Polygonal valuations

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Abstract

We develop a valuation theory for generalized polygons similar to the existing theory for dense near polygons. This valuation theory has applications for the study and classification of generalized polygons that have full subpolygons as subgeometries.

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1 Introduction and overview

Valuations of dense near polygons have been introduced by De Bruyn and Vandecasteele [4] for the purpose of classifying dense near polygons. Each such valuation is a map from the point set of a dense near polygon $\mathcal{S}$ to the set of nonnegative integers satisfying certain nice properties. They are intended to axiomatize certain maps which naturally arise when $\mathcal{S}$ is fully and isometrically embedded into a larger dense near polygon. The advance in classifying dense near polygons which has taken place in the last decade (realized by the author and some of his collaborators) can largely be credited to the study of valuations as axiomatic objects.

Valuations of dense near polygons have found other applications than the ones they were originally designed for (namely, for the classification of dense near polygons). They have shown to be valuable tools for the study of isometric embeddings, the construction of new hyperplanes of dual polar spaces and even the classification of certain classes of hyperplanes of dual polar spaces.

One of the reasons why valuations can be useful for the classification of dense near polygons is the fact that dense near polygons are known to contain full proper sub-near-polygons. Indeed, by Shult and Yanushka [7, Proposition 2.5], we know that every dense near polygon of diameter at least two contains full convex subgeometries that are generalized quadrangles. This result was later generalized by Brouwer and Wilbrink [1, Theorem 4] who showed the existence of full convex sub-near-polygons of any feasible diameter.

One can wonder whether similar ideas can be useful for classifying near polygons that are not dense. The problem here is that such near polygons do not necessarily contain "suitable sub-near-polygons". One way to avoid this problem is however to presuppose the existence of such sub-near-polygons.

The aim of the present paper is to develop a valuation theory for generalized polygons that will be suitable to study and classify generalized polygons that contain a particular generalized polygon as a proper full subgeometry. The theory developed here will indeed have applications. One of the important open problems in the theory of generalized polygons is the problem regarding the uniqueness of the generalized octagon of order $(2,4)$. In another paper [3], we will use the theory developed in this paper to show the uniqueness of the generalized octagon of order $(2,4)$, assuming that it has at least one suboctagon of order $(2,1)$. This generalized octagon of order $(2,4)$ belongs to the family of the Ree-Tits octagons introduced by Tits [9], using a class of simple groups discovered by Ree [6].

In Section 3, we introduce the notion of a polygonal valuation of a generalized $2d$-gon and study some of its basic properties. The notion of "polygonal valuation" will be the counterpart of the notion "valuation" in the theory of dense near polygons. In De Bruyn [2], we introduced the notion of neighboring valuations. The corresponding notion of "neighboring polygonal valuations" will be discussed in Section 4. In Section 5, we take a closer look to the special case of generalized polygons with three points on each line. The study of polygonal valuations and neighboring polygonal valuations will lead to the notion of the "valuation geometry" of a generalized $2d$-gon $\mathcal{S}$. This valuation geometry, which we introduce in Section 6, carries information on how $\mathcal{S}$ can be fully embedded into a larger generalized $2d$-gon.

2 Generalized polygons

Throughout this paper, d will be a nonnegative integer distinct from 0 and 1. A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *generalized* $2d$*-gon* if it satisfies the following three properties:

(GP1) $\mathcal{S}$ is a *partial linear space*, i.e. every two distinct points of $\mathcal{S}$ are incident with at most one line;

(GP2) if $\{A_1, A_2\} \subseteq \mathcal{P} \cup \mathcal{L}$, then there exists a subgeometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ of $\mathcal{S}$ isomorphic to an ordinary $2d$-gon for which $\{A_1, A_2\} \subseteq \mathcal{P}' \cup \mathcal{L}'$;

(GP3) $\mathcal{S}$ has no subgeometries that are ordinary m-gons with $m \in \{3, 4, \ldots, 2d-1\}$.

Recall that a point-line geometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is called a *subgeometry of* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathrm{I}' = \mathrm{I} \cap (\mathcal{P} \times \mathcal{L})$. If $\{x \in \mathcal{P} \,|\, x \mathrm{~I~} L\} = \{x \in \mathcal{P}' \,|\, x \mathrm{~I}' L\}$ for every line L of $\mathcal{L}'$, then the subgeometry $\mathcal{S}'$ of $\mathcal{S}$ is called *full*.

Generalized $2d$-gons were introduced by Tits in [8]. For an extensive study of generalized polygons, see Van Maldeghem [11]. A generalized $2d$-gon is said to be of *order*

(s,t) if every line is incident with precisely $s+1$ points and if every point is incident with precisely $t+1$ lines.

Every generalized $2d$-gon belongs to the class of near $2d$-gons as introduced by Shult and Yanushka [7]. Such a *near $2d$-gon* is a partial linear space of diameter d having the property that for every point p and every line L, there exists a unique point on L nearest to p. Here, distances $\mathrm{d}(\cdot,\cdot)$ are measured in the collinearity graph of the geometry. A generalized $2d$-gon can alternatively be defined as a point-line geometry $\mathcal{S}$ that satisfies the following three properties:

(GP1') $\mathcal{S}$ is a near $2d$-gon;

(GP2') every point of $\mathcal{S}$ is incident with at least two lines;

(GP3') for every two points x and y at distance $i \in \{1,2,\ldots,d-1\}$ from each other, there exists a unique point collinear with y at distance $i-1$ from x.

In the sequel, we will often regard a generalized $2d$-gon as a near $2d$-gon that satisfies the properties (GP2') and (GP3') above. This point of view will have some advantages. A natural consequence of this point of view is that we adopt the convention that distances will always be measured in the collinearity graph of the geometry. (It is common in the theory of generalized $2d$-gons to measure distances in the incidence graph.)

For every point x of a generalized $2d$-gon $\mathcal{S}$ and for every $i \in \mathbb{N}$, we denote by $\Gamma_i(x)$ the set of points of $\mathcal{S}$ at distance i from x. If $\mathcal{S}'$ is a full sub-$2d$-gon of a generalized $2d$-gon $\mathcal{S}$, then $\mathcal{S}'$ is isometrically embedded into $\mathcal{S}$. This means that for every two points x and y of $\mathcal{S}'$, the distance $\mathrm{d}_{\mathcal{S}'}(x,y)$ between x and y in $\mathcal{S}'$ is equal to the distance $\mathrm{d}_{\mathcal{S}}(x,y)$ between x and y in $\mathcal{S}$.

3 Definition and basic properties of polygonal valuations

As told in Section 1, valuations of dense near polygons were introduced by De Bruyn and Vandecasteele [4] for the purpose of classifying dense near polygons. Such a valuation is a certain nice map from the point set of a dense near polygon $\mathcal{S}$ to the set of nonnegative integers, and provides information on how the dense near polygon can be isometrically embedded into a larger dense near polygon. We now introduce a similar notion for generalized $2d$-gons, which we call *polygonal valuations*. As we will see later, the polygonal valuations of a given generalized $2d$-gon give information on how this generalized polygon can be fully embedded into another generalized $2d$-gon. The notion of polygonal valuation should not be confused with the notion of a valuation of a generalized polygon, as introduced in Van Maldeghem [10].

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon. A map $f : \mathcal{P} \rightarrow \mathbb{N}$ is called a *polygonal valuation* if the following three conditions are satisfied.

(PV1) There exists at least one point with f-value 0.

(PV2) Every line L of $\mathcal{S}$ contains a unique point x_L with smallest f-value and every other point of L has f-value $f(x_L)+1$.

(PV3) If M is the maximal value attained by f and x is a point of $\mathcal{S}$ for which $f(x) < M$, then there is at most one line through x containing a (necessary unique) point with f-value $f(x)-1$.

Two polygonal valuations f_1 and f_2 of $\mathcal{S}$ are called *isomorphic* if there exists an automorphism θ of $\mathcal{S}$ such that $f_2 = f_1 \circ \theta$. Suppose f is a polygonal valuation of $\mathcal{S}$. Then we denote by $\mathcal{O}_f$ the set of points with f-value 0 and by $\mathcal{M}_f$ the set of all points x of $\mathcal{S}$ that are not collinear with a point having f-value $f(x)-1$. We denote by M_f the maximal value attained by f. Clearly, $\mathcal{O}_f \subseteq \mathcal{M}_f$ and $M_f \in \{1,2,\ldots,d\}$.

