Isometric embeddings of the near polygons $\mathbb{H}_n$ and $\mathbb{G}_n$ into dual polar spaces

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Abstract

We prove that for every $n \in \mathbb{N} \setminus \{0,1\}$ there exists up to isomorphism a unique isometric embedding of the near polygon $\mathbb{H}_n$ into the dual polar space $DW(2n-1,2)$ and a unique isometric embedding of the near polygon $\mathbb{G}_n$ into the dual polar space $DH(2n-1,4)$.

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1 The main result

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, (possibly empty) line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *partial linear space* if every two distinct points are incident with at most one line. If x_1 and x_2 are two points of a partial linear space $\mathcal{S}$, then the distance $\mathrm{d}(x_1, x_2)$ between x_1 and x_2 will be measured in the collinearity graph of $\mathcal{S}$. A set X of points of $\mathcal{S}$ is called a *subspace* if every line having two of its points in X has all its points in X. If X is a subspace of $\mathcal{S}$, then we denote by $\widetilde{X}$ the subgeometry of $\mathcal{S}$ induced on X by those lines of $\mathcal{S}$ that have all their points in X. If X is a set of points of $\mathcal{S}$ such that the smallest subspace $S(X)$ of $\mathcal{S}$ containing X coincides with the whole point set, then X is called a *generating set* of $\mathcal{S}$.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two partial linear spaces. An *embedding* of $\mathcal{S}_1$ into $\mathcal{S}_2$ is an injective mapping ϵ from the point set of $\mathcal{S}_1$ to the point set of $\mathcal{S}_2$ satisfying the following two properties:

- ϵ maps every line of $\mathcal{S}_1$ into a line of $\mathcal{S}_2$;
- ϵ maps distinct lines of $\mathcal{S}_1$ into distinct lines of $\mathcal{S}_2$.

An embedding ϵ of $\mathcal{S}_1$ into $\mathcal{S}_2$ will be denoted by $\epsilon : \mathcal{S}_1 \to \mathcal{S}_2$. An embedding $e : \mathcal{S}_1 \to \mathcal{S}_2$ is called *full* if it maps lines of $\mathcal{S}_1$ to full lines of $\mathcal{S}_2$. The embedding ϵ is called *isometric* if it preserves the distances between points. Two embeddings $\epsilon : \mathcal{S}_1 \to \mathcal{S}_2$ and $\epsilon' : \mathcal{S}_1 \to \mathcal{S}_2'$ of the same partial linear space $\mathcal{S}_1$ are called *isomorphic* if there exists an isomorphism θ from $\mathcal{S}_2$ to $\mathcal{S}_2'$ such that $\epsilon' = \theta \circ \epsilon$.

Suppose ϵ is an embedding of the partial linear space $\mathcal{S}_1$ into the partial linear space $\mathcal{S}_2$. If θ_1 and θ_2 are automorphisms of respectively $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\theta_2 \circ \epsilon = \epsilon \circ \theta_1$, then we say that θ_1 *lifts (through ϵ) to* θ_2. If every automorphism of $\mathcal{S}_1$ lifts (through ϵ) to an automorphism of $\mathcal{S}_2$, then ϵ is called a *homogeneous embedding*.

In the present paper, we will meet two classes of dual polar spaces of rank $n \geq 2$. On the one hand, the symplectic dual polar space $DW(2n-1,q)$ associated with a symplectic polarity of the projective space $\mathrm{PG}(2n-1,q)$ and on the other hand the Hermitian dual polar space $DH(2n-1,q^2)$ associated with a nonsingular Hermitian variety $H(2n-1,q^2)$ of $\mathrm{PG}(2n-1,q^2)$. For $q \neq 2$, Cooperstein [4, 5] showed that the dual polar space $DW(2n-1,q)$ has a generating set of size $\frac{1}{n+2}\binom{2n+2}{n+1} = \binom{2n}{n} - \binom{2n}{n-2}$ and that the dual polar space $DH(2n-1,q^2)$ has a generating set of size $\binom{2n}{n}$. Neither of these results remain valid for $q=2$ and $n \geq 3$. However in each of the two cases, it is still true that there exists a set X of points in the dual polar space of the above-mentioned size such that $\widetilde{S(X)}$ is a nice subgeometry.

(1) By Blokhuis and Brouwer [1, Section 2] and Brouwer et al. [2, Section 5], we know that there exists a set X of $\frac{1}{n+2}\binom{2n+2}{n+1}$ points of $DW(2n-1,2)$ such that $\widetilde{S(X)}$ is isomorphic to the near $2n$-gon $\mathbb{H}_n$.

(2) By De Bruyn [8, Theorem 1.2], we know that there exists a set X of $\binom{2n}{n}$ points of $DH(2n-1,4)$ such that $\widetilde{S(X)}$ is isomorphic to the near $2n$-gon $\mathbb{G}_n$.

Explicit constructions of the near $2n$-gons $\mathbb{H}_n$ and $\mathbb{G}_n$ as well as the dual polar spaces $DW(2n-1,q)$ and $DH(2n-1,q^2)$ will be given in Section 2.

In each of the above cases, the embedding ϵ which realizes an isomorphism between $\mathcal{S} \in \{\mathbb{H}_n, \mathbb{G}_n\}$ and $\widetilde{S(X)}$ is isometric. One can wonder whether this embedding is the unique isometric embedding of $\mathcal{S}$ into the corresponding dual polar space. The following theorem, which is the main result of this paper, answers this question affirmatively.

Theorem 1.1 *Let $n \in \mathbb{N} \setminus \{0,1\}$.*

(1) *Up to isomorphism, there is a unique isometric embedding of $\mathbb{H}_n$ into $DW(2n-1,2)$. Every isometric embedding ϵ of $\mathbb{H}_n$ into $DW(2n-1,2)$ is homogeneous. More precisely, every isomorphism of $\mathbb{H}_n$ lifts through ϵ to precisely one automorphism of $DW(2n-1,2)$.*

(2) *Up to isomorphism, there is a unique isometric embedding of $\mathbb{G}_n$ into $DH(2n-1,4)$. Every isometric embedding ϵ of $\mathbb{G}_n$ into $DH(2n-1,4)$ is homogeneous. More precisely, every automorphism of $\mathbb{G}_n$ lifts through ϵ to precisely one automorphism of $DH(2n-1,4)$.*

One of the motivations for studying isometric embeddings of $\mathbb{H}_n$ into $DW(2n-1,2)$ and of $\mathbb{G}_n$ into $DH(2n-1,4)$ is the theory of valuations of dense near polygons introduced by De Bruyn and Vandecasteele [11]. This theory of valuations is important for obtaining classification results regarding dense near polygons. If $\epsilon : \mathcal{S}_1 \to \mathcal{S}_2$ is an isometric embedding between two dense near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$, then ϵ will give rise to valuations of $\mathcal{S}_1$. The valuations of a given dense near polygon $\mathcal{S}$ thus provide information on how $\mathcal{S}$ can be isometrically embedded into another dense near polygon. Isometric embeddings are also often useful for obtaining classification results regarding valuations themselves. This was the case in the paper [9] of the author where isometric embeddings of $\mathbb{G}_n$ into $DH(2n-1,4)$ have been used to obtain a complete classification of all valuations of the near $2n$-gon $\mathbb{G}_n$ and this will also be the case in the paper [10] where isometric embeddings of $\mathbb{H}_n$ into $DW(2n-1,2)$ will be used to obtain a complete classification of certain valuations of the near $2n$-gon $\mathbb{H}_n$.

2 Dense near polygons

The aim of this section is to collect some known definitions and properties regarding (dense) near polygons that will be useful during the proof of Theorem 1.1. Proofs of these properties can be found in the literature, see e.g. the book [7] of the author. We will also prove a number of new facts regarding isometric embeddings between dense near polygons.

2.1 Near polygons

A *near polygon* is a partial linear space with the property that for every point x and every line L, there exists a unique point $\pi_L(x)$ on L nearest to x. If d is the maximal distance between two points of a near polygon $\mathcal{S}$, then $\mathcal{S}$ is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called *generalized quadrangles*.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. A subspace X of $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of X is also contained in X. Clearly, the whole point set $\mathcal{P}$ is a convex subspace of $\mathcal{S}$ and the intersection of any number of (convex) subspaces of $\mathcal{S}$ is again a (convex) subspace of $\mathcal{S}$. If X is a non-empty convex subspace, then $\widetilde{X}$ itself is also a near polygon. If $*_1, *_2, \ldots, *_k$ are $k \geq 1$ objects of $\mathcal{S}$, each being a point or a nonempty set of points, then $\langle *_1, *_2, \ldots, *_k \rangle$ denotes the smallest convex subspace of $\mathcal{S}$ containing $*_1, *_2, \ldots, *_k$. The set $\langle *_1, *_2, \ldots, *_k \rangle$ is well-defined since it equals the intersection of all convex subspaces containing $*_1, *_2, \ldots, *_k$.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors. If x and y are two points of a dense near $2n$-gon $\mathcal{S}$ at distance δ from each other, then by Shult and Yanushka [14, Proposition 2.5] and Brouwer and Wilbrink [3, Theorem 4], $\langle x, y \rangle$ is the unique convex subspace of diameter δ containing x and y. The convex subspace $\langle x, y \rangle$ is called a *max* if $\delta = n - 1$. A max M of a dense near polygon $\mathcal{S}$ is called *big* if every point x of $\mathcal{S}$ not contained in M is collinear with a (necessarily unique) point $\pi_M(x)$ of M. If M is a big max of $\mathcal{S}$ and x is a point not contained in M, then $\mathrm{d}(x, y) = 1 + \mathrm{d}(\pi_M(x), y)$

for every point y of M. If M_1 and M_2 are two disjoint big maxes of $\mathcal{S}$, then the map $x \mapsto \pi_{M_2}(x)$ defines an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a dense near polygon with three points on each line, and that M is a big max of $\mathcal{S}$. For every point x of M, we define $\mathcal{R}_M(x) := x$. For every point x outside M, let $\mathcal{R}_M(x)$ denote the third point on the line through x and $\pi_M(x)$. The map $\mathcal{R}_M : \mathcal{P} \to \mathcal{P}$ defines an automorphism of $\mathcal{S}$ and is called the *reflection about* $\mathcal{S}$. So, if F is a convex subspace of $\mathcal{S}$, then $\mathcal{R}_M(F)$ is a convex subspace of the same diameter as F. If F is a big max, then also $\mathcal{R}_M(F)$ is a big max.

