

A characterisation result on a particular class of non-weighted minihypers

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Abstract

We present a characterisation of $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$ -minihypers, q square, $q = p^h$, $p > 3$ prime, $h \geq 1$, $q \geq 1217$, for $\epsilon_0 + \epsilon_1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. This improves a characterisation result of S. Ferret and L. Storme [?], involving more Baer subgeometries contained in the minihyper.

Keywords: minihypers, Griesmer bound, blocking sets

1 Introduction

Let $\text{PG}(n, q)$ be the n -dimensional projective space over the finite field \mathbb{F}_q of order q . A *weight function* w of $\text{PG}(n, q)$ is a mapping from the point set of $\text{PG}(n, q)$ to the set of non-negative integers. For a point P , the integer $w(P)$ is called the *weight* of the point P , and for a set M of points, its *weight* is the sum of the weights of its points. The sum of the weights of all points of $\text{PG}(n, q)$ is the *total weight* of w .

Definition 1.1 *An $\{f, m; n, q\}$ -minihyper, $f \geq 1$, $n \geq 2$, is a pair (F, w) , where F is a set of points of $\text{PG}(n, q)$, w is a weight function of $\text{PG}(n, q)$, and*

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$P \in F \iff w(P) > 0$; the total weight of w is f , and m is the minimum weight of the hyperplanes of $\text{PG}(N, q)$, i.e.

$$\sum_{P \in \pi} w(P) \geq m,$$

for any hyperplane π of $\text{PG}(n, q)$, with equality for at least one hyperplane of $\text{PG}(n, q)$.

Of course, the set F is determined by the weight function w . When the range of w is $\{0, 1\}$, the converse is true and then the minihyper is identified with F and is called a *non-weighted minihyper*. Thus, a non-weighted $\{f, m; n, q\}$ -minihyper is a set F of f points of $\text{PG}(n, q)$ such that every hyperplane contains at least m points of F . This is the definition of a minihyper given by Hamada and Tamari in [?] and it was generalised to the definition of a weighted minihyper in [?]. (Weighted) minihypers can also be called (*weighted*) *multiple blocking sets*. In the sequel, we will always clearly distinguish between weighted and non-weighted minihypers and use the term *minihyper* to denote a non-weighted minihyper.

As (weighted) minihypers are (weighted) multiple blocking sets with respect to the hyperplanes of a projective space, the study of these objects fits completely in the rich literature on the study of blocking sets and generalisations. But (weighted) minihypers are also a geometric interpretation of linear codes meeting the Griesmer bound. This connection is described in detail in several references, such as [?, ?].

Denote the number of points in an i -dimensional projective space $\text{PG}(i, q)$ by θ_i , i.e. $\theta_i = \frac{q^{i+1}-1}{q-1}$. The following characterisation theorem was shown by Hamada, Hellesteth and Maekawa.

Theorem 1.2 ([?, ?]) *A $\{\sum_{i=0}^s \epsilon_i \theta_i, \sum_{i=0}^s \epsilon_i \theta_{i-1}; n, q\}$ -minihyper is the union of pairwise disjoint ϵ_i projective subspaces of dimension i , for $i = 0, \dots, s$, if $\sum_{i=0}^s \epsilon_i = h < \sqrt{q} + 1$.*

In [?], Ferret and Storme proved that increasing h to $2\sqrt{q} - 1$ allows one Baer subgeometry in the minihyper.

Theorem 1.3 ([?]) *Let F be a $\{\sum_{i=0}^{k-2} \epsilon_i \theta_i, \sum_{i=0}^{k-2} \epsilon_i \theta_{i-1}; k-1, q\}$ -minihyper, q square, $q = p^h$, h even, p prime, where $\sum_{i=0}^{k-2} \epsilon_i \leq \min\{2\sqrt{q} - 1, c_p q^{5/9}\}$, $c_p = 2^{-1/3}$, $q \geq 2^{14}$, when $p = 2, 3$, and where $\sum_{i=0}^{k-2} \epsilon_i \leq \min\{2\sqrt{q} - 1, q^{6/9}/(1 + q^{1/9})\}$, $q \geq 2^{12}$, when $p > 3$.*

Then F consists of the union of pairwise disjoint

- (1) ϵ_{k-2} spaces $\text{PG}(k-2, q)$, ϵ_{k-3} spaces $\text{PG}(k-3, q)$, \dots , ϵ_0 points, or
- (2) one subgeometry $\text{PG}(2l+1, \sqrt{q})$, for some integer l with $1 \leq l \leq k-2$, ϵ_{k-2} spaces $\text{PG}(k-2, q)$, \dots , ϵ_{l+1} spaces $\text{PG}(l+1, q)$, $\epsilon_l - \sqrt{q} - 1$ spaces $\text{PG}(l, q)$, ϵ_{l-1} spaces $\text{PG}(l-1, q)$, \dots , ϵ_0 points, or
- (3) one subgeometry $\text{PG}(2l, \sqrt{q})$, for some integer l with $1 \leq l \leq k-2$, ϵ_{k-2} spaces $\text{PG}(k-2, q)$, \dots , ϵ_{l+1} spaces $\text{PG}(l+1, q)$, $\epsilon_l - 1$ spaces $\text{PG}(l, q)$, $\epsilon_{l-1} - \sqrt{q}$ spaces $\text{PG}(l-1, q)$, ϵ_{l-2} spaces $\text{PG}(l-2, q)$, \dots , ϵ_0 points.

