New upper bounds on the sizes of caps in $PG(N, 5)$ and $PG(N, 7)$

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Abstract

Let $m_2(N, q)$ denote the size of the largest caps in $PG(N, q)$ and let $m'_2(N, q)$ denote the size of the second largest complete caps in $PG(N, q)$. Presently, it is known that $m_2(4, 5) \leq 111$ and that $m_2(4, 7) \leq 316$. Via computer searches for caps in $PG(4, 5)$ using the result of Abatangelo, Larato and Korchmáros that $m'_2(3, 5) = 20$, we improve the first upper bound to $m_2(4, 5) \leq 88$. Computer searches in $PG(3, 7)$ show that $m'_2(3, 7) = 32$ and this latter result then improves the upper bound on $m_2(4, 7)$ to $m_2(4, 7) \leq 238$. We also present the known upper bounds on $m_2(N, 5)$ and $m_2(N, 7)$ for $N > 4$.

1 Introduction

An *n-cap* in the projective space $PG(N, q)$ of dimension N over the finite field of order q is a set of n points, no three of which are collinear. A cap is called *complete* when it is not contained in a larger cap of the same projective space. The largest size of caps in $PG(N, q)$ is denoted by $m_2(N, q)$. The size of the second largest complete caps in $PG(N, q)$ is denoted by $m'_2(N, q)$. Thus any n-cap with $n > m'_2(N, q)$ can be extended to a cap of size $m_2(N, q)$.

Presently, only the following exact values of $m_2(N, q)$ are known. In $PG(2, q)$, q odd, there are at most $(q+1)$-caps [8]. In $PG(2, q)$, q even, there are at most $(q+2)$-caps [8]. In $PG(3, q)$, $q > 2$, the maximal size of a cap is $q^2 + 1$ [8, 32], and in $PG(N, 2)$, the maximal size of a cap is 2^N [8].

In some spaces $PG(N, q)$, a complete characterization of the $m_2(N, q)$-caps is known. Namely, in $PG(2, q)$, q odd, every $(q + 1)$-cap is a conic

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[33, 34]. In $PG(2,q)$, q even, $q \geq 16$, distinct types of $(q+2)$-caps exist; see [27] for a list of the known infinite classes of $(q+2)$-caps. In $PG(3,q)$, q odd, every (q^2+1)-cap is an elliptic quadric [3, 30]. In $PG(3,q)$, $q = 2^h$, h odd, $h \geq 3$, at least one type of (q^2+1)-caps different from the elliptic quadrics exists, called the *Tits ovoid* [38]. In $PG(N,2)$, every 2^N-cap is the complement of a hyperplane [35].

Apart from these results which are valid either for arbitrary q or for arbitrary dimension N, some sporadic results are known. Namely, the maximal size of a cap in $PG(4,3)$ is 20 [31], the maximal size of a cap in $PG(5,3)$ is 56 [20], and the maximal size of a cap in $PG(4,4)$ is 41 [14].

Regarding the characterizations, exactly 9 types of 20-caps exist in $PG(4,3)$ [22], the 56-cap in $PG(5,3)$ is projectively unique [21], and there are exactly 2 distinct types of 41-caps in $PG(4,4)$ [13].

In the other cases, only upper bounds on the sizes of caps in $PG(N,q)$ are known. We refer to [27] for a list of the known results. We also wish to state the following result published in [4, 5] which gives the best upper bounds on the size of caps in $PG(N,q)$, for large enough N.

Theorem 1.1 *For $q > 3$ and $N \geq 3$,*

$$m_2(N,q) \leq q^N \cdot \frac{N+1}{N^2} + q^{N-1} \cdot \frac{3 \cdot N}{2(N-1)^2}.$$

The following tables show for small values of q and N the known values of $m'_2(N,q)$. Table 1 is [27, Table 2.4]. For the exact references for Table 1, we refer to [27, Table 2.4].

q	7	8	9	11	13	16	17	19	23	25	27	29
$m'_2(2,q)$	6	6	8	10	12	13	14	14	17	21	22	24

Table 1: $m'_2(2,q)$ in small planes

N \ q	3	4	5	7
3	8	14	20	32
4	19	40		
5	48			

Table 2: $m'_2(N,q)$

For the values of Table 2, we refer to [17] for $(N,q) = (3,3)$, [25] for $(N,q) = (3,4)$, [1] for $(N,q) = (3,5)$, [37] for $(N,q) = (4,3)$, [16] for $(N,q) =$

(4, 4), and [2] for $(N, q) = (5, 3)$. The latter value $m'_2(3, 7) = 32$ is presented in this article (Theorem 3.5).

Apart from these results, it is also known that

(1) $m'_2(2, 2^{2h}) = 2^{2h} - 2^h + 1$ for $h > 1$ [7, 18, 28],

(2) $m'_2(N, 2) = 2^{N-1} + 2^{N-3}$, $N \geq 3$ [10].

There exists a 66-cap in $PG(4, 5)$ [15], and a result of Gronchi [19] shows that $m_2(4, 5) \leq 111$. So presently,

$$66 \leq m_2(4, 5) \leq 111.$$

We will lower the upper bound to 88 by using computer searches using geometrical arguments which include the result of Abatangelo, Larato and Korchmáros that $m'_2(3, 5) = 20$ [1]. This then leads to

$$66 \leq m_2(4, 5) \leq 88.$$

Presently, from [15, 19],

$$132 \leq m_2(4, 7) \leq 316.$$

We will improve this to

$$132 \leq m_2(4, 7) \leq 238.$$

We obtain this improvement by using computer searches which determine the precise value of $m'_2(3, 7)$. Our computer searches show that

$$m'_2(3, 7) = 32.$$

2 Caps in $PG(N, 5)$

Presently, the following results on caps in $PG(3, 5)$ and $PG(4, 5)$ are known:

(a) since $m'_2(3, 5) = 20$, every 21-cap in $PG(3, 5)$ is a subset of an elliptic quadric, and

(b) $66 \leq m_2(4, 5) \leq 111$.

We will improve the upper bound on $m_2(4, 5)$ to $m_2(4, 5) \leq 88$. This upper bound will be obtained by eliminating the existence of 89-caps in $PG(4, 5)$ by means of computer searches.

We first prove a number of results which are useful for the computer searches.

Lemma 2.1 *For every* 19*-cap in* $PG(3,5)$*, there is a plane intersecting this cap in a conic.*

Proof: Assume that K is a 19-cap such that every plane intersects K in at most 5 points. Then an elementary counting shows that every bisecant to K lies in exactly one plane sharing 4 points with K and in five planes sharing 5 points with K. This implies that the number of bisecants must be a multiple of 6 which is the number of bisecants in a 4-plane.