2.2 Dual polar spaces

Suppose Π is a thick polar space of rank $n \geq 2$ (Tits [15]). With Π, there is associated a *dual polar space* Δ *of rank* n. This is the point-line geometry whose points are the $(n-1)$-dimensional singular subspaces of Π, whose lines are the $(n-2)$-dimensional singular subspaces of Π and whose incidence relation is reverse containment. The dual polar space Δ is a dense near $2n$-gon. If P_1 and P_2 are two $(n-1)$-dimensional singular subspaces of Π, then the distance between the points P_1 and P_2 of Δ is equal to $n - 1 - \dim(P_1 \cap P_2)$. There exists a bijective correspondence between the points of Π and the maxes of Δ. If p is a point of Π, then the set of maximal singular subspaces of Π containing p is a max M_p of Δ. We say that p is *the point of* Π *corresponding to* M_p and that M_p is *the max of* Δ *corresponding to* p. Every max of Δ is big. If a max M and a convex subspace F of diameter δ of Δ have a point in common, then either $F \subseteq M$ or $F \cap M$ is a convex subspace of diameter $\delta - 1$. In the present paper, we will meet two classes of dual polar spaces.

Let ζ be a symplectic polarity of the projective space $\mathrm{PG}(2n-1, q)$, where $n \geq 2$ and q is a prime power. The subspaces of $\mathrm{PG}(2n-1, q)$ totally isotropic for ζ define a polar space $W(2n-1, q)$ whose associated dual polar space will be denoted by $DW(2n-1, q)$. A line of $\mathrm{PG}(2n-1, q)$ that is not totally isotropic with respect to ζ is called a *hyperbolic line of* $W(2n-1, q)$.

Let $H(2n-1, q^2)$ be a nonsingular Hermitian variety of the projective space $\mathrm{PG}(2n-1, q^2)$, where $n \geq 2$ and q is a prime power. The subspaces of $\mathrm{PG}(2n-1, q^2)$ contained in $H(2n-1, q^2)$ define a polar space whose associated dual polar space will be denoted by $DH(2n-1, q^2)$. A line of $\mathrm{PG}(2n-1, q^2)$ intersecting $H(2n-1, q^2)$ in precisely $q+1$ points is called a *hyperbolic line of* $H(2n-1, q^2)$.

Suppose Π is one of the polar spaces $W(2n-1, q)$ or $H(2n-1, q^2)$ and that $L = \{x_1, x_2, \ldots, x_{q+1}\}$ is the set of $q+1$ points of Π contained in some hyperbolic line of Π. Let Δ be the dual polar space associated with Π and for every $i \in \{1, 2, \ldots, q+1\}$, let M_i denote the max of Δ corresponding to x_i. Then the maxes $M_1, M_2, \ldots, M_{q+1}$ are mutually disjoint and each of the $|M_1| = |M_2|$ lines meeting M_1 and M_2 intersects every M_i, $i \in \{1, 2, \ldots, q+1\}$, in precisely one point.

2.3 The dense near $2n$-gon $\mathbb{H}_n$

Let $n \in \mathbb{N}$. With every set X of size $2n+2$, there is associated a point-line geometry $\mathbb{H}_n(X)$: the points of $\mathbb{H}_n(X)$ are the partitions of X in $n+1$ subsets of size 2; the lines of $\mathbb{H}_n(X)$ are the partitions of X in $n-1$ subsets of size 2 and 1 subset of size 4; a point p of $\mathbb{H}_n(X)$ is incident with a line L of $\mathbb{H}_n(X)$ if and only if the partition corresponding to p is a refinement of the partition corresponding to L. By Brouwer et al. [2], $\mathbb{H}_n(X)$ is a dense near $2n$-gon with three points on each line. The isomorphism class of the geometry $\mathbb{H}_n(X)$ is obviously independent of the set X of size $2n+2$. We will denote by $\mathbb{H}_n$ any suitable representative of this isomorphism class. The near polygon $\mathbb{H}_0$ consists of a unique point, the near polygon $\mathbb{H}_1$ is a line of size 3 and the near polygon $\mathbb{H}_2$ is isomorphic to the generalized quadrangle $W(2)$ described in Payne and Thas [13, Section 3.1].

Let P_1 and P_2 be two points of $\mathbb{H}_n(X)$, i.e. P_1 and P_2 are two partitions of X in $n+1$ subsets of size 2. Let Γ_{P_1,P_2} denote the graph with vertices the elements of X, with two distinct vertices i and j adjacent whenever $\{i,j\}$ is contained in $P_1 \cup P_2$. Then the distance between P_1 and P_2 in the near polygon $\mathbb{H}_n(X)$ is equal to $n+1-C$ where C is the number of connected components of Γ_{P_1,P_2}.

Suppose $n \geq 2$. There exists a bijective correspondence between the subsets of size 2 of X and the maxes of $\mathbb{H}_n(X)$. If Y is a subset of size 2 of X, then the set of all partitions P of X in $n+1$ subsets of size 2 such that $Y \in P$ is a max M_Y of $\mathbb{H}_n(X)$. We say that M_Y is *the max of* $\mathbb{H}_n(X)$ *corresponding to* Y and that Y is *the subset of size 2 of* X *corresponding to* M_Y. If M is a max of $\mathbb{H}_n(X)$, then $\widetilde{M} \cong \mathbb{H}_{n-1}$.

Suppose M_1 and M_2 are two distinct big maxes of $\mathbb{H}_n(X)$, $n \geq 2$. Let $\{x_i, y_i\}$, $i \in \{1,2\}$, be the subset of size 2 of X corresponding to M_i. If $|\{x_1,y_1\} \cap \{x_2,y_2\}| = 1$, say $x_1 = x_2$ and $y_1 \neq y_2$, then M_1 and M_2 are disjoint and the subset of size 2 of X corresponding to the big max $\mathcal{R}_{M_1}(M_2)$ is equal to $\{y_1, y_2\}$. If $\{x_1,y_1\} \cap \{x_2,y_2\} = \emptyset$, then $M_1 \cap M_2 \neq \emptyset$.

Every permutation of X determines in a natural way a permutation of the point set of $\mathbb{H}_n(X)$ defining an automorphism of $\mathbb{H}_n(X)$, and every automorphism of $\mathbb{H}_n(X)$ is obtained in this way.

2.4 The dense near $2n$-gon $\mathbb{G}_n$

Let $n \in \mathbb{N} \setminus \{0,1\}$, let V be a $2n$-dimensional vector space over $\mathbb{F}_4$ and let $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$ be an ordered basis of V. The set of all points $\langle \sum_{i=1}^{2n} X_i \bar{e}_i \rangle$ of $\mathrm{PG}(V)$ that satisfy the equation $\sum_{i=1}^{2n} X_i^3 = 0$ is a nonsingular Hermitian variety $H(V,B) \cong H(2n-1,4)$ of $\mathrm{PG}(V)$. We denote the dual polar space associated with $H(V,B)$ by $DH(V,B) \cong DH(2n-1,4)$. The *B-support* S_p of a point $p = \langle \sum_{i=1}^{2n} X_i \bar{e}_i \rangle$ of $\mathrm{PG}(V)$ is the set of all $i \in \{1,2,\ldots,2n\}$ for which $X_i \neq 0$. The number of elements of S_p is called the *B-weight* of p. Let Y denote the set of all $(n-1)$-dimensional subspaces of $H(V,B)$ generated by n points with B-weight 2 (whose B-supports are mutually disjoint). By De Bruyn [6], Y is a subspace of $DH(V,B)$ and $\mathbb{G}_n(V,B) := \widetilde{Y}$ is a dense near $2n$-gon with three points on each line. The isomorphism class of the geometry $\mathbb{G}_n(V,B)$ is independent of the $2n$-dimensional vector space V and the ordered basis B of V. We will denote by $\mathbb{G}_n$ any suitable representative of this isomorphism class. By [6], the generalized quadrangle $\mathbb{G}_2$ is isomorphic to the generalized quadrangle $Q^-(5,2)$ described in Payne and Thas [13, Section 3.1]. By convention, $\mathbb{G}_1$ is the line with three points and $\mathbb{G}_0$ is the near 0-gon.

If P_1 and P_2 are two points of $\mathbb{G}_n(V,B)$, then the distance between P_1 and P_2 in $\mathbb{G}_n(V,B)$ is equal to the distance between P_1 and P_2 in the dual polar space $DH(V,B)$.

If $n \geq 3$, then there exists a bijective correspondence between the points of $\mathrm{PG}(V)$ with B-weight 2 and the big maxes of $\mathbb{G}_n(V,B)$. If p is a point of $\mathrm{PG}(V)$ with B-weight 2, then the points of $\mathbb{G}_n(V,B)$ which, regarded as maximal singular subspaces of $H(V,B)$, contain p form a big max M_p of $\mathbb{G}_n(V,B)$. We will say that p is *the point of* $H(2n-1,4)$ *corresponding to* M_p and that M_p is *the big max of* $\mathbb{G}_n(V,B)$ *corresponding to* p. If M is a big max of $\mathbb{G}_n(V,B)$, then $\widetilde{M} \cong \mathbb{G}_{n-1}$.