In this article, in Theorems ?? and ??, we give a characterisation of these minihypers with $h < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ and $s = 1$. In particular, we show that these minihypers will contain subspaces as well as Baer subgeometries.

2 Preliminaries

We explain formally the notion of removing a point P from a weighted minihyper (F, w) . Suppose that (F, w) is a weighted minihyper. With the notation $P \in (F, w)$, we always mean $w(P) > 0$, equivalently $P \in F$.

Suppose that we have two weighted sets (F_1, w_1) and (F_2, w_2) in $\text{PG}(n, q)$, where $w_1(P) \geq w_2(P)$ for all points P of $\text{PG}(n, q)$. Then we can define the new weighted set $(F, w) = (F_1, w_1) - (F_2, w_2)$ defined by the weight function w , with $w : \text{PG}(n, q) \rightarrow \mathbb{N} : P \mapsto w(P) = w_1(P) - w_2(P)$. When the weights $w_2(P)$ of all the points P of $\text{PG}(n, q)$ are equal to zero or one, we simply write this difference as $(F, w) = (F_1, w_1) - F_2$.

For instance, suppose that $P \in (F, w)$ and define $w' : \text{PG}(n, q) \rightarrow \mathbb{N} : w'(R) = w(R)$ for any point $R \in \text{PG}(n, q) \setminus \{P\}$ and $w'(P) = w(P) - 1$. Then w' determines a new set F' and $(F, w) - \{P\}$ is by definition (F', w') . This is the weighted minihyper in which the point P is removed once from (F, w) . It is clear that $F' = F \setminus \{P\}$ when (F, w) is a non-weighted minihyper.

We can easily extend the notion of removing points from (F, w) to removing sets $M \subseteq F$ from (F, w) by defining $w' : \text{PG}(n, q) \rightarrow \mathbb{N} : w'(R) = w(R)$ for any point $R \in \text{PG}(n, q) \setminus M$ and $w'(P) = w(P) - 1$ for $P \in M$.

Removing points or sets from a minihyper (F, w) can, under certain circumstances, yield a minihyper as expected, as is shown in the next lemma.

Lemma 2.1 *Let (F, w) be a weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n, q\}$ -minihyper, $n \geq 3$, with $2\epsilon_1 + \epsilon_0 < q + 2$, containing a line L . Then $(F, w) - L$ is a weighted $\{(\epsilon_1 - 1)(q + 1) + \epsilon_0, \epsilon_1 - 1; n, q\}$ -minihyper.*

Proof. We have to show that $(F, w) - L$ is a weighted $\{(\epsilon_1 - 1)(q + 1) + \epsilon_0, \epsilon_1 - 1; n, q\}$ -minihyper. The essential part is to show that any hyperplane intersects $(F, w) - L$ in at least $\epsilon_1 - 1$ points. So consider the line L and an arbitrary hyperplane π . Since $|\pi \cap (F, w)| \geq \epsilon_1$, it follows immediately that $|\pi \cap ((F, w) - L)| \geq \epsilon_1 - 1$ if π intersects L in exactly one point.

We are left with the case $L \subset \pi$. If $|\pi \cap (F, w)| \geq q + \epsilon_1$, then clearly, $|\pi \cap ((F, w) - L)| \geq \epsilon_1 - 1$. So suppose that $|\pi \cap (F, w)| < q + \epsilon_1$.

Consider an $(n - 3)$ -dimensional space Ω in π skew to F . The minihyper (F, w) is projected from Ω on a weighted $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; 2, q\}$ -minihyper (F', w') . The projection of L is a line L' contained in (F', w') . By [?, Theorem 2.2], we can reduce the weight of every point of L' by one to obtain an $(\epsilon_1 - 1)$ -fold blocking set F'' in this plane. But then L' is still blocked at least $\epsilon_1 - 1$ times. So π is blocked at least $q + \epsilon_1$ times by F . \square

As we may consider minihypers containing no lines, the following lemma will provide information on their size. It is proved in [?] and it is a generalisation of a result from [?].

Lemma 2.2 *A weighted $\{f, t; 2, q\}$ -minihyper (B, w) , with $1 \leq t < q - 1$ and $q \geq 3$, contains a line or satisfies $f \geq tq + \sqrt{tq} + 1$.*

We first of all wish to characterise weighted $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; 3, q\}$ -minihypers, having weighted points with total weight of the weighted points at most $\frac{2\epsilon_1^2}{q}$ and where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, as a sum of A lines, B isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C Baer subgeometries $\text{PG}(3, \sqrt{q})$, where $A + B + C(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points.

With an *isolated* Baer subplane $\text{PG}(2, \sqrt{q})$ contained in F , we mean a Baer subplane $\text{PG}(2, \sqrt{q})$ contained in F , but not contained in a 3-dimensional Baer subgeometry $\text{PG}(3, \sqrt{q})$, completely contained in F .