Proposition 3.1 *Let f be a polygonal valuation of a generalized $2d$-gon $\mathcal{S}$, and let $x \in \mathcal{O}_f$. Then $f(y) = d(x,y)$ for every point y at distance at most M_f from x.*

Proof. We will prove this by induction on the distance $\mathrm{d}(x,y)$. Obviously, the proposition holds if $\mathrm{d}(x,y)=0$ and by Property (PV2) the proposition also holds if $\mathrm{d}(x,y)=1$. So, we may suppose that $\mathrm{d}(x,y) = i \in \{2,3,\ldots,M_f\}$. Let z be a point of $\Gamma_1(y) \cap \Gamma_{i-1}(x)$ and let z' be the unique point of $\Gamma_1(z) \cap \Gamma_{i-2}(x)$. By the induction hypothesis, $f(z') = i-2$ and $f(z) = i-1 < M_f$. By Property (PV3), zz' is the unique line through z containing a point with f-value $i-2$. This implies that $f(y) = f(z)+1 = i = \mathrm{d}(x,y)$ as we needed to prove. ■

We now describe some classes of polygonal valuations.

(a) Let x be a point of $\mathcal{S}$. For every point y of $\mathcal{S}$, we define $f(y) := \mathrm{d}(x,y)$. Then f is a polygonal valuation of $\mathcal{S}$ with $M_f = d$ and $\mathcal{O}_f = \mathcal{M}_f = \{x\}$. Every polygonal valuation that can be obtained in this way is called *classical*.

(b) Let O be a set of points of $\mathcal{S}$ meeting each line in a singleton. For every point y of O, we define $f(y)=0$. For every point y of $\mathcal{S}$ not contained in O, we define $f(y)=1$. Then f is a polygonal valuation of $\mathcal{S}$ with $M_f = 1$ and $\mathcal{O}_f = \mathcal{M}_f = O$. Every polygonal valuation that can be obtained in this way is called *ovoidal*.

(c) Let x be a point of $\mathcal{S}$ and O a set of points at distance d from x such that every line of $\mathcal{S}$ at distance $d-1$ from x has a unique point in common with O. If y is a point of $\mathcal{S}$ at distance at most $d-1$ from x, then we define $f(y) := \mathrm{d}(x,y)$. If y is a point of $\mathcal{S}$ at distance d from x, then we define $f(y) := d-2$ if $y \in O$ and $f(y) = d-1$ otherwise. Then f is a polygonal valuation of $\mathcal{S}$ with $M_f = d-1$. If $d=2$, then $\mathcal{O}_f = \mathcal{M}_f = \{x\} \cup O$. If d ≥ 3, then $\mathcal{O}_f = \{x\}$ and $\mathcal{M}_f = \{x\} \cup O$. Every polygonal valuation that can be obtained in the above way is called *semi-classical*. For generalized quadrangles, the notions of ovoidal polygonal valuations and semi-classical polygonal valuations coincide.

The ovoidal polygonal valuations belong to a larger family of polygonal valuations which we will now describe. A *distance-j-ovoid* ($2 \leq j \leq d$) of $\mathcal{S}$ is a set X of points satisfying:

(O1) $\mathrm{d}(x,y) \geq j$ for every two distinct points x and y of X;

(O2) there exist two points of X at distance j from each other;

(O3) for every point a of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(a,x) \leq \frac{j}{2}$;

(O4) for every line L of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(L,x) \leq \frac{j-1}{2}$.

The notion of a distance-j-ovoid was introduced in Offer and Van Maldeghem [5]. The following is immediately clear from the properties (O1), (O3) and (O4) above.

(O4) If j is odd, then for every point a of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(a,x) \leq \frac{j-1}{2}$.

(O5) If j is even, then for every line L of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(L,x) \leq \frac{j-2}{2}$.

Proposition 3.2 *Let X be a distance-j-ovoid of $\mathcal{S}$ where $j \in \{2,\dots,d\}$ is even. For every point x of $\mathcal{S}$, define $f(x) := d(x,X)$. Then f is a polygonal valuation of $\mathcal{S}$ with $M_f = \frac{j}{2}$ and $\mathcal{O}_f = \mathcal{M}_f = X$.*

Proof. Clearly, f satisfies Property (PV1). We now also prove that f satisfies Property (PV2).

Let L be an arbitrary line of $\mathcal{S}$. Let $x_L \in L$ and $x^* \in X$ such that $\delta := \mathrm{d}(L,X) = \mathrm{d}(x_L,x^*)$. By Property (O5), $\delta \leq \frac{j-2}{2}$. Let y be an arbitrary point of L distinct from x_L. Since x_L is the unique point of L nearest to $x^* \in X$, we have $\mathrm{d}(x^*,y) = \mathrm{d}(x^*,x_L)+1 = \delta+1$. Now, suppose that $\mathrm{d}(y,X) \leq \mathrm{d}(x_L,X)$ and let y^* be a point of X such that $\mathrm{d}(y,X) = \mathrm{d}(y,y^*)$. Then $\mathrm{d}(y,y^*) = \mathrm{d}(y,X) \leq \mathrm{d}(x_L,X) = \delta \leq \frac{j-2}{2}$. Since $\mathrm{d}(y,x^*) = \delta+1$, we have $y^* \neq x^*$. Now, there exists a path of length at most $\frac{j-2}{2}+1+\frac{j-2}{2} = j-1$ in $\mathcal{S}$ that connects the points x^* and y^* and passes through the collinear points x_L and y. This clearly is impossible by Property (O1). Hence, f satisfies Property (PV2).

Let M_f denote the maximal value attained by f. Then $M_f \leq \frac{j}{2}$ by Property (O3). If u is the point in the middle of a shortest path connecting two points of X at distance j from each other, then $\mathrm{d}(u,X) = \frac{j}{2}$ by Property (O1). Hence, $M_f = \frac{j}{2}$.

We now also prove that f satisfies Property (PV3). Let x be an arbitrary point with $0 \neq f(x) = \mathrm{d}(x,X) \leq M_f - 1 = \frac{j-2}{2}$. Then by Property (O1), there exists a unique $x^* \in X$ such that $\mathrm{d}(x,X) = \mathrm{d}(x,x^*)$ and every other point of X lies at distance at least $\frac{j+2}{2}$ from x. Since $0 \neq \mathrm{d}(x,x^*) \leq \frac{j-2}{2} < d$, there exists a unique line L^* through x containing a point at distance $\mathrm{d}(x,x^*)-1$ from x^*. Now, suppose L is a line through x containing a point y at distance $\mathrm{d}(x,X)-1$ from X. Let y^* be a point of X such that $\mathrm{d}(y,y^*) = \mathrm{d}(y,X) = \mathrm{d}(x,X)-1 = \mathrm{d}(x,x^*)-1$. Since $\mathrm{d}(y^*,x) \leq \mathrm{d}(y^*,y)+\mathrm{d}(y,x) = \mathrm{d}(x,x^*) \leq \frac{j-2}{2}$, we necessarily have $y^* = x^*$. So, L is a line through x containing a point at distance $\mathrm{d}(x,x^*)-1$ from $y^* = x^*$. This implies that $L = L^*$. So, Property (PV3) is satisfied.

Finally, it is clear that $\mathcal{O}_f = \mathcal{M}_f = X$. ■

Any polygonal valuation that can be obtained as described in Proposition 3.2 is called *distance-j-ovoidal*. The distance-2-ovoidal polygonal valuations are precisely the ovoidal polygonal valuations.

The notions of classical polygonal valuation, semi-classical polygonal valuation, ovoidal polygonal valuation and distance-j-ovoidal polygonal valuation all have their counterparts in the theory of valuations of dense near polygons, see De Bruyn and Vandecasteele [4].

Proposition 3.3 *Suppose f is a polygonal valuation of a generalized $2d$-gon $\mathcal{S}$. Then:*
(1) *f is an ovoidal polygonal valuation if and only if $M_f = 1$;*
(2) *f is a semi-classical polygonal valuation if and only if $M_f = d-1$;*
(3) *f is a classical polygonal valuation if and only if $M_f = d$.*

Proof. Above, we already mentioned the values of M_f in case f is classical, semi-classical or ovoidal.

(1) Suppose $M_f = 1$. Then by Property (PV2) every line contains a unique point of $\mathcal{O}_f$ and any other point of that line has f-value 1. So, f is indeed an ovoidal polygonal valuation.

(2) Suppose $M_f = d-1$ and let $x \in \mathcal{O}_f$. Then by Proposition 3.1, $f(y) = \mathrm{d}(x,y)$ for every point y of $\mathcal{S}$ for which $\mathrm{d}(x,y) \leq d-1$. Now, let O denote the set of those points at distance d from x that have value $d-2$. Since $M_f = d-1$ and every line L at distance $d-1$ from x contains a point with f-value $d-1$, namely the unique point in $\Gamma_{d-1}(x) \cap L$, the line L must contain a unique point of O by (PV2). Every other point of L has f-value $d-1$. It is now clear that f is a semi-classical polygonal valuation of $\mathcal{S}$.

(3) Suppose $M_f = d$ and let $x \in \mathcal{O}_f$. Then Proposition 3.1 implies that $f(y) = \mathrm{d}(x,y)$ for every point y of $\mathcal{S}$. So, f is a classical polygonal valuation. ■

The following is an immediate consequence of Proposition 3.3.