Now, suppose that M_1 and M_2 are two distinct big maxes of $\mathbb{G}_n(V,B)$, $n \geq 3$, and that x_i, $i \in \{1,2\}$, is the point with B-weight 2 of $\mathrm{PG}(V)$ corresponding to M_i. We can distinguish the following three cases.

- $x_1 = \langle \bar{e}_{i_1} + \alpha \bar{e}_{i_2} \rangle$ and $x_2 = \langle \bar{e}_{i_1} + \beta \bar{e}_{i_2} \rangle$ for some $i_1, i_2 \in \{1,2,\ldots,2n\}$ with $i_1 \neq i_2$ and some $\alpha, \beta \in \mathbb{F}_4^*$ with $\alpha \neq \beta$. Then M_1 and M_2 are disjoint. If γ is the unique element in $\mathbb{F}_4 \setminus \{0,\alpha,\beta\}$ and $M_3 = \mathcal{R}_{M_1}(M_2)$, then $\langle \bar{e}_{i_1} + \gamma \bar{e}_{i_2} \rangle$ is the point with B-weight 2 corresponding to M_3.

- $x_1 = \langle \bar{e}_{i_1} + \alpha \bar{e}_{i_2} \rangle$ and $x_2 = \langle \bar{e}_{i_2} + \beta \bar{e}_{i_3} \rangle$ for some mutually distinct $i_1, i_2, i_3 \in \{1,2,\ldots,2n\}$ and some $\alpha, \beta \in \mathbb{F}_4^*$. Then M_1 and M_2 are disjoint. If $M_3 = \mathcal{R}_{M_1}(M_2)$, then $\langle \bar{e}_{i_1} + \alpha\beta \bar{e}_{i_3} \rangle$ is the point with B-weight 2 corresponding to M_3.

- $x_1 = \langle \bar{e}_{i_1} + \alpha \bar{e}_{i_2} \rangle$ and $x_2 = \langle \bar{e}_{i_3} + \beta \bar{e}_{i_4} \rangle$ for some mutually distinct $i_1, i_2, i_3, i_4 \in \{1,2,\ldots,2n\}$ and some $\alpha, \beta \in \mathbb{F}_4^*$. Then $M_1 \cap M_2 \neq \emptyset$.

If σ is a permutation of $\{1,2,\ldots,2n\}$, if ψ is an automorphism of $\mathbb{F}_4$ and if $\lambda_i \in \mathbb{F}_4^*$ for every $i \in \{1,2,\ldots,2n\}$, then the unique semi-linear map of V with associated field automorphism ψ that maps $\bar{e}_i$ to $\lambda_i \cdot \bar{e}_{\sigma(i)}$ for every $i \in \{1,2,\ldots,2n\}$ induces an automorphism of $\mathbb{G}_n(V,B)$. If $n \geq 3$, then every automorphism of $\mathbb{G}_n(V,B)$ is obtained in this way. (This is not true if $n = 2$.)

Proposition 2.1 *The natural inclusion defines a full homogeneous isometric embedding* ϵ^* *of* $\mathbb{G}_n(V,B)$ *into* $DH(V,B)$.

Proof. We have already remarked above that the inclusion maps preserves distances. It remains to show that every automorphism of $\mathbb{G}_n(V,B)$ lifts through ϵ^* to an automorphism of $DH(V,B)$. This is obviously the case if $n = 2$ since both geometries then coincide. If $n \geq 3$, then the claim follows from the fact that every automorphism of $\mathbb{G}_n(V,B)$ is induced by an automorphism of $\mathrm{PG}(V)$ stabilizing the Hermitian variety $H(V,B)$. ■

Proposition 2.2 *Only the trivial automorphism of* $DH(V,B)$ *fixes each point of its subgeometry* $\mathbb{G}_n(V,B)$.

Proof. Obviously, this is the case if $n = 2$ since $DH(V, B)$ and $\mathbb{G}_n(V, B)$ then coincide. Suppose therefore that $n \geq 3$. Suppose also that θ is an automorphism of $DH(V, B)$ fixing each point of $\mathbb{G}_n(V, B)$. Then θ is induced by an automorphism η of $\mathrm{PG}(V)$.

We prove that $\eta(\langle \bar{e}_i + \lambda \bar{e}_j \rangle) = \langle \bar{e}_i + \lambda \bar{e}_j \rangle$ for all $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$ and all $\lambda \in \mathbb{F}_4^*$. Since θ fixes each point of $\mathbb{G}_n(V, B)$, η fixes each $(n-1)$-dimensional subspace of $\mathrm{PG}(V)$ of the form $\langle \bar{e}_{\sigma(1)} + \lambda \bar{e}_{\sigma(2)}, \bar{e}_{\sigma(3)} + \bar{e}_{\sigma(4)}, \ldots, \bar{e}_{\sigma(2n-1)} + \bar{e}_{\sigma(2n)} \rangle$, where σ is some permutation of $\{1, 2, \ldots, 2n\}$ satisfying $\sigma(1) = i$ and $\sigma(2) = j$. Hence, η stabilizes the intersection of all these subspaces. Since this intersection is $\langle \bar{e}_{\sigma(1)} + \lambda \bar{e}_{\sigma(2)} \rangle = \langle \bar{e}_i + \lambda \bar{e}_j \rangle$, we have $\eta(\langle \bar{e}_i + \lambda \bar{e}_j \rangle) = \langle \bar{e}_i + \lambda \bar{e}_j \rangle$.

Hence, η stabilizes the line $\langle \bar{e}_i, \bar{e}_j \rangle$ for all $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$. If i, j, k are distinct elements of $\{1, 2, \ldots, 2n\}$, then since η stabilizes the lines $\langle \bar{e}_i, \bar{e}_j \rangle$ and $\langle \bar{e}_i, \bar{e}_k \rangle$, η fixes the point $\langle \bar{e}_i \rangle$.

So, η fixes each point of $\mathrm{PG}(V)$ with B-weight 1 or 2. This implies that η is the identity and hence that θ is the trivial automorphism of $DH(V, B)$. ■

Corollary 2.3 *Let ϵ^* denote the isometric embedding of $\mathbb{G}_n(V, B)$ into $DH(V, B)$ induced by the inclusion map. Then every automorphism θ of $\mathbb{G}_n(V, B)$ lifts through ϵ^* to precisely one automorphism of $DH(V, B)$.*

2.5 Isometric embeddings between dense near polygons

The following proposition, which was proved in Huang [12, Corollary 3.3], is often useful for verifying whether a given embedding between two dense near polygons is isometric.

Proposition 2.4 ([12]) *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two dense near polygons with respective distance functions $d_1(\cdot, \cdot)$ and $d_2(\cdot, \cdot)$ and respective diameters n_1 and n_2. Let ϵ be a map from $\mathcal{P}_1$ to $\mathcal{P}_2$ satisfying the following for any two points x and y of $\mathcal{P}_1$: if $d_1(x, y) = 1$, then also $d_2(\epsilon(x), \epsilon(y)) = 1$. Then ϵ is an isometric embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$ if and only if there exist points x^* and y^* in $\mathcal{S}_1$ satisfying $d_1(x^*, y^*) = d_2(\epsilon(x^*), \epsilon(y^*)) = n_1$.*

The following three propositions provide (structural) information regarding isometric embeddings between dense near polygons. We shall need that information later.

Proposition 2.5 *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two dense near polygons and let ϵ be an isometric embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$. Then for every nonempty convex subspace F of $\mathcal{S}_1$, there exists a unique nonempty convex subspace $\overline{F}$ of $\mathcal{S}_2$ satisfying:*

- *$\overline{F}$ and F have the same diameter;*
- *$\overline{F} \cap \epsilon(\mathcal{P}_1) = \epsilon(F)$.*

If F_1 and F_2 are two distinct nonempty convex subspaces of $\mathcal{S}_1$, then $\overline{F_1}$ and $\overline{F_2}$ are distinct.

Proof. Let δ be the diameter of F and let x, y be two points of F at maximal distance δ from each other. Then the points $\epsilon(x)$ and $\epsilon(y)$ of $\mathcal{S}_2$ lie at distance δ from each other and hence are contained in a unique convex subspace F' of diameter δ of $\mathcal{S}_2$. Note that if $\overline{F}$ is a convex subspace of diameter δ of $\mathcal{S}_2$ such that $\overline{F} \cap \epsilon(\mathcal{P}_1) = \epsilon(F)$, then $\overline{F} = F'$ since $\overline{F}$ contains the points $\epsilon(x)$ and $\epsilon(y)$.

Since F' is a convex subspace of diameter δ of $\mathcal{S}_2$, the set $F'' := \epsilon^{-1}(\epsilon(\mathcal{P}_1) \cap F')$ is a convex subspace of $\mathcal{S}_1$ of diameter at most δ. Since F'' contains the points x and y which lie at distance δ from each other, the diameter of F'' equals δ and so F'' has to coincide with F. So, we have $F' \cap \epsilon(\mathcal{P}_1) = \epsilon(F)$.