We will focus on the existence of the isolated Baer subgeometries $\text{PG}(2, \sqrt{q})$ and the Baer subgeometries $\text{PG}(3, \sqrt{q})$ contained in (F, w) . To find Baer subgeometries completely contained in (F, w) , we will use a result of Barát and Storme [?]. This paper contains a lot of results on multiple (weighted) blocking sets in projective spaces, and we state a related result required in this article.

Theorem 2.3 *Let B be an s -fold blocking set in $\text{PG}(2, q)$, $q = p^h$, p prime, $h \geq 1$, having points with multiplicities. Assume that $|B| \leq s(q+1) + c$ where*

1. $h > 1$, $c < c_p q^{2/3}$ and $s < \min(c_p q^{1/6}, q^{1/4}/2)$ where $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$,
2. $q = p^2$, $s < q^{1/4}/2$ and $c < q^{3/4}/2$,

and assume that B has at least $(s-2)(q + \sqrt{q} + 1) + 16\sqrt{q} + 8q^{1/4}$ simple points in (1) and at least $(s-2)(q + \sqrt{q} + 1) + 16\sqrt{q} + 16q^{1/6}$ simple points in (2).

Then B contains the sum of s Baer subplanes and/or lines.

This theorem is proved in exactly the same way as [?, Theorem 3.10], but using two new arguments. First of all, the $t \pmod{p}$ result on small weighted minimal t -fold blocking sets in $\text{PG}(2, q)$, $q = p^h$, p prime, $h \geq 1$ [?, Theorem 2.13], that is now known, has to be used. Secondly, by [?, Theorem 2.2], if a line L is contained in the s -fold blocking set B , then we can reduce the weight of every point of L by one to obtain an $(s-1)$ -fold blocking set F'' in this plane. By using these two new arguments, the upper bound $s(q+1) + c - (s-1)(s-2)/2$ in [?, Theorem 3.10] can be replaced by the upper bound $s(q+1) + c$.

Suppose that (F, w) is a weighted minihyper in $\text{PG}(n, q)$ and consider any subspace π of $\text{PG}(n, q)$. Then $(F, w) \cap \pi$ is the weighted minihyper (F', w') induced in the subspace π , where $F' := F \cap \pi$ and w' is the function w restricted to the points of π . The following theorem provides useful information on intersections of weighted minihypers with subspaces.

Theorem 2.4 ([?, Result 2.9]) *Let (F, w) be a weighted $\{\sum_{i=0}^{n-1} \epsilon_i \theta_i, \sum_{i=0}^{n-1} \epsilon_i \theta_{i-1}; n, q\}$ -minihyper satisfying $n \geq 1$, $\sum_{i=0}^{n-1} \epsilon_i = h \leq q$. Then every r -space π_r , $1 \leq r \leq n$, not contained in (F, w) , intersects (F, w) in a weighted $\{\sum_{i=0}^{r-1} m_i \theta_i, \sum_{i=0}^{r-1} m_i \theta_{i-1}; r, q\}$ -minihyper $(F, w) \cap \pi_r$, satisfying $\sum_{i=0}^{r-1} m_i \leq h$.*

Lemma 2.5 *Let (F, w) be a weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n, q\}$ -minihyper, with $\epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, containing no lines and having at most $q^{1/6}/2$ multiple points. If a plane π intersects (F, w) in a weighted $\{m_1(q+1) + m_0, m_1; 2, q\}$ -minihyper, with $m_1 \geq 1$, then $(F, w) \cap \pi$ contains a sum of m_1 Baer subplanes.*

Proof. By Theorem ??, we know that $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. The intersection of π with F does not contain lines, since F does not contain lines, so $|\pi \cap (F, w)| \geq m_1 q + \sqrt{m_1 q} + 1$ by Lemma ??, which implies $\sqrt{m_1 q} + 1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. Hence, $m_1 < q^{1/6}/2$. By Theorem ??, $\pi \cap (F, w)$ contains a sum of m_1 Baer subplanes. \square

3 Minihypers in three dimensions

To obtain the desired characterisation, we will use an inductive argument on the dimension $n \geq 3$. In this inductive step, we will require a characterisation of weighted minihypers in three dimensions, which we will obtain in this section.

The following theorem also plays a crucial role.

Theorem 3.1 ([?, Theorem 3.1]) *Let B be a t -fold blocking set in $\text{PG}(n, q)$, $q = p^h$, p prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq + c_p q^{2/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$. Then B contains a union of t pairwise disjoint lines and/or Baer subplanes.*

We assume that (F, w) is a weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 3, q\}$ -minihyper, $q = p^h$, $p > 3$ prime, $h \geq 1$, $q \geq 1217$, where the total weight of the multiple points is at most $\frac{2\epsilon_1^2}{q}$ and with $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, so $\eta < \frac{q^{1/12}}{2}$, and we assume that (F, w) does not contain a line of $\text{PG}(3, q)$. The preceding Theorem ?? characterises these minihypers for $\epsilon_1 < q^{1/6}/2$, so from now on, we assume that $\epsilon_1 \geq q^{1/6}/2$.

