(p)$, it follows that $T(p)$ intersects X in just p.

Now consider a point $x \in X$ not on O. Since $T(x)$ meets X in just X, the spaces $T(x)$ and ξ are complementary. Consider a point $a \in X \setminus \{x\}$. The space $[a, x]$ meets $T(x)$ in the plane $T_x([a, x])$ and so the projection of a from $T(x)$ onto ξ coincides with the unique intersection point $O \cap [a, x]$. This implies that the image of the projection of $X \setminus \{x\}$ from $T(x)$ onto ξ coincides with O. Denote the projection operator by ρ_3 for further reference. The previous argument now easily implies that the image under ρ_3 of any member of Θ not containing x coincides with O. By varying x we deduce that all members of Θ are projectively equivalent.

Note that the projections ρ_3 and ρ_4 are in a certain sense "opposite". Indeed, we can choose $W_4 = T(x)$ and then the kernel of one projection is the image of the other. Let ζ be a members of Θ containing x. Put $z = \zeta \cap \xi$. Then $\zeta \cap T(x)$ equals $T_x(X(\zeta))$, and hence, since $\langle z, T_x(X(\zeta))\rangle = \zeta$

and $z \in \xi$, the image under ρ_4 of ζ coincide with $\zeta \cap T_x(X(\zeta))$. But, with the above notation, the latter plane is spanned by x and L_z. It follows that the spread $\mathcal{S}$ is projectively equivalent with the set of lines arising from the planes $T_x(O^*)$, with O^* ranging through the set of members of Θ containing x, by projecting from x inside $T(x)$. Consequently,

Remark 3.3 *The tangent planes at x cover the whole 4-dimensional tangent space $T(x)$.*

Now we return to the André representation above. Considering a plane π in W_4 intersecting Σ in a line not belonging to the spread $\mathcal{S}$, we obtain a subplane $\mathcal{P}'$ of $\mathcal{P}$. Taking inverse images with respect to ρ_4, we see that $\mathcal{P}'$ lives in a 6-dimensional space $\pi^{\rho_4^{-1}}$ containing O and that the lines of $\mathcal{P}'$ not contained in ξ correspond to plane sections of members of Θ (and we call such plane sections *ovals*). Since $\mathcal{P}'$ is determined by two such ovals, and since an oval always projects from one of its points b onto a line (modulo the point b), we can obtain the same subplane $\mathcal{P}'$ by projecting from another elliptic space containing a line of $\mathcal{P}$. Intersecting now the two 6-spaces thus obtained, we see that $\mathcal{P}'$ is contained in a 5-space Ω, and the lines of $\mathcal{P}'$ are plane ovals. Moreover, $\mathcal{P}'$ generates Ω (indeed, the 6-space $\pi^{\rho_4^{-1}}$ is generated by $\mathcal{P}'$ and ξ, hence by $\mathcal{P}'$ and one further point of O; so $\langle \mathcal{P}' \rangle$ has codimension at most one in $\pi^{\rho_4^{-1}}$). Consequently the plane $\mathcal{P}'$ has a Veronesean embedding in Ω, which implies that $\mathbb{K}$ is commutative and that all ovals above are conics.

It is now easy to see that all nontrivial plane sections of members of Θ (a plane section is nontrivial if it contains at least two points of X) using planes contained in elliptic spaces are conics. We claim that this, in turn, implies that all ovoids of X are elliptic quadrics. Indeed, consider the ovoid O and consider two plane section C_1 and C_2 intersecting in two distinct points x_1 and x_2. Put $\pi_i = \langle C_i \rangle$, $i = 1, 2$. Let $L_{i,j}$ be the tangent to C_i in π_i at the point x_j, $i, j \in \{1, 2\}$. Put $z_i = L_{i,1} \cap L_{i,2}$, $i = 1, 2$, and set $\alpha_j = \langle L_{1,j}, L_{2,j} \rangle$, $j = 1, 2$. Let y be some point of O not contained in $C_1 \cup C_2$. We claim that O is the unique ovoid of ξ containing C_1, C_2 and y and such that $(*)$ every plane intersecting S in at least two points intersects S in a plane conic. Indeed, it suffices to show that the intersection of O with an arbitrary line through y is determined by C_1, C_2, y, the property $(*)$ and the definition of ovoid. Let L be any line through y and suppose first that L is not incident with either z_1 or z_2, and that it does not meet both $L_{1,1}$ and $L_{2,2}$, and neither both $L_{1,2}$ and $L_{2,1}$ (and let us call the lines through y not satisfying these conditions exceptional). Let y_i be the intersection of L with π_i, $i = 1, 2$. The conditions on L imply that for some $i \in \{1, 2\}$ none of the lines y_1x_i and y_2x_i is tangent to C_1 and C_2, respectively. Hence they intersect C_1 and C_2, respectively, in the point x_i and two further points u_1 and u_2, respectively. Hence in the plane $\alpha := \langle L, x_i \rangle$ we already count four points of O. Hence $\alpha \cap O$ is a conic C. The tangent to C at x_i is the line obtained by intersecting α with $\langle L_{1,i}, L_{2,i} \rangle$. Hence C is uniquely determined and so is the intersection of O with L.