Suppose now that F_1 and F_2 are two nonempty convex subspaces of $\mathcal{S}_1$. If $\overline{F_1} = \overline{F_2}$, then $F_1 = \epsilon^{-1}(\epsilon(\mathcal{P}_1) \cap \overline{F_1}) = \epsilon^{-1}(\epsilon(\mathcal{P}_1) \cap \overline{F_2}) = F_2$. ■

Proposition 2.6 *Let $\epsilon^* : \mathcal{S}_1 \to \mathcal{S}_2$ be a full homogeneous isometric embedding between two dense near polygons $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$. Suppose $\epsilon : \mathcal{S}_1 \to \mathcal{S}_2$ is a full isometric embedding such that there exists an automorphism θ of $\mathcal{S}_2$ such that $\epsilon^*(\mathcal{P}_1) = \theta(\epsilon(\mathcal{P}_1))$. Then ϵ is isomorphic to ϵ^*.*

Proof. We have $\widetilde{\epsilon^*(\mathcal{P}_1)} \cong \mathcal{S}_1$, $\widetilde{\epsilon(\mathcal{P}_1)} \cong \mathcal{S}_1$ and θ defines an isomorphism between $\widetilde{\epsilon(\mathcal{P}_1)}$ and $\widetilde{\epsilon^*(\mathcal{P}_1)}$. Now, $(\epsilon^*)^{-1} \circ \theta \circ \epsilon$ is an isomorphism θ' of $\mathcal{S}_1$. So, $\epsilon^* \circ \theta' = \theta \circ \epsilon$. Since ϵ^* is homogeneous, there exists an automorphism θ'' of $\mathcal{S}_2$ such that $\theta'' \circ \epsilon^* = \epsilon^* \circ \theta'$. But then $\epsilon = \theta^{-1} \circ \theta'' \circ \epsilon^*$, showing that ϵ is isomorphic to ϵ^*. ■

Proposition 2.7 *Let $\epsilon : \mathcal{S}_1 \to \mathcal{S}_2$ be an isometric embedding between two dense near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$ of the same diameter n. If M_1 and M_2 are two disjoint big maxes of $\mathcal{S}_1$, then $\overline{M_1}$ and $\overline{M_2}$ are two disjoint maxes of $\mathcal{S}_2$.*

Proof. Let x_1 and x_1' be two points of M_1 at maximal distance $n-1$ from each other. Let x_2 and x_2' denote the points of M_2 collinear with x_1 and x_1', respectively. Then $\mathrm{d}(x_2, x_2') = n-1$ since the map $x \mapsto \pi_{M_2}(x)$ defines an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$. Let L be the line of $\mathcal{S}_2$ spanned by the points $\epsilon(x_1)$ and $\epsilon(x_2)$. Since $\pi_{M_2}(x_1) = x_2$ and $\pi_{M_2}(x_1') = x_2'$, we have $\mathrm{d}(x_1, x_2') = 1 + \mathrm{d}(x_2, x_2') = n$ and $\mathrm{d}(x_2, x_1') = \mathrm{d}(x_2, x_2') + 1 = n$. Hence, $\mathrm{d}(\epsilon(x_1), \epsilon(x_2')) = \mathrm{d}(\epsilon(x_2), \epsilon(x_1')) = n$. So, $\overline{M_1}$ contains neither of the points $\epsilon(x_2)$ and $\epsilon(x_2')$, and $\overline{M_2}$ contains neither of the points $\epsilon(x_1)$ and $\epsilon(x_1')$.

Let u be an arbitrary point of $\overline{M_1}$. As $\overline{M_1}$ is convex and $\pi_L(u)$ is on a shortest path between $\epsilon(x_1) \in \overline{M_1}$ and $u \in \overline{M_1}$, the point $\pi_L(u)$ is also contained in $\overline{M_1}$. If $\pi_L(u) \neq \epsilon(x_1)$, this would imply that the line L is completely contained in $\overline{M_1}$, in contradiction with the fact that $\epsilon(x_2) \notin \overline{M_1}$.

So, we should have that $\pi_L(u) = \epsilon(x_1)$ for every $u \in \overline{M_1}$. In a similar way one proves that $\pi_L(u) = \epsilon(x_2)$ for every $u \in \overline{M_2}$. Since $\epsilon(x_1) \neq \epsilon(x_2)$, this implies that $\overline{M_1}$ and $\overline{M_2}$ are disjoint. ■

3 An isometric homogeneous embedding of $\mathbb{H}_n$ into $DW(2n-1,2)$

We will now define an isometric embedding of $\mathbb{H}_n$ into $DW(2n-1,2)$ for every $n \in \mathbb{N} \setminus \{0,1\}$. The construction is due to Brouwer et al. [2, p. 356].

Let $n \geq 2$ and let V be a $(2n+2)$-dimensional vector space over $\mathbb{F}_2$ with basis $B = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n+2}\}$. We denote by W the set of all vectors of the form $\sum_{i=1}^{2n+2} X_i \bar{e}_i$ where $X_1 + X_2 + \cdots + X_{2n+2} = 0$. Put $\bar{e} := \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n+2}$ and $R := \langle \bar{e} \rangle$. Clearly, R is a subspace of W. On the quotient vector space W/R, we will now define a certain alternating bilinear form. For any two vectors $\bar{x} = \sum_{i=1}^{2n+2} X_i \bar{e}_i$ and $\bar{y} = \sum_{i=1}^{2n+2} Y_i \bar{e}_i$ of W, we put $f(\bar{x}+R, \bar{y}+R) := \sum_{i=1}^{2n+2} X_i Y_i$. Then:

(1) f is well-defined, i.e. replacing $\bar{x}$ by $\bar{x}+\bar{e}$ and/or $\bar{y}$ by $\bar{y}+\bar{e}$ does not alter the value of $f(\bar{x}+R, \bar{y}+R)$;

(2) f is bilinear;

(3) f is alternating since $f(\bar{x}+R, \bar{x}+R) = \sum_{i=1}^{2n+2} X_i^2 = \left(\sum_{i=1}^{2n+2} X_i\right)^2 = 0$ if $\bar{x} = \sum_{i=1}^{2n+2} X_i \bar{e}_i \in W$;

(4) f is nondegenerate. Indeed, if $\bar{x} = \sum_{i=1}^{2n+2} X_i \bar{e}_i \in W \setminus R$, then $X_{i_1} \neq X_{i_2}$ for some $i_1, i_2 \in \{1, 2, \ldots, 2n+2\}$ and hence $f(\bar{x}+R, \bar{e}_{i_1} + \bar{e}_{i_2} + R) \neq 0$.

The nondegenerate alternating bilinear form f determines a symplectic polarity ζ of $\mathrm{PG}(W/R)$ and we denote by $W(2n-1,2)$ and $DW(2n-1,2)$ the corresponding polar and dual polar spaces. For every point $p = \{\{\bar{b}_1, \bar{b}_2\}, \{\bar{b}_3, \bar{b}_4\}, \ldots, \{\bar{b}_{2n+1}, \bar{b}_{2n+2}\}\}$ of $\mathbb{H}_n := \mathbb{H}_n(B)$, let $\epsilon^*(p)$ be the $(n-1)$-dimensional subspace $\mathrm{PG}(\langle \bar{b}_1+\bar{b}_2, \bar{b}_3+\bar{b}_4, \ldots, \bar{b}_{2n+1}+\bar{b}_{2n+2}\rangle / R)$ of $\mathrm{PG}(W/R)$. Clearly, $\epsilon^*(p)$ is totally isotropic with respect to ζ and hence is a point of $DW(2n-1,2)$. It is straightforward to verify that ϵ^* is injective.

Suppose $L = \{p_1, p_2, p_3\}$ is some line of $\mathbb{H}_n$. Then

$$\begin{aligned}
p_1 &= \{\{\bar{b}_1, \bar{b}_2\}, \ldots, \{\bar{b}_{2n-3}, \bar{b}_{2n-2}\}, \{\bar{b}_{2n-1}, \bar{b}_{2n}\}, \{\bar{b}_{2n+1}, \bar{b}_{2n+2}\}\}, \\
p_2 &= \{\{\bar{b}_1, \bar{b}_2\}, \ldots, \{\bar{b}_{2n-3}, \bar{b}_{2n-2}\}, \{\bar{b}_{2n-1}, \bar{b}_{2n+1}\}, \{\bar{b}_{2n}, \bar{b}_{2n+2}\}\}, \\
p_3 &= \{\{\bar{b}_1, \bar{b}_2\}, \ldots, \{\bar{b}_{2n-3}, \bar{b}_{2n-2}\}, \{\bar{b}_{2n-1}, \bar{b}_{2n+2}\}, \{\bar{b}_{2n}, \bar{b}_{2n+1}\}\}
\end{aligned}$$

for some vectors $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{2n+2}$ of V such that $\{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{2n+2}\} = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n+2}\}$. Clearly, the points

$$\begin{aligned}
\epsilon^*(p_1) &= \mathrm{PG}(\langle \bar{b}_1+\bar{b}_2, \ldots, \bar{b}_{2n-3}+\bar{b}_{2n-2}, \bar{b}_{2n-1}+\bar{b}_{2n}, \bar{b}_{2n+1}+\bar{b}_{2n+2}\rangle/R), \\
\epsilon^*(p_2) &= \mathrm{PG}(\langle \bar{b}_1+\bar{b}_2, \ldots, \bar{b}_{2n-3}+\bar{b}_{2n-2}, \bar{b}_{2n-1}+\bar{b}_{2n+1}, \bar{b}_{2n}+\bar{b}_{2n+2}\rangle/R), \\
\epsilon^*(p_3) &= \mathrm{PG}(\langle \bar{b}_1+\bar{b}_2, \ldots, \bar{b}_{2n-3}+\bar{b}_{2n-2}, \bar{b}_{2n-1}+\bar{b}_{2n+2}, \bar{b}_{2n}+\bar{b}_{2n+1}\rangle/R)
\end{aligned}$$

are incident with the line $\mathrm{PG}(\langle \bar{b}_1+\bar{b}_2, \ldots, \bar{b}_{2n-3}+\bar{b}_{2n-2}, \bar{b}_{2n-1}+\bar{b}_{2n}+\bar{b}_{2n+1}+\bar{b}_{2n+2}\rangle/R)$ of $DW(2n-1,2)$, showing that ϵ^* is a full embedding.