Remark 3.2 *As indicated in the preceding paragraph, we assume that $q = p^h$, $p > 3$ prime, $h \geq 1$, and that $q \geq 1217$. The condition $q \geq 1217$ follows from the inequality at the end of the proof of Lemma ??.*

The other inequalities in the proofs are valid for $q \geq 1217$.

In the main theorems, we repeat that $q = p^h$, $p > 3$ prime, $h \geq 1$, and that $q \geq 1217$, to give the correct statements of the theorems.

Projecting the minihyper (F, w) from a point $R \notin F$ onto a plane gives a weighted ϵ_1 -fold blocking set B in this plane. This set B can contain lines, and we will distinguish two cases.

First we assume that B does not contain a line.

Lemma 3.3 *If B does not contain a line, then $\epsilon_1 < \frac{q^{1/6}}{2}$ and (F, w) is an ϵ_1 -fold blocking multiset containing a sum of ϵ_1 Baer subplanes and lines.*

Proof. The set B is a weighted ϵ_1 -fold blocking set in $\text{PG}(2, q)$ of size $\epsilon_1(q+1) + \epsilon_0 < \epsilon_1 q + \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, containing no lines. Lemma ?? implies that $\epsilon_1 < \frac{q^{1/6}}{2}$. In this case, there are no multiple points since $\frac{2\epsilon_1^2}{q} < 1$.

So (F, w) is an ϵ_1 -fold blocking set satisfying the conditions of Theorem ??, and by assumption not containing lines. Using Theorem ??, we conclude that (F, w) contains a union of ϵ_1 pairwise disjoint Baer subplanes. Hence, (F, w) is the sum of ϵ_1 Baer subplanes and points. \square

An upper bound on the number of secants to F through a point R , not in F , will be very useful. The next lemma yields such an upper bound.

Lemma 3.4 *There exists in $\text{PG}(3, q)$ a point not in F lying on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ secants to F , containing at least two points of F of weight one.*

Proof. We count the number of points of $\text{PG}(3, q)$, not contained in F , on secants to F through two points of weight one. Here, $|F| \leq \epsilon_1 q + \eta\sqrt{q}$, but we subtract $\frac{q^{1/6}}{2}$ from $|F|$, since there can be up to $\frac{2\epsilon_1^2}{q} < \frac{q^{1/6}}{2}$ multiple points in (F, w) :

$$\begin{aligned} & (\epsilon_1 q + \eta\sqrt{q} - \frac{q^{1/6}}{2})(\epsilon_1 q + \eta\sqrt{q} - \frac{q^{1/6}}{2} - 1) \frac{(q-1)}{2} \\ \leq & \frac{\epsilon_1^2 q^3 + 2\eta\epsilon_1 q^2 \sqrt{q} + \eta^2 q^2 - \epsilon_1 q^2 - \eta q \sqrt{q} - \epsilon_1^2 q^2 - 2\eta\epsilon_1 q \sqrt{q} - \eta^2 q + \epsilon_1 q + \eta\sqrt{q}}{2} \\ \leq & \frac{\epsilon_1^2 q^3 + 2\eta\epsilon_1 q^2 \sqrt{q} + \eta^2 q^2 - \epsilon_1^2 q^2 - 2\eta\epsilon_1 q \sqrt{q} - \eta^2 q}{2}. \end{aligned}$$

Since we can assume that $\epsilon_1 \geq q^{1/6}/2$, also $\epsilon_1 \geq \eta$, so the upper bound further simplifies to

$$\frac{\epsilon_1^2 q^3 + 2\epsilon_1 \eta q^2 \sqrt{q}}{2} \leq \frac{\epsilon_1^2 q^3 + 2\eta^2 q^3}{2},$$

where the last inequality follows from $\epsilon_1 \leq \eta\sqrt{q}$.

There are $\theta_3 - |F| \geq q^3$ points in $\text{PG}(3, q)$ not contained in F , hence, we find a point R , not in F , lying on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ such secants to F . \square

Lemma 3.5 *If B contains at least one line, then $\sqrt{q} - q^{1/6} \leq \epsilon_1$.*

Proof. Consider a point R of $\text{PG}(3, q) \setminus F$ lying on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ secants to F , containing at least two simple points of F . The minihyper F is projected from R onto a weighted point set in a plane containing a line L . The plane $\langle R, L \rangle$ intersects F in at least a 1-fold blocking set since $|\langle R, L \rangle \cap F| \geq q + 1$ (Theorem ??). So Lemma ?? implies that $\langle R, L \rangle \cap F$ contains a Baer subplane having a Baer subline on a line through R . This Baer subline has at most $\frac{q^{1/6}}{2}$ distinct multiple points of F , so is counted at least $\frac{1}{2}(\sqrt{q} - \frac{q^{1/6}}{2})^2$ times as a secant in the previous lemma. This number must be smaller than or equal to the total number of such secants to F through R , so

$$\begin{aligned} (\sqrt{q} - \frac{q^{1/6}}{2})^2 &\leq \epsilon_1^2 + 2\eta^2 \\ \Leftrightarrow q - \sqrt{q}q^{1/6} + \frac{q^{1/3}}{4} - q^{1/6} &\leq \epsilon_1^2, \quad \text{since } \eta < \frac{q^{1/12}}{2} \\ \Rightarrow (\sqrt{q} - q^{1/6})^2 &\leq q - \sqrt{q}q^{1/6} + \frac{q^{1/3}}{4} - q^{1/6} \leq \epsilon_1^2. \end{aligned}$$

This last equation holds if $q \geq 4$ and then we have the assertion. \square

From now on, we assume that $\epsilon_1 \geq \sqrt{q} - q^{1/6}$.