Corollary 3.4 (1) *Every polygonal valuation of a generalized quadrangle is either classical or ovoidal.*

(2) *Every polygonal valuation of a generalized hexagon is either classical, ovoidal or semi-classical.*

We have seen above that if f is a polygonal valuation of a generalized $2d$-gon $\mathcal{S}$, then $\mathcal{O}_f \subseteq \mathcal{M}_f$ and that equality occurs if f is classical or distance-j-ovoidal for some even j. The following proposition says that these are the only two instances in which we have equality.

Proposition 3.5 *Let f be a polygonal valuation of a generalized $2d$-gon $\mathcal{S}$ with $\mathcal{O}_f = \mathcal{M}_f$. Then f is either classical or distance-j-ovoidal for some even j.*

Proof. By Property (PV2), we have $f(x) \leq \mathrm{d}(x, \mathcal{O}_f)$ for every point x of $\mathcal{S}$. On the other hand, since $\mathcal{O}_f = \mathcal{M}_f$, there exists for every point x of $\mathcal{S}$ a path of length $f(x)$ connecting x with a point of $\mathcal{O}_f$. So, we have that $f(x) = \mathrm{d}(x, \mathcal{O}_f)$ for every point x of $\mathcal{S}$. If $\mathcal{O}_f$ is a singleton, then f necessarily is a classical polygonal valuation. So, in the sequel, we may suppose that $|\mathcal{O}_f| \geq 2$. Let δ be the minimal distance between two distinct points of $\mathcal{O}_f$.

We prove that δ is even. Suppose $\delta = 2j+1$ for some nonnegative integer j. Let x and y be two points of $\mathcal{O}_f$ at minimal distance $2j+1$ from each other and consider a shortest path between x and y. Let u and v be the two neighboring points in the middle of this path. Then $\mathrm{d}(x,u) = j$, $u \sim v$ and $\mathrm{d}(v,y) = j$. Since the distance between two distinct points of $\mathcal{O}_f$ is at least $2j+1$, we have $f(u) = \mathrm{d}(u,\mathcal{O}_f) = j$ and $f(v) = \mathrm{d}(v,\mathcal{O}_f) = \mathrm{d}(v,y) = j$. Since the line uv contains two points with f-value j, the line uv contains a unique point w with f-value $j-1$. So, $\mathrm{d}(w,\mathcal{O}_f) = j-1$. This implies that there exists a path of length $\mathrm{d}(x,u) + \mathrm{d}(u,w) + \mathrm{d}(w,y) = 2j$ between two distinct points of $\mathcal{O}_f$, clearly a contradiction.

So, $\delta = 2j$ is even. Let x and y be two points of $\mathcal{O}_f$ at distance $2j$ from each other, consider a shortest path between x and y, and let u, v and w denote the points in the middle of this path. Then $\mathrm{d}(x,u) = j-1$, $\mathrm{d}(x,v) = \mathrm{d}(y,v) = j$, $\mathrm{d}(y,w) = j-1$ and $u \sim v \sim w$. Since the distance between two points of $\mathcal{O}_f$ is at least $2j$, we have $f(u) = \mathrm{d}(u,\mathcal{O}_f) = \mathrm{d}(u,x) = j-1$, $f(v) = \mathrm{d}(v,\mathcal{O}_f) = \mathrm{d}(v,x) = \mathrm{d}(v,y) = j$ and $f(w) = \mathrm{d}(w,\mathcal{O}_f) = \mathrm{d}(w,y) = j-1$. So, through v there are two lines vu and vw containing a point with f-value $f(v) - 1 = j-1$. This implies that $M_f = j$.

We now prove that $\mathcal{O}_f$ is a distance-$2j$-ovoid. Clearly, Properties (O1) and (O2) are satisfied. Property (O3) follows from the equality $M_f = j$ and the fact that $f(x) = \mathrm{d}(x,\mathcal{O}_f)$ for every point x of $\mathcal{S}$. Property (O4) follows from the equality $M_f = j$, the fact that every line contains a unique point with smallest f-value and the fact that $f(x) = \mathrm{d}(x,\mathcal{O}_f)$ for every point x of $\mathcal{S}$.

Since $\mathcal{O}_f$ is a distance-$2j$-ovoid and $f(x) = \mathrm{d}(x,\mathcal{O}_f)$ for every point x of $\mathcal{S}$, the polygonal valuation f is distance-$2j$-ovoidal. ∎

Proposition 3.6 *Let f be a polygonal valuation of a finite generalized $2d$-gon $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ of order (s,t), and let n_i, $i \in \{0,1,\ldots,M_f\}$, denote the total number of points with f-value i. Then $\sum_{i=0}^{M_f} \frac{n_i}{(-s)^i} = 0$.*

Proof. For every line L of $\mathcal{S}$, we have $\sum_{x \in L} \frac{1}{(-s)^{f(x)}} = 0$ since L contains a unique point x_L such that $f(x) = f(x_L) + 1$ for every point x of L distinct from x_L. So, we have $0 = \sum_{L \in \mathcal{L}} \sum_{x \in L} \frac{1}{(-s)^{f(x)}} = (t+1) \cdot \sum_{x \in \mathcal{P}} \frac{1}{(-s)^{f(x)}} = (t+1) \cdot \sum_{i=0}^{M_f} \frac{n_i}{(-s)^i}$. ∎

Proposition 3.7 *Let $\mathcal{S}$ be a finite generalized $2d$-gon of order (s,t) having v points. If f is a polygonal valuation of $\mathcal{S}$ with $M_f = 2$, then $|\mathcal{M}_f| = \frac{v}{s+1} - st \cdot |\mathcal{O}_f|$.*

Proof. Let n_i, $i \in \{0,1,2\}$, be the number of points of $\mathcal{S}$ with f-value i. Then $n_0 = |\mathcal{O}_f|$, $n_0 + n_1 + n_2 = v$ and $n_0 - \frac{n_1}{s} + \frac{n_2}{s^2} = 0$ (recall Proposition 3.6). Hence,

$$\begin{aligned} n_1 &= \frac{v}{s+1} + (s-1) \cdot |\mathcal{O}_f|, \\ n_2 &= \frac{sv}{s+1} - s \cdot |\mathcal{O}_f|. \end{aligned}$$

The number of points with f-value 1 that are collinear with a (necessarily unique) point of $\mathcal{O}_f$ is equal to $|\mathcal{O}_f| \cdot s(t+1)$. Hence,

$$|\mathcal{M}_f| = |\mathcal{O}_f| + \frac{v}{s+1} + (s-1) \cdot |\mathcal{O}_f| - |\mathcal{O}_f| \cdot s(t+1) = \frac{v}{s+1} - st \cdot |\mathcal{O}_f|.$$

∎

Proposition 3.8 *Let $\mathcal{S}$ be a finite generalized $2d$-gon of order (s,t) having v points. If f is a polygonal valuation of $\mathcal{S}$ with $M_f = 2$, then $|\mathcal{O}_f| \leq \frac{v}{(s+1)(st+1)}$ with equality if and only if either ($d = 2$ and f is a classical polygonal valuation) or ($d \geq 3$ and f is a distance-4-ovoidal polygonal valuation).*

Proof. By Proposition 3.7, we have $|\mathcal{M}_f| = \frac{v}{s+1} - st \cdot |\mathcal{O}_f|$. Since $\mathcal{O}_f \subseteq \mathcal{M}_f$, we have $|\mathcal{O}_f| \leq \frac{v}{(s+1)(st+1)}$, with equality if and only if $\mathcal{O}_f = \mathcal{M}_f$. If $d = 2$, then $\frac{v}{(s+1)(st+1)} = 1$ and Proposition 3.3(3) implies that f is a classical polygonal valuation of $\mathcal{S}$. If $d \geq 3$, then f is not a classical polygonal valuation since $M_f = 2$, and Proposition 3.5 implies that we have equality if and only if f is a distance-4-ovoidal polygonal valuation. ■

Proposition 3.9 *If f is a polygonal valuation of a generalized $2d$-gon $\mathcal{S}$, then the set of all points of $\mathcal{S}$ with non-maximal f-value is a hyperplane of $\mathcal{S}$.*

Proof. This is an immediate consequence of Property (PV2). ■

The hyperplane of a generalized $2d$-gon $\mathcal{S}$ that is associated with a polygonal valuation f of $\mathcal{S}$ (see Proposition 3.9) is denoted by H_f. We call H_f a *hyperplane of valuation type*. The following proposition basically shows that a polygonal valuation is uniquely determined by its associated hyperplane.

Proposition 3.10 *If f is a polygonal valuation of a generalized $2d$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, then $f(x) = M_f - \mathrm{d}(x, \mathcal{P} \setminus H_f)$ for every point x of $\mathcal{S}$.*

Proof. If $k = \mathrm{d}(x, \mathcal{P} \setminus H_f)$, then there exists a path of length k connecting x with a point y of $\mathcal{P} \setminus H_f$. Since $f(y) = M_f$, we have $f(x) \geq M_f - k = M_f - \mathrm{d}(x, \mathcal{P} \setminus H_f)$ by successive application of (PV2).