Now let L be one of the four exceptional lines through y. Since this is a finite number, the tangent plane to O at y can be identified with the points we already traced. So we can consider a plane distinct from that tangent plane, containing L. It contains infinitely many points of O, and hence the corresponding conic section is uniquely determined. Our claim is proved.

Note that in the previous paragraph, the only defining property of "ovoid" that we used was the fact the planes generated by $L_{1,1}, L_{2,1}$ and by $L_{1,2}, L_{2,2}$ intersect O in just x_1 and x_2, respectively. In fact, it suffices to assume, for the above reasoning to work, that this intersection is a degenerate conic.

Now any quadric through C_1, C_2, y satisfies the above conditions, namely, every plane intersects the quadric in a conic, and the planes generated by $L_{1,1}, L_{2,1}$ and by $L_{1,2}, L_{2,2}$ intersect the quadric in a degenerate conic. Hence if we show that there is a unique such quadric, then O must necessarily be an elliptic quadric.

We introduce coordinates. We may assume that C_1 contains the points $x_1 := (1,0,0,0)$, $x_2 := (0,0,1,0)$ and $(1,1,1,0)$ and that z_1 has coordinates $(0,1,0,0)$. Also, C_2 can be assumed to contain, besides x_1 and x_2, the point $(1,0,1,1)$ and z_2 can be chosen as $(0,0,0,1)$. Then C_1 has equations

$$\left\{ \begin{array}{rcl} X_1^2 & = & X_0X_2, \\ X_3 & = & 0, \end{array} \right.$$

whereas C_2 has equations

$$\left\{ \begin{array}{rcl} X_3^2 & = & X_0X_2, \\ X_1 & = & 0. \end{array} \right.$$

Hence the equation of a generic quadric through C_1 and C_2, distinct from the union of the two planes π_1 and π_2, is $X_1^2 + kX_1X_3 + X_3^2 = X_0X_2$, with $k \in \mathbb{K}$. Since y is not contained in $\pi_1 \cup \pi_2$, there is a unique $k \in \mathbb{K}$ such that the coordinates of y satisfy the above equation.

So we have shown that O is an elliptic quadric. Since the rest of the proof of Theorem 4.1 in [1] holds for arbitrary fields $\mathbb{K}$ we conclude that, if $\mathbb{L}$ is the quadratic extension of $\mathbb{K}$ with the roots of $x^2 + kx + 1$, then $\mathcal{P}$ is isomorphic to $\mathsf{PG}(2, \mathbb{L})$. □

3.3 The general case

In Section 5 of [1] in the induction argument from index 2 to index n the only point where the finiteness assumption is used is the starting point of the induction which we proved here in Theorem 3.2 and the fact that the tangent planes at x cover the whole 4-dimensional tangent space $T(x)$, which we proved for the general case in Remark 3.3. Finally, we exclude the possibility of $\mathcal{P}$ being infinite-dimensional. By Remark 3.3 we can proceed as in [1] to prove that in a cap of index r the tangent space at a point has dimension $2r$, the general version of Proposition 3.1 of [1]. This immediately shows that $\mathcal{P}$ is finite-dimensional.

Hence Theorem 2.4 is proved.

