A classification result on weighted $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers

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Abstract

We classify all $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers, $\delta \le 2p^2 - 4p$, $p = p_0^h \ge 11$, $h \ge 1$, for a prime number $p_0 \ge 7$, with excess $e \le p^3 - 4p$ when $\mu = 1$ and with excess $e \le p^2 + p$ when $\mu > 1$. For $N \ge 4$, p non-square, such a minihyper is a sum of μ-dimensional spaces $PG(\mu, p^3)$ and of at most one (possibly projected) subgeometry $PG(3\mu + 2, p)$; except for one special case when $\mu = 1$. When p is a square, also (possibly projected) Baer subgeometries $PG(2\mu + 1, p^{3/2})$ can occur.

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1 Introduction

Let $PG(N, q)$ be the N-dimensional projective space over the finite field of order q.

Definition 1 (Hamada and Tamari [12]) *An $\{f, m; N, q\}$-minihyper is a pair (F, w), where F is a subset of the point set of $PG(N, q)$ and w is a weight*

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function $w : PG(N,q) \to \mathbb{N} : x \mapsto w(x)$, satisfying
(1) $w(x) > 0 \Leftrightarrow x \in F$,
(2) $\sum_{x \in F} w(x) = f$, *and*
(3) $\min(|(F,w) \cap H| = \sum_{x \in H} w(x) || H \in \mathcal{H}) = m$; *where $\mathcal{H}$ denotes the set of hyperplanes.*

In the case that w is a mapping onto $\{0,1\}$, the minihyper (F,w) can be identified with the set F and is simply denoted by F.

The excess *of a point of a minihyper (F,w) is the weight of the point minus one.*

The excess *e of a minihyper (F,w) is the number $\sum_{x \in F}(w(x)-1)$.*

Let $v_s = (q^s - 1)/(q-1)$.

Minihypers in finite projective spaces were first introduced to study linear codes meeting the Griesmer bound. The Griesmer bound states that if there exists an $[n,k,d;q]$ code for given values of k, d and q, then

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k,d),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x [7,19].

Suppose that there exists a linear $[n,k,d;q]$ code meeting the Griesmer bound $(d \geq 1, k \geq 3)$, then we can write d in an unique way as $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^i$ such that $\theta \geq 1$ and $0 \leq \epsilon_i < q$.

Using this expression for d, the Griesmer bound for an $[n,k,d;q]$ code can be expressed as: $n \geq \theta v_k - \sum_{i=0}^{k-2} \epsilon_i v_{i+1}$.

Let $\bar{E}(t,q)$ denote the set of all ordered tuples $(\zeta_0, \ldots, \zeta_{t-1})$ of integers ζ_i such that $(\zeta_0, \ldots, \zeta_{t-1}) \neq (0, \ldots, 0)$ and either: (a) $0 \leq \zeta_0 \leq q-1, 0 \leq \zeta_1 \leq q-1, \ldots, 0 \leq \zeta_{t-1} \leq q-1$, or (b) $\zeta_0 = q, 0 \leq \zeta_1 \leq q-1, \ldots, 0 \leq \zeta_{t-1} \leq q-1$, or (c) $\zeta_0 = \ldots = \zeta_{\lambda-1} = 0$, $\zeta_\lambda = q$, $0 \leq \zeta_{\lambda+1} \leq q-1, \ldots, 0 \leq \zeta_{t-1} \leq q-1$ for some integer $\lambda \in \{1, \ldots, t-1\}$. Let $E(t,q)$ denote the set of all ordered tuples $(\zeta_0, \ldots, \zeta_{t-1})$ of integers ζ_i such that $(\zeta_0, \ldots, \zeta_{t-1}) \neq (0, \ldots, 0)$ and $0 \leq \zeta_0 \leq q-1, 0 \leq \zeta_1 \leq q-1, \ldots, 0 \leq \zeta_{t-1} \leq q-1$.

In the next paragraph, we suppose that $(\epsilon_0, \ldots, \epsilon_{k-2}) \in E(k-1,q)$.

Hamada and Helleseth [10] showed that there is a one-to-one correspondence between the set of all non-equivalent $[n,k,d;q]$ codes meeting the Griesmer bound and the set of all projectively distinct $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihypers (F,w), such that $1 \leq w(p) \leq \theta$ for every point $p \in F$.