It remains to show that $\mathrm{d}(x, \mathcal{P} \setminus H_f) \leq M_f - f(x)$. We will prove this by downwards induction on the value $f(x)$. If $f(x) = M_f$, then $x \in \mathcal{P} \setminus H_f$ and we are done. So, we may suppose that $f(x) < M_f$. By Property (PV3), there exists a line L through x not containing points with f-value $f(x) - 1$. If y is an arbitrary point of $L \setminus \{x\}$, then $f(y) = f(x) + 1$ by Property (PV2). By the induction hypothesis, $\mathrm{d}(y, \mathcal{P} \setminus H_f) \leq M_f - f(y) = M_f - f(x) - 1$. Hence, $\mathrm{d}(x, \mathcal{P} \setminus H_f) \leq M_f - f(x)$ by the triangle inequality. ■

Corollary 3.11 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon. Then $f \mapsto H_f$ defines a bijective map between the set of polygonal valuations of $\mathcal{S}$ and the set of hyperplanes of valuation type of $\mathcal{S}$.*

Proof. If H is a hyperplane of valuation type of $\mathcal{S}$, if M is the maximal distance from a point of $\mathcal{S}$ to $\mathcal{P} \setminus H$ and if f is a valuation of $\mathcal{S}$ such that $H_f = H$, then by Proposition 3.10, $f(x) = M - \mathrm{d}(x, \mathcal{P} \setminus H)$. ■

Clearly, two polygonal valuations f_1 and f_2 of a generalized $2d$-gon, are isomorphic if and only if their corresponding hyperplanes H_{f_1} and H_{f_2} are isomorphic.

Proposition 3.12 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon having precisely two lines through each of its points. Let H be a hyperplane of $\mathcal{S}$ and let M denote the maximal distance from a point of $\mathcal{S}$ to $\mathcal{P} \setminus H$. Then the map $f : \mathcal{P} \to \mathbb{N}; x \mapsto M - d(x, \mathcal{P} \setminus H)$ is a polygonal valuation if and only if f satisfies Property (PV2).*

Proof. By definition, f satisfies Property (PV1). Suppose now that f satisfies Property (PV2). We prove that f also satisfies Property (PV3). Let x be a point of $\mathcal{S}$ with non-maximal f-value. Then $x \notin \mathcal{P} \setminus H$. So, there exists a line L through x containing a point at distance $\mathrm{d}(x, \mathcal{P} \setminus H) - 1$ from $\mathcal{P} \setminus H$. By the triangle inequality, we have $\mathrm{d}(y, \mathcal{P} \setminus H) \leq \mathrm{d}(x, \mathcal{P} \setminus H)$, or equivalently, that $f(y) \geq f(x)$ for every $y \in L$. So, there is at most one line through x containing a (necessarily unique) point with f-value $f(x) - 1$. ■

Proposition 3.13 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon having at least three points on each line. If f_1 and f_2 are two polygonal valuations of $\mathcal{S}$ such that $H_{f_1} \subseteq H_{f_2}$, then $H_{f_1} = H_{f_2}$ and $f_1 = f_2$.*

Proof. For every $i \in \{1, 2\}$, put $C_i := \mathcal{P} \setminus H_{f_i}$. Then $C_2 \subseteq C_1$. Suppose $H_{f_1} \neq H_{f_2}$. Then C_2 is properly contained in C_1. Let x be a point of $C_1 \setminus C_2$ at smallest distance from C_2. Since $x \notin C_2$, there exists by Properties (PV2) and (PV3) a line L through x such that every point of $L \setminus \{x\}$ has f_2-value $f_2(x) + 1$. By Property (PV2) and the fact that $|L| \geq 3$, there exists a point $y \in L \setminus \{x\}$ with f_1-value $f_1(x) = M_{f_1}$. For such a point y, we have $y \in C_1$ and $\mathrm{d}(y, C_2) = \mathrm{d}(x, C_2) - 1$ by Proposition 3.10. Since x is a point of $C_1 \setminus C_2$ at smallest distance from C_2, this implies that $y \in C_2$ and $\mathrm{d}(x, C_2) = 1$. Hence, $f_2(x) = M_{f_2} - 1$ and every point of $L \setminus \{x\}$ belongs to C_2. By (PV2), there exists a point $z \in L \setminus \{x\}$ with f_1-value $f_1(x) - 1$. For this point z, we have $z \in C_2 \setminus C_1$, in contradiction with $C_2 \subseteq C_1$. Hence, $C_2 = C_1$ and $H_{f_1} = H_{f_2}$. By Corollary 3.11, we then have $f_1 = f_2$. ■

Proposition 3.14 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon having at least three points on each line. If f_1 and f_2 are two polygonal valuations of $\mathcal{S}$, then there exists a point x of $\mathcal{S}$ that is not contained in $H_{f_1} \cup H_{f_2}$.*

Proof. Put $C_i := \mathcal{P} \setminus H_{f_i}$, $i \in \{1, 2\}$. Let x be a point of C_1 at smallest distance from C_2. Suppose $x \notin C_2$. Then there exists a point y collinear with x at distance $\mathrm{d}(x, C_2) - 1$ from x. By Proposition 3.10, $f_2(y) = f_2(x) + 1$. So, if L denotes the line xy, then every point of $L \setminus \{x\}$ has f_2-value $f_2(x) + 1$ and lies at distance $\mathrm{d}(x, C_2) - 1$ from C_2 (again by Proposition 3.10). By (PV2) and the fact that $|L| \geq 3$, there exists a point $u \in L \setminus \{x\}$ belonging to C_1. Now, $u \in C_1$ with $\mathrm{d}(u, C_2) < \mathrm{d}(x, C_2)$, clearly a contradiction. So, the point x must belong to $C_1 \cap C_2$. ■

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a generalized $2d$-gon. A map $f : \mathcal{P} \to \mathbb{Z}$ is called a *semi-valuation* if it satisfies Property (PV2). If f is a semi-valuation of $\mathcal{S}$, then as before we denote by M_f the maximal value attained by f. Two semi-valuations f_1 and f_2 of $\mathcal{S}$ are called *equivalent* if there exists an $\epsilon \in \mathbb{Z}$ such that $f_2(x) = f_1(x) + \epsilon$ for every point x of $\mathcal{S}$. The equivalence class containing the semi-valuation f of $\mathcal{S}$ will be denoted by $[f]$. If f is a semi-valuation of $\mathcal{S}$, then the set of points of $\mathcal{S}$ with non-maximal f-value is a hyperplane H_f of $\mathcal{S}$. If f_1 and f_2 are two equivalent semi-valuations of $\mathcal{S}$, then $H_{f_1} = H_{f_2}$.

Proposition 3.15 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon having at least three points on each line. Let f_1 be a semi-valuation of $\mathcal{S}$ with minimal value 0 and f_2 a polygonal valuation of $\mathcal{S}$ such that $H_{f_1} = H_{f_2}$. Then $f_1 = f_2$.*

Proof. We prove by induction on $i = \mathrm{d}(x, \mathcal{P} \setminus H_{f_1}) = \mathrm{d}(x, \mathcal{P} \setminus H_{f_2})$ that $f_1(x) - M_{f_1} = f_2(x) - M_{f_2}$. Clearly, this holds if $i = 0$. So, suppose that $i \geq 1$. Let L be an arbitrary line through x containing a point at distance $i-1$ from $\mathcal{P} \setminus H_{f_1} = \mathcal{P} \setminus H_{f_2}$. By Proposition 3.10, $f_2(y) = f_2(x) + 1$. Hence, all points of $L \setminus \{x\}$ have value $f_2(x) + 1$ and lie (again by Proposition 3.10) at distance $i - 1$ from $\mathcal{P} \setminus H_{f_1} = \mathcal{P} \setminus H_{f_2}$. By the induction hypothesis, all points of $L \setminus \{x\}$ have f_1-value $f_2(x) + 1 - M_{f_2} + M_{f_1}$. Since f_1 is a semi-valuation and $|L| \geq 3$, we have $f_1(x) = f_2(x) - M_{f_2} + M_{f_1}$, as we needed to prove.

Since $f_1(x) - M_{f_1} = f_2(x) - M_{f_2}$ for every point x of $\mathcal{S}$, the semi-valuations f_1 and f_2 are equivalent. Since both have minimal value 0, they must coincide. ■

4 Neighboring semi-valuations

Two semi-valuations f_1 and f_2 of a generalized $2d$-gon are called *neighboring semi-valuations* if there exists an $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$ for every point x of $\mathcal{S}$. The notion of "neighboring semi-valuation" was introduced in De Bruyn [2] for arbitrary connected partial linear spaces. The following proposition, which is easy to prove, is taken form [2] (Lemma 2.1 and Corollary 2.3), where its validity was shown for any connected partial linear space (assuming that f_1 and f_2 attain minimal values for Claim (1)).

