On large maximal partial ovoids of the parabolic quadric $Q(4,q)$

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Abstract

We use the representation $T_2(\mathcal{O})$ for $Q(4,q)$ to show that maximal partial ovoids of $Q(4,q)$ of size q^2-1, $q=p^h$, $h>1$, do not exist. Although this was known before, we give a slightly alternative proof, also resulting in more combinatorial information of the known examples for q prime.

keywords: maximal partial ovoid, generalized quadrangle, parabolic quadric.

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This paper is based on joint work with András Gács, started in the autumn of 2008, as a continuation of the work in [4]. His unfortunate and sudden death prevented us from continuing our joint work on the geometrical interpretation of the results. I would like to dedicate this work to András.

1 Introduction

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of objects called points and lines, and for which $\mathrm{I}\subseteq(\mathcal{P}\times\mathcal{B})\cup(\mathcal{B}\times\mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1+t$ lines ($t\geqslant 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1+s$ points ($s\geqslant 1$) and two distinct lines are incident with at most one point;

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(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

The integers s and t are the parameters of the GQ and $\mathcal{S}$ is said to have order (s,t). If $s = t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s,t), then $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t+1)(st+1)$ (see e.g. [8]).

An *ovoid* of a GQ $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that every line is incident with exactly one point of the ovoid. An ovoid of a GQ of order (s,t) has necessarily size $1+st$. A *partial ovoid* of a GQ is a set $\mathcal{K}$ of points such that every line contains *at most* one point of $\mathcal{K}$. A partial ovoid $\mathcal{K}$ is called *maximal* if and only if $\mathcal{K} \cup \{P\}$ is not a partial ovoid for any point $P \in \mathcal{P} \setminus \mathcal{K}$, in other words, if $\mathcal{K}$ cannot be extended. It is clear that any partial ovoid of a GQ of order (s,t) contains $1 + st - \rho$ points, $\rho \geq 0$, with $\rho = 0$ if and only if $\mathcal{K}$ is an ovoid.

It is a natural question to study *extendability* of partial ovoids, i.e. can one alway extend a partial ovoid of size $1 + st - \epsilon$ (e.g. to an ovoid) if ϵ is not too big? The following theorem is a typical example.

Theorem 1 ([8, 2.7.1]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t). Any partial ovoid of size $st - \rho$, $0 \leq \rho \leq \frac{t}{s}$ is contained in a uniquely defined ovoid of $\mathcal{S}$.*

Note that if no ovoids of a particular GQ exist, then Theorem 1 implies an upper bound on the size of partial ovoids. The following theorem deals with the limit situation, and will be of use in Section 3.

Theorem 2 ([8, 2.7.2]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t). Let $\mathcal{K}$ be a maximal partial ovoid of size $st - t/s$ of $\mathcal{S}$. Let $\mathcal{B}'$ be the set of lines incident with no point of $\mathcal{K}$, and let $\mathcal{P}'$ be the set of points on at least one line of $\mathcal{B}'$ and let I' be the restriction of I to points of $\mathcal{P}'$ and lines of $\mathcal{B}'$. Then $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order $(s, \rho = t/s)$.*

Consider the parabolic quadric $\mathrm{Q}(4,q)$ in the 4-dimensional projective space $\mathrm{PG}(4,q)$. This quadric is the set of points and lines that are totally isotropic with relation to a non-singular quadratic form on $\mathrm{PG}(4,q)$, which is, up to coordinate transform, unique, and its points and lines constitute an example of a generalized quadrangle of order q.

It is known, (see e.g. [8]) that this GQ has ovoids. A particular example of an ovoid is any elliptic quadric $\mathrm{Q}^-(3,q)$ contained in it and obtained by a hyperplane section of $\mathrm{Q}(4,q)$. When q is prime, these are the only ovoids [1]; when q is a prime power, other examples are known, see e.g. [5] for a list of references. The classification of ovoids of $\mathrm{Q}(4,q)$, for q prime, is essentially due to the computation of intersection numbers (modulo p) of a hypothetical ovoid with elliptic quadrics, and the use of this information in a combinatorial argument.