More precisely, the link is described in the following way. Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n, k, d; q]$ code, meeting the Griesmer bound. We look at a column of G as being the coordinates of a point in $PG(k-1, q)$. Let the point set of $PG(k-1, q)$ be $\{s_1, \ldots, s_{v_k}\}$. Let $m_i(G)$ denote the number of columns in G defining s_i. Let $m(G) = \max\{m_i(G) || i = 1, 2, \ldots, v_k\}$. Then $\theta = m(G)$ is uniquely determined by the code C and we call it the *maximum multiplicity* of the code. Define the weight function $w : PG(k-1, q) \to \mathbb{N}$ as $w(s_i) = \theta - m_i(G)$, $i = 1, 2, \ldots, v_k$. Let $F = \{s_i \in PG(k-1, q) || w(s_i) > 0\}$, then (F, w) is a $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper with weight function w.

Minihypers also have many applications in finite geometries [2,4–6]. A class of minihypers which is crucial in the study of *maximal partial μ-spreads* and *minimal μ-covers* in finite projective spaces $PG(N, q)$, where $(\mu+1)|(N+1)$, is the class of $\{\delta v_{\mu+1}, \delta v_\mu; N, q\}$-minihypers. These have been used by Govaerts and Storme [4,5] to study the extendability of maximal partial μ-spreads in $PG(N, q)$, $(\mu+1)|(N+1)$, of small deficiency δ; by Ferret and Storme [2] to study the extendability of maximal partial 1-spreads in $PG(3, q)$ of small deficiency δ; and by Govaerts, Storme and Van Maldeghem [6] to obtain results on other types of substructures in finite incidence structures.

This article improves the results of [5]. By using the recent results on the classification of the smallest minimal blocking sets B in $PG(2, p^3)$, new classification results on $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers are obtained.

This article presents the results for $\{\delta v_2, \delta v_1; N, p^3\}$-minihypers, $N > 3$, and for $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers, $\mu > 1$. In a first paper [3], the $\{\delta v_2, \delta v_1; 3, p^3\}$-minihypers were discussed.

The easiest way to construct weighted minihypers is to construct a *sum* of certain geometrical objects.

Consider a number of geometrical objects, such as subspaces $PG(d, q = p^h)$ of $PG(N, q = p^h)$, subgeometries $PG(d, p^t)$ of $PG(N, q = p^h)$, where $t|h$, and projected subgeometries $PG(d, p^t)$ in $PG(N, q = p^h)$, where $t|h$. In the first two cases, a point of respectively $PG(d, q)$ or $PG(d, p^t)$ has *weight* one, while all the other points not belonging to respectively $PG(d, q)$ or $PG(d, p^t)$ have weight zero. In the latter case, let Π be a projected $PG(d, p^t)$. The *weight* of a point s of Π is the number of points s' of $PG(d, p^t)$ that are projected onto s; all other points s not belonging to Π have weight zero.

Then the *sum* of these subspaces and (projected) subgeometries is the weighted set (F, w), where the weight $w(s)$ of a point s of (F, w) is the sum of all the weights of s in the subspaces and (projected) subgeometries of (F, w).

We will also speak of a minihyper (F, w) in which a line R is deleted. This means that we consider the following minihyper $(F, w) \setminus R$:

(1) if $r \in R$, then the weight of r in the minihyper $(F, w) \setminus R$ is $w(r) - 1$,
(2) if $r \notin R$, then the weight of r in the minihyper $(F, w) \setminus R$ is $w(r)$.

The following characterization results on $\{\delta v_2, \delta v_1; 3, p^3\}$-minihypers were obtained in [3].