4 An application of Hermitian Veronesean caps

We now show Theorem 2.5. We start with the case of index 2. Recall that we assume that $|\mathbb{K}| > 2$. At a crucial point in the proof, it will become clear why this theorem does not hold for $|\mathbb{K}| = 2$ and a counterexample naturally pops up.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a projective plane. Let $\mathsf{PG}(d, \mathbb{K})$ be a d-dimensional projective space over the skew field $\mathbb{K}$, with $d \geq 8$. A projective space of dimension l in a projective space of dimension d has *codimension* $d - l$. Suppose that $\mathcal{P}$ is a spanning subset of the point set of $\mathsf{PG}(d, \mathbb{K})$ and that every element of $\mathcal{L}$ induces an ovoid in some solid of $\mathsf{PG}(d, \mathbb{K})$. In the sequel, we identify a line of $\mathcal{S}$ with the set of points incident with it. Our aim is to prove that $\mathbb{K}$ is commutative and that there is a quadratic extension $\mathbb{L}$ of $\mathbb{K}$ such that $\mathcal{S} \cong \mathsf{PG}(2, \mathbb{L})$ and $\mathcal{P}$ is the Hermitian Veronesean cap of $\mathsf{PG}(2, \mathbb{L})$.

In order to do so, we need to show that two lines of $\mathcal{S}$ generate a six-dimensional subspace of $\mathsf{PG}(d, \mathbb{K})$, and that the tangent planes $T_x(O)$ at a fixed point x to a variable ovoid corresponding with a line of $\mathcal{S}$ containing x, are contained in a four-space. We first need some preliminary lemma's.

Lemma 4.1 *Every triangle of lines of $\mathcal{S}$ generates a subspace of codimension at most* 1.

Proof Let L_1, L_2, L_3 be such a triangle of lines. Let x_{ij} be the intersection of L_i with L_j (this implies $x_{ij} = x_{ji}$), for $i, j \in \{1, 2, 3\}$, $i \neq j$. Let x be a point of $\mathcal{S}$ not contained in $\langle L_1, L_2, L_3 \rangle$. Let L be any line of $\mathcal{S}$ through x. If L is not incident with x_{12}, x_{23}, x_{31}, then L meets $L_1 \cup L_2 \cup L_3$ in three non-collinear points, say y_1, y_2, y_3. Since x is not contained in $\langle L_1, L_2, L_3 \rangle$, we see that $\langle L \rangle = \langle y_1, y_2, y_3, x \rangle \subseteq \langle L_1, L_2, L_3, x \rangle$.

Since $|\mathbb{K}| > 2$, it is clear that the number of points of any line M not contained in a given subspace is either 0 or exceeds 5. But from the foregoing it follows that every line of $\mathcal{S}$ not through x has at most 3 points not in $\langle L_1, L_2, L_3, x \rangle$, namely the ones potentially lying on lines through x and x_{12}, x_{13} or x_{31}. Hence every such line is contained in $\langle L_1, L_2, L_3, x \rangle$. This implies that $\mathsf{PG}(d, \mathbb{K})$ coincides with $\langle L_1, L_2, L_3, x \rangle$ and so $\langle L_1, L_2, L_3 \rangle$ has codimension 0 or 1 in $\mathsf{PG}(d, \mathbb{K})$. $\square$

Remark 4.2 If $|\mathbb{K}| = 2$ if follows from the proof of the foregoing lemma that at most six points of $\mathcal{S}$ are not contained in $\langle L_1, L_2, L_3, x \rangle$. Indeed, the only possible ones lie on the three lines through x and x_{12}, x_{13} and x_{23} respectively. Each such line contains five points and intersects $\langle L_1, L_2, L_3 \rangle$ in at least two points. In case this maximum is attained these six points must form a hyperoval in $\mathcal{S}$. This situation really occurs. Indeed, let $|\mathbb{K}| = 2$. There is an example in dimension 10, and it is the largest one. To see this, we note that the points of every line add up to zero. Hence we can try to find the universal embedding by relating to each point of $\mathsf{PG}(2, 4)$ a generator of an elementary abelian 2-group, and then factor out the subgroup generated by the sums of points corresponding to the lines. This subgroup is exactly the code of $\mathsf{PG}(2,4)$ generated by the lines, which has dimension 10, see page 722 of [2]. Hence we obtain an eleven-dimensional vector space over $\mathsf{GF}(2)$, and hence this "universal" example lives in 10-dimensional projective space. Now, with previous notation, the subspace $\langle L_1, L_2, L_3, x\rangle$ is at most 9-dimensional, and it easily follows from the arguments above that it is at most 1-codimensional. Consequently, $\langle L_1, L_2, L_3, x\rangle$ has dimension 9 and the points of $\mathcal{S}$ not in $\langle L_1, L_2, L_3, x\rangle$ form a hyperoval.