But the number of bisecants to a 19-cap is 171 and this is not a multiple of 6. $\square$

Lemma 2.2 *For every* 84*-cap in* $PG(4,5)$*, there is at least one plane intersecting this cap in a conic.*

Proof: Through every bisecant to a 84-cap K, there is at least one plane intersecting this cap in at least 5 points. Denote this plane by π. Consider the six solids through π. Then there is at least one solid through π sharing at least 19 points with K. The preceding lemma now shows that there is at least one plane sharing a conic with K. $\square$

Let K be a cap in $PG(4,5)$, let π be a plane intersecting K in a conic, and let π_1 and π_2 be two solids through π both sharing at least 21 points with K. These solids intersect K in subsets of elliptic quadrics. Denote these two elliptic quadrics in π_1 and π_2 respectively by Q_1 and Q_2.

These two 3-dimensional elliptic quadrics Q_1 and Q_2 define a pencil of six 4-dimensional quadrics pairwise intersecting in $Q_1 \cup Q_2$. We now determine which quadrics precisely occur within this pencil.

Lemma 2.3 *The two* 3*-dimensional elliptic quadrics* Q_1 *and* Q_2 *in the solids* π_1 *and* π_2 *define a pencil of six* 4*-dimensional quadrics consisting of the solid pair* $\pi_1 \cup \pi_2$*, three non-singular parabolic quadrics, and two cones with base a non-singular* 3*-dimensional elliptic quadric and a point as vertex.*

Proof: These two elliptic quadrics Q_1 and Q_2 together contain $26+20 = 46$ points since they intersect in a conic. One of the quadrics in the pencil defined by Q_1 and Q_2 is $\pi_1 \cup \pi_2$ containing 281 points.

Now $|PG(4,5) \setminus (\pi_1 \cup \pi_2)| = 500$.

Assume that, besides $\pi_1 \cup \pi_2$, the pencil defined by Q_1 and Q_2 contains x non-singular 4-dimensional parabolic quadrics and y cones with base a non-singular 3-dimensional elliptic quadric and a point as vertex. Then

$$\begin{cases} x+y & = & 5 \\ x(156-46)+y(131-46) & = & 500, \end{cases}$$

where 156 is the cardinality of a non-singular 4-dimensional parabolic quadric and where 131 is the cardinality of a cone with base a non-singular 3-dimensional elliptic quadric and a point as vertex.

This implies $(x, y) = (3, 2)$. □

We now state a lemma involving a particular size for the cap K. The main goal of this lemma is to present some of the ideas used in the computer searches, and to motivate the following subsections. The ideas of this lemma will also be used for other sizes of caps. We will present these analogous results, by referring to this lemma.

Lemma 2.4 *Let K be a 67-cap in $PG(4,5)$ intersecting at least one plane π in a conic. Let S_1 and S_2 be two solids through π with $|S_1 \cap K| \geq 24$ and $|S_2 \cap K| \geq 21$, and let Q_1 and Q_2 be the two elliptic quadrics containing the intersections $S_1 \cap K$ and $S_2 \cap K$.*

Then there exists a 4-dimensional non-singular parabolic quadric Q through Q_1 and Q_2 containing at least two points of $K \setminus (S_1 \cup S_2)$ if $|S_1 \cap K| = 26$, and containing at least one point of $K \setminus (S_1 \cup S_2)$ if $|S_1 \cap K| \in \{24, 25\}$.

Proof: Suppose that $|S_1 \cap K| = 26$, then $|(Q_1 \cup Q_2) \cap K| = x \geq 26 + 15 = 41$.

A quadratic cone with base a non-singular 3-dimensional elliptic quadric $Q^-(3,5)$ has at most 52 points in common with K, so the two quadratic cones in the pencil defined by Q_1 and Q_2 contain at most $x + 2(52 - x) \leq 63$ points of K. So at least 4 points of K lie on one of the parabolic quadrics contained in the pencil. So one of those three parabolic quadrics contains at least 2 points of $K \setminus (Q_1 \cup Q_2)$.

A similar argument discusses the case $|S_1 \cap K| \in \{24, 25\}$. □

For a 4-dimensional parabolic quadric Q through $Q_1 \cup Q_2$, the set $Q \setminus (Q_1 \cup Q_2)$ contains $156 - 46 = 110$ points. So, when performing a computer search for such a 67-cap K, we need to find at least 1 or 2 points of K within a set of 110 points.

We now describe the starting configurations for the computer searches which will eliminate the existence of particular caps K in $PG(4,5)$, intersecting at least one plane π in a conic, and such that at least two solids through π intersect K in at least 21 points.

2.1 Two general starting configurations

Consider a non-singular 4-dimensional parabolic quadric $Q = Q(4,5)$ and consider a plane π intersecting Q in a non-singular conic. This plane is the polar plane of a bisecant or external line to Q [26, Theorem 22.6.6]. This shows that under the group $\mathrm{PGO}(5,5)$ stabilizing Q, there are exactly two orbits of planes intersecting Q in a non-singular conic.

A plane π intersecting Q in a non-singular conic corresponding to a bisecant polar line of Q, lies in six solids intersecting Q in respectively two tangent cones, two elliptic and two hyperbolic quadrics. A plane π intersecting Q in a non-singular conic corresponding to an external polar line of Q, lies in six solids intersecting Q in respectively three elliptic and three hyperbolic quadrics.

2.2 A plane corresponding to a bisecant polar line

Let $Q : X_1^2 - X_0X_2 + X_3X_4 = 0$. Let $\pi : X_3 = X_4 = 0$, then π is the polar plane of the bisecant $\langle e_3 = (0,0,0,1,0), e_4 = (0,0,0,0,1)\rangle$ to Q.

Let $C = \pi \cap Q$, then C lies in two elliptic quadrics, namely in the elliptic quadrics

$$\left\{ \begin{array}{rcl} X_3 & = & 2X_4 \\ X_1^2 - X_0X_2 + 2X_4^2 & = & 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} X_3 & = & 3X_4 \\ X_1^2 - X_0X_2 + 3X_4^2 & = & 0. \end{array} \right.$$

Using the subgroup G of the stabilizer group $\mathrm{PGO}(5,5)$ which fixes the pair $\{e_3, e_4\}$ and Q, it is possible to assume that $|K \cap (X_3 - 2X_4 = 0)| \geq |K \cap (X_3 - 3X_4 = 0)| \geq 21$.

2.3 A plane corresponding to an external polar line

Let $Q : X_1^2 - X_0X_2 + X_4^2 - 3X_3^2 = 0$. Let $\pi : X_3 = X_4 = 0$, then π is the polar plane of the external line $\langle e_3, e_4\rangle$ to Q.

Let $C = \pi \cap Q$, then C lies in three elliptic quadrics contained in Q.