Theorem 2 (Ferret and Storme [3]) *A $\{\delta(p^3 + 1), \delta; 3, p^3\}$-minihyper (F, w), p non-square, $p = p_0^h$, $p \geq 11$, p_0 prime, $h \geq 1$, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, and with excess $e \leq p^3$, is either:*
(1) a sum of lines and of at most one projected $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) a sum of lines and of a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega, w) \setminus R$, where Ω is a $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, *and where R is the line contained in Ω.*

Theorem 3 (Ferret and Storme [3]) *A $\{\delta(p^3 + 1), \delta; 3, p^3\}$-minihyper (F, w), $p = p_0^h$, p_0 prime, $h \geq 2$ even, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, and with excess $e \leq p^3$, is either:*
(1) a sum of lines, (projected) $PG(3, p^{3/2})$, and of at most one projected $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) a sum of lines, (projected) $PG(3, p^{3/2})$, and of a $\{(p^2 + p)(p^3 + 1), p^2 + p; 3, p^3\}$-minihyper $(\Omega, w) \setminus R$, where Ω is a $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, *and where R is the line contained in Ω.*

Crucial in the classification results of the preceding theorems are the recent classification results on non-trivial minimal blocking sets in $PG(2, p^3)$.

Definition 4 *A* blocking set *of $PG(2, q)$ is a set of points intersecting every line of $PG(2, q)$ in at least one point.*

A blocking set is called minimal *when no proper subset of it is still a blocking set; and we call a blocking set* non-trivial *when it contains no line.*

A blocking set of $PG(2, q)$ is called small *when it has less than $3(q + 1)/2$ points.*

If $q = p^h$, p prime, we call the exponent *E of the minimal blocking set B the maximal integer E such that every line intersects B in 1 modulo p^E points.*

From a result of Szőnyi [21], it follows that $E \geq 1$ for every small non-trivial minimal blocking set in $PG(2, q)$.

Remark 5 In [21], it is proven that, if E is the exponent of a small non-trivial minimal blocking set in $PG(2, q)$, $q = p^h$, p prime, then $1 \leq E \leq h/2$, and the size of the blocking set must lie in certain intervals depending on p^E. We note that the bounds given in [21] are improved in [15] and in [17].

The results of [21] have been used to classify all non-trivial small minimal blocking sets of $PG(2, q)$, $q = p^h$, of exponent $E \geq h/3$.

Theorem 6 (Polverino, Polverino and Storme [16–18]) *The smallest minimal blocking sets in $PG(2, p^3)$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, with exponent $E \geq h$, are:*
(1) *a line,*
(2) *a Baer subplane of cardinality $p^3 + p^{3/2} + 1$, when p is a square,*
(3) *a set of cardinality $p^3 + p^2 + 1$, equivalent to*

$$\{(x, T(x), 1) || x \in GF(p^3)\} \cup \{(x, T(x), 0) || x \in GF(p^3) \setminus \{0\}\},$$

with T the trace function from $GF(p^3)$ to $GF(p)$,
(4) *a set of cardinality $p^3 + p^2 + p + 1$, equivalent to*

$$\{(x, x^p, 1) || x \in GF(p^3)\} \cup \{(x, x^p, 0) || x \in GF(p^3) \setminus \{0\}\}.$$

This result is also the complete classification of all small minimal non-trivial blocking sets in $PG(2, p^3)$, p prime, $p \geq 7$.

From the intervals for the sizes of minimal non-trivial blocking sets in $PG(2, p^3)$, the following result follows.

Theorem 7 *In $PG(2, p^3)$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h \geq 1$, every non-trivial blocking set B of size at most $p^3 + 2p^2 - 4p$ contains a minimal blocking set of one of the types described in Theorem 6.*

Remark 8 (1) The minimal blocking set of size p^3+p^2+1 has a unique point, called the *vertex*, lying on exactly $p + 1$ lines containing $p^2 + 1$ points of the blocking set. These $p + 1$ lines form a dual $PG(1, p)$. All other lines intersect the blocking set in 1 or in $p + 1$ points.

Furthermore, these $(p^2 + 1)$-sets which are the intersection of the blocking set with these $(p^2 + 1)$-secants are equivalent to the set $\{\infty\} \cup \{x \in GF(p^3) || x + x^p + x^{p^2} = 0\}$, with ∞ corresponding with the vertex of the blocking set.

Later on, we will refer to the point corresponding with ∞ as being the *special point* of this $(p^2 + 1)$-set.

The lines sharing $p+1$ points with this blocking set intersect the blocking set in a subline $PG(1, p)$.

(2) The minimal blocking set of size p^3+p^2+p+1 has p^2+p+1 points in common with exactly one line; all other lines intersect the blocking set in 1 or in $p+1$ points.