We now continue with the general case. The spans of two lines of $\mathcal{S}$ cannot meet in a plane, since otherwise the span of these two lines and any third line not through the intersection point would be contained in a subspace of dimension $4 + 2 = 6$, contradicting Lemma 4.1. Also, if $d = 9$, then the spans of two lines of $\mathcal{S}$ cannot meet in a line by a similar argument. Finally, d cannot be larger than 9, as a triangle of lines cannot generate a subspace of dimension > 8.

We now analyze the case where a point x_0 is contained in a subspace $\langle L_0\rangle$, where L_0 is a line of $\mathcal{S}$ not incident with x_0 in $\mathcal{S}$. By the previous paragraph this can only happen if $d = 8$. So in the next proof, we may restrict to $d = 8$.

Lemma 4.3 *If $x_0 \notin \mathcal{P}\backslash L_0$, then $x_0 \notin \langle L_0\rangle$.*

Proof Let x_0 and L_0 be as above and suppose by way of contradiction that $x_0 \in \langle L_0\rangle$. Let $\tilde{L}$ be any line of $\mathcal{S}$ not through x_0. Let x be the intersection of $\tilde{L}$ with L_0. Let L' be any line of $\mathcal{S}$ through x_0 but not incident with x. Then $\langle L'\rangle$ shares at least one plane with $\langle \tilde{L}, L_0\rangle$, and so the dimension of $\langle L_0, \tilde{L}, L'\rangle$ is at most one more than the dimension of $\langle L_0, \tilde{L}\rangle$. By Lemma 4.1, it follows that the dimension of $\langle L_0, \tilde{L}\rangle$ is at least six, and hence equal to six. Suppose there is another point $y_0 \in \mathcal{P}$ contained in $\langle L_0\rangle$ not incident with L_0. Then the line L'' through x_0 and y_0 passes through x by the previous argument. But then $\langle \tilde{L}, L', L''\rangle$ is a six-dimensional space, contradicting Lemma 4.1. Hence it follows that x_0 is the only element of $\mathcal{P}$ in $\langle L_0\rangle \setminus L_0$.

Consider the projection of $\mathcal{P} \setminus (L_0 \cup \{x_0\})$ from $\langle L_0\rangle$ onto some 4-dimensional space U skew to $\langle L_0\rangle$. The points of every line L not through x_0 not in L_0 are mapped onto an affine plane α_L of U. The points of every line M through x_0 different from x_0 and from $x_M := M \cap L_0$ are mapped onto the points of a line λ_M, and the point fibers are plane ovals through x_0 and x_M (with x_0 and x_M themselves omitted).

Now we fix a line L not through x_0 and distinct from L_0, and a line M through x_0. We choose L and M such that L, M and L_0 are not concurrent. Then α_L and λ_M meet in a point z_M (they share the projection of the intersection $L \cap M$, but not more as otherwise L_0, L, M would be contained in a 6-dimensional space). Hence $\langle \alpha_L, \lambda_M\rangle$ is a solid Σ. Now let M' be any line of $\mathcal{S}$ through x_0. We claim that $\lambda_{M'}$ is contained in Σ.

First we assume that L_0, L, M' are not concurrent. Then α_L and $\lambda_{M'}$ share a unique point $z_{M'}$. We choose two arbitrary points u, v in α_L with the only restriction that the line $\langle u, v\rangle$ of U

does not contain any of the points $z_M, z_{M'}$ (this is possible). Let u_L be the unique point of L mapping down to u. Then we can choose a point $\tilde{v} \in \mathcal{P}$ in the fiber of v such that $\tilde{v}$ is not on L, and such that the line $L' := u_L\tilde{v}$ of $\mathcal{S}$ contains neither x_M nor $x_{M'}$. Then $\alpha_{L'}$ contains both u and v and intersects λ_M in the projection of $L' \cap M$. Hence $\alpha_{L'}$ belongs to Σ. Now $\lambda_{M'}$ shares the projection of $M' \cap L'$ with $\alpha_{L'}$ (which is different from $z_{M'}$ as otherwise $\alpha_{L'} = \alpha_L$, which would imply that L_0, L, L' are contained in a 6-dimensional space), and it also contains $z_{M'}$; hence it is entirely contained in Σ.