Since $(\epsilon_1^2 + 2\eta^2)/2 \leq \epsilon_1^2$ and since we will need in Lemma ?? a lot of points R , not in F , lying on a small number of secants to F , we will look for points R , not in F , lying on at most ϵ_1^2 secants to F containing at least two simple points of F .

Lemma 3.6 *Let R be a point of $\text{PG}(3, q) \setminus F$ lying on at most ϵ_1^2 secants to F , containing at least two simple points of F . Then R lies on a line containing a Baer subline of F which is contained in at least $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of F , containing at least $\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1$ points of F .*

Proof. The projection of F from R is a weighted ϵ_1 -fold blocking set B in a plane, containing lines. Let x be the number of lines contained in B , where some lines can be counted more than once in this weighted ϵ_1 -fold blocking set. It follows from Lemma ?? that the x lines contained in B can be removed from B to obtain a new weighted $(\epsilon_1 - x)$ -fold blocking set B' , containing no lines. Denote $\epsilon_1 - x$ by ϵ'_1 . By Lemma ??, for an ϵ'_1 -fold blocking set B of

size $\epsilon'_1(q+1) + \epsilon_0$ without lines, necessarily $\epsilon'_1 < \frac{q^{1/6}}{2}$, so B must contain at least $\epsilon_1 - \epsilon'_1 > \epsilon_1 - \frac{q^{1/6}}{2}$ lines.

For each such line $L \subset B$, let m_1 be its multiplicity as a line in the weighted set B . Then the plane $\langle R, L \rangle$ intersects F in an $\{m_1(q+1) + m_0, m_1; 2, q\}$ -minihyper, with $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ (Theorem ??). This plane $\langle R, L \rangle$ contains m_1 Baer subplanes of F (Lemma ??), and, for each such Baer subplane in $\langle R, L \rangle \cap F$, there is a line through R containing a Baer subline of this Baer subplane.

A Baer subline is counted at least $\frac{1}{2}(\sqrt{q} - \frac{q^{1/6}}{2})^2$ times as a secant in Lemma ???. The point R lies on at most $\epsilon_1^2 \leq \eta^2(q - q^{2/3} + \frac{q^{1/3}}{4})$ secants, hence R lies on at most $2\eta^2$ different lines containing a Baer subline of F . There are at least $\epsilon_1 - \frac{q^{1/6}}{2}$ Baer sublines, in Baer subplanes of F , on lines through R . So some Baer subline lies in at least $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of F . These Baer subplanes of F contain at least $\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1$ points of F . \square

Remark 3.7 *We will denote these Baer subplanes, contained in F , through a common Baer subline on a line through R as flags of Baer subplanes corresponding to R . The next lemma shows that we can find several flags which leads to the fact that they must intersect each other in a certain minimum number of points.*

Lemma 3.8 (1) *There are more than $8\eta^2$ points R of $\text{PG}(3, q) \setminus F$, defining different flags of Baer subplanes.*

(2) *There are two such flags intersecting each other in at least $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points of F .*

Proof. (1) We find the required points one by one. There exists at least one such point (Lemmas ?? and ??).

Suppose that we have already found $8\eta^2$ points R not in F with a corresponding flag of Baer subplanes, as in the previous lemma. Is there another point of $\text{PG}(3, q) \setminus F$ lying on at most ϵ_1^2 secants to F , containing at least two simple points of F ? The number of points in these $8\eta^2$ flags, in the corresponding planes $\text{PG}(2, q)$, is at most

$$8\eta^2\left(\left(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}\right)q^2 + q + 1\right) = 4\epsilon_1 q^2 - 2q^{1/6}q^2 + 8\eta^2(q + 1).$$

We count these points in the corresponding planes $\text{PG}(2, q)$ to assure that the new flag is different from the ones we already have. There are at least $q^3 + q^2 + q + 1 - \epsilon_1(q + 1) - \epsilon_0 - 4\epsilon_1 q^2 + 2q^{1/6}q^2 - 8\eta^2(q + 1)$ points in $\text{PG}(3, q)$ not in F and not in the extended flags. If all these points lie on more than ϵ_1^2 secants to F , then the number of incidences on the remaining secants is larger than $(\epsilon_1^2 q^3 + 2\eta^2 q^3)/2$, the total number of incidences on secants to F we had in Lemma ???. So there is still another point $P \notin F$ on at most ϵ_1^2 secants to F .