The subgroup G of $\mathrm{PGO}(5,5)$ which fixes the line $\langle e_3, e_4\rangle$ and fixes the quadric Q acts as the symmetric group S_3 on the three hyperplanes π_1, π_2, π_3 through π intersecting Q in an elliptic quadric. So it is possible to select the two hyperplanes π_1 and π_2 through π for which $|K\cap\pi_1| \geq |K\cap\pi_2| \geq |K\cap\pi_3|$, without losing generality.

For $\pi_1 : X_4 = 0$ was selected and for $\pi_2 : X_3 = X_4$.

2.4 The computer search results

The preceding ideas were used to perform a computer search for caps in $PG(4,5)$. This led to the following results.

Theorem 2.5 (a) *There is no* 67*-cap* K *in* $PG(4,5)$ *for which there exist two solids* S_1 *and* S_2*, where* $\pi = S_1 \cap S_2$ *has a conic in common with* K*, where* S_1 *has at least* 24 *points in common with* K *and where* S_2 *has at least* 21 *points in common with* K*.*

(b) *There is no* 84*-cap* K *in* $PG(4,5)$ *for which there exist two solids* S_1 *and* S_2*, where* $\pi = S_1 \cap S_2$ *has a conic in common with* K*, and where* S_1 *and* S_2 *have at least* 21 *points in common with* K*.*

These latter computer searches used the ideas of Lemma 2.4. Let $Q^-(3,5)_1$ be the elliptic quadric containing $K \cap S_1$ and let $Q^-(3,5)_2$ be the elliptic quadric containing $K \cap S_2$. In Case (a), it was possible to assume that there is a parabolic quadric through $Q^-(3,5)_1$ and $Q^-(3,5)_2$ containing at least 2 points of $K \setminus (S_1 \cup S_2)$. In Case (b), it was possible to assume that there is a parabolic quadric through $Q^-(3,5)_1$ and $Q^-(3,5)_2$ containing at least 6 points of $K \setminus (S_1 \cup S_2)$.

We now present further computer search results. We first explain a particular notation.

Let K be a cap of $PG(4,5)$ intersecting at least one plane π in a conic. Let $S_1, \ldots, S_6$ be the hyperplanes through π. Assume that $|S_i \cap K| = s_i$. Then we say that K contains a *conic plane of type* $(s_1, \ldots, s_6)$.

Note that by Lemma 2.2, every 84-cap intersects at least one plane in a conic.

Theorem 2.6 *In* $PG(4,5)$*,*

(a) *there is no* 82*-cap having a conic plane of type* $(25,18,18,18,17,16)$*,*

(a) *there is no* 84*-cap having a conic plane of type* $(24,19,19,19,18,15)$*,*

(c) *there is no* 84*-cap having a conic plane of type* $(22,20,20,20,19,13)$*,*

(d) *there is no* 84*-cap having a conic plane of type* $(20,20,20,18,18,18)$*, and*

(e) *there is no* 89*-cap having a conic plane of type* $(23,20,19,19,19,19)$*.*

The preceding lemmas now imply that there are no 89-caps in $PG(4,5)$.

Theorem 2.7

$$m_2(4,5) \leq 88.$$

Proof: Assume that there is a 89-cap K in $PG(4,5)$. Then there is at least one plane π sharing a conic with K (Lemma 2.2). The results of Theorem 2.5 show that π does not lie in two hyperplanes sharing at least 21 points with K.

We now use the results of Theorem 2.6. Consider all possible types $(s_1,\ldots,s_6)$, with $s_1 \geq s_2 \geq s_3 \geq s_4 \geq s_5 \geq s_6$, for the conic plane π.

Then $s_1 \geq 20$, and by assumption, $s_2 \leq 20$. All possible types for the conic plane lead to a contradiction.

For instance, assume that the type is $(s_1,\ldots,s_6) = (26,20,20,20,20,13)$. Then, by deleting 4 points, not in π, of the cap in the 26-hyperplane and one point, not in π, in a 20-hyperplane, a 84-cap of conic type $(22,20,20,20,19,13)$ is obtained. This contradicts Theorem 2.6 (c). □

Corollary 2.8

$$66 \leq m_2(4,5) \leq 88.$$

2.5 Bounds on $m_2(N,5)$

We now present the known bounds on $m_2(N,5)$, $N > 4$.

Theorem 2.9 *For* $5 \leq N \leq 9$,

$$m_2(N,5) \leq 4 \cdot 5^{N-2} - 2 \cdot 5^{N-3} - \frac{7}{2} \cdot 5^{N-4} + \frac{3}{2}.$$

For $10 \leq N \leq 12$,

$$m_2(N,5) \leq \frac{5^N \cdot (N+1)}{N^2} + 4 \cdot 5^{N-3} - 2 \cdot 5^{N-4} - \frac{7}{2} \cdot 5^{N-5} + \frac{3}{2}.$$

For $N \geq 13$,

$$m_2(N,5) \leq \frac{5^N \cdot (N+1)}{N^2} + \frac{3 \cdot N \cdot 5^{N-1}}{2 \cdot (N-1)^2}.$$

Proof: The first formula arises from the formula of Hill [21]. The second formula arises from the bound of Bierbrauer-Edel on caps in affine spaces [6] plus the formula of Hill for a cap in a hyperplane in $PG(N,q)$. The third formula is from Theorem 1.1. □

3 Caps in $PG(N,7)$

In this section, we show that

(a) $m'_2(3,7) = 32$, so every 33-cap in $PG(3,7)$ is a subset of an elliptic quadric, and

(b) $132 \leq m_2(4,7) \leq 238$.

3.1 The determination of $m'_2(3,7)$

We describe how the exact value of $m'_2(3,7)$ was determined.

It is known that every 7-cap in $PG(2,7)$ is contained in a conic [24, Theorem 10.28]. The computer searches for complete n-caps K in $PG(3,7)$, with $n \geq 33$, first of all relied on this property.

We started from a bisecant L to K lying in two planes π_1 and π_2 sharing at least 7 points with K. These latter planes intersect K in subsets of conics C_1 and C_2. Two conics, which share two distinct points and which lie in distinct planes, define a pencil of quadrics in $PG(3,7)$. The pencils of quadrics in $PG(3,q)$ were classified by Bruen and Hirschfeld. In [9, Theorem 4.4], they showed that there exist precisely two distinct pencils of quadrics intersecting in two distinct conics in two distinct planes, where these two conics share two distinct points. Their results [9, p. 262, Cases 3(c)(i) and 3(c)(iii)] imply that we can assume that C_1 and C_2 are one of the following:

$$\left\{ \begin{array}{rcl} X_0X_1 & = & 0 \\ X_0^2 + X_1^2 + X_2X_3 & = & 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} X_0X_1 & = & 0 \\ X_0^2 - X_1^2 + X_2X_3 & = & 0. \end{array} \right.$$

We now prove that for caps of size at least 37, this bisecant L and these latter two planes π_1 and π_2 really exist.