Applying Theorem 1 to the *GQ* $Q(4, q)$ implies that a partial ovoid of size q^2 cannot be maximal. It is shown in [4] that maximal partial ovoids of $Q(4, q)$, $q = p^h$, p an odd prime, $h > 1$, do not exist. The natural question arises if maximal partial ovoids exist when $h = 1$. Curiously, this is the case when $p \in \{3, 5, 7, 11\}$, but no examples are known for $q > 11$, [9]. In this paper we give a slightly alternative proof of the non-existence result for $h > 1$. Further, we compute the intersection numbers of a hypothetical maximal partial ovoid of size q^2-1 with elliptic quadrics embedded in $Q(4, q)$, for q a prime. This yields structural information on the existing examples, and it is our hope that this information could contribute to finally proving their uniqueness and non-existence for $p > 11$.

2 Non-existence for $q > p$

We follow almost the same approach as in [4]. Therefore we need to introduce an alternative representation of the GQ $Q(4, q)$.

An *oval* of $PG(2, q)$ is a set of $q+1$ points $\mathcal{C}$, such that no three points of $\mathcal{C}$ are collinear. When q is odd, it is known that all ovals of $PG(2, q)$ are conics. When q is even, several other examples and infinite families are known, see e.g. [2]. The GQ $T_2(\mathcal{C})$ is defined as follows. Let $\mathcal{C}$ be an oval of $PG(2, q)$, embed $PG(2, q)$ as a plane in $PG(3, q)$ and denote this plane by π_∞. Points are defined as follows:

(i) the points of $PG(3, q) \setminus PG(2, q)$;

(ii) the planes π of $PG(3, q)$ for which $|\pi \cap \mathcal{C}| = 1$;

(iii) one new symbol (∞).

Lines are defined as follows:

(a) the lines of $PG(3, q)$ which are not contained in $PG(2, q)$ and meet $\mathcal{C}$ (necessarily in a unique point);

(b) the points of $\mathcal{C}$.

Incidence between points of type (i) and (ii) and lines of type (a) and (b) is the inherited incidence of $PG(3, q)$. In addition, the point (∞) is incident with no line of type (a) and with all lines of type (b). It is straightforward to show that this incidence structure is a GQ of order q. The following theorem (see e.g. [8]) allows us to use this representation.

Theorem 3 *The GQs $T_2(\mathcal{C})$ and $\mathrm{Q}(4,q)$ are isomorphic if and only if $\mathcal{C}$ is a conic of the plane $\mathrm{PG}(2,q)$.*

From now on we suppose that $\mathcal{C}$ is a conic. Let $\mathcal{K}$ be a maximal partial ovoid of size k of $T_2(\mathcal{C})$. Since $\mathrm{Q}(4,q) \cong T_2(\mathcal{C})$ has a collineation group acting transitively on the points (see e.g. [7]), we can suppose that $(\infty) \in \mathcal{K}$. This implies that $\mathcal{K}$ contains no points of type (ii). It is clear that no two points of type (i) of $\mathcal{K}$ determine a line meeting π_∞ in a point of $\mathcal{C}$. Hence the existence of $\mathcal{K}$ implies the existence of a set U of $k-1$ points of type (i) such that no two points determine a line meeting π_∞ in $\mathcal{C}$. It is easy to see that the converse is also true: from a set U of $k-1$ points in $\mathrm{PG}(3,q) \setminus \pi_\infty$ with the property that all lines joining at least two points of U are disjoint from $\mathcal{C}$, we can find a maximal partial ovoid $\mathcal{K}$ of $T_2(\mathcal{C})$ of size k by adding (∞) to U. The maximality of $\mathcal{K}$ is equivalent to the maximality of U.

Hence the existence of a maximal partial ovoid of size q^2-1 of $\mathrm{Q}(4,q)$, is equivalent with the existence of a set U of q^2-2 affine points, not determining the points of a conic at infinity. In [4], it is shown that such a set U can always be extended when $q > p$. In fact, only the assumption that at least $p+2$ points are not determined is used. In this paper we will assume that the points of a conic are not determined and that U is not extendable, compute the range of a certain polynomial, and find a contradiction when $q > p$. In the third section, we will describe the use of this particular polynomial to compute the intersection numbers modulo p of the point set U with planes of $\mathrm{AG}(3,q)$. This will yield intersection numbers modulo p of the maximal partial ovoid of size q^2-1 with elliptic quadrics embedded in $\mathrm{Q}(4,q)$.

From now on, let $\mathcal{K}$ denote a partial ovoid of $\mathrm{Q}(4,q)$, $q = p^h$, p an odd prime and $h \geq 1$. Let U denotes the point set of $\mathrm{PG}(3,q) \setminus \pi_\infty$ corresponding with the partial ovoid of $\mathrm{Q}(4,q)$.