(2) Take $8\eta^2$ such points R and suppose that the union of the $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes through the Baer subline of a flag corresponding to a point R share for two such points R and R' at most $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points of F . Then

$$\begin{aligned} |F| &\geq \sum_{i=1}^{8\eta^2} \left(\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1 - (i-1) \frac{q}{16\eta^2} \left(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2} \right) \right) \\ &\geq 8\eta^2 \left(\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1 \right) + \frac{(8\eta^2)^2}{2} \frac{q}{16\eta^2} \left(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2} \right) \\ &\geq 3\epsilon_1 q - \frac{3}{2} q^{7/6} + 8\eta^2(\sqrt{q} + 1). \end{aligned}$$

This is false since $\epsilon_1 \geq \sqrt{q} - q^{1/6}$. □

We have found different points R and R' with a corresponding flag of Baer subplanes. We now build with them a Baer subgeometry $\text{PG}(3, \sqrt{q})$ contained in F .

Lemma 3.9 *The minihyper F contains a Baer subgeometry $\text{PG}(3, \sqrt{q})$ if $\epsilon_1 \geq \sqrt{q} - q^{1/6}$.*

Proof. Let R and R' be two points of $\text{PG}(3, q) \setminus F$ corresponding with a flag of $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of F , where these two flags share at least $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points. Denote those two flags by f_R and $f_{R'}$. Then some Baer subplane $\pi_{R'}$ of $f_{R'}$ shares at least $\frac{q}{16\eta^2}$ points with the Baer subplanes of f_R . If this Baer subplane $\pi_{R'}$ shares at most two points with every Baer subplane of f_R , then $\frac{q}{16\eta^2} \leq 2(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$, which is false since $\epsilon_1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. So this Baer subplane $\pi_{R'}$ shares a Baer subline with some Baer subplane of f_R . Denote by l the Baer subline of the flag f_R . This Baer subplane $\pi_{R'}$

cannot pass through l , since then this Baer subplane $\pi_{R'}$ only shares this subline l with all these Baer subplanes of the flag f_R , but $\frac{q}{16\eta^2} > \sqrt{q} + 1$.

We wish to find a lower bound on the number of Baer subplanes of f_R , sharing a Baer subline with the Baer subplane $\pi_{R'}$. Two distinct Baer sublines share at most two points. We subtract two for every of the $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of f_R from $\frac{q}{16\eta^2}$ and then divide by $\sqrt{q} - 1$. The quotient is at least $\frac{q^{1/2}}{16\eta^2} + \frac{1}{16\eta^2} - \frac{1}{\eta}$, hence this Baer subplane $\pi_{R'}$ shares a Baer subline with at least $\frac{q^{1/2}}{16\eta^2} + \frac{1}{16\eta^2} - \frac{1}{\eta}$ Baer subplanes of f_R . Take this Baer subplane $\pi_{R'}$ and consider a Baer subplane π_R of the flag f_R which shares a Baer subline with $\pi_{R'}$. Together they define a Baer subgeometry Ω isomorphic to $\text{PG}(3, \sqrt{q})$. Every Baer subplane of f_R intersecting $\pi_{R'}$ in a Baer subline shares l and this Baer subline with Ω . Two intersecting Baer sublines define a Baer subplane in a unique way, so these Baer subplanes then lie completely in this Baer subgeometry Ω .

Consider an arbitrary Baer subplane π of Ω not through l . Then π shares at least $\frac{q^{1/2}}{16\eta^2} + \frac{1}{16\eta^2} - \frac{1}{\eta}$ Baer sublines with F , so shares at least $\frac{q}{16\eta^2} + \frac{q^{1/2}}{16\eta^2} - \frac{q^{1/2}}{\eta} + 1$ points with F . Consider the plane over \mathbb{F}_q of this Baer subplane π . This plane intersects F in an $\{m_1(q+1) + m_0, m_1; 2, q\}$ -minihyper, with $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 = \eta(q^{1/2} - q^{1/6})$, which contains m_1 Baer subplanes (Lemma ??). Suppose this Baer subplane π is not contained in F . It contains already at least $\frac{q}{16\eta^2} + \frac{q^{1/2}}{16\eta^2} - \frac{q^{1/2}}{\eta} + 1$ points of F . By [?, Lemma 4.4], we have that

$$|\pi \cap F| \leq m_0 + m_1(\sqrt{q} + 1) \leq 2\eta(q^{1/2} - q^{1/6}),$$

where the upper bound is obtained in the following way. First of all, $m_0 + m_1 \leq \eta(q^{1/2} - q^{1/6})$. Secondly, every one of the m_1 Baer subplanes contained in the $\{m_1(q+1) + m_0, m_1; 2, q\}$ -minihyper has size $q + \sqrt{q} + 1$ and contributes $\sqrt{q} + 1$ to the sum $m_1 + m_0$. Hence, also $m_1\sqrt{q} \leq \eta(q^{1/2} - q^{1/6})$.

But $\frac{q}{16\eta^2} + \frac{q^{1/2}}{16\eta^2} - \frac{q^{1/2}}{\eta} + 1 > 2\eta(q^{1/2} - q^{1/6})$ if $q \geq 1217$, so this Baer subplane π lies completely in F . As a consequence, this Baer subgeometry Ω defined by π_R and $\pi_{R'}$ lies completely in F . \square

Lemma 3.10 *Let F be an $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 3, q\}$ -minihyper, with $2\epsilon_1 + \epsilon_0 < q + 2$, containing a subgeometry $\text{PG}(3, \sqrt{q})$. Then $F \setminus \text{PG}(3, \sqrt{q})$ is an $\{(\epsilon_1 - \sqrt{q} - 1)(q+1) + \epsilon_0, \epsilon_1 - \sqrt{q} - 1; 3, q\}$ -minihyper.*

Proof. A plane π either intersects a Baer subgeometry $\text{PG}(3, \sqrt{q})$ in a subline $\text{PG}(1, \sqrt{q})$ or a subplane $\text{PG}(2, \sqrt{q})$. We only have to discuss the case that $\pi \cap \text{PG}(3, \sqrt{q})$ is a subplane $\text{PG}(2, \sqrt{q})$ of size $q + \sqrt{q} + 1$.