Lemma 3.1 *Every bisecant of an n-cap K in $PG(3,7)$ of size at least 37 lies in at least two planes π_1 and π_2 containing at least 7 points of K.*

Proof: A bisecant lies in 8 planes; so one of those planes contains at least $2+35/8 > 6$ points of K. Denote this plane by π_1. Then there is still a second plane π_2 through the bisecant containing at least $2+29/7 > 6$ points of K. $\square$

We determined the stabilizer group G of the two possible configurations $C_1 \cup C_2$. The stabilizer group G has in both cases transformations interchanging C_1 and C_2, acts in both cases transitively on the 6 points in $C_1 \setminus C_2$, so if $|C_1 \cap K| = 7$, then these results show that it is possible to select, without losing generality, the unique point of $C_1 \setminus K$. Once this point r is selected, the stabilizer group $H = G_r$ has two orbits on $C_2 \setminus C_1$.

For the different cases, $(|C_1 \cap K|, |C_2 \cap K|) = (8,7)$, and $|C_1 \cap K| = |C_2 \cap K| = 7$, representatives were determined, and then, also for the case $|C_1 \cap K| = |C_2 \cap K| = 8$, computer searches were performed to find the size of the largest complete caps extending these starting configurations.

These computer searches showed:

Lemma 3.2 (i) *There is no complete n-cap K, with $33 \leq n \leq 49$, sharing 7 points with C_1 and C_2, and containing the two points of $C_1 \cap C_2$.*

(ii) *There is no complete n-cap K, with $34 \leq n \leq 49$, sharing 8 points with C_1 and 7 points with C_2.*

(iii) *There is no complete n-cap K, with $35 \leq n \leq 49$, sharing 8 points with C_1 and C_2.*

This now implies the following result.

Lemma 3.3 $m'_2(3,7) \leq 34$.

Proof: The preceding two lemmas already imply that $m'_2(3,7) \leq 36$ (Lemma 3.1).

Assume that there is a complete 36-cap K in $PG(3,7)$, then the preceding lemma implies that a bisecant lies in at most one plane sharing at least 7 points with K. This then implies that every bisecant lies in exactly one plane sharing 8 points with K. So the number of bisecants $36 \cdot 35/2$ to a 36-cap must be a multiple of $8 \cdot 7/2$, which is the number of bisecants to a 8-cap in a plane. This is however false.

Assume that there is a complete 35-cap K in $PG(3,7)$, then the preceding lemma implies that every bisecant lies in either:

(a) one plane sharing 8 points with K, one plane sharing 5 points with K, and in 6 planes sharing 6 points with K, or

(b) one plane sharing 7 points with K and 7 planes sharing 6 points with K.

Let u be the number of planes containing 8 points of K, let v be the number of planes sharing 7 points with K, and let w be the number of planes sharing 5 points with K. By counting the bisecants in two ways, we obtain:

$$\begin{aligned} u \cdot 8 \cdot 7/2 + v \cdot 7 \cdot 6/2 &= 35 \cdot 34/2 \\ w \cdot 5 \cdot 4/2 + v \cdot 7 \cdot 6/2 &= 35 \cdot 34/2. \end{aligned}$$

The unique solution to this system of equations, consisting of non-negative integers, is $(u, v, w) = (10, 15, 28)$.

We now count the number N of ordered pairs (π, p), where π is a plane containing 8 or 7 points of K, where $p \in K$, and where $p \in \pi$. Necessarily $N = 10 \cdot 8 + 15 \cdot 7 = 185$.

On the other hand, let $n(p)$ be the number of planes through $p \in K$ containing 8 or 7 points of K. As two such planes through p have no other point of K in common, $n(p) < 6$.

So $N = \sum_{p \in K} n(p) \leq 35 \cdot 5 = 175$. A contradiction is obtained. $\square$

To prove that $m'_2(3,7) = 32$, we still have to exclude the existence of complete 33- and 34-caps. Computer searches gave the following results.

Lemma 3.4 (1) *There is no complete* 33*-cap* K *in* $PG(3,7)$ *having a bisecant lying simultaneously in a plane* π_1 *which shares* 8 *points with* K *and lying in a plane* π_2 *which shares* 7 *points with* K.
(2) *There is no complete* 33*-cap or complete* 34*-cap* K *in* $PG(3,7)$ *having a bisecant lying in two planes which share* 8 *points with* K.

Theorem 3.5 $m'_2(3,7) = 32$.

Proof: Assume that there exists a complete 34-cap K. The preceding computer search results show that a bisecant lies in either:

(a) one plane sharing 8 points with K, one plane sharing 4 points with K, and six planes sharing 6 points with K,

(b) one plane sharing 8 points with K, two planes sharing 5 points with K, and five planes sharing 6 points with K,

(c) one plane sharing 7 points with K, one plane sharing 5 points with K, and six planes sharing 6 points with K,

(d) eight planes sharing 6 points with K.

Let a, b, c, d denote respectively the number of bisecants of type (a), (b), (c) and (d). Let s_i be the number of incident ordered pairs (bisecant L, plane containing i points of K). This number s_i is a multiple of $i(i-1)/2$. Then the following equations are valid:

$$\begin{aligned} a + b &= s_8 \\ c &= s_7 \\ 6a + 5b + 6c + 8d &= s_6 \\ 2b + c &= s_5 \\ a &= s_4 \\ a + b + c + d &= 34 \cdot 33/2. \end{aligned}$$

Let h_i be the number of planes containing i points of K, then $h_i = s_i/(i(i-1)/2)$.

Count the number N of pairs (π, p), where $p \in K$, where π is a plane containing 7 or 8 points of K, and where $p \in \pi$. Then $N = 8h_8 + 7h_7$.

On the other hand, for $p \in K$, let $n(p)$ be the number of planes through p containing 8 or 7 points of K. As two such planes through p do not share a second point of K, necessarily $n(p) \leq 33/6 < 6$. So

$$8h_8 + 7h_7 \leq 34 \cdot 5.$$

Using the same counting method as in the previous paragraph, but now only for the planes π containing 8 points of K, we obtain

$$8h_8 \leq 34 \cdot 4,$$

and the same counting argument, but now for the planes π containing 7 or 4 points of K, implies

$$7h_7 + 4h_4 \leq 34 \cdot 11.$$

Moreover

$$h_4 + h_5 + h_6 + h_7 + h_8 \leq (7^4 - 1)/(7 - 1).$$

The planes containing less than 4 points of K contain 0 or 1 points of K.

There are 171 solutions $(h_4, h_5, h_6, h_7, h_8)$ to the equations above. Consider these solutions, together with the possible solutions for h_0 and h_1. It is sufficient to calculate

$$\sum_{i=0}^{8} h_i = \frac{7^4 - 1}{7 - 1},$$

$$\sum_{i=0}^{8} i h_i = 34 \cdot \frac{7^3 - 1}{7 - 1},$$

to obtain a contradiction for all of these solutions.