Now we assume that M' is incident with $L_0 \cap L$. We may replace L with a line L' as introduced in the previous paragraph, and the claim is proved.

This claim immediately implies that the entire projection is contained in Σ, and so $\mathcal{P}$ is contained in a 7-dimensional space, a contradiction. This completes the proof. $\square$

The next lemma is trivial if $d = 9$ by Lemma 4.1. Hence we may assume $d = 8$.

Lemma 4.4 *The space generated by two lines of $\mathcal{S}$ is 6-dimensional, so* (H2) *holds.*

Proof Suppose, by way of contradiction, that two lines L_0, M generate a 5-space (it is impossible that $\langle L_0, M\rangle$ is 4-dimensional, as this would imply that $\langle L_0, M, N\rangle$ is at most 6-dimensional, for every line N not incident with $L \cap M$). We again consider the projection from $\langle L_0\rangle$ of $\mathcal{P} \setminus L_0$ onto a suitable 4-dimensional subspace U skew to $\langle L_0\rangle$. Now, $M \setminus \{L_0 \cap M\}$ is projected onto an affine line λ_M and the fibers are pointed plane ovals.

Let L be any line of $\mathcal{S}$ not incident with $L_0 \cap M$. If the projection of $L \setminus \{L_0 \cap L\}$ were a line λ_L, then, since λ_L and λ_M have a point in common (namely, the projection of $L \cap M$), the lines L_0, L, M would be contained in a 6-space, a contradiction. Hence the projection of $L \setminus \{L_0 \cap L\}$ is an affine plane α_L. So the projection is injective on lines not incident with $L_0 \cap M$.

We now claim that every line M' of $\mathcal{S}$ incident with $L_0 \cap M$ is projected onto an affine line (where we do not project $L_0 \cap M'$ of course). Indeed, let M' be such a line and let x' be the intersection of M' with L. Also, let x be the intersection of M with L. Let u and u' be the projection of x and x', respectively. We choose a point w on the line $\langle u, u'\rangle$ inside α_L (this is possible since we assume that there are at least 4 points on a line in the projective space $\mathsf{PG}(8, \mathbb{K})$). Let y be a point in the fiber of u distinct from x. Let z be the point of L projected onto w. Then the line $yz \in \mathcal{L}$ does not contain $L_0 \cap M$ and is hence projected onto an affine plane α. This plane α contains u and w and hence intersects α_L in either an affine line or an affine line minus a point.

If u' is in the intersection of α and α_L, then at least two points are projected onto u'. By this non-injectivity the join in $\mathcal{S}$ of these two points must be a line through $L_0 \cap M$. If this line M'' were not M' then the space $\langle L_0, M', M''\rangle$ would be too small, contradicting Lemma 4.1.

So we may assume that u' is not in α. Let p be the unique point on $\langle u, u'\rangle$ not in α_L. Since u' is not in α it follows that p belongs to α. We can now vary the point y, and by the foregoing, we may assume that the point p belongs each time to the corresponding affine plane. Hence the fiber of the point p contains at least two elements, say q and q'. The line $qq' \in \mathcal{L}$ must be incident with $L_0 \cap M$ and projects onto a(n affine) line. But this affine line must have a point in common with α_L. This implies that this affine line is contained in $\langle \alpha_L \rangle$, which implies that L_0, L, qq' is contained in a 6-space, a contradiction by Lemma 4.1.

But now, similarly as in the previous lemma, one shows that the projection of $\mathcal{P} \setminus L_0$ is contained in the 3-space generated by λ_M and α_L. This contradiction proves that $\langle L_0, M \rangle$ is 6-dimensional. $\square$

From now on, we assume $d \in \{8, 9\}$ again.