Proposition 3.1 *The embedding ϵ^* is a full isometric homogeneous embedding of $\mathbb{H}_n$ into $DW(2n-1,2)$.*

Proof. To show that ϵ^* is isometric, it suffices by Proposition 2.4 to show that there exist opposite points p_1 and p_2 in $\mathbb{H}_n$ which are mapped by ϵ^* to opposite points $\epsilon^*(p_1)$ and $\epsilon^*(p_2)$ of $DW(2n-1,2)$. Take $p_1 = \{\{\bar{e}_1, \bar{e}_2\}, \{\bar{e}_3, \bar{e}_4\}, \ldots, \{\bar{e}_{2n+1}, \bar{e}_{2n+2}\}\}$ and $p_2 = \{\{\bar{e}_2, \bar{e}_3\}, \{\bar{e}_4, \bar{e}_5\}, \ldots, \{\bar{e}_{2n}, \bar{e}_{2n+1}\}, \{\bar{e}_{2n+2}, \bar{e}_1\}\}$. Since the subspace of W generated by $\langle \bar{e}_1 + \bar{e}_2, \bar{e}_3 + \bar{e}_4, \ldots, \bar{e}_{2n+1} + \bar{e}_{2n+2} \rangle$ and $\langle \bar{e}_2 + \bar{e}_3, \ldots, \bar{e}_{2n} + \bar{e}_{2n+1}, \bar{e}_{2n+2} + \bar{e}_1 \rangle$ equals W, the points $\epsilon^*(p_1)$ and $\epsilon^*(p_2)$ are indeed opposite points of $DW(2n-1,2)$.

The fact that ϵ^* is homogeneous follows from the fact that every automorphism of $\mathbb{H}_n = \mathbb{H}_n(B)$ is induced by a permutation of B and that every permutation of B extends in a unique way to an element of $GL(V)$ which induces a linear map of W/R leaving the form f invariant (and hence determines an automorphism of $DW(2n-1,2)$). ■

Proposition 3.2 *Only the trivial automorphism of $DW(2n-1,2)$ fixes each point of the image of ϵ^*.*

Proof. Let $\mathcal{P}$ denote the point set of $\mathbb{H}_n$. Suppose θ is an automorphism of $DW(2n-1,2)$ which fixes each point of $\epsilon^*(\mathcal{P})$. Then θ is induced by an element η of $GL(W/R)$ which leaves the form f invariant.

We prove that $\eta(\bar{e}_i + \bar{e}_j + R) = \bar{e}_i + \bar{e}_j + R$ for all $i, j \in \{1, 2, \ldots, 2n+2\}$ with $i \neq j$. Since θ fixes each point of $\epsilon^*(\mathcal{P})$, η stabilizes each n-dimensional subspace of W/R of the form $\langle \bar{e}_{\sigma(1)} + \bar{e}_{\sigma(2)}, \bar{e}_{\sigma(3)} + \bar{e}_{\sigma(4)}, \ldots, \bar{e}_{\sigma(2n+1)} + \bar{e}_{\sigma(2n+2)} \rangle / R$, where σ is some permutation of $\{1, 2, \ldots, 2n+2\}$ satisfying $\sigma(1) = i$ and $\sigma(2) = j$. So, η also stabilizes the intersection of all these subspaces. Since this intersection is equal to $\langle \bar{e}_i + \bar{e}_j + R \rangle$, we have $\eta(\bar{e}_i + \bar{e}_j + R) = \bar{e}_i + \bar{e}_j + R$.

Now, since every vector of W/R can be written as a sum of vectors of the form $\bar{e}_i + \bar{e}_j + R$, we necessarily have that η is the trivial element of $GL(W/R)$. So, θ is the trivial automorphism of $DW(2n-1,2)$. ■

The following is an immediate consequence of Proposition 3.1 and 3.2.

Corollary 3.3 *Every automorphism of $\mathbb{H}_n$ lifts through ϵ^* to precisely one automorphism of $DW(2n-1,2)$.*

4 Isometric embeddings of $\mathbb{H}_n$ into $DW(2n-1,2)$

Let $n \geq 2$, let V be a $2n$-dimensional vector space over the field $\mathbb{F}_2$ equipped with a nondegenerate alternating bilinear form f, and let ζ denote the symplectic polarity of PG(V) corresponding to f. With ζ there is associated a polar space $W(2n-1,2)$ and a dual polar space $DW(2n-1,2)$. Put $\mathbb{H}_n := \mathbb{H}_n(X)$ where $X = \{1, 2, \ldots, 2n+2\}$. For every subset $\{i, j\}$ of size 2 of X, let $M_{i,j}$ denote the big max of $\mathbb{H}_n$ corresponding to $\{i, j\}$. Recall that if i, j and k are three distinct elements of X, then $\mathcal{R}_{M_{i,j}}(M_{i,k}) = M_{j,k}$.

Now, suppose that ϵ is an isometric embedding of $\mathbb{H}_n$ into $DW(2n-1,2)$. For every convex subspace F of $\mathbb{H}_n$, there exists by Proposition 2.5 a unique convex subspace $\overline{F}$ of $DW(2n-1,2)$ having the same diameter as F and containing $\epsilon(F)$.

Lemma 4.1 *If M_1 and M_2 are two disjoint big maxes of $\mathbb{H}_n$, then also the maxes $\overline{M_1}$ and $\overline{M_2}$ of $DW(2n-1,2)$ are disjoint, and $\mathcal{R}_{\overline{M_2}}(\overline{M_1}) = \overline{\mathcal{R}_{M_2}(M_1)}$.*

Proof. By Proposition 2.7, the maxes $\overline{M_1}$ and $\overline{M_2}$ are disjoint. Put $M_3 := \mathcal{R}_{M_2}(M_1)$. Since every point of M_3 lies on a line joining a point of M_1 with a point of M_2, we have $\epsilon(M_3) \subseteq \mathcal{R}_{\overline{M_2}}(\overline{M_1})$. Since $\epsilon(M_3)$ and $\mathcal{R}_{\overline{M_2}}(\overline{M_1})$ have the same diameter, we have $\overline{M_3} = \mathcal{R}_{\overline{M_2}}(\overline{M_1})$. ■

For every big max M of $\mathbb{H}_n$, let x_M denote the unique point of $W(2n-1,2)$ corresponding to the max $\overline{M}$ of $DW(2n-1,2)$.

Lemma 4.2 (1) *If M_1 and M_2 are two distinct big maxes of $\mathbb{H}_n$ which meet each other, then $\overline{M_1} \cap \overline{M_2} \neq \emptyset$ and hence $x_{M_1}x_{M_2}$ is a singular line of $W(2n-1,2)$.*

(2) *If M_1 and M_2 are two disjoint big maxes of $\mathbb{H}_n$ and $M_3 = \mathcal{R}_{M_1}(M_2)$, then $\{x_{M_1}, x_{M_2}, x_{M_3}\}$ is a hyperbolic line of $W(2n-1,2)$.*

Proof. (1) Since $\epsilon(M_1 \cap M_2) \subseteq \overline{M_1} \cap \overline{M_2}$, we have $\overline{M_1} \cap \overline{M_2} \neq \emptyset$. Hence, $x_{M_1}x_{M_2}$ is a singular line of $W(2n-1,2)$.

(2) By Lemma 4.1, $\overline{M_1}$ and $\overline{M_2}$ are disjoint and $\mathcal{R}_{\overline{M_2}}(\overline{M_1}) = \overline{M_3}$. So, $\{x_{M_1}, x_{M_2}, x_{M_3}\}$ is a hyperbolic line of $W(2n-1,2)$. ■

Lemma 4.3 *Let x be a point of $\mathbb{H}_n$ and let $M_1, M_2, \dots, M_{n+1}$ denote the $n+1$ big maxes of $\mathbb{H}_n$ containing x. Then $\langle x_{M_1}, x_{M_2}, \dots, x_{M_n} \rangle = \langle x_{M_1}, x_{M_2}, \dots, x_{M_{n+1}} \rangle$ is the maximal singular subspace of $W(2n-1,2)$ corresponding to the point $\epsilon(x)$ of $DW(2n-1,2)$.*

Proof. Observe that $\epsilon(x) \in \overline{M_i}$ for every $i \in \{1,2,\dots,n+1\}$. We prove that the diameter of $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_j}$ is equal to $n-j$ for every $j \in \{1,2,\dots,n\}$. Suppose this claim is not valid and let i be the smallest value of j for which this is the case. Then $i \neq 1$ and $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_{i-1}} \subseteq \overline{M_i}$. Now, there exists a point $y \in (M_1 \cap M_2 \cap \cdots \cap M_{i-1}) \setminus M_i$, since there exists a partition of X in $n+1$ subsets of size 2 containing all subsets of size 2 corresponding to $M_1, M_2, \dots, M_{i-1}$, but not the subset of size 2 corresponding to M_i. Clearly, $\epsilon(y) \in \overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_{i-1}}$ and hence $\epsilon(y) \in \overline{M_i}$, contradicting Proposition 2.5.