Lemma 1 *If a plane $\pi \neq \pi_\infty$ of $\mathrm{PG}(3,q)$, meets $\mathcal{C}$ in at least one point, then $|\pi \cap U| \leq q$.*

Proof. Let $P \in \mathcal{C} \cap \pi$. If $|\pi \cap U| > q$, then at least one of the q lines of π through P must contain two points of U. This contradicts the fact that U does not determine any point of $\mathcal{C}$. Hence, $|\pi \cap U| \leq q$. □

We choose π_∞ to be the plane with equation $X_3 = 0$. Then any line l of π_∞ is determined by the equation $yX_0+zX_1+wX_2 = 0$, $(y,z,w) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$. We denote such a line as $l(y,z,w)$. Any plane $\pi \neq \pi_\infty$ through $l(y,z,w)$ is determined by the equation $yX_0 + zX_1 + wX_2 + xX_3 = 0$. We denote such a plane as $\pi(x,y,z,w)$.

The point set $U = \{(a_i, b_i, c_i, 1) : i = 1, \dots, q^2-2\} \subset \mathrm{PG}(3, q) \setminus \pi_\infty$. We define the Rédei polynomial associated to the point set U as follows:

$$R(X,Y,Z,W) = \prod_{i=1}^{q^2-2}(X+a_iY+b_iZ+c_iW) = X^{q^2-2}+\sum_{i=1}^{q^2-2}\sigma_i(Y,Z,W)X^{q^2-2-i}.$$

Here $\sigma_i(Y, Z, W)$ is the i-th elementary symmetric polynomial of the multi-set $\{a_iY + b_iZ + c_iZ : i\}$ and is either zero or has degree i.

Lemma 2 *Suppose that the line $l(y, z, w)$ meets $\mathcal{C}$ in at least one point. Then $R(X, y, z, w) \mid (X^q - X)^q$.*

Proof. If $x \in \mathbb{F}_q$ is a root of $R(X, y, z, w) = 0$, then its multiplicity equals $|\pi(x, y, z, w) \cap U| \leq q$ (the latter by Lemma 1). Since $|U| = q^2-2$, each of the q planes $\pi(x, y, z, w)$, $x \in \mathbb{F}_q$, contains points of U, hence $R(X, y, z, w) = 0$ has each element $x \in \mathbb{F}_q$ as root, with multiplicity at most q, and the lemma follows. □

Since we suppose that q is odd and $|U| = q^2-2$, after the affine translation

$$a_i \mapsto a_i - \frac{\sum a_i}{q^2-2}, \qquad b_i \mapsto b_i - \frac{\sum b_i}{q^2-2}, \qquad c_i \mapsto c_i - \frac{\sum c_i}{q^2-2},$$

not affecting the (non)-determined points at infinity, we may assume that $\sum a_i = \sum b_i = \sum c_i = 0$, which is equivalent to $\sigma_1(Y, Z, W) \equiv 0$.

Lemma 3 *If a line $l(y, z, w)$ has at least one common point with $\mathcal{C}$, then*

$$R(X,y,z,w)(X^2 - \sigma_2(y,z,w)) = (X^q - X)^q. \tag{1}$$

Proof. From Lemma 2 we know that

$$R(X,y,z,w)(X-S)(X-S') = (X^q - X)^q,$$

where S and S' are not necessarily different and depend on y, z, w. Considering the first three terms on both sides and taking into account that $\sigma_1(Y, Z, W) \equiv 0$, we have $(X - S)(X - S') = X^2 - \sigma_2(y, z, w)$. □

Lemma 4 *Suppose that the line $l(y, z, w)$ meets $\mathcal{C}$ in at least one point. Then*

$$\sigma_{2l+1}(y,z,w) = 0, 0 \leq l \leq \frac{q^2-3}{2},$$

$$\sigma_{2l}(y,z,w) = \sigma_2^l(y,z,w), 0 \leq l \leq \frac{q^2-q-2}{2},$$

$$\sigma_{q^2-q+2k}(y,z,w) = \sigma_2^{\frac{q^2-q+2k}{2}}(y,z,w) - \sigma_2^k(y,z,w), 0 \leq k \leq \frac{q-3}{2}.$$

Proof. Computation of both right-hand and left-hand sides of Equation (1), and the use of $\sigma_1(Y, Z, W) \equiv 0$ proves the lemma. $\square$