Next we construct quadratic subveroneseans. Let C_1 and C_2 be two plane ovals contained in the lines L_1 and L_2, respectively, of $\mathcal{S}$, and suppose that $C_1 \cap C_2 = \{x\}$ is a point of $\mathcal{S}$. Let L be a line of $\mathcal{S}$ incident with a point x_i of C_i, for $i = 1, 2$, and with $x_1 \neq x \neq x_2$. We project $\mathcal{P} \setminus L$ from $\langle L \rangle$ onto a $(d-4)$-dimensional subspace U skew to $\langle L \rangle$. By Lemma 4.4, this projection is injective. The points of every line M of $\mathcal{S}$ (except for the point $M \cap L$) are mapped onto the points of an affine plane α_M (bijectively). Let p be a point of $\langle \alpha_M \rangle \setminus \alpha_M$. We claim that

$(*)$ p is not the projection of any element of $\mathcal{P} \setminus L$.

Indeed, assume by way of contradiction that $q \in \mathcal{P}$ is mapped onto p. Take an arbitrary point y of M. Then the projection of the line yq of $\mathcal{S}$ intersects α_M in at least two points, contradicting injectivity of the projection operator on $\mathcal{P} \setminus L$.

Now it is clear that $C_1 \cup C_2$ is mapped onto the union of two intersecting affine lines, say $\lambda_1 \cup \lambda_2$, with λ_i the projection of C_i, $i = 1, 2$. Let π be the plane of $\mathsf{PG}(d, \mathbb{K})$ generated by $\lambda_1 \cup \lambda_2$ and let p_i^∞ be the unique non-affine point of the projective extension of λ_i, $i = 1, 2$. Let z_i be an arbitrary point of $C_i \setminus (L \cup \{x\})$, $i = 1, 2$. Then the projection of the oval on $z_1 z_2$ determined by z_1, z_2 and $z_1 z_2 \cap L$ is projected onto an affine line $\lambda_{z_1 z_2}$ in π. Let z^∞ be the unique point on $\langle \lambda_{z_1 z_2} \rangle$ not in $\lambda_{z_1 z_2}$. If $z^\infty \notin \langle p_1^\infty, p_2^\infty \rangle$, then we choose distinct z_i' and z_i'' on C_i, $i = 1, 2$, such that, with corresponding notation, z^∞ belongs to $\langle \lambda_{z_1' z_2'} \rangle$ and to $\langle \lambda_{z_1'' z_2''} \rangle$. By the claim $(*)$, this implies $z^\infty = z'^\infty = z''^\infty$. Hence $\langle p_1^\infty, p_2^\infty \rangle \cap (\lambda_{z_1' z_2'} \cup \lambda_{z_1'' z_2''})$ contains at least two elements, and this contradicts $(*)$. Hence the points in π that are a projection of points of $\mathcal{S}$ not in L form an affine plane α. Each affine line of α corresponds with a plane oval, and all these form an affine subplane $\mathcal{A}$ of $\mathcal{S}$.

Lemma 4.5 *The abstract projective closure of $\mathcal{A}$ is a projective subplane $\mathcal{S}'$ of $\mathcal{S}$, which forms a quadratic Veronesean.*

Proof For this, we have to show that three parallel affine lines of $\mathcal{A}$ meet in a point of $\mathcal{S}$, and that all points thus obtained are collinear in $\mathcal{S}$. We start by showing that, if D_1 and D_2 are two distinct ovals corresponding with two parallel affine lines of α, then $D_1 \cap D_2$ is nonempty and is contained in L. Indeed, let K_i be the line of $\mathcal{S}$ containing D_i, $i = 1, 2$. Let t be the intersection point of K_1 and K_2. If $t \notin L$, then the projections of K_1 and K_2 would share an affine line, a contradiction. Hence $t \in L$, and since D_1 and D_2 both contain a point of L (as they are projected onto affine lines), we must necessarily have $t \in D_1 \cap D_2$. Since t is uniquely determined by any of D_1 or D_2 only, we automatically have that three lines of $\mathcal{S}$ which induce parallel lines in $\mathcal{A}$ meet in a point of $\mathcal{S}$. Since all these points lie on L, our claim is proved.