This (p^2+p+1)-set which is the intersection of the blocking set with the (p^2+p+1)-secant is equivalent to $\{x \in GF(p^3) || x^{p^2+p+1} = 1\}$. The $(p+1)$-secants intersect the blocking set in a subline $PG(1,p)$.

(3) These two blocking sets are also characterized [13] as being a projected $PG(3,p)$ in the plane $PG(2,p^3)$. Namely, embed the plane $PG(2,p^3)$ in a 3-dimensional space $PG(3,p^3)$. Consider a subgeometry $PG(3,p)$ of $PG(3,p^3)$ and a point r not belonging to this subgeometry $PG(3,p)$ and not belonging to the plane $PG(2,p^3)$.

Project $PG(3,p)$ from r onto $PG(2,p^3)$.

If the point r belongs to a line of the subgeometry $PG(3,p)$, then this $PG(3,p)$ is projected onto the blocking set of size p^3+p^2+1; else we obtain the blocking set of size p^3+p^2+p+1.

(4) To simplify the notations in this article, every set of p^2+1 collinear points projectively equivalent to the set $\{\infty\} \cup \{x || x^{p^2}+x^p+x=0\}$ will be called a *(p^2+1)-set*, and every set of p^2+p+1 collinear points projectively equivalent to the set $\{x || x^{p^2+p+1}=1\}$ will be called a *(p^2+p+1)-set*.

We will also use recent results on *blocking sets K with respect to hyperplanes* of $PG(N,q)$; these are sets of points intersecting every hyperplane in at least one point; hence $\{|B|, 1; N, p^3\}$-minihypers.

Theorem 9 (Storme and Weiner [20]) *In $PG(N,p^3)$, $p = p_0^h$, $h \geq 1$, p_0 prime, $p_0 \geq 7$, $N \geq 3$, a minimal blocking set K with respect to hyperplanes, of cardinality at most p^3+p^2+p+1, is either:*

(1) *a line;*

(2) *a Baer subplane when p is a square;*

(3) *a minimal blocking set of cardinality p^3+p^2+1 in a plane of $PG(N,p^3)$;*

(4) *a minimal blocking set of cardinality p^3+p^2+p+1 in a plane of $PG(N,p^3)$;*

(5) *a subgeometry $PG(3,p)$ in a 3-dimensional subspace of $PG(N,p^3)$.*

Remark 10 Since we are considering weighted minihypers (F,w), we wish to distinguish between the following notations. The notation $F \cap H$ means the intersection of the two sets F and H; so all points of $F \cap H$ are counted with weight one. The notation $(F,w) \cap H$ is the weighted minihyper in H consisting of the point set $F \cap H$, where each point of H is given the same weight as in the minihyper (F, w).

We will use the following results on minihypers. Note that these results were stated for minihypers which are sets of points, they can however easily be extended to weighted minihypers.

Theorem 11 (Ferret and Storme [1], Hamada and Helleseth [11]) *Let (F, w) be a $\{\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=1}^{t-1} \epsilon_i v_i; t, q\}$-minihyper where $t \geq 2$, $h \geq 2$, $q \geq h$, $0 \leq \epsilon_i \leq q-1$, $\sum_{i=0}^{t-1} \epsilon_i = h$.*
(1) If for a hyperplane H of $PG(t, q)$, $|(F, w) \cap H| = \sum_{i=1}^{t} m_i v_i$, $(m_1, \ldots, m_t) \in \bar{E}(t, q)$, then $(F, w) \cap H$ is a $\{\sum_{i=1}^{t} m_i v_i, \sum_{i=1}^{t} m_i v_{i-1}; t-1, q\}$-minihyper in H.
(2) There does not exist a hyperplane H in $PG(t, q)$ such that $|(F, w) \cap H| = \sum_{i=1}^{t} m_i v_i$ for any $(m_1, \ldots, m_t) \in \bar{E}(t, q)$ such that $\sum_{i=1}^{t} m_i > h$.
(3) In the case $\epsilon_0 = 0$ and $q \geq 2h-1$, there is no hyperplane H in $PG(t, q)$ such that $|(F, w) \cap H| = \sum_{i=1}^{t} m_i v_i$ for any $(m_1, \ldots, m_t) \in \bar{E}(t, q)$ such that $\sum_{i=1}^{t} m_i < h$.