Since $M_i \cap M_j \neq \emptyset$, $x_{M_i}x_{M_j}$ is a singular line of $W(2n-1,2)$ for all $i,j \in \{1,2,\dots,n+1\}$ with $i \neq j$. Hence, $\langle x_{M_1}, x_{M_2}, \dots, x_{M_n} \rangle$ and $\langle x_{M_1}, x_{M_2}, \dots, x_{M_n}, x_{M_{n+1}} \rangle$ are singular subspaces of $W(2n-1,2)$. If α is a maximal singular subspace of $W(2n-1,2)$ containing $\langle x_{M_1}, x_{M_2}, \dots, x_{M_n} \rangle$, then α regarded as point of $DW(2n-1,2)$ is contained in each of the maxes $\overline{M_1}, \overline{M_2}, \dots, \overline{M_n}$. Since $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_n}$ has diameter 0, $\epsilon(x)$ is the unique point in $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_n}$. It follows that $\alpha = \epsilon(x) = \langle x_{M_1}, x_{M_2}, \dots, x_{M_n} \rangle$. Hence, also $\epsilon(x) = \langle x_{M_1}, x_{M_2}, \cdots, x_{M_{n+1}} \rangle$. ■

Corollary 4.4 *Let x and y be two opposite points of $\mathbb{H}_n$. Let M_i, $i \in \{1,2,\dots,n+1\}$, denote the $n+1$ big maxes of $\mathbb{H}_n$ containing x. Let N_i, $i \in \{1,2,\dots,n+1\}$, denote the $n+1$ big maxes of $\mathbb{H}_n$ containing y. Then $\langle x_{M_1}, x_{M_2}, \dots, x_{M_n}, x_{N_1}, x_{N_2}, \dots, x_{N_n} \rangle = \mathrm{PG}(V)$.*

Proof. The points $\epsilon(x)$ and $\epsilon(y)$ are opposite points of $DW(2n-1,2)$. Hence, by Lemma 4.3 $\langle x_{M_1}, x_{M_2}, \dots, x_{M_n} \rangle$ and $\langle x_{N_1}, x_{N_2}, \dots, x_{N_n} \rangle$ are disjoint maximal singular subspaces of $W(2n-1,2)$. So, these maximal singular subspaces generate the whole projective space $\mathrm{PG}(V)$. ■

For all $i,j \in \{1,2,\ldots,2n+2\}$ with $i \neq j$, let $x_{i,j}$ denote the point of $W(2n-1,2)$ corresponding with the max $\overline{M_{i,j}}$ of $DW(2n-1,2)$. By Lemma 4.2(2), we have

Corollary 4.5 *If i, j and k are three distinct elements of $\{1,2,\ldots,2n+2\}$, then the points $x_{i,j}$, $x_{j,k}$ and $x_{i,k}$ form a line of* $\mathrm{PG}(V)$.

Lemma 4.6 *We have* $\mathrm{PG}(V) = \langle x_{1,2}, x_{1,3}, \ldots, x_{1,2n+1}\rangle$.

Proof. Let x and y be two opposite points of $\mathbb{H}_n$. Let M_i (respectively N_i), $i \in \{1,2,\ldots,n+1\}$, denote the $n+1$ big maxes of $\mathbb{H}_n$ containing x (respectively y). Without loss of generality, we may suppose that there exist $a,b \in \{1,2,\ldots,2n+1\}$ such that $M_{n+1} = M_{a,2n+2}$ and $N_{n+1} = M_{b,2n+2}$.

By Corollary 4.5, we have $\langle x_{1,2}, x_{1,3}, \ldots, x_{1,2n+1}\rangle = \langle x_{i,j} \,|\, 1 \leq i < j \leq 2n+1\rangle$. By Corollary 4.4, the subspace $\langle x_{i,j} \,|\, 1 \leq i < j \leq 2n+1\rangle$ should coincide with $\mathrm{PG}(V)$. ■

Now, let $\{\bar{e}_{1,2}, \bar{e}_{1,3}, \ldots, \bar{e}_{1,2n+1}\}$ be a basis of V such that $x_{1,i} = \langle \bar{e}_{1,i}\rangle$ for every $i \in \{2,3,\ldots,2n+1\}$. We put $\bar{e}_{1,1}$ equal to the zero vector of V.

Lemma 4.7 (1) *For all $i,j \in \{2,3,\ldots,2n+1\}$, $f(\bar{e}_{1,i}, \bar{e}_{1,j})$ is equal to 0 if $i=j$ and equal to 1 otherwise.*

(2) *If $\mathcal{P}$ is the point set of $\mathbb{H}_n$, then $\epsilon(\mathcal{P})$ consists of all maximal singular subspaces of $W(2n-1,2)$ of the form $\langle \bar{e}_{1,i_1} + \bar{e}_{1,i_2}, \bar{e}_{1,i_3} + \bar{e}_{1,i_4}, \ldots, \bar{e}_{1,i_{2n-1}} + \bar{e}_{1,i_{2n}}\rangle$ where $\{i_1, i_2, \ldots, i_{2n}\}$ is some subset of size $2n$ of $\{1,2,\ldots,2n+1\}$.*

Proof. (1) Since f is an alternating form we have $f(\bar{e}_{1,i}, \bar{e}_{1,i}) = 0$ for every $i \in \{2,3,\ldots,2n+1\}$. If $i,j \in \{2,3,\ldots,2n+1\}$ with $i \neq j$, then $\overline{M_{1,i}}$ and $\overline{M_{1,j}}$ are disjoint since $M_{1,i}$ and $M_{1,j}$ are disjoint. Hence, $f(\bar{e}_{1,i}, \bar{e}_{1,j}) = 1$.

(2) By Corollary 4.5, $x_{i,j} = \langle \bar{e}_{1,i} + \bar{e}_{1,j}\rangle$ for all $i,j \in \{1,2,\ldots,2n+1\}$. The claim then follows from Lemma 4.3. ■

Proposition 4.8 *Let $\mathcal{P}$ denote the point set of $\mathbb{H}_n$. If ϵ_1 and ϵ_2 are two isometric embeddings of $\mathbb{H}_n$ into $DW(2n-1,2)$, then there exists an automorphism θ of $DW(2n-1,2)$ such that $\theta(\epsilon_1(\mathcal{P})) = \epsilon_2(\mathcal{P})$.*

Proof. By Lemma 4.7, there exists for every $k \in \{1,2\}$ a basis $\{\bar{e}^k_{1,2}, \bar{e}^k_{1,3}, \ldots, \bar{e}^k_{1,2n+1}\}$ of V for which the following hold:

(1) for all $i,j \in \{2,3,\ldots,2n+1\}$, $f(\bar{e}^k_{1,i}, \bar{e}^k_{1,j})$ is equal to 0 of $i=j$ and equal to 1 otherwise;

(2) $\epsilon_k(\mathcal{P})$ consists of all maximal singular subspaces of $W(2n-1,2)$ of the form $\langle \bar{e}^k_{1,i_1} + \bar{e}^k_{1,i_2}, \bar{e}^k_{1,i_3} + \bar{e}^k_{1,i_4}, \ldots, \bar{e}^k_{1,i_{2n-1}} + \bar{e}^k_{1,i_{2n}}\rangle$ where $\{i_1, i_2, \ldots, i_{2n}\}$ is some subset of size $2n$ of $\{1,2\ldots,2n+1\}$.

Here, $\bar{e}^1_{1,1}$ and $\bar{e}^2_{1,1}$ denote the null vector. Now, the linear automorphism of V determined by $\bar{e}^1_{1,i} \mapsto \bar{e}^2_{1,i}$, $\forall i \in \{2, 3, \ldots, 2n+1\}$, leaves the form f invariant and hence determines an automorphism θ of $DW(2n-1, 2)$. Clearly, $\theta(\epsilon_1(\mathcal{P})) = \epsilon_2(\mathcal{P})$. ■

The following corollary, which is precisely Theorem 1.1(1), is a consequence of Propositions 2.6, 3.1, 4.8 and Corollary 3.3.

Corollary 4.9 *Up to isomorphism, there is a unique isometric embedding of* $\mathbb{H}_n$ *into* $DW(2n-1, 2)$. *Every isometric embedding* ϵ *of* $\mathbb{H}_n$ *into* $DW(2n-1, 2)$ *is homogeneous. More precisely, every automorphism of* $\mathbb{H}_n$ *lifts through* ϵ *to precisely* 1 *automorphism of* $DW(2n-1, 2)$.

5 Isometric embeddings of $\mathbb{G}_n$ into $DH(2n-1, 4)$

Let $n \geq 3$. Let V be a $2n$-dimensional vector space over $\mathbb{F}_4$ and let $B^* = (\bar{e}^*_1, \bar{e}^*_2, \ldots, \bar{e}^*_{2n-1}, \bar{e}^*_{2n})$ be a given ordered basis of V. Put $\mathbb{G}_n := \mathbb{G}_n(V, B^*)$. For every point p of $\mathrm{PG}(V)$ with B^*-weight 2, let M_p denote the big max of $\mathbb{G}_n$ corresponding to p. If $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$, let $\mathcal{M}_{i,j} = \{M_p \,|\, p = \langle \bar{e}^*_i + a\bar{e}^*_j \rangle$ for some $a \in \mathbb{F}^*_4\}$. The following hold:

(1) For every big max M of $\mathbb{G}_n$, there exists precisely one subset $\{i, j\}$ of size 2 of $\{1, 2, \ldots, 2n\}$ such that $M \in \mathcal{M}_{i,j}$.

(2) Let $\{i, j\} \in \{1, 2, \ldots, 2n\}$ with $i \neq j$. If M_1, M_2 and M_3 are the three elements of $\mathcal{M}_{i,j}$, then M_1, M_2 and M_3 are mutually disjoint and $M_3 = \mathcal{R}_{M_1}(M_2)$.

(3) Let M and M' be two big maxes of $\mathbb{G}_n$ and let $\{i, j\}$ and $\{i', j'\}$ be the unique subsets of size 2 of $\{1, 2, \ldots, 2n\}$ such that $M \in \mathcal{M}_{i,j}$ and $M' \in \mathcal{M}_{i',j'}$. If $\{i, j\} \cap \{i', j'\} = \emptyset$, then $M \cap M' \neq \emptyset$. If $|\{i, j\} \cap \{i', j'\}| = 1$, say $j = i'$ and $i \neq j'$, then $M \cap M' = \emptyset$ and $\mathcal{R}_M(M') = \mathcal{R}_{M'}(M) \in \mathcal{M}_{i,j'}$.