Proposition 4.1 *Suppose f_1 and f_2 are two neighboring semi-valuations of a generalized $2d$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ and let m_i, $i \in \{1, 2\}$, denote the minimal value attained by f_i. Then:*

(1) *If $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$, $\forall x \in \mathcal{P}$, then $|m_1 - m_2 + \epsilon| \leq 1$.*

(2) *If f_1 and f_2 are equivalent, then there are precisely three $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$ for every point x of $\mathcal{S}$. These three possible values of ϵ are consecutive integers.*

(3) *Suppose f_1 and f_2 are not equivalent. Then there exists a unique $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$ for every point x of $\mathcal{S}$. There also exists a line L of $\mathcal{S}$ such that the unique point x_1 of L with smallest f_1-value is distinct from the unique point x_2 of L with smallest f_2-value. Moreover, $\epsilon = f_2(x_2) - f_1(x_1)$.*

Proposition 4.2 *Let f_1 and f_2 be two neighboring polygonal valuations of a generalized $2d$-gon $\mathcal{S}$. Then the following holds.*

(1) *If $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$, then $\epsilon \in \{-1, 0, 1\}$.*

(2) *If every line of* $\mathcal{S}$ *contains at least three points, then* $|(f_1(x)-M_{f_1})-(f_2(x)-M_{f_2})| \leq 1$ *for every point* x *of* $\mathcal{S}$.

Proof. Claim (1) is an immediate consequence of Proposition 4.1(1). We now give a proof for the second claim.

If $f_1 = f_2$, then $M_{f_1} = M_{f_2}$ and hence $(f_1(x) - M_{f_1}) - (f_2(x) - M_{f_2}) = 0$ for every point x of $\mathcal{S}$. So, we may suppose that $f_1 \neq f_2$. Then f_1 and f_2 are not equivalent and hence by Proposition 4.1 there exists a unique $\epsilon \in \mathbb{Z}$ (necessarily belonging to $\{-1, 0, 1\}$) such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$ for every point x of $\mathcal{S}$. Since $f_1 \neq f_2$, we have $H_{f_1} \setminus H_{f_2} \neq \emptyset \neq H_{f_2} \setminus H_{f_1}$ by Proposition 3.13. If $x \in H_{f_1} \setminus H_{f_2}$ and $y \in H_{f_2} \setminus H_{f_1}$, then $f_1(x) - f_2(x) + \epsilon \leq M_{f_1} - 1 - M_{f_2} + \epsilon$ and $f_1(y) - f_2(y) + \epsilon \geq M_{f_1} - M_{f_2} + 1 + \epsilon$. Since the numbers $f_1(x) - f_2(x) + \epsilon$ and $f_1(y) - f_2(y) + \epsilon$ belong to the set $\{-1, 0, 1\}$, we necessarily have $\epsilon = M_{f_2} - M_{f_1}$. ∎

The following is an immediate consequence of Proposition 4.2.

Corollary 4.3 *Let* $\mathcal{S}$ *is a generalized* $2d$*-gon having at least three points on each line, and let* f_1 *and* f_2 *be two neighboring polygonal valuations of* $\mathcal{S}$. *Then* $|M_{f_1} - M_{f_2}| \leq 1$.

Proposition 4.4 *Let* $\mathcal{S}$ *be a generalized* $2d$*-gon having at least three points on each line, and let* f_1, f_2 *be two neighboring polygonal valuations of* $\mathcal{S}$ *such that* $M_{f_2} = M_{f_1} - 1$. *Then* $\mathcal{O}_{f_1} \subseteq \mathcal{O}_{f_2}$.

Proof. By Proposition 4.2, we have $|f_1(x) - f_2(x) - 1| \leq 1$ for every point x of $\mathcal{S}$. If $x \in \mathcal{O}_{f_1}$, then $|f_1(x) - f_2(x) - 1| = |-f_2(x) - 1| \leq 1$ implies that $f_2(x) = 0$, i.e. $x \in \mathcal{O}_{f_2}$. So, $\mathcal{O}_{f_1} \subseteq \mathcal{O}_{f_2}$. ∎

Let f_i, $i \in I$, be a collection of mutually distinct semi-valuations of a generalized $2d$-gon $\mathcal{S}$, where I is some index set of size at least two. We say that the set $\{f_i \,|\, i \in I\}$ is an *L-set of semi-valuations* of $\mathcal{S}$ if the following property is satisfied:

> For every point x of $\mathcal{S}$, there exists a (necessarily unique) $i \in I$ such that $f_j(x) - M_{f_j} = f_i(x) - M_{f_i} + 1$ for every $j \in I \setminus \{i\}$.

The letter "L" in L-set is meant to be an abbreviation for the word "Line". Indeed, as we will see in Proposition 6.2, if $\mathcal{S}$ is a subpolygon of another generalized $2d$-gon, then with every line of the latter, there corresponds some L-set of polygonal valuations of $\mathcal{S}$.

Proposition 4.5 *If* $\mathcal{F} = \{f_i \,|\, i \in I\}$ *is an* L*-set of semi-valuations of a generalized* $2d$*-gon* $\mathcal{S}$, *then any two distinct elements of* $\mathcal{F}$ *are neighboring semi-valuations.*

Proof. Let $i_1, i_2 \in I$ such that $f_{i_1} \neq f_{i_2}$ and let x be an arbitrary point of $\mathcal{S}$. Since $\mathcal{F}$ is an L-set of semi-valuations of $\mathcal{S}$, we have $|(f_{i_1}(x) - M_{f_{i_1}}) - (f_{i_2}(x) - M_{f_{i_2}})| \leq 1$. So, f_{i_1} and f_{i_2} are neighboring semi-valuations of $\mathcal{S}$. ∎

Proposition 4.6 *Suppose $\mathcal{S}$ is a generalized $2d$-gon with at least three points on each line. Let f_i, $i \in I$, be a collection of mutually distinct semi-valuations of $\mathcal{S}$ such that $\{f_i \,|\, i \in I\}$ is an L-set of semi-valuations of $\mathcal{S}$. Then for every $j \in I$, the equivalence class $[f_j]$ containing the semi-valuation f_j is uniquely determined by the semi-valuations f_i, $i \in I \setminus \{j\}$.*

Proof. Let x be an arbitrary point of $\mathcal{S}$. There are two possibilities.

• If all numbers $f_k(x) - M_{f_k}$, $k \in I \setminus \{j\}$, are equal to a certain number N, then $f_j(x) - M_{f_j} = N - 1$.

• If not all numbers $f_k(x) - M_{f_k}$, $k \in I \setminus \{j\}$, are equal, then $f_j(x) - M_{f_j}$ is equal to the maximum of these values.

So, the function $f_j - M_{f_j} : \mathcal{P} \to \mathbb{Z}; x \mapsto f_j(x) - M_{f_j}$ is uniquely determined by the semi-valuations f_i, $i \in I \setminus \{j\}$. In other words, the equivalence class $[f_j]$ is uniquely determined by the semi-valuations f_i, $i \in I \setminus \{j\}$. ■

Proposition 4.7 *Let f_i, $i \in I$, be a collection of mutually distinct semi-valuations of a generalized $2d$-gon $\mathcal{S}$ such that $\{f_i \,|\, i \in I\}$ is an L-set of semi-valuations of $\mathcal{S}$. Then:*

(1) *The hyperplanes H_{f_i}, $i \in I$, cover the whole set of points of $\mathcal{S}$.*

(2) *If i_1, i_2 and i_3 are mutually distinct elements of I, then $H_{f_{i_1}} \cap H_{f_{i_2}} = H_{f_{i_1}} \cap H_{f_{i_3}}$.*

Proof. (1) Let x be an arbitrary point of $\mathcal{S}$. Then there exists a unique $i \in I$ such that $f_j(x) - M_{f_j} = f_i(x) - M_{f_i} + 1$ for every $j \in I \setminus \{i\}$. Since $f_j(x) - M_{f_j} \leq 0$, we necessarily have $f_i(x) < M_{f_i}$. So, $x \in H_{f_i}$.