Assume that there exists a complete 33-cap K. The preceding computer search results show that a bisecant lies in either:

(a) one plane sharing 8 points with K, one plane sharing 3 points with K, and six planes sharing 6 points with K,

(b) one plane sharing 8 points with K, one plane sharing 4 points with K, one plane sharing 5 points with K, and five planes sharing 6 points with K,

(c) one plane sharing 8 points with K, three planes sharing 5 points with K, and four planes sharing 6 points with K,

(d) one plane sharing 7 points with K, one plane sharing 4 points with K, and six planes sharing 6 points with K,

(e) one plane sharing 7 points with K, two planes sharing 5 points with K, and five planes sharing 6 points with K,

(f) one plane sharing 5 points with K and seven planes sharing 6 points with K.

Let a, b, c, d, e, f denote respectively the number of bisecants of type (a), (b), (c), (d), (e) and (f). Using the same notations s_i, h_i, $n(p)$ as above, the following equations are obtained:

$$\begin{aligned} a+b+c &= s_8 \\ d+e &= s_7 \\ 6a+5b+4c+6d+5e+7f &= s_6 \\ b+3c+2e+f &= s_5 \\ b+d &= s_4 \\ a &= s_3 \\ a+b+c+d+e+f &= 33\cdot 32/2. \end{aligned}$$

Count the number N of pairs (π, p), where $p \in K$, where π is a plane containing 7 or 8 points of K, and where $p \in \pi$. Then $N = 8h_8 + 7h_7$.

The same argument as for the complete 34-caps gives $n(p) \leq 32/6 < 6$, so

$$8h_8 + 7h_7 \leq 33 \cdot 5.$$

Similarly, the same counting methods as in the previous paragraph imply

$$8h_8 \leq 33 \cdot 4,$$

$$7h_7 + 3h_3 \leq 33 \cdot 16,$$

$$4h_4 \leq 33 \cdot 10,$$

$$3h_3 + 4h_4 \leq 33 \cdot 16,$$

$$7h_7 \leq 33 \cdot 5.$$

Moreover

$$h_3 + h_4 + h_5 + h_6 + h_7 + h_8 \leq (7^4 - 1)/(7-1).$$

The planes containing less than 3 points of K contain 0 or 1 points of K.

Proceeding as for the complete 34-caps, all solutions $(h_0, h_1, h_3, h_4, \ldots, h_8)$ lead to a contradiction.

So $m_2'(3,7) \le 32$. During the computer searches, complete 32-caps were found. So

$$m_2'(3,7) = 32.$$

□

3.2 Caps in $PG(4,7)$

We now use the preceding result to improve the known upper bound $m_2(4,7) \le 316$ to $m_2(4,7) \le 238$. This is achieved by eliminating the existence of 239-caps. We will rely on geometrical arguments and on computer search results. The results of the preceding theorem already imply the following lemma.

Lemma 3.6 *Let K be a 215-cap of $PG(4,7)$. Then every plane of $PG(4,7)$ intersects K in a subset of a conic.*

Proof: The only caps in $PG(2,7)$ not contained in a conic, are complete 6-caps [24, p. 376]. Assume that a plane intersects K in a complete 6-cap, then every solid through this plane intersects K in at most a 32-cap. So $|K| \le 6 + 8 \cdot 26 = 214$. □

To eliminate the existence of 239-caps in $PG(4,7)$, we will prove that if there is a 239-cap K in $PG(4,7)$, then there is a 4-dimensional parabolic quadric $Q(4,7)$ or a cone $rQ^-(3,7)$, with vertex r and a non-singular 3-dimensional elliptic quadric $Q^-(3,7)$ as base, containing at least 101 points of K. This is however impossible since such quadrics in $PG(4,7)$ contain at most 100-caps, as is shown by the following lemma.

Lemma 3.7 *A non-singular 4-dimensional parabolic quadric in $PG(4,q)$ and a cone $rQ^-(3,q)$, with vertex r and a non-singular 3-dimensional elliptic quadric $Q^-(3,q)$ as base, contain at most $2(q^2+1)$-caps.*

Proof: Every line of the quadratic cone $rQ^-(3,q)$ contains at most 2 points of a cap, so such a quadric trivially contains at most $2(q^2+1)$-caps. To prove the result for a 4-dimensional parabolic quadric $Q(4,q)$, we note that every generator of $Q(4,q)$ contains at most 2 points of a cap, and that every point of $Q(4,q)$ lies on $q+1$ generators of $Q(4,q)$. So if K is a cap contained in $Q(4,q)$, then a double counting argument implies that

$$|K|(q+1) \le 2(q^3 + q^2 + q + 1),$$

where $q^3 + q^2 + q + 1$ is the number of generators of $Q(4, q)$. This implies that $|K| \leq 2(q^2 + 1)$. $\square$

Remark 3.8 The preceding upper bound on the size of caps in the 4-dimensional parabolic quadric $Q(4, q)$ of $PG(4, q)$ is sharp since $Q(4, q)$ contains $2(q^2 + 1)$-caps.

This follows from results of Drudge [11] and Ebert [12].

They constructed for respectively q even and for q odd sets of $2(q^2+1)$ lines of $PG(3, q)$ doubly covering the points of $PG(3, q)$. These latter $2(q^2 + 1)$ lines are totally isotropic lines of a symplectic polarity of $PG(3, q)$. This implies that under the Klein correspondence, the Plücker coordinates of these $2(q^2 + 1)$ lines define $2(q^2 + 1)$ points of a 4-dimensional parabolic quadric $Q(4, q)$ on the Klein quadric. Since these lines doubly cover the points of $PG(3, q)$, the corresponding Plücker coordinates define a $2(q^2 + 1)$-cap on this 4-dimensional parabolic quadric.

To find a quadric containing at least 101 points of a 239-cap K, we first of all use the arguments of Nagy and Szőnyi [29]. We first of all determine a first solid α_1 intersecting K in a subset of an elliptic quadric Q_1. We consider a plane π of α_1 having a large number of points in common with $\alpha_1 \cap K$. We then determine a second solid α_2 through π intersecting K in a subset of an elliptic quadric Q_2.

The two 3-dimensional quadrics Q_1 and Q_2 determine a pencil of eight 4-dimensional quadrics. One of those 4-dimensional quadrics is the union $\alpha_1 \cup \alpha_2$. The other seven 4-dimensional quadrics are non-singular 4-dimensional parabolic quadrics, or are cones $rQ^-(3, 7)$. Every point of $PG(4, 7) \setminus (\alpha_1 \cup \alpha_2)$ belongs to exactly one of those quadrics. We will select one point r of $K \setminus (\alpha_1 \cup \alpha_2)$. This determines one quadric Q of the pencil of quadrics defined by Q_1 and Q_2. We will show that this latter quadric Q contains at least 101 points of K; thus giving us the desired contradiction.