If π still contains $\epsilon_1 - \sqrt{q} - 1$ other points of F , then removing this Baer subgeometry $\text{PG}(3, \sqrt{q})$ from F causes no problem for the plane π . So from now on, we assume that $q + \sqrt{q} + 1 \leq |\pi \cap F| < q + \epsilon_1$.

We select a point R of $\pi \setminus F$. Project π and F from R onto a plane. Then we obtain an ϵ_1 -fold blocking multiset B in this plane containing a line L , which is the projection of $\pi \cap F$. By [?, Theorem 2.2], we can reduce the weight of every point of L by one to obtain an $(\epsilon_1 - 1)$ -fold blocking set B' in this plane. But then L is still blocked at least $\epsilon_1 - 1$ times by B' . So π is blocked at least $q + \epsilon_1$ times by F . \square

Theorem 3.11 *Let F be a weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 3, q\}$ -minihyper, having weighted points with total weight at most $\frac{2\epsilon_1^2}{q}$ and where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, then F contains a sum of A lines, B isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C Baer subgeometries $\text{PG}(3, \sqrt{q})$, where $A + B + C(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points.*

Proof. If $\epsilon_1 < q^{1/6}/2$, this is the result mentioned in Theorem ??, so from now on assume that $\epsilon_1 \geq q^{1/6}/2$.

If F contains A lines, we can remove these lines from F , and then apply the arguments to F minus these A lines (Lemma ??). We denote the minihyper that remains again by F . Let R be a point not in the minihyper F on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ secants to F , containing at least two points of F of weight one (Lemma ??). Projecting F from R onto a plane not through R gives a weighted ϵ_1 -fold blocking set B in this plane. If B does not contain lines, Lemma ?? states that F is the sum of ϵ_1 lines and Baer subplanes $\text{PG}(2, \sqrt{q})$, and possibly some extra points. If B does contain lines, we find a Baer subgeometry $\text{PG}(3, \sqrt{q})$ contained in F , which can be removed from F to obtain a new $\{(\epsilon_1 - \sqrt{q} - 1)(q + 1) + \epsilon_0, \epsilon_1 - \sqrt{q} - 1; 3, q\}$ -minihyper, see Lemma ?. Repeating the previous arguments with this minihyper gives us the assertion. \square

4 Larger dimensions

We now characterise non-weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n, q\}$ -minihypers F , $q = p^h$, $p > 3$ prime, $h \geq 1$, $q \geq 1217$, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, by induction on the dimension n . We suppose that every $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n-1, q\}$ -minihyper, with $n \geq 4$, is a pairwise disjoint union of A lines, B isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C Baer subgeometries $\text{PG}(3, \sqrt{q})$, with $A + B + C(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points. As in the 3-dimensional case, we start by using Lemma ?? to remove the lines contained in F .

We want to project F onto a hyperplane in such a way that the number of multiple points appearing in the projection is as small as possible.

Lemma 4.1 *For $n = 4$, there is a point $R \notin F$ lying on at most $\frac{\epsilon_1^2}{q}$ secants to F . In larger dimensions n , there are points $R \notin F$ only lying on tangents to F .*

Proof. The number of points on secants to F is at most

$$\frac{(\epsilon_1(q+1) + \epsilon_0)^2}{2}(q-1) = \frac{\epsilon_1(q^3 + q^2 - q - 1) + 2\epsilon_1\epsilon_0(q^2 - 1) + \epsilon_0^2(q-1)}{2}.$$

Now $\epsilon_1\epsilon_0, \epsilon_0^2 < \frac{q^{7/6}}{4}$. For $n \geq 5$, this number is smaller than the number of points in $\text{PG}(n, q) \setminus F$. In this case, there exists at least one point R only lying on tangents to F .

For $n = 4$, we divide by $q^4 + q^3 \leq \theta_4 - |F|$. This gives a point R lying on at most

$$\frac{\epsilon_1^2}{2q} + \frac{2q^{7/6}(q^2 - 1)/2 + q^{7/6}(q - 1)/2}{2(q^4 + q^3)} \leq \frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}}$$

secants to F . Either $\frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}} < 1$ and then R lies on zero secants to F or either $\frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}} \geq 1$, then $\frac{\epsilon_1^2}{2q} \geq \frac{1}{2q^{5/6}}$. In both cases, $\frac{\epsilon_1^2}{q}$ can be used as an upper bound on the number of secants to F through R . \square

In the case of $n = 4$, projecting from a point $R \notin F$ as in the previous lemma gives a weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n-1, q\}$ -minihyper with the total weight of the multiple points at most $\frac{2\epsilon_1^2}{q}$. This explains the bound in Theorem ??.