Hence we see that $\mathcal{S}'$ is a subplane of $\mathcal{S}$ contained in a 6-dimensional subspace W of $\mathsf{PG}(d, \mathbb{K})$. However, if we consider a different line L, then we obtain the same subplane contained in a different 6-dimensonal space W' (indeed different since it will now not contain all points of L but only an oval). Hence it is now easy to see that $\mathcal{S}'$ spans a 5-dimensional subspace V. Consequently, $\mathcal{S}'$ forms a quadratic Veronesean. $\square$

Lemma 4.6 *Condition* (H3) *holds.*

Proof Now let $x \in \mathcal{P}$ be an arbitrary point and let L_1, L_2 be two distinct lines of $\mathcal{S}$ through x. Then the tangent planes at x in $\langle L_i \rangle$, $i = 1, 2$, to L_i together span a 4-space Ξ. We show that, if L is an arbitrary line of $\mathcal{S}$ through x, then the tangent plane at x to L in $\langle L \rangle$ is contained in Ξ. This will follow if we show that an arbitrary tangent line T at x to L in $\langle L \rangle$ is contained in Ξ. Therefore, let C be an arbitrary oval on L through x with T tangent to C at x, and let L' be any line of $\mathcal{S}$ not through x but containing a point y of C. Let C' be the oval on L' containing y and a point of each of L_1 and L_2. Then C, C' are contained in a unique subplane inducing a quadratic subveronesean $\mathcal{V}$ on $\mathcal{S}$, as shown in the previous paragraphs. Note that $\mathcal{V}$ contains a conic lying on L_1 and one on L_2. The line T is contained in the plane spanned by the tangent lines at x to the conics of $\mathcal{V}$ on L_1 and L_2, and hence T is contained in Ξ. $\square$

This completes the proof of the fact that $\mathcal{S}$ is a Hermitian cap; in particular $d = 8$ and the case $d = 9$ cannot occur after all. Indeed, in this case, one can give a short proof for it. Suppose $d = 9$, consider a line L and a point p not on L and project $\mathcal{P} \setminus L$ from $\langle L \rangle$ onto a 5-dimensional space skew to $\langle L \rangle$. Then, since we also assume that X has the structure of a projective plane, it follows that $\mathcal{P} \setminus L$ is projected into the 4-dimensional space which is the projection of the tangent space through p. Hence $d = 8$.

Now, as for the general case, similarly to the finite case, it suffices to show that, if we consider a subset Y of X corresponding to the point set of a plane of $\mathsf{PG}(n, \mathbb{L})$, then Y generates a subspace of dimension 8. But the proof of the finite case, see Section 4 of [7], applies verbatim.

5 Another application

We now define the following object. Let $\mathbb{L}$ be a quadratic extension of the field $\mathbb{K}$. Let V be a vector space of dimension n over $\mathbb{L}$. Consider V as a vector space W of dimension $2n$ over $\mathbb{K}$, and let $\mathfrak{L}$ be the set of 2-dimensional subspaces of W arising from the vector lines of V. Then consider the line Grassmannian in $\mathsf{PG}(W)$. The image $\mathcal{G}(\mathfrak{L})$ of $\mathfrak{L}$ is precisely the Hermitian Veronesean variety of index $n-1$.

Indeed, there exists $d \in \mathbb{K}$ such that the equation $x^2+x+d=0$ has no solution in $\mathbb{K}$ and two solutions in $\mathbb{L}$. Note that $\mathfrak{L}$ has the natural structure of a projective space $\mathfrak{P}$ isomorphic to $\mathsf{PG}(V)$. A line of $\mathfrak{P}$ corresponds to the set of members of $\mathfrak{L}$ arising from vector lines of V contained in a vector plane π. This vector plane over $\mathbb{L}$ becomes a 4-space over $\mathbb{K}$. We now coordinatize the situation as follows. Each element x of $\mathbb{L}$ can be written as a couple (x_1, x_2), where $x = x_1 + ix_2$, with $i^2+i+d=0$. Note that $i(x_1+ix_2) = -dx_2 + i(x_1-x_2)$. Then a vector of V with coordinates $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ can be given the coordinates $(x_1^{(1)}, x_2^{(1)}, , x_1^{(2)}, x_2^{(2)}, , \ldots, x_1^{(n)}, x_2^{(n)})$ in W. For π we can now take the set of vectors with coordinates $(x, y, 0, 0, \ldots, 0)$, $x, y \in \mathbb{L}$, and the vector plane in W corresponding to the above vector $(x, y, 0, \ldots, 0)$ is spanned by the vectors $(x_1, x_2, y_1, y_2, \ldots, 0)$ and $(-dx_2, x_1-x_2, -dy_2, y_1-y_2, 0, \ldots, 0)$. The image of this vector plane under the line Grassmannian in $\mathsf{PG}(W)$ is the point with coordinates