Corollary 1

$$\sigma_{2l+1}(Y, Z, W) \equiv 0,\ 0 \leq l \leq \frac{q-1}{2},$$

$$\sigma_{2l}(Y, Z, W) \equiv \sigma_2^l(Y, Z, W),\ 0 \leq l \leq \frac{q-1}{2}.$$

Proof. Consider any line $l(y, z, w)$ meeting $\mathcal{C}$ in at least one point. By Lemma 4, the equations of the corollary are true after subsituting $Y = y$, $Z = z$, $W = w$. But for each point $P \in \mathcal{C}$, each line $l(y, z, w)$ on P gives a substitution for which the equations are true. Dually, this means that the points of at least $q + 1$ different lines are a solution of the equations of the corollary. Since the degree of each equation is at most q, by the theorem of Bézout, each curve represented by an equation must contain $q + 1$ lines as a component. But then its degree must be at least $q + 1$. Hence, the polynomials are identically zero. $\square$

We define now the polynomials S_j, $j = 0, \ldots, q-1$ as follows.

$$S_j(Y, Z, W) := \sum_{i=1}^{q^2-2} (a_i Y + b_i Z + c_i W)^j\,.$$

The Newton identities describe a relation between the polynomials $S_j(Y, Z, W)$ and $\sigma_i(Y, Z, W)$ as follows:

$$k\sigma_k(Y, Z, W) \equiv \sum_{j=1}^{k} (-1)^{j-1} S_j(Y, Z, W)\sigma_{k-1}(Y, Z, W)\,.$$

Lemma 5

$$S_{2l+1}(Y, Z, W) \equiv 0,\ 0 \leq l \leq \frac{q-1}{2},$$

$$S_{2l}(Y, Z, W) \equiv -2\sigma_2^l(Y, Z, W),\ 0 \leq l \leq \frac{q-1}{2}.$$

Proof. Using Corollary 1, the Newton identities, the fact that $S_1(Y, Z, W) \equiv \sigma_1(Y, Z, W)$, $\sigma_0 = 1$, and induction, the lemma follows. $\square$

Lemma 6 *If $\sigma_2(Y, Z, W)$ is reducible then the set U is extendable.*

Proof. Suppose that $\sigma_2(Y,Z,W)$ is reducible. By equation (1), $\sigma_2(y,z,w)$ must be a square for any (y,z,w) such that $l(y,z,w)$ meets $\mathcal{C}$ in at least one point. So there are triples (y,z,w), contained in a line (the dual of the pencil of lines through a point $P \in \mathcal{C}$) for which $\sigma_2(y,z,w)$ is a square. It follows that $\sigma_2(Y,Z,W) = (AY + BZ + CW)^2$. Now define $U^* := U \cup \{(A,B,C,1),(-A,-B,-C,1)\}$. Consider any point $P \in \mathcal{C}$ and any line $l(y,z,w)$ on P. From Equation (1) it follows that each plane on l now contains exactly q points of U^*. But if P is a determined by U^*, then there exists a line m on P containing $r \geq 2$ points of U^*. But all $q+1$ planes on m contain exactly q points of U^*, so $q^2 = |U^*| = r + (q+1)(q-r)$, a contradiction. Hence, U^* does not determine the points of $\mathcal{C}$. $\square$

Theorem 4 *If U is not extendable, then $q = p$.*

Proof. We define

$$\begin{aligned}
\chi(X,Y,Z,W) &:= \sum_{i=1}^{q^2-2}(X + a_iY + b_iZ + c_iW)^{q-1} \\
&= \sum_{i=1}^{q^2-2}\sum_{j=0}^{q-1}\binom{q-1}{j}X^{q-1-j}(a_iY + b_iZ + c_iW)^j \\
&= \sum_{j=0}^{q-1}(-1)^j X^{q-1-j}S_j(Y,Z,W) \\
&= -2\sum_{k=0}^{\frac{q-1}{2}} X^{q-1-2k}\sigma_2^k(Y,Z,W) = -2\frac{X^{q+1} - \sigma_2^{\frac{q+1}{2}}(Y,Z,W)}{X^2 - \sigma_2(Y,Z,W)},
\end{aligned}$$

where we used Lemma 5 to obtain the second last equality. If U is not extendable, then $\sigma_2(Y,Z,W)$ is not reducible. So the range of $\sigma_2(Y,Z,W)$ is the complete field $\mathbb{F}_q$, so for each non-square $\nu \in \mathbb{F}_q$, we can find a triple (y,z,w) such that $\sigma_2(y,z,w) = \nu$. Then $\sigma_2^{\frac{q+1}{2}}(y,z,w) = -\sigma_2(y,z,w)$ and

$$\chi(X,y,z,w) = -2\frac{X^{q+1} + \sigma_2(y,z,w)}{X^2 - \sigma_2(y,z,w)} \tag{2}$$