This will be achieved in the following way.

1. Select a fixed point p of $K \cap \pi$.
2. Consider all solids through the line pr. We will show that there are at least two solids α_3 and α_3^* through pr satisfying the following conditions:
   (a) α_3 and α_3^* intersect K in subsets of elliptic quadrics Q_3 and Q_3^*,
   (b) both α_3 and α_3^* intersect Q_1 and Q_2 in distinct conics containing at least 5 points of K.

The elliptic quadrics Q_3 and Q_3^* then share two distinct conics with Q, and also share the point r with Q. From Bézout's theorem, Q_3 and Q_3^* are contained in Q.

By the lower bounds on $|\alpha_1 \cap K|$, $|\alpha_2 \cap K|$, $|\alpha_3 \cap K|$, and $|\alpha_3^* \cap K|$, following from the size 239 of the 239-cap, it then follows that Q contains a 101-cap, which is impossible.

To achieve this contradiction, we rely on the following computer search results.

Lemma 3.9 (i) *A point p of a 50-, 49-, 48-, or 47-cap in $PG(3,7)$ lies on exactly one plane intersecting this latter cap in at most 4 points.*

(ii) *A point p of a 46-, 45-, or 44-cap in $PG(3,7)$ lies on at most two planes intersecting this latter cap in at most 4 points.*

(iii) *A point p of a 43- or 42-cap in $PG(3,7)$ lies on at most three planes intersecting this latter cap in at most 4 points.*

(iv) *A point p of a 41-, 40-, 39-, 38-, 37-, or 36-cap in $PG(3,7)$ lies on respectively at most four, six, seven, eight, ten, eleven planes intersecting this latter cap in at most 4 points.*

The following result is also valid.

Lemma 3.10 *Every 32-cap in $PG(3,7)$ has a 7- or 8-plane.*

Proof: Assume that there are at most 6-planes, then a bisecant lies in $(6,6,6,6,6,6,6,4)$- or $(6,6,6,6,6,6,5,5)$-planes.

Let a be the number of bisecants of the first type and let b be the number of bisecants of the second type. Then,

$$\begin{aligned} 7a + 6b &= 15 \cdot h_6 \\ 2b &= 10 \cdot h_5 \\ a &= 6 \cdot h_4 \\ a + b &= 496. \end{aligned}$$

Then the second equation implies that $5|b$ and then the first two equations imply that $5|a$. But then the fourth equation implies that 5 divides 496. This is false. $\square$

Other computer searches led to the following conclusions. In these computer searches, we relied on the fact that caps in $PG(3,7)$, of size at least 33, are subsets of elliptic quadrics (Theorem 3.5).

Lemma 3.11 (i) *There are* 33*-caps in* $PG(3,7)$ *having at most* 7*-planes.*

(ii) *All* 34*-caps in* $PG(3,7)$ *have at least one* 8*-plane.*

(iii) *Every* 35*-cap in* $PG(3,7)$ *contains a pair of different* 8*-planes intersecting in a bisecant of the* 35*-cap. This latter property is not always valid for a* 34*-cap in* $PG(3,7)$*.*

This led to the following conclusions.

Theorem 3.12 *In* $PG(4,7)$*, every*

(i) 174*-cap* K *has at least one* 6*-,* 7*-, or* 8*-plane,*

(ii) 207*-cap* K *has at least one solid sharing at least* 32 *points with* K*, and so has a* 7*- or* 8*-plane,*

(iii) 216*-cap* K *has at least one solid sharing at least* 34 *points with* K*, and so has an* 8*-plane,*

(iv) 219*-cap* K *has at least one solid* α_1 *sharing at least* 34 *points with* K *and at least one solid* α_2 *sharing at least* 33 *points with* K*, and where* $\alpha_1 \cap \alpha_2$ *shares* 8 *points with* K*.*

Proof: (i) If there are no 6-, 7-, or 8-planes, then counting the number of points of K in the planes through a bisecant to K gives $|K| \leq 2+57\cdot 3 = 173$.

(ii) A 207-cap has a 6-, 7-, or 8-plane. If all solids through a 6-plane contain at most 31 points of K, then $|K| \leq 6 + 8 \cdot (31-6) = 206$. A similar counting argument can be done for 7- and 8-planes. It follows that at least one solid shares at least 32 points with K. Hence, a 207-cap has a 7- or 8-plane.

(iii) A 216-cap has a 7- or 8-plane. If they do not lie in a solid sharing at least 34 points with K, then $|K| \leq 215$. So a 216-cap K has at least one solid intersecting K in at least 34 points, and so K has an 8-plane (Lemma 3.11).

(iv) A 219-cap has an 8-plane π. Assume that π lies in a 50-hyperplane and that all other hyperplanes through π share at most 32 points with K, then $|K| \leq 50 + 7 \cdot (32-8) = 218$. So a 219-cap has an 8-plane and through this plane pass at least two solids sharing at least 33 points with K. By the preceding paragraph, at least one hyperplane intersects K in at least 34 points, and this hyperplane has an 8-plane, so starting from this plane, this part is also proven. □

Lemma 3.13 *Let* π *be a plane intersecting a cap* K *of* $PG(4,7)$ *in at least* 5 *points, and let* α_1 *and* α_2 *be solids through* π *intersecting* K *in subsets of elliptic quadrics* Q_1 *and* Q_2*. Let* p *be a point of* $\pi \cap K$*, and let* r *be a point of* $K \setminus (\alpha_1 \cup \alpha_2)$*. Let* α_3 *and* α_3^* *be two solids, different from* $\langle \pi, r\rangle$*, through* *pr intersecting K in subsets of elliptic quadrics Q_3 and Q_3^*, and intersecting α_1 and α_2 in planes containing at least 5 points of K.*

Let $n_1 = |\alpha_1 \cap K|$, $n_2 = |\alpha_2 \cap K|$, $n_3 = |\alpha_3 \cap K|$, and $n_3^* = |\alpha_3^* \cap K|$.

Then

$$n_1 + n_2 + n_3 + n_3^* - 45 \leq 100.$$

Proof: The two elliptic quadrics Q_1 and Q_2 define a pencil of 4-dimensional quadrics. Exactly one of those quadrics Q contains the point r. The elliptic quadric Q_3 shares two distinct conics with Q and also shares the point r with Q. By Bézout's theorem, Q_3 is contained in Q. Similarly, Q_3^* is contained in Q.

We now use the generalized inclusion-exclusion principle to find a lower bound on $|Q \cap K|$. This latter lower bound is

$$n_1 + n_2 + n_3 + n_3^* - 6 \cdot 8 + a_3 - a_4,$$

with a_3 the sum of the intersection sizes of the intersections of three of those solids with K, and with a_4 the intersection size of the intersection of all four solids with K. The negative contribution $6 \cdot 8$ comes from the fact that planes which are the intersection of two distinct solids share at most 8 points with K.