(2) By symmetry, it suffices to prove the inclusion $H_{f_{i_1}} \cap H_{f_{i_2}} \subseteq H_{f_{i_3}}$. Suppose to the contrary that there exists an $x \in (H_{f_{i_1}} \cap H_{f_{i_2}}) \setminus H_{f_{i_3}}$. There exists a unique $i \in I$ such that $f_j(x) - M_{f_j} = f_i(x) - M_{f_i} + 1$ for every $j \in I \setminus \{i\}$. If $f_i(x) = M_{f_i}$, then $f_j(x) - M_{f_j} \geq 1$ which is impossible. If $f_i(x) \leq M_{f_i} - 2$, then $f_{i_3}(x) - M_{f_{i_3}} \leq -1$, in contradiction with $x \notin H_{f_{i_3}}$. Hence, $f_i(x) = M_{f_i} - 1$. This however implies that at least one of $f_{i_1}(x) - M_{f_{i_1}}$, $f_{i_2}(x) - M_{f_{i_2}}$ is equal to 0, in contradiction with $x \in H_{f_{i_1}} \cap H_{f_{i_2}}$. ■

Proposition 4.8 *Let f_i, $i \in I$, be a collection of mutually distinct semi-valuations of a generalized $2d$-gon $\mathcal{S}$ such that $\mathcal{F} = \{f_i \,|\, i \in I\}$ is an L-set of semi-valuations of $\mathcal{S}$. Then there exists a line L of $\mathcal{S}$ such that $|\mathcal{F}| = |L|$.*

Proof. For every $i \in I$, we put $M_i := M_{f_i}$. Let j^* be an arbitrary element of I and let x^* be a point of $\mathcal{S}$ such that $f_{j^*}(x^*) = M_{j^*}$. Let i^* be the unique element of I such that $f_j(x^*) - M_j = f_{i^*}(x^*) - M_{i^*} + 1$ for every $j \in I \setminus \{i^*\}$. Then $i^* \neq j^*$. Taking $j = j^*$, we see that $f_{i^*}(x^*) = M_{i^*} - 1$. By Property (PV3), there exists a line L through x^* not containing a point with f_{i^*}-value $f_{i^*}(x^*) - 1 = M_{i^*} - 2$. By Property (PV2), we then know that $f_{i^*}(y) = f_{i^*}(x^*) + 1 = M_{i^*}$ for every $y \in L \setminus \{x^*\}$. Observe also that $f_j(x^*) = M_j$ for every $j \in I \setminus \{i^*\}$.

Consider now the map $\phi : L \to I$ that maps each point $y \in L$ to the unique element $\phi(y)$ of I such that $f_j(y) - M_j = f_{\phi(y)}(y) - M_{\phi(y)} + 1$ for every $j \in I \setminus \{\phi(y)\}$. We prove that ϕ is a bijection.

• Observe that $\phi(x^*) = i^*$.

• Suppose $\phi(y) = i^*$ for some $y \in L \setminus \{x^*\}$. Since $f_{i^*}(y) = M_{i^*}$, we have that $f_j(y) - M_j = f_{i^*}(y) - M_{i^*} + 1 = 1$ for every $j \in I \setminus \{i^*\}$, which is impossible. Hence, $\phi(y) \neq i^*$ for every $y \in L \setminus \{x^*\}$. This implies that $0 = f_{i^*}(y) - M_{i^*} = f_{\phi(y)}(y) - M_{\phi(y)} + 1$, i.e. $f_{\phi(y)}(y) = M_{\phi(y)} - 1$ for every $y \in L \setminus \{x^*\}$.

• We prove that ϕ is injective. Let $i \in I$ and suppose $y_1, y_2 \in L$ such that $\phi(y_1) = \phi(y_2) = i$. If $i = i^*$, then we know by the previous paragraph that $y_1 = y_2 = x^*$. So, we may suppose that $i \neq i^*$ and $y_1 \neq x^* \neq y_2$. By the previous paragraph, we also know that $f_i(y_1) = f_i(y_2) = M_i - 1$. Since $f_i(x^*) = M_i$, the points y_1 and y_2 must coincide with the unique point of L with smallest f_i-value.

• We prove that ϕ is surjective. Let $i \in I$. Since $\phi(x^*) = i^*$, we may suppose that $i \neq i^*$. Then $f_i(x^*) = M_i$. Let y denote the unique point of L such that $f_i(y) = M_i - 1$. Then $y \neq x^*$. In order to show that $\phi(y) = i$, we must show that $f_j(y) = M_j$ for every $j \in I \setminus \{i\}$. Clearly, this holds if $j = i^*$. So, we may suppose that $j \neq i^*$. Since $f_j(x^*) = M_j$, we must show that the case $f_j(y) = M_j - 1$ is impossible. Suppose $f_j(y) = M_j - 1$. Then $y \in H_{f_j} \cap H_{f_i}$. By Proposition 4.7, also $y \in H_{f_{i^*}}$. This is impossible since $f_{i^*}(y) = M_{i^*}$. ∎

Let f_i, $i \in I$, be a collection of mutually distinct polygonal valuations of $\mathcal{S}$. We say that the set $\mathcal{F} = \{f_i \,|\, i \in I\}$ is *admissible* if the following holds for all $i_1, i_2 \in I$ with $i_1 \neq i_2$, for every $x \in \mathcal{M}_{f_{i_1}}$ and every $y \in \mathcal{M}_{f_{i_2}}$:

(1) if f_{i_1} and f_{i_2} are classical, then $\mathrm{d}(x, y) = 1$;

(2) if $x = y$, then $(f_{i_1}(x) - M_{f_{i_1}}) - (f_{i_2}(x) - M_{f_{i_2}}) \in \{-1, 0, 1\}$;

(3) if $x \neq y$ and at least one of f_{i_1}, f_{i_2} is not classical, then $\mathrm{d}(x, y) + f_{i_1}(x) + f_{i_2}(y) - M_{f_{i_1}} - M_{f_{i_2}} + 1 \geq 0$.

Observe that if $x = y$, then since $f_{i_1} \neq f_{i_2}$, we necessarily have that at least one of f_{i_1}, f_{i_2} is not classical.

Proposition 4.9 *Let $\mathcal{S}$ be a generalized $2d$-gon and let f_1 and f_2 be two distinct polygonal valuations of $\mathcal{S}$ such that $\{f_1, f_2\}$ is admissible. Then f_1 and f_2 are two neighboring polygonal valuations of $\mathcal{S}$.*

Proof. Suppose first that f_1 and f_2 are classical. If $\mathcal{M}_{f_1} = \{x\}$ and $\mathcal{M}_{f_2} = \{y\}$, then $\mathrm{d}(x, y) = 1$. For every point z of $\mathcal{S}$, we have $|f_1(z) - f_2(z)| = |\,\mathrm{d}(x, z) - \mathrm{d}(y, z)| \leq \mathrm{d}(x, y) = 1$ by the triangle inequality. So, f_1 and f_2 are indeed neighboring polygonal valuations. So, in the sequel we may assume that at least one of f_1, f_2 is not classical.

Suppose f_1 and f_2 are not neighboring polygonal valuations. Then there exists a point x such that $\alpha := (f_1(x) - M_{f_1}) - (f_2(x) - M_{f_2}) \notin \{-1, 0, 1\}$. Consider a geodesic path of length $f_1(x) - f_1(y_1)$ between x and a point $y_1 \in \mathcal{M}_{f_1}$ and a geodesic path of length $f_2(x) - f_2(y_2)$ between x and a point $y_2 \in \mathcal{M}_{f_2}$.

If $y_1 = y_2$, then $f_1(x) - f_1(y_1) = f_2(x) - f_2(y_2) = f_2(x) - f_2(y_1)$ and hence $\alpha = (f_1(y_1) - M_{f_1}) - (f_2(y_1) - M_{f_2}) \notin \{-1, 0, 1\}$, in contradiction with condition (2).

If $y_1 \neq y_2$, then by condition (3) and the fact that $f_i(x) \leq M_{f_i}$, $i \in \{1,2\}$, we find that $0 \leq \mathrm{d}(y_1,y_2)+f_1(y_1)+f_2(y_2)-M_{f_1}-M_{f_2}+1 \leq \mathrm{d}(x,y_1)+\mathrm{d}(x,y_2)+f_1(y_1)+f_2(y_2)-M_{f_1}-M_{f_2}+1 = (f_1(x)-M_{f_1})+(f_2(x)-M_{f_2})+1 \leq 1$. So, $f_1(x)-M_{f_1}$ and $f_2(x)-M_{f_2}$ belong to the set $\{-1,0\}$, but that is impossible since $\alpha = (f_1(x)-M_{f_1})-(f_2(x)-M_{f_2}) \notin \{-1,0,1\}$.

So, also in the case that at least one of f_1, f_2 is not classical, we have that f_1 and f_2 are neighboring semi-valuations. ∎

5 The case of generalized $2d$-gons with three points on each line

In this subsection, we study polygonal valuations of generalized $2d$-gons having precisely three points on each of its lines.