(4) Let i, j and k be three distinct elements of $\{1, 2, \ldots, 2n\}$. Then any two distinct elements M_1 and M_2 of $\mathcal{M}_{i,j} \cup \mathcal{M}_{i,k} \cup \mathcal{M}_{j,k}$ are disjoint. Moreover, also $\mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$ belongs to $\mathcal{M}_{i,j} \cup \mathcal{M}_{i,k} \cup \mathcal{M}_{j,k}$. The point-line geometry with points the elements of $\mathcal{M}_{i,j} \cup \mathcal{M}_{i,k} \cup \mathcal{M}_{j,k}$ and with lines all the subsets $\{M_1, M_2, \mathcal{R}_{M_1}(M_2)\}$, where M_1 and M_2 are two distinct elements of $\mathcal{M}_{i,j} \cup \mathcal{M}_{i,k} \cup \mathcal{M}_{j,k}$ is isomorphic to the affine plane of order 3.

Now, suppose that ϵ is an isometric embedding of $\mathbb{G}_n$ into $DH(2n-1, 4) = DH(V, B^*)$. If ϵ is the inclusion map, then ϵ coincides with the natural embedding ϵ^* of $\mathbb{G}_n = \mathbb{G}_n(V, B^*)$ into $DH(2n-1, 4) = DH(V, B^*)$. Let $H(2n-1, 4)$ denote the nonsingular Hermitian variety $H(V, B^*)$ of $\mathrm{PG}(V)$. For every max M of $\mathbb{G}_n$, let $\overline{M}$ denote the unique max of $DH(2n-1, 4)$ containing $\epsilon(M)$.

Lemma 5.1 *If* M_1 *and* M_2 *are two disjoint big maxes of* $\mathbb{G}_n$, *then also the maxes* $\overline{M_1}$ *and* $\overline{M_2}$ *of* $DH(2n-1, 4)$ *are disjoint. Moreover,* $\mathcal{R}_{\overline{M_2}}(\overline{M_1}) = \overline{\mathcal{R}_{M_2}(M_1)}$.

Proof. By Proposition 2.7, the maxes $\overline{M_1}$ and $\overline{M_2}$ are disjoint. Put $M_3 := \mathcal{R}_{M_2}(M_1)$. Since through every point of $\overline{M_3}$, there is a line joining a point of $\overline{M_1}$ with a point of $\overline{M_2}$, we have $\epsilon(M_3) \subseteq \mathcal{R}_{\overline{M_2}}(\overline{M_1})$. Since $\epsilon(M_3)$ and $\mathcal{R}_{\overline{M_2}}(\overline{M_1})$ have the same diameter, we necessarily have $\overline{M_3} = \mathcal{R}_{\overline{M_2}}(\overline{M_1})$. ■

Let $\{i, j\}$ be a subset of size 2 of $\{1, 2, \ldots, 2n\}$. If $\mathcal{M}_{i,j} = \{M_1, M_2, M_3\}$, then by Lemma 5.1, $\overline{M_1}$, $\overline{M_2}$ and $\overline{M_3}$ are mutually disjoint maxes of $DH(2n-1, 4)$ satisfying $\overline{M_3} = \mathcal{R}_{\overline{M_2}}(\overline{M_1})$. So, if x_i, $i \in \{1, 2, 3\}$, denotes the point of $H(2n-1, 4)$ corresponding to $\overline{M_i}$, then there exists a hyperbolic line $L_{i,j}$ of $H(2n-1, 4)$ such that $L_{i,j} \cap H(2n-1, 4) = \{x_1, x_2, x_3\}$.

Lemma 5.2 *Let i, j and k be three distinct elements of $\{1, 2, \ldots, 2n\}$. Then $L_{i,j}$ intersects $L_{i,k}$ in a point of $\mathrm{PG}(V) \setminus H(2n-1, 4)$.*

Proof. Put $\mathcal{M}_{i,j} = \{M_1, M_2, M_3\}$, $\mathcal{M}_{i,k} = \{M_4, M_5, M_6\}$ and $\mathcal{M}_{j,k} = \{M_7, M_8, M_9\}$. Let x_i, $i \in \{1, 2, \ldots, 9\}$, denote the point of $H(2n-1, 4)$ corresponding to $\overline{M_i}$. Recall that an affine plane can be defined on the set $\mathcal{M}_{i,j} \cup \mathcal{M}_{i,k} \cup \mathcal{M}_{j,k}$. This affine plane is generated by the points M_1, M_2 and M_4. Hence, there exists a plane α of $\mathrm{PG}(V)$ containing the points $x_1, x_2, \ldots, x_9$. Since $L_{i,j} \cap H(2n-1, 4) = \{x_1, x_2, x_3\}$, $L_{i,k} \cap H(2n-1, 4) = \{x_4, x_5, x_6\}$ and $L_{i,j} \cup L_{i,k} \subseteq \alpha$, $L_{i,j} \cap L_{i,k}$ is a singleton not contained in $H(2n-1, 4)$. ■

Lemma 5.3 *For every $i \in \{1, 2, \ldots, 2n\}$, there exists a unique point $x_i^* \in \mathrm{PG}(V) \setminus H(2n-1, 4)$ such that $x_i^* \in L_{i,j}$ for every $j \in \{1, 2, \ldots, 2n\} \setminus \{i\}$.*

Proof. In view of Lemma 5.2, we need to show that $L_{i,j_1} \cap L_{i,j_2} = L_{i,j_1} \cap L_{i,j_3}$ for any three distinct elements j_1, j_2 and j_3 of $\{1, 2, \ldots, 2n\} \setminus \{i\}$.

Suppose the contrary. Then $u_1 \neq u_2 \neq u_3 \neq u_1$, where u_1, u_2 and u_3 are the unique elements in $L_{i,j_1} \cap L_{i,j_2}$, $L_{i,j_1} \cap L_{i,j_3}$ and $L_{i,j_2} \cap L_{i,j_3}$, respectively. Notice that $L_{i,j_1} \setminus H(2n-1, 4) = \{u_1, u_2\}$, $L_{i,j_2} \setminus H(2n-1, 4) = \{u_1, u_3\}$ and $L_{i,j_3} \setminus H(2n-1, 4) = \{u_2, u_3\}$. Now, take a $j_4 \in \{1, 2, \ldots, 2n\} \setminus \{i, j_1, j_2, j_3\}$. Since L_{i,j_4} intersects each of the lines L_{i,j_1}, L_{i,j_2}, L_{i,j_3} in a point outside $H(2n-1, 4)$, L_{i,j_4} contains at least two of the points u_1, u_2, u_3. So, L_{i,j_4} must coincide with one of the lines L_{i,j_1}, L_{i,j_2}, L_{i,j_3}, a contradiction, since $\mathcal{M}_{i,j_4}$ is distinct from each of the sets $\mathcal{M}_{i,j_1}$, $\mathcal{M}_{i,j_2}$ and $\mathcal{M}_{i,j_3}$. ■

Lemma 5.4 *If $i_1, i_2 \in \{1, 2, \ldots, 2n\}$ with $i_1 \neq i_2$, then $x_{i_1}^* \neq x_{i_2}^*$.*

Proof. Let i_3 be an element of $\{1, 2, \ldots, 2n\}$ distinct from i_1 and i_2. Put $\mathcal{M}_{i_1,i_2} = \{M_1, M_2, M_3\}$, $\mathcal{M}_{i_1,i_3} = \{M_4, M_5, M_6\}$ and $\mathcal{M}_{i_2,i_3} = \{M_7, M_8, M_9\}$. Let x_i, $i \in \{1, 2, \ldots, 9\}$, denote the point of $H(2n-1, 4)$ corresponding to $\overline{M_i}$. Notice that the points x_1, x_2, x_3 are contained in a line U containing $x_{i_1}^*$ and $x_{i_2}^*$. Similarly, the points x_4, x_5, x_6 are contained in a line U' containing $x_{i_1}^*$ and the points x_7, x_8, x_9 are on a line U'' containing $x_{i_2}^*$. As indicated in the proof of Lemma 5.2, the points $x_1, x_2, \ldots, x_9$ are contained in a plane α of $\mathrm{PG}(V)$. Since $M_1, M_2, \ldots, M_9$ are mutually disjoint, the points $x_1, x_2, \ldots, x_9$ are mutually noncollinear on $H(2n-1, 4)$ by Lemma 5.1. Hence, $\alpha \cap H(2n-1, 4)$ is a unital of α which is equal to $\{x_1, x_2, \ldots, x_9\}$. Now, every point of $\alpha \setminus H(2n-1, 4)$

is contained in precisely 2 lines which contain three points of $\alpha \cap H(2n-1,4)$. Since $x^*_{i_1} \in \alpha \setminus H(2n-1,4)$ is contained in the lines U and U', $x^*_{i_1} \notin U''$. Since $x^*_{i_2} \in U''$, we have $x^*_{i_1} \neq x^*_{i_2}$. ■

The following is an immediate corollary of Lemmas 5.3 and 5.4.