It is now easy to see that the range of $\chi(X,Y,Z,W)$ will contain at least $\frac{q+1}{2}$ different elements of $\mathbb{F}_q$. On the other hand,

$$\chi(x,y,z,w) = q^2 - 2 - |U \cap \pi(x,y,z,w)| \bmod p\,,$$

for any 4-tuple $(x,y,z,w) \notin \{(1,0,0,0),(0,0,0,0)\}$. So the right hand side is necessarily an element of $\mathbb{F}_p$, a contradiction with the range of $\chi(X,Y,Z,W)$ if $q > p$. $\square$

3 The intersection numbers for q a prime

Suppose now that $q = p$, p an odd prime. We consider the possibilities of $\chi(X, Y, Z, W)$. Consider a plane $\pi(x, y, z, w)$.

(a) Suppose that $\sigma_2(y, z, w) = 0$. Then $\chi(X, y, z, w) = -2X^{q-1}$, hence $\chi(x, y, z, w) = 0$ if $x = 0$ and $\chi(x, y, z, w) = -2$ if $x \neq 0$.

(b) Suppose that $\sigma_2(y, z, w)$ is a square different from 0. If $x^2 \neq \sigma_2(y, z, w)$ then $\chi(x, y, z, w) = -2$. If $x^2 = \sigma_2(y, z, w)$ then $\chi(x, y, z, w) = -1$.

(c) Suppose that $\sigma_2(y, z, w)$ is a non-square. Then

$$\chi(x, y, z, w) = -2\frac{x^2 + \sigma_2(y, z, w)}{x^2 - \sigma_2(y, z, w)} \neq 0$$

Lemma 7 *The curve $\sigma_2(Y, Z, W) = 0$ is the dual of $\mathcal{C}$.*

Proof. Theorem 2 ensures that the set of lines of $\mathrm{Q}(4, q)$, not meeting $\mathcal{K}$, is the set of lines of a hyperbolic quadric embedded as a hyperplane section in $\mathrm{Q}(4, q)$. We denote this hyperbolic quadric as Q^+. Since $\mathcal{K} = \{(\infty)\} \cup U$, clearly $(\infty) \notin \mathrm{Q}^+$, and from the proof of Theorem 3 in [8], it follows that Q^+ is represented in $T_2(\mathcal{C})$ as a hyperbolic quadric meeting π_∞ in $\mathcal{C}$. We denote this quadric as Q^+_T. The hyperbolic quadric Q^+_T contains exactly $q+1$ points of type (ii). Consider such a point, represented by the plane π. The two lines of type (a) of Q^+_T incident with π, are contained in π, and do not meet U. But the other $q-1$ lines of $T_2(\mathcal{C})$, incident with π, do meet U in exactly one point. Hence the plane π must contain exactly $q-2$ points of U. If π is represented by the 4-tuple (x, y, z, w), then $\chi(x, y, z, w) = q^2 - 2 - |\pi \cap U| \bmod q$. So if $|\pi \cap U| = q-2$, then $\chi(x, y, z, w) = 0$ and by the above overview of the range of χ, the planes $\pi(x, y, z, w)$ that represent a point of type (ii) of Q^+_T, are exactly those for which $\sigma_2(y, z, w) = 0 = x$. But the planes that represent points of type (ii) of Q^+_T are planes that meet $\mathcal{C}$ in a tangent line. Hence, $\sigma_2(y, z, w) = 0$ if and only if $l(y, z, w)$ is a tangent line to $\mathcal{C}$. $\square$

Corollary 2 *A plane $\pi(x, y, z, w)$ represents an elliptic quadric containing (∞) if and only if $\sigma_2(y, z, w)$ is a non-square.*

Proof. From the proof of Theorem 3, it follows that an elliptic quadric containing (∞) is represented in $T_2(\mathcal{C})$ by a plane meeting π_∞ in a line external to $\mathcal{C}$. The Corollary now follows from Lemma 7. $\square$