Proposition 5.1 *Let $\mathcal{S}$ be a generalized $2d$-gon having precisely three points on each of its lines. Then any two distinct polygonal valuations f_1 and f_2 of $\mathcal{S}$ are contained in at most one L-set of semi-valuations with minimal value 0. If f_3 is a semi-valuation of $\mathcal{S}$ such that $\{f_1,f_2,f_3\}$ is an L-set of semi-valuations, then H_{f_3} equals the complement $\overline{H_{f_1}\Delta H_{f_2}}$ of the symmetric difference $H_{f_1}\Delta H_{f_2}$ of H_{f_1} and H_{f_2}.*

Proof. Since f_1 and f_2 are distinct, $H_{f_1} \neq H_{f_2}$ by Corollary 3.11. By Proposition 4.8, every L-set of semi-valuations of $\mathcal{S}$ contains precisely three elements. The fact that there exists at most one semi-valuation f_3 of $\mathcal{S}$ with minimal value 0 such that $\{f_1,f_2,f_3\}$ is an L-set of semi-valuations follows from Proposition 4.6. If f_3 is a semi-valuation of $\mathcal{S}$ such that $\{f_1,f_2,f_3\}$ is an L-set of semi-valuations, then by Proposition 4.7, H_{f_3} equals the complement of the symmetric difference of H_{f_1} and H_{f_2}. ∎

Suppose $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ is a generalized $2d$-gon having precisely three points on each of its lines.

For every two maps $f_1 : \mathcal{P} \to \mathbb{Z}$ and $f_2 : \mathcal{P} \to \mathbb{Z}$ such that $|f_1(x)-f_2(x)| \leq 1$, $\forall x \in \mathcal{P}$, we define a new map $f_1 \diamond f_2 : \mathcal{P} \to \mathbb{Z}$ as follows. If $f_1(x) = f_2(x)$, then we define $f_1 \diamond f_2(x) := f_1(x)-1 = f_2(x)-1$. If $|f_1(x)-f_2(x)| = 1$, then we define $f_1 \diamond f_2(x) := \max\{f_1(x), f_2(x)\}$. Clearly, $f_2 \diamond f_1 = f_1 \diamond f_2$. Since $|f_1(x) - f_1 \diamond f_2(x)| \leq 1$ and $|f_2(x) - f_1 \diamond f_2(x)| \leq 1$ for every point x of $\mathcal{S}$, also $f_1 \diamond (f_1 \diamond f_2)$ and $f_2 \diamond (f_1 \diamond f_2)$ are defined. It is straightforward to verify that $f_1 \diamond (f_1 \diamond f_2) = f_2$ and $f_2 \diamond (f_1 \diamond f_2) = f_1$.

If f_1 and f_2 are two semi-valuations of $\mathcal{S}$ such that $|f_1(x)-f_2(x)| \leq 1$, $\forall x \in \mathcal{P}$, then one readily sees that also $f_1 \diamond f_2$ is a semi-valuation of $\mathcal{S}$. Suppose f_1 and f_2 are two neighboring semi-valuations of $\mathcal{S}$. Let $g_1 \in [f_1]$ and $g_2 \in [f_2]$ such that $|g_1(x)-g_2(x)| \leq 1$ for every point x of $\mathcal{S}$. Then $f_1 * f_2$ denotes the unique element of $[g_1 \diamond g_2]$ with minimal value 0. Obviously, $f_1 * f_2$ is independent of the chosen representatives $g_1 \in [f_1]$ and $g_2 \in [f_2]$. If f is a semi-valuation of $\mathcal{S}$ with minimal value 0, then $g_1 * g_2 = f$ for all $g_1, g_2 \in [f]$.

Proposition 5.2 *Let $\mathcal{S}$ be a generalized 2d-gon having precisely three points on each of its lines. Let f_1 and f_2 be two distinct neighboring polygonal valuations of $\mathcal{S}$ and let f_3 be the semi-valuation $f_1 * f_2$ of $\mathcal{S}$. Let $f_1' \in [f_1]$, $f_2' \in [f_2]$ and $f_3' \in [f_3]$ such that $f_3' = f_1' \diamond f_2'$. Then:*

(1) *If M_i, $i \in \{1,2,3\}$, denotes the maximal value attained by f_i', then $M_1 = M_2 = M_3$.*

(2) *$\{f_1, f_2, f_3\}$ is an L-set of semi-valuations.*

(3) *$H_{f_3} = \overline{H_{f_1} \Delta H_{f_2}}$.*

Proof. (1) Since $f_1' = f_2' \diamond f_3'$, $f_2' = f_1' \diamond f_3'$ and $f_3' = f_1' \diamond f_2'$, we have $M_1 \leq \max(M_2, M_3)$, $M_2 \leq \max(M_1, M_3)$ and $M_3 \leq \max(M_1, M_2)$. So, we have that $M_i \leq M_j = M_k$, where i, j, k are such that $\{i, j, k\} = \{1, 2, 3\}$. Suppose $H_{f_j'} \setminus H_{f_k'} \neq \emptyset$ and let x be an arbitrary point of $H_{f_j'} \setminus H_{f_k'}$. Then since $f_j'(x) \leq M_j - 1$ and $f_k'(x) = M_k = M_j$, we have $f_j'(x) = M_j - 1$ and $f_i'(x) = M_j$. Hence, $M_i \geq M_j$ and $M_1 = M_2 = M_3$. In a similar way, one proves that $M_1 = M_2 = M_3$ if the set $H_{f_k}' \setminus H_{f_j'}$ is nonempty. Consider now the case where $H_{f_j'} = H_{f_k'}$, i.e. $H_{f_j} = H_{f_k}$. Then $f_j = f_k$ be Proposition 3.15. But this implies $f_1 = f_2 = f_3$, clearly a contradiction.

(2) By the definition of the $\diamond$-operator and the fact that $M_1 = M_2 = M_3$, we have that $\{f_1', f_2', f_3'\}$ is an L-set of semi-valuations of $\mathcal{S}$. Hence, also $\{f_1, f_2, f_3\}$ is an L-set of semi-valuations of $\mathcal{S}$.

(3) Claim (3) follows from Claim (2) and Proposition 5.1. ∎

Proposition 5.3 *Suppose $\mathcal{S}$ is a generalized 2d-gon having precisely three points on each of its lines. Let H_1, H_2 and H_3 be three mutually distinct hyperplanes of valuation type of $\mathcal{S}$ such that $H_3 = \overline{H_1 \Delta H_2}$. Let f_i, $i \in \{1,2,3\}$, be the unique polygonal valuation of $\mathcal{S}$ for which $H_{f_i} = H_i$. If $\{f_1, f_2, f_3\}$ is admissible, then $\{f_1, f_2, f_3\}$ is an admissible L-set.*

Proof. By Proposition 4.9, f_1 and f_2 are neighboring polygonal valuations of $\mathcal{S}$. Put $f_3' := f_1 * f_2$. Then f_3' is a semi-valuation of $\mathcal{S}$ and by Proposition 5.2(3), we have $H_{f_3'} = \overline{H_{f_1} \Delta H_{f_2}} = H_{f_3}$. By Proposition 3.15, we have $f_3' = f_3$. So, $\{f_1, f_2, f_3\}$ is an L-set of semi-valuations of $\mathcal{S}$ by Proposition 5.2(2). ∎

6 Generalized $2d$-gons containing a sub-$2d$-gon

In this subsection, we show how polygonal valuations and admissible L-sets of polygonal valuations naturally arise in some concrete situations. The propositions and corollary of this subsection offer an indication of how a theory of polygonal valuations can be helpful for classifying those generalized $2d$-gons that contain a particular generalized $2d$-gon as subgeometry.

Proposition 6.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized 2d-gon and let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a full sub-2d-gon of $\mathcal{S}$. Let x be a point of $\mathcal{S}$ and put $m := \min\{d(x,y) \,|\, y \in \mathcal{P}'\}$. For every point $y \in \mathcal{P}'$, we define $f_x(y) := d(x,y) - m$. Then:*

(1) *f_x is a polygonal valuation of $\mathcal{S}'$ with $M_{f_x} = d - m$.*

(2) *The polygonal valuation f_x is classical if and only if x is a point of $\mathcal{S}'$, semi-classical if and only if $d(x, \mathcal{P}') = 1$ and ovoidal if and only if $d(x, \mathcal{P}') = d - 1$.*

(3) *If x_1 and x_2 are two distinct collinear points of $\mathcal{S}$, then the polygonal valuations f_{x_1} and f_{x_2} are distinct.*

Proof. (1) That f_x satisfies Property (PV1) is an immediate consequence of the definition of f_x. The fact that $\mathcal{S}$ is a near $2d$-gon implies that f_x also satisfies Property (PV2).

In order to prove that f_x satisfies Property (PV3), consider a point y of $\mathcal{S}'$ for which $f_x(y) < M$, where M denotes the maximal value attained by f_x. We need to prove that there is at most one line through y containing a point with f_x-value $f_x(y) - 1$. Obviously, this is the case if $f_x(y) = 0$. So, we may suppose that $f_x(y) > 0$. This implies that also $\mathrm{d}(x, y) > 0$. For every point z of $\mathcal{S}'$ satisfying $f_x(z) = M$, we have $\mathrm{d}(x, y) < d(x, z)$ and hence $\mathrm{d}(x, y) < d - 1$. Since $0 < \mathrm{d}(x, y) < d - 1$, there exists a unique line L through y containing a point at distance $\mathrm{d}(x, y) - 1$ from x. If L is a line of $\mathcal{S}'$, then there is a unique line of $\mathcal{S}'$ through y (namely L) containing a point with value $f_x(y) - 1$. If L is not a line of $\mathcal{S}'$, then there is no such line.