Since $\pi \not\subset \alpha_3$, necessarily $\alpha_1 \cap \alpha_2 \cap \alpha_3 \cap \alpha_3^*$ is at most a line through p, so $a_4 \leq 2$. It is also trivial that the intersection of three of the solids $\alpha_1, \alpha_2, \alpha_3, \alpha_3^*$ shares at least a_4 points with K, so the lower bound becomes

$$n_1 + n_2 + n_3 + n_3^* - 6 \cdot 8 + 4 \cdot a_4 - a_4.$$

Since $a_4 \geq 1$, necessarily, by Lemma 3.7,

$$100 \geq |Q \cap K| \geq n_1 + n_2 + n_3 + n_3^* - 45.$$

□

We now present the ideas leading to the exclusion of 239-caps in $PG(4,7)$. These ideas are based on the results of the preceding lemma. In the following description of the method, we assume the size of the cap K to be large enough to get the desired contradiction. The precise value for the size of K is given in Table 3.

Part 1. Let $y \geq 33$ be the maximal size of a solid intersection of an n-cap K in $PG(4,7)$. We also assume that $x \geq 33$, with $x \leq y$, is the largest size of a solid intersection of K, intersecting a y-solid in a plane sharing at least 5 points with K (Theorem 3.12).

Let α_1 be a hyperplane sharing y points with K, and consider a plane π of α_1 sharing at least 5 points with K. We select π and α_1 in such a way that $|\pi \cap K| \geq 5$, $|\alpha_1 \cap K| = y$, and $|\alpha_2 \cap K| = x$, where α_2 is a second solid through π.

Consider the notations and geometrical setting of the preceding lemma. From Lemma 3.9, we know the upper bound a_y on the number of solids through pr intersecting α_1 in a plane sharing at most 4 points with K. These latter, at most a_y, solids share at most y points with K. All the remaining solids through pr intersect α_1 in a plane sharing at least 5 points with K, and so share at most x points with K.

From Lemma 3.9, we know the upper bound a_x on the number of solids through pr intersecting α_2 in a plane sharing at most 4 points with K. At most a_x solids through pr intersect α_1 in a plane sharing at least 5 points with K, but intersect α_2 in a plane sharing at most 4 points with K. These latter, at most a_x, solids cannot be used to play the role of the solids α_3 and α_3^*, and contain at most x points of K.

The solid $\langle \pi, r \rangle$ also cannot be used to play the role of one of the solids α_3 and α_3^*. This latter solid also contains at most x points of K. There still remain $57 - a_y - a_x - 1$ solids through pr.

The remaining solids through pr can be used to play the role of the solids α_3 or α_3^* if they contain more than 32 points of K. Suppose that the largest solid intersection of these latter solids with K is equal to n_3 and that the second largest solid intersection of these latter solids with K is equal to n_3^*. We first of all assume that $x \geq n_3 \geq n_3^* > 32$; the case $n_3 \geq 33$ and $n_3^* \leq 32$ is discussed in Part 3, while the case $n_3 \leq 32$ is discussed in Part 4.

We count in all three cases the number of ordered pairs (s, α), where $s \in K$, where α is a solid through pr, and where $s \in \alpha$. This number is equal to

$$(|K| - 2) \cdot 8 + 2 \cdot 57. \qquad (1)$$

Part 2. Assume that $n_3 \geq n_3^* > 32$.

Then, by Lemma 3.13,

$$\begin{aligned} y + x + n_3 + n_3^* &\leq 145 \\ y + x + 2n_3^* &\leq 145 \end{aligned}$$

which implies that

$$n_3^* \leq \lfloor \frac{145 - y - x}{2} \rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

Note that, by the assumptions on n_3 and n_3^*, this case only occurs if effectively $33 \leq (145 - y - x)/2$.

All the remaining solids through pr contain at most n_3^* points of K. So, from Part 1,

$$\begin{aligned} & a_y \cdot y + (a_x+1) \cdot x + n_3 + n_3^* + (57 - a_y - a_x - 3) \cdot \lfloor \frac{145 - y - x}{2} \rfloor \\ \leq\ & (a_y - 1)y + a_x x + y + x + n_3 + n_3^* + (57 - a_y - a_x - 3) \lfloor \frac{145 - y - x}{2} \rfloor \\ \leq\ & (a_y - 1)y + a_x x + 145 + (57 - a_y - a_x - 3) \lfloor \frac{145 - y - x}{2} \rfloor \qquad (2) \end{aligned}$$

is an upper bound for (1).

Part 3. If $n_3 \geq 33$, but all the remaining solids through pr intersect K in at most $n_3^* \leq 32$ points, then we cannot use formula (2) since we are not sure that the solid α_3^* exists intersecting K in a subset of an elliptic quadric, so we cannot rely on Lemma 3.13.

We have in this case the following upper bound on the number of incidences (s, α), where $s \in K$ and where α is a solid through pr:

$$\begin{aligned} & a_y \cdot y + (a_x + 1) \cdot x + n_3 + (57 - a_y - a_x - 2) \cdot 32 \\ \leq\ & a_y \cdot y + (a_x + 2) \cdot x + (57 - a_x - a_y - 2) \cdot 32, \qquad (3) \end{aligned}$$

since $x \geq n_3$.

Part 4. There still remains one case, namely $n_3 \leq 32$. Then all the remaining solids through pr share at most 32 points with K, so (3) again is an upper bound for (1).

Lemma 3.14 *Let K be a cap in $PG(4,7)$ intersecting a plane π in at least five points, and intersecting two solids α_1 and α_2 through π in subsets of elliptic quadrics Q_1 and Q_2 in α_1 and α_2.*

Assume that $|(\alpha_1 \cup \alpha_2) \cap K| = z$. Then

$$z + \frac{|K| - z}{7} \leq 100.$$

Proof: The two elliptic quadrics Q_1 and Q_2 define a pencil of 4-dimensional quadrics in $PG(4,7)$. One of these quadrics is the union $\alpha_1 \cup \alpha_2$. The other seven 4-dimensional quadrics of this pencil of quadrics are non-singular parabolic quadrics or cones with a point as vertex and a non-singular 3-dimensional elliptic quadric as base. These quadrics contain at most 100-caps of $PG(4,7)$ (Lemma 3.7).

Since one of those seven quadrics contains at least $z + (|K| - z)/7$ points, necessarily

$$z + \frac{|K| - z}{7} \leq 100.$$

□

Remark 3.15 First of all, using Lemma 3.14, we eliminated some pairs (y, x) for $|K| = 239$. For the remaining pairs (y, x), Table 3 shows the smallest value of $|K|$ for which the upper bounds (2) and (3) give a contradiction, when compared to the exact value $(|K| - 2) \cdot 8 + 2 \cdot 57$.