Corollary 3 *If an elliptic quadric* $Q^- \subset Q(4,q)$ *contains one point of* $\mathcal{K}$*, then*

$$|Q^- \cap \mathcal{K}| \bmod p \in \{-1 + 2\frac{x^2+\nu}{x^2-\nu} \parallel \nu \text{ running over the non-squares, } x \in \mathbb{F}_q\}$$

Proof. If an elliptic quadric contains a point of $\mathcal{K}$, we can choose it to be the point (∞). Then

$$|\pi(x,y,z,w) \cap U| \bmod q = -2 - \chi(x,y,z,w) = -2 + 2\frac{x^2+\nu}{x^2-\nu}, \qquad (3)$$

$\nu = \sigma_2(y,z,w)$, which is non-square. □

Consider now any point $P \in Q(4,q) \setminus Q^+$. Then $P^\perp \cap Q^+$ is a conic C_P, and $C_P^\perp = \{P, P'\}$, $P \neq P' \in Q(4,q) \setminus Q^+$. We call P' the antipode of P. Consider now the point ∞, this is collinear with the points of type (ii) of Q_T^+. But for each point of type (ii) of Q_T^+, represented by a plane $\pi(x,y,z,w)$, we have seen that $x = 0$. Hence the point $(0,0,0,1)$ is contained in the planes representing the points of type (ii) of Q_T^+, so, the points of type (ii) of Q_T^+ are collinear with $(0,0,0,1)$. Hence, the point $(0,0,0,1)$ is the antipode of the point (∞).

Lemma 8 *If an elliptic quadric* $Q^- \subset Q(4,q)$ *contains a point of* $\mathcal{K}$ *and its antipode, then* $|Q^- \cap \mathcal{K}| \equiv -3 \bmod q$.

Proof. A point and its antipode are non-collinear, and the collineation group of $Q(4,q)$ acts transitively on the pairs of non-collinear points. So in the $T_2(\mathcal{C})$ representation, if an elliptic quadric contains a point of $\mathcal{K}$, this can be chosen (∞) while its antipode can be chosen to be the point $(0,0,0,1)$. For a plane $\pi(x,y,z,w)$ containing $(0,0,0,1)$, we have $x = 0$. The lemma now follows from Corollary 3. □

We remark that the computed intersection numbers (modulo q) do not exclude elliptic quadrics that contain no point of $\mathcal{K}$. We list the range for intersection numbers modulo q found in Corollary 3 for $q \in \{5,7,11\}$. Recall that these numbers are valid for elliptic quadrics containing at least one point of $\mathcal{K}$. Hence 0 means a positive multiple of q in reality.

- $q = 5$: $\{0,2,3\}$
- $q = 7$: $\{2,3,4,6\}$
- $q = 11$: $\{0,4,5,8,9,10\}$.

We used an explicit description of the known examples ([3]) to compute the intersection numbers with all elliptic quadrics. We list the results. In this list, for $q = 5$ and $q = 11$, we see that there are elliptic quadrics containing no point of $\mathcal{K}$. However this 0 is **not** related to a 0 in the above list.

- $q = 5$: $\{0, 2, 3, 5, 8, 12\}$
- $q = 7$: $\{2, 3, 4, 6, 9, 10, 18\}$
- $q = 11$: $\{0, 4, 5, 8, 9, 10, 11, 15, 16, 20, 30\}$.

As a final remark, we notice that the number of different intersection numbers is relatively large compared with q. On the other hand, an elliptic quadric containing a point of $\mathcal{K}$ and its antipode always meets $\mathcal{K}$ in $-3 \mod q$ points. In the above list, we notice for each q only two different intersection numbers corresponding to $-3 \mod q$. This might suggest that pairs point-antipode play a special role, and indeed, for the known examples, it is true that when a point belongs to $\mathcal{K}$, then also its antipode belongs to $\mathcal{K}$, [3, Theorem 12]. Unfortunately, the above combinatorial information seems too weak to prove such a characterisation. It is our feeling that such a characterisation could be helpful in proving the non-existence for larger q. We note that in [3], where a completely different approach is used, a comparable conclusion on the pairs point-antipode is made. Finally, we also mention the work in [6], where the non-existence for larger q is shown under the extra assumption that $(q^2 - 1)^2$ divides the order of the automorphism group of the maximal partial ovoid.

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