$$\begin{array}{ll} p_{01} = x_1^2 - x_1x_2 + dx_2^2, & p_{12} = -x_1y_1 + x_2y_1 - dx_2y_2, \\ p_{02} = -dx_1y_2 + dx_2y_1, & p_{13} = -x_1y_2 + x_2y_1, \\ p_{03} = x_1y_1 - x_1y_2 + dx_2y_2, & p_{23} = y_1^2 - y_1y_2 + dy_2^2. \end{array}$$

We observe that $p_{02} = dp_{13}$ and $p_{13} = p_{12} + p_{03}$, hence we may concentrate on the coordinates $p_{01}, p_{12}, p_{13}, p_{23}$. One easily verifies that these satisfy the equation $p_{01}p_{23} = p_{12}^2 - p_{12}p_{13} + dp_{13}^2$ and hence all these points are contained in an elliptic quadric Q. Conversely, suppose the four field elements $a, b, c, e \in \mathbb{K}$ satisfy $ab = c^2 - ce + de^2$ (where we think of a, b, c, e as potential values for $p_{23}, p_{01}, p_{12}, p_{13}$, respectively), then $(a, b) \neq (0, 0)$. Suppose without loss of generality that $a \neq 0$, then we may even assume $a = 1$. Put $y_1 = 1$, $y_0 = 0$, $x_1 = e - c$ and $x_2 = e$. Then $b = c^2 - ce + de^2 = x_1^2 - x_1x_2 + dx_2^2$, and so the corresponding point on Q belongs to $\mathcal{G}(\mathfrak{L})$. It follows that the lines of $\mathfrak{P}$ are elliptic quadrics in solids of $\mathsf{PG}(W)$.

As clearly every point of $\mathcal{G}(\mathfrak{L})$ contained in $\langle Q \rangle$ belongs to Q (since every other point has to involve an additional coordinate p_{ij}, with $\{i, j\} \not\subseteq \{0, 1, 2, 3\}$), left to show is the fact that the dimension of he span of $\mathcal{G}(\mathfrak{L})$ is at least $(n-1)(n+1)$. We show this by induction on n. For $n = 2$, this is proved above, as Q spans a 3-space.

Now suppose that $n > 2$. Consider a basis of size n in V and let V' be the subspace generated by the first $n-1$ basis vectors. Let W' be the corresponding subspace of W, let $\mathfrak{L}'$ be the corresponding subset of $\mathfrak{L}$ and let $\mathcal{G}(\mathfrak{L}')$ be the corresponding image on the line Grassmannian. The the induction hypothesis implies that $\langle \mathcal{G}(\mathfrak{L}') \rangle$ has dimension at least $(n-2)n$. Now consider the $2n-1$ points of V with coordinates $(0, 0, \ldots, 0, 1)$, $(1, 0, 0, \ldots, 0, 1)$, $(0, 1, 0, 0, \ldots, 0, 1)$,..., $(0, 0, \ldots, 0, 1, 1)$, $(i, 0, 0, \ldots, 0, 1)$, $(0, i, 0, 0, \ldots, 0, 1)$,..., $(0, 0, \ldots, 0, i, 1)$. Then it is a routine exercise to verify that the corresponding points in $\mathcal{G}(\mathfrak{L})$ generate a $(2n-2)$-dimensional projective subspace skew to $\langle \mathcal{G}(\mathfrak{L}')$ (this follows from the fact that the first point gives rise to the Grassmann coordinate $p_{2n-1,2n}$, the next $n-1$ point additionally to $p_{2k-1,2n} - p_{2k,2n-1}$, $1 \leq k < n$, and the corresponding points on the Grassmannian of the last $n-1$ vectors involve $p_{2k,2n} + p_{2k,2n-1} + p_{2k-1,2n-1}$, $1 \leq k < n$).

Hence we checked (H1*), (H2*) and (H3*) and we are done.

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