Theorem 4.2 *Let F be a non-weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n, q\}$ -minihyper, $q = p^h$, $p > 3$ prime, $h \geq 1$, $q \geq 1217$, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, then F is the union of pairwise disjoint A lines, B isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C Baer subgeometries $\text{PG}(3, \sqrt{q})$, with $A + B + C(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points.*

Proof. We prove this result by induction on n . Theorem ?? proves this theorem for $n = 3$. If F contains lines, these lines can be removed from F (Lemma ??); so we assume that F does not contain any lines.

Project F from a point R , lying only on tangents to F or on at most ϵ_1^2/q secants to F if $n = 4$, onto a hyperplane π not through R . We obtain a (weighted if $n = 4$) $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n-1, q\}$ -minihyper F' which is the sum of A' lines, B' isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C' Baer subgeometries $\text{PG}(3, \sqrt{q})$, with $A' + B' + C'(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ points.

Case I: assume that F' contains a line L .

The plane $\langle R, L \rangle$ intersects F in at least a 1-fold blocking set, which contains a Baer subplane, since by assumption, F does not contain lines (Lemma ??).

Since $\langle R, L \rangle$ shares a Baer subplane with F , R lies on a Baer subline to this Baer subplane, but then R lies on a $(\sqrt{q} + 1)$ -secant to F , which is false for $n > 4$. For $n = 4$, this line is projected onto a point of F' with weight up to $\sqrt{q} + 1 > \frac{q^{1/6}}{2} > \frac{2\epsilon_1^2}{q}$, which is false. So this case cannot occur.

Case II: assume that F' contains an isolated Baer subplane $\text{PG}(2, \sqrt{q})$.

Denote this Baer subplane $\text{PG}(2, \sqrt{q})$ by ω . The 3-space $\langle R, \omega \rangle$ intersects F in an $\{m_1(q+1) + m_0, m_1; 3, q\}$ -minihyper, with $m_1 \geq 1$ (Lemma ??), so $\langle R, \omega \rangle \cap F$ contains by the induction hypothesis on n the union of points, isolated Baer subgeometries $\text{PG}(2, \sqrt{q})$ and Baer subgeometries $\text{PG}(3, \sqrt{q})$, which are all pairwise disjoint. Assume $\langle R, \omega \rangle$ contains a Baer subgeometry $\text{PG}(3, \sqrt{q})$ and consider the conjugate point $R^{\sqrt{q}}$ of R w.r.t. $\text{PG}(3, \sqrt{q})$. The line $RR^{\sqrt{q}}$ intersects $\text{PG}(3, \sqrt{q})$ in a Baer subline, which is false since the projection of this Baer subline would lead to a projected point in F' with weight at least $\sqrt{q} + 1$. So $\langle R, \omega \rangle \cap F$ contains points and isolated Baer subplanes. One of these Baer subplanes $\text{PG}(2, \sqrt{q})$ is projected from R onto

ω .

Case III: assume that F' contains a Baer subgeometry $\text{PG}(3, \sqrt{q})$.

Consider two Baer subplanes ω_1 and ω_2 in $\text{PG}(3, \sqrt{q})$. By the arguments of case II, we find Baer subplanes ω'_1 and ω'_2 contained in F projected onto ω_1 and ω_2 respectively. Since there are less than $\frac{q^{1/6}}{2}$ multiple points in the intersection line of ω_1 and ω_2 , this projected Baer subline $\omega_1 \cap \omega_2$ must be the projection of a Baer subline contained in F , which must be equal to the intersection line of ω'_1 and ω'_2 . So ω'_1 and ω'_2 span a Baer subgeometry $\text{PG}(3, \sqrt{q})$. The 3-space over \mathbb{F}_q defined by this Baer subgeometry shares two intersecting Baer subplanes with F . By the induction hypothesis, this Baer subgeometry $\text{PG}(3, \sqrt{q})$ must be contained in F .

Conclusion: Let F be a non-weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n, q\}$ -minihyper, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$.

By assumption, if F contains a line L , then this line L can be removed from F , so that $F \setminus L$ is an $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; n, q\}$ -minihyper (Lemma ??). From now on, assume that all the lines contained in F are removed from F . By induction on $\epsilon_1 + \epsilon_0$, these lines contained in F are pairwise disjoint among each other.

The preceding cases I, II, and III show that the projection of F , from a suitably selected point R , onto a hyperplane leads to a minihyper in this hyperplane containing B' isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C' Baer subgeometries $\text{PG}(3, \sqrt{q})$, with $B' + C'(\sqrt{q} + 1) = \epsilon_1$, and that these isolated Baer subplanes and 3-dimensional Baer subgeometries contained in the projected minihyper arise from B' isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C' Baer subgeometries $\text{PG}(3, \sqrt{q})$ contained in the minihyper F .

If lines again are permitted to be contained in F , the preceding arguments therefore show that F is the union of pairwise disjoint A lines, B isolated Baer subplanes $\text{PG}(2, \sqrt{q})$ and C Baer subgeometries $\text{PG}(3, \sqrt{q})$, with $A + B + C(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points. \square

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