For the values for $|K|, y, x$, which are preceded by !, the size of $|K|$ arises from formula (3).

	$\|K\|$	y	x		$\|K\|$	y	x		$\|K\|$	y	x
!	225	48	36	!	226	44	36	!	229	41	40
!	226	47	37	!	227	43	41	!	229	41	39
!	225	47	36	!	228	43	40		233	41	38
!	227	46	38	!	228	43	39		233	41	37
!	227	46	37	!	228	43	38	!	230	40	40
!	226	46	36	!	228	43	37		235	40	39
!	227	45	39		232	43	36		234	40	38
!	227	45	38	!	226	42	42		239	40	37
!	227	45	37	!	227	42	41		235	39	39
!	226	45	36	!	228	42	40		239	39	38
!	227	44	40	!	228	42	39		239	39	37
!	227	44	39	!	228	42	38		239	38	38
!	227	44	38		232	42	37		243	38	37
!	227	44	37	!	228	41	41		243	37	37

Table 3

The preceding table shows that 239-caps cannot exist unless $(y, x) = (38, 37)$ or $(y, x) = (37, 37)$.

This latter case is eliminated by the following ideas.

Assume that there is a 239-cap K in $PG(4, 7)$ having at most 37-solids. Through an 8-plane of K (Theorem 3.12), there are seven 37-solids and one 36-solid.

Consider a 36-solid α_0. In α_0, we find a pair π_1, π_2 of 8-planes intersecting in the bisecant L of K (Lemma 3.11). Let α_i, $i = 0, \ldots, 7$, be the 36-solid α_0 and the seven 37-solids through π_1, and let β_i, $i = 0, \ldots, 7$, be the 36-solid $\beta_0 = \alpha_0$ and the seven 37-solids through π_2.

Consider the planes through L of a solid β_i, $i > 0$. At least five of those planes through L share more than four points with K. One of those planes is π_2 which lies in $\beta_0 = \alpha_0$. The other planes of β_i through L correspond to the intersections of the solids α_j, $j > 0$, with β_i. This shows that each of the solids β_i, $i = 1, \ldots, 7$, intersects at least four of the 37-solids α_i, $i > 0$, in a plane containing at least five points of K.

This implies that it is possible to find two solids $\beta_i, \beta_{i'}$, with $i, i' > 0$, and two solids $\alpha_j, \alpha_{j'}$, with $j, j' > 0$, such that $|\beta_i \cap \alpha_j \cap K|, |\beta_i \cap \alpha_{j'} \cap K|, |\beta_{i'} \cap \alpha_j \cap K|, |\beta_{i'} \cap \alpha_{j'} \cap K| \geq 5$.

Select a point r of $\pi_2 \setminus L$ belonging to K. The solid intersections $\alpha_j \cap K, \alpha_{j'} \cap K$ and the point r define a unique 4-dimensional quadric Q, also containing the elliptic quadrics containing the solid intersections $\beta_i \cap K$ and $\beta_{i'} \cap K$.

Taking into account that three of the solids $\alpha_j, \alpha_{j'}, \beta_i, \beta_{i'}$ intersect in the bisecant L to K, the generalized inclusion-exclusion principle applied to the 37-solids $\alpha_j, \alpha_{j'}, \beta_i, \beta_{i'}$ shows that Q contains at least

$$4 \cdot 37 - 6 \cdot 8 + 4 \cdot 2 - 2 = 106$$

points of K. This is false (Lemma 3.7).

So there is no 239-cap in $PG(4, 7)$ having at most 37-solids.

Assume that $(y, x) = (38, 37)$ for a 239-cap K. Consider an 8-plane in a 38-solid (Lemma 3.11). Through this 8-plane, there either pass:

(a) one 38-solid, six 37-solids, and one 35-solid, or
(b) one 38-solid, five 37-solids, and two 36-solids.

Consider the first possibility (a). Let α_0 be the 35-solid. In α_0, we again find a pair π_1, π_2 of 8-planes intersecting in the bisecant L of K (Lemma 3.11). Let α_i, $i = 0, \ldots, 7$, be the 35-solid α_0, the six 37-solids and the 38-solid through π_1, and let β_i, $i = 0, \ldots, 7$, be the 35-solid $\beta_0 = \alpha_0$, the six 37-solids and the 38-solid through π_2.

Consider the planes through L of a solid β_i, $i > 0$. At least five of those planes through L share more than four points with K. One of those planes is π_2 which lies in $\beta_0 = \alpha_0$. The other planes of β_i through L correspond to the intersections of the solids α_j, $j > 0$, with β_i. This shows that each of the solids β_i, $i = 1, \ldots, 7$, intersects at least four of the 37-solids and the 38-solid α_i, $i > 0$, in a plane containing at least five points of K.

This leads again to the contradiction as obtained for 239-caps having at most 37-solids.

Consider now possibility (b), where we assume that there are no 35-solids. Recall that every 35-solid has at least one 8-plane (Lemma 3.11).

Consider a 36-solid α_0. We proceed as before, but it could happen that $\{|\alpha_j \cap K|, |\alpha_{j'} \cap K|\} = \{36, 37\}$ or that $\{|\beta_i \cap K|, |\beta_{i'} \cap K|\} = \{36, 37\}$. This does not impose any problems in the arguments of the case $(y, x) = (37, 37)$. The inclusion-exclusion principle still implies that the quadric Q would contain at least 104 points of K, which is impossible (Lemma 3.7).

So also the case $(y, x) = (38, 37)$ does not occur for 239-caps in $PG(4, 7)$.

This leads to the following improvement to the known upper bound on $m_2(4, 7)$. In $PG(4, 7)$, a 132-cap exists [15].

Corollary 3.16

$$132 \leq m_2(4, 7) \leq 238.$$

3.3 Bounds on $m_2(N, 7)$

We now present the known bounds on $m_2(N, 7)$, $N > 4$.

Theorem 3.17 *For* $5 \leq N \leq 12$*,*

$$m_2(N, 7) \leq 5 \cdot 7^{N-2} - \frac{25}{3} \cdot 7^{N-4} + \frac{4}{3}.$$

For $13 \leq N \leq 17$*,*

$$m_2(N, 7) \leq \frac{7^N \cdot (N+1)}{N^2} + 5 \cdot 7^{N-3} - \frac{25}{3} \cdot 7^{N-5} + \frac{4}{3}.$$

For $N \geq 18$*,*

$$m_2(N, 7) \leq \frac{7^N \cdot (N+1)}{N^2} + \frac{3 \cdot N \cdot 7^{N-1}}{2 \cdot (N-1)^2}.$$

Proof: The first formula arises from the formula of Hill [21]. The second formula arises from the bound of Bierbrauer-Edel on caps in affine spaces [6] plus the formula of Hill for a cap in a hyperplane of $PG(N, q)$. The third formula is from Theorem 1.1. $\square$

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