Corollary 5.5 *For all $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$, $L_{i,j} = x^*_i x^*_j$.*

Lemma 5.6 *Let u be a point of $\mathbb{G}_n$ and let $M_1, M_2, \ldots, M_n$ denote the n big maxes of $\mathbb{G}_n$ containing u. Let x_i, $i \in \{1, 2, \ldots, n\}$, denote the point of $H(2n-1,4)$ corresponding to $\overline{M_i}$. Then $\langle x_1, x_2, \ldots, x_n \rangle$ is the maximal singular subspace of $H(2n-1,4)$ corresponding to the point $\epsilon(u)$ of $DH(2n-1,4)$.*

Proof. Observe that $\epsilon(u) \in \overline{M_i}$ for every $i \in \{1, 2, \ldots, n\}$. We prove that the diameter of $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_j}$ is equal to $n-j$ for every $j \in \{1, 2, \ldots, n\}$. Suppose that this claim is not valid and let i be the smallest value of j for which this is the case. Then $i \neq 1$ and $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_{i-1}} \subseteq \overline{M_i}$. Now, there exists a point $v \in (M_1 \cap M_2 \cap \cdots \cap M_{i-1}) \setminus M_i$. Indeed, if y_j, $j \in \{1, 2, \ldots, n\}$, denotes the point of $\mathrm{PG}(V)$ with B^*-weight 2 corresponding to M_j, then there exists a maximal singular subspace of $H(2n-1,4)$ containing $y_1, y_2, \ldots, y_{i-1}$, but not y_i. For any choice of $v \in (M_1 \cap M_2 \cap \cdots \cap M_{i-1}) \setminus M_i$, we have $\epsilon(v) \in \overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_{i-1}}$ and hence $\epsilon(v) \in \overline{M_i}$, contradicting Proposition 2.5.

So we know that $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_n}$ consists of a unique point. Since $\epsilon(u) \in \overline{M_i}$ for every $i \in \{1, 2, \ldots, n\}$, we have $\overline{M_1} \cap \overline{M_2} \cap \cdots \cap \overline{M_n} = \{\epsilon(u)\}$. Since $\epsilon(u) \in \overline{M_i}$, $i \in \{1, 2, \ldots, n\}$, $\epsilon(u)$ can be regarded as a maximal singular subspace of $H(2n-1,4)$ containing the point x_i. If α is a maximal singular subspace of $H(2n-1,4)$ containing $\langle x_1, x_2, \ldots, x_n \rangle$, then α regarded as point of $DH(2n-1,4)$ is contained in each of the maxes $\overline{M_1}, \overline{M_2}, \ldots, \overline{M_n}$. It follows that $\alpha = \epsilon(u) = \langle x_1, x_2, \ldots, x_n \rangle$. ■

Lemma 5.7 *The points $x^*_1, x^*_2, \ldots, x^*_{2n}$ generate* $\mathrm{PG}(V)$.

Proof. Let u be an arbitrary point of $\mathbb{G}_n$ and let $M_1, M_2, \ldots, M_n$ denote the n big maxes of $\mathbb{G}_n$ containing u. For every $i \in \{1, 2, \ldots, n\}$, let M'_i be an arbitrary element of the set $\mathcal{M}_{j,k} \setminus \{M_i\}$, where $\{j, k\}$ is the unique subset of size 2 of $\{1, 2, \ldots, 2n\}$ such that $M_i \in \mathcal{M}_{j,k}$. If y_i and y'_i are the points of $\mathrm{PG}(V)$ with B^*-weight 2 corresponding to M_i and M'_i, respectively, then y_i and y'_i are distinct and have the same B^*-supports. Hence, the n maxes $M'_1, M'_2, \ldots, M'_n$ intersect in a unique point $u' = \langle y'_1, y'_2, \ldots, y'_n \rangle$ of $\mathbb{G}_n$. This point u' is opposite to $u = \langle y_1, y_2, \ldots, y_n \rangle$. Hence, also the points $\epsilon(u)$ and $\epsilon(u')$ of $DH(2n-1,4)$ are opposite.

For every $i \in \{1, 2, \ldots, 2n\}$, let x_i and x'_i denote the points of $H(2n-1,4)$ corresponding to $\overline{M_i}$ and $\overline{M'_i}$, respectively. Notice that if $M_i \in \mathcal{M}_{j,k}$, then $\langle x^*_j, x^*_k \rangle = \langle x_i, x'_i \rangle$. Also, if $M_{i_1} \in \mathcal{M}_{j_1,k_1}$ and $M_{i_2} \in \mathcal{M}_{j_2,k_2}$ with $i_1 \neq i_2$, then $\{j_1, k_1\} \cap \{j_2, k_2\} = \emptyset$. It follows that $\langle x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n \rangle = \langle x^*_1, x^*_2, \ldots, x^*_{2n} \rangle$.

By Lemma 5.6, $\epsilon(u)$ and $\epsilon(u')$ considered as maximal subspaces of $H(2n-1,4)$ are respectively equal to $\langle x_1, x_2, \ldots, x_n \rangle$ and $\langle x'_1, x'_2, \ldots, x'_n \rangle$. Since $\epsilon(u)$ and $\epsilon(u')$ are opposite points of $DH(2n-1,4)$, we have $\mathrm{PG}(V) = \langle \epsilon(u), \epsilon(u') \rangle = \langle x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_n \rangle = \langle x^*_1, x^*_2, \ldots, x^*_{2n} \rangle$. ∎

Now, choose an ordered basis $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$ in V such that $x^*_i = \langle \bar{e}_i \rangle$ for every $i \in \{1, 2, \ldots, 2n\}$.

Lemma 5.8 (1) *No point of* $\mathrm{PG}(V)$ *with B-weight* 1 *belongs to $H(2n-1,4)$.*

(2) *The points of* $\mathrm{PG}(V)$ *with B-weight* 2 *are precisely the points of $H(2n-1,4)$ corresponding to the maxes $\overline{M}$ of $DH(2n-1,4)$, where M is a max of $\mathbb{G}_n$.*

(3) *With respect to the reference system B, the Hermitian variety $H(2n-1,4)$ has equation $X_1^3 + X_2^3 + \ldots + X_{2n}^3 = 0$.*

Proof. (1) By Lemma 5.3, every point of $\mathrm{PG}(V)$ with B-weight 1 does not belong to $H(2n-1,4)$.

(2) Let p be an arbitrary point of $\mathrm{PG}(V)$ with B-weight 2. Then there exist $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$ such that $p \in x^*_i x^*_j \setminus \{x^*_i, x^*_j\}$. So, $p \in L_{i,j}$ and p is the point of $H(2n-1,4)$ corresponding to a $\overline{M}$, where M is one of the three maxes of $\mathcal{M}_{i,j}$.

Conversely, let M be a big max of $\mathbb{G}_n$ and let $\{i, j\}$ be the unique subset of size 2 such that $M \in \mathcal{M}_{i,j}$. If p is the point of $H(2n-1,4)$ corresponding to $\overline{M}$, then $p \in L_{i,j} = x^*_i x^*_j$ and hence $p \in x^*_i x^*_j \setminus \{x^*_i, x^*_j\}$. So, p has B-weight 2.

(3) Let $\sum_{1 \leq i,j \leq 2n} a_{ij} X_i X_j^2 = 0$ be an equation of $H(2n-1,4)$ with respect to the reference system $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$. Without loss of generality, we may suppose that $a_{ij}^2 = a_{ji}$ for all $i, j \in \{1, 2, \ldots, 2n\}$. Since no point of weight 1 belongs to $H(2n-1,4)$, we have $a_{ii} \neq 0$ for all $i \in \{1, 2, \ldots, 2n\}$. Hence, $a_{ii} = 1$ since $a_{ii}^2 = a_{ii}$. Since $\langle \bar{e}_i + k\bar{e}_j \rangle \in H(2n-1,4)$, we have $a_{ii} + a_{jj} + (ka_{ji}) + (ka_{ji})^2 = 0$ for all $k \in \mathbb{F}_4^*$ and all $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$. This implies that $a_{ij} = 0$ if $i \neq j$. ∎

Lemma 5.9 *Let $\mathcal{P}$ denote the point set of $\mathbb{G}_n$. The points of $\epsilon(\mathcal{P})$ are precisely the maximal singular subspaces of $H(2n-1,4)$ which are generated by n points with B-weight* 2 *whose B-supports are mutually disjoint.*

Proof. Let p be an arbitrary point of $\mathbb{G}_n$. Then p is contained in precisely n big maxes $M_1, M_2, \ldots, M_n$ of $\mathbb{G}_n$. Let x_i, $i \in \{1, 2, \ldots, n\}$, denote the point of B-weight 2 of $H(2n-1,4)$ corresponding to $\overline{M_i}$. Since $\overline{M_i}$ and $\overline{M_j}$ meet each other, the B-supports of x_i and x_j are disjoint for all $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$. Hence, $\langle x_1, x_2, \ldots, x_n \rangle$ is a maximal singular subspace of $H(2n-1,4)$. By Lemma 5.6, $\langle x_1, x_2, \ldots, x_n \rangle$ is the maximal singular subspace of $H(2n-1,4)$ corresponding to the point $\epsilon(p)$ of $DH(2n-1,4)$. Hence, every point of $\epsilon(\mathcal{P})$ is a maximal singular subspace of $H(2n-1,4)$ that is generated by n points with B-weight 2 whose B-supports are mutually disjoint. The claim now follows from the fact that there are as many points in $\mathbb{G}_n$ as there are maximal singular subspaces of $H(2n-1,4)$ which are generated by n points with B-weight 2 whose B-supports are mutually disjoint. ∎

Proposition 5.10 *The embedding ϵ is isomorphic to ϵ^*.*

Proof. Recall that ϵ^* is a full isometric homogeneous embedding of $\mathbb{G}_n$ into $DH(2n-1,4)$. So, in view of Proposition 2.6, it suffices to prove that there exists an automorphism θ of $DH(2n-1,4)$ mapping $\epsilon^*(\mathcal{P})$ to $\epsilon(\mathcal{P})$. But such an automorphism is induced by the unique linear map of V that maps the ordered basis B^* to the ordered basis B. ∎

Theorem 1.1(2) is an immediate consequence of Corollary 2.3 and Proposition 5.10. Observe also that Theorem 1.1(2) is valid for $n=2$ since $\mathbb{G}_2 \cong DH(3,2) \cong Q^-(5,2)$.

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