This finishes the proof that f_x is a polygonal valuation of $\mathcal{S}'$. In order to prove that the maximal value attained by f_x is equal to $d - m$, we need to prove that there exists a point u in $\mathcal{S}'$ at (maximal) distance d from x. Let u be one of the points of $\mathcal{S}'$ at maximal distance from x. Then $x \neq u$. If $\mathrm{d}(x, u) < d-1$, then there exists a unique line through u containing a point at distance $\mathrm{d}(x, u) - 1$ from x. Every other line L through x contains a point at distance $\mathrm{d}(x, u)+1$ from x. In particular, this holds if we take L such that it is a line of $\mathcal{S}'$. So, we have our desired contradiction and we can conclude that $\mathrm{d}(x, u) = d$, i.e. that the maximal value attained by f_x is equal to $d - m$.

(2) Claim (2) of the lemma follows from Claim (1) and Proposition 3.3.

(3) Suppose the polygonal valuations f_{x_1} and f_{x_2} are equal. Then by Claim (1), the points x_1 and x_2 have the same distance δ from $\mathcal{P}'$. For every point x' of $\mathcal{S}'$, we have $\mathrm{d}(x_1, x') = f_{x_1}(x') + \delta = f_{x_2}(x') + \delta = \mathrm{d}(x_2, x')$. So, $\delta \neq 0$ and x_1, x_2 are not contained in $\mathcal{S}'$. Now, let y be an arbitrary point of $\mathcal{O}_{f_{x_1}} = \mathcal{O}_{f_{x_2}}$ and let x_3 be the unique point of the line x_1x_2 at smallest distance from y. Since $\mathrm{d}(x_1, y) = \mathrm{d}(x_2, y) = \delta$, we have $\mathrm{d}(x_3, y) = \delta - 1$. The point x_3 is contained in a geodesic path from x_1 to y. The points x_3 and y are connected by a unique geodesic path γ of length $\delta - 1$. Let γ' be a geodesic path of length $d - \delta + 1$ in $\mathcal{S}'$ starting from y and ending in a point z. Let u denote the neighbor of z contained in this geodesic path γ'. Now, the concatenation of the paths γ' and γ is a geodesic path γ'' of maximal length d. The geodesic path γ'' and the point x_1 are contained in a unique ordinary sub-$2d$-gon of $\mathcal{S}$. The lines x_1x_3 and zu are opposite lines in this ordinary subpolygon. So, the unique point of the line zu at distance $d - 1$ from x_1 is distinct from the unique point of zu at distance $d-1$ from x_2. This contradicts the fact that $\mathrm{d}(x_1, x') = \mathrm{d}(x_2, x')$ for every point x' of $\mathcal{S}'$. So, the polygonal valuations f_{x_1} and f_{x_2} are distinct. ∎

If x_1 and x_2 are two noncollinear points of $\mathcal{S}$, then the polygonal valuations f_{x_1} and f_{x_2} need not to be distinct.

Proposition 6.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon and let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a full sub-$2d$-gon of $\mathcal{S}$. For every line L of $\mathcal{S}$, put $\mathcal{F}_L := \{f_y \,|\, y \in L\}$, where f_y is the polygonal valuation of $\mathcal{S}'$ as defined in Proposition 6.1. Then $\mathcal{F}_L$ is an admissible L-set of polygonal valuations.*

Proof. For every point x of $\mathcal{S}'$ and every point y of L, we have

$$\mathrm{d}(x, y) = f_y(x) + \mathrm{d}(y, \mathcal{P}') = (f_y(x) - M_{f_y}) + d. \tag{1}$$

So, since $\mathcal{S}$ is a near polygon, there exists a unique point $y^* \in L$ such that $f_y(x) - M_{f_y} = f_{y^*}(x) - M_{f_{y^*}} + 1$ for every $y \in L \setminus \{y^*\}$. This proves that $\mathcal{F}_L$ is an L-set of polygonal valuations.

We now also prove that $\mathcal{F}_L$ is admissible. Let y_1 and y_2 be two distinct points of L, let x_1 be an arbitrary point of $\mathcal{M}_{f_{y_1}}$ and let x_2 be an arbitrary point of $\mathcal{M}_{f_{y_2}}$. We distinguish three cases.

(1) Suppose f_{y_1} and f_{y_2} are classical. Then $y_1, y_2 \in \mathcal{P}'$ by Proposition 6.1(2). In this case we have $x_1 = y_1$, $x_2 = y_2$ and hence $\mathrm{d}(x_1, x_2) = 1$.

(2) Suppose $x_1 = x_2$. Then by equation (1), we have $|(f_{y_1}(x_1) - M_{f_{y_1}}) - (f_{y_2}(x_1) - M_{f_{y_2}})| = |\,\mathrm{d}(x_1, y_1) - \mathrm{d}(x_1, y_2)| \leq \mathrm{d}(y_1, y_2) = 1$. So, $(f_{y_1}(x_1) - M_{f_{y_1}}) - (f_{y_2}(x_1) - M_{f_{y_2}}) \in \{-1, 0, 1\}$.

(3) Suppose $x_1 \neq x_2$ and at least one of f_{y_1}, f_{y_2} is not classical. By Proposition 6.1(2), we then know that at least one of y_1, y_2 is not a point of $\mathcal{S}'$. Consider the following paths:
- the geodesic path γ_1 of length 1 connecting y_1 and y_2;
- the geodesic path γ_2 of length $\mathrm{d}(y_1, x_1) = (f_{y_1}(x_1) - M_{f_{y_1}}) + d$ connecting y_1 and x_1;
- the geodesic path γ_3 of length $\mathrm{d}(y_2, x_2) = (f_{y_2}(x_2) - M_{f_{y_2}}) + d$ connecting y_2 to x_2;
- a geodesic path γ_4 of length $\mathrm{d}(x_1, x_2)$ connecting x_1 and x_2 completely contained in $\mathcal{S}'$.

These geodesic paths define a closed path γ of length $\mathrm{d}(x_1, x_2) + f_{y_1}(x_1) + f_{y_2}(x_2) - M_{f_{y_1}} - M_{f_{y_2}} + 2d + 1$. Since at least one of y_1, y_2 is not a point of $\mathcal{S}'$, the part of γ corresponding to γ_1, γ_2 and γ_3 is not contained in $\mathcal{S}'$. In fact, only the end points of that part are also points of $\mathcal{S}'$. Since the part of γ corresponding to γ_4 is completely contained in $\mathcal{S}'$, the closed path γ defines a cycle of length at most $\mathrm{d}(x_1, x_2) + f_{y_1}(x_1) + f_{y_2}(x_2) - M_{f_{y_1}} - M_{f_{y_2}} + 2d + 1$. This implies that $\mathrm{d}(x_1, x_2) + f_{y_1}(x_1) + f_{y_2}(x_2) - M_{f_{y_1}} - M_{f_{y_2}} + 2d + 1 \geq 2d$, i.e. $\mathrm{d}(x_1, x_2) + f_{y_1}(x_1) + f_{y_2}(x_2) - M_{f_{y_1}} - M_{f_{y_2}} + 1 \geq 0$. So, $\mathcal{F}_L$ is admissible. ■

For every generalized $2d$-gon $\mathcal{S}$, we can consider the following point-line geometry $\mathcal{V}_{\mathcal{S}}$:

- the points of $\mathcal{V}_{\mathcal{S}}$ are the polygonal valuations of $\mathcal{S}$;
- the lines of $\mathcal{V}_{\mathcal{S}}$ are the admissible L-sets of polygonal valuations of $\mathcal{S}$;

- incidence is containment.

The point-line geometry $\mathcal{V}_{\mathcal{S}}$ is called the *valuation geometry* of $\mathcal{S}$. By Propositions 6.1 and 6.2, the valuation geometry $\mathcal{V}_{\mathcal{S}}$ provides information on how $\mathcal{S}$ can be embedded as a full subgeometry into a larger generalized $2d$-gon. More precisely, we can say the following.

Corollary 6.3 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon and let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a full sub-$2d$-gon of $\mathcal{S}$. For every point x of $\mathcal{S}$, let f_x denote the polygonal valuation of $\mathcal{S}'$ as defined in Proposition 6.1. Then the map $\theta : x \mapsto f_x$ between the point sets of $\mathcal{S}$ and $\mathcal{V}_{\mathcal{S}'}$ maps every line of $\mathcal{S}$ to a full line of $\mathcal{V}_{\mathcal{S}'}$.*

The map θ in Corollary 6.3 needs not to be injective, nor surjective.

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