Theorem 12 (Hamada [9]) *If there exists a $\{\sum_{i=0}^{N-1} \epsilon_i v_{i+1}, \sum_{i=1}^{N-1} \epsilon_i v_i; N, q\}$-minihyper (F, w) for some ordered set $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1})$ in $E(N, q)$, then for $1 \leq n < N$:*

(1) $|(F, w) \cap \Omega| \geq \sum_{i=n-1}^{N-1} \epsilon_i v_{i+1-n}$ for any $(N-n)$-dimensional subspace Ω in $PG(N, q)$ and equality holds for some $(N-n)$-dimensional subspace Ω in $PG(N, q)$.

(2) In the special case $n = 2$, $|(F, w) \cap \Delta| \geq \sum_{i=1}^{N-1} \epsilon_i v_{i-1}$ for any $(N-2)$-dimensional subspace Δ in $PG(N, q)$ and $|(F, w) \cap G| = \sum_{i=1}^{N-1} \epsilon_i v_{i-1}$ for some $(N-2)$-dimensional subspace G in $PG(N, q)$. Let H_j, $j = 1, 2, \ldots, q+1$, be the $q+1$ hyperplanes in $PG(N, q)$ that contain G. Then $(F, w) \cap H_j$ is a $\left\{\delta_j + \sum_{i=1}^{N-1} \epsilon_i v_i, \sum_{i=1}^{N-1} \epsilon_i v_{i-1}; N-1, q\right\}$-minihyper in H_j for $j = 1, 2, \ldots, q+1$, where the δ_j are some non-negative integers such that $\sum_{j=1}^{q+1} \delta_j = \epsilon_0$.

2 Projected $PG(5, p)$ in $PG(3, p^3)$

From the results of Theorem 2 and 3 on $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers, such minihypers might contain projected subgeometries $PG(5, p) \equiv \Omega$.

To obtain the classification results on the $\{\delta(p^3+1), \delta; N, p^3\}$-minihypers, $N > 3$, and on the $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers, $\mu > 1$, we will use the descriptions of these projected $PG(5, p)$. A detailed description of the different types of projected subgeometries $PG(5, p) \equiv \Omega$ was given in [3]. We now repeat the properties of these projected subgeometries which will be used in the next sections.

Consider a subgeometry $\Lambda = PG(5,p)$ naturally embedded in $PG(5,p^3)$. Let L be a line of $PG(5,p^3)$ skew to Λ. Then L has two conjugate lines with respect to Λ. We will always denote these conjugate lines by L^p and L^{p^2}.

We now present the detailed descriptions of such projected $PG(5,p)$.

Case 1. Suppose that Ω is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Then every projected point s in Ω has weight one, a line of $PG(3,p^3)$ sharing at least two points with Ω shares a $PG(1,p)$ or a (p^2+p+1)-set with Ω, and planes either share a (p^2+p+1)-set, a subplane $PG(2,p)$, or a minimal blocking set of size p^3+p^2+p+1 with Ω.

Case 2. Suppose that Ω is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

Then the 4-dimensional space $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ is called the *special* 4-space of Λ, and similarly, its projection is called the *special* projected 4-space of Ω. We will denote this special 4-space $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ by $\mathcal{P}$.

Then for exactly one point r of L, $\dim\langle r, r^p, r^{p^2}\rangle = 1$. This line $M = \langle r, r^p, r^{p^2}\rangle$ is projected from L onto a point m of Ω of weight $p+1$.

Every plane π of Λ passing through M, and not lying in $\mathcal{P}$, is projected from L onto a (p^2+1)-set with special point m. Each such plane π lies in p^2+p+1 solids of Λ which are projected onto planar minimal blocking sets of size p^3+p^2+1; thus implying that m lies in $p^4+p^3+p^2$ planes of $PG(3,p^3)$ sharing a 1-fold blocking set of size p^3+p^2+1 with Ω.

Let s be a point of Ω not lying in the special 4-space of Ω. Assume that s is the projection of $s' \in \Lambda$. Then each solid $\langle r, r^p, r^{p^2}, s'\rangle \cap \Lambda$, with $r \in L \setminus M$, is projected from L onto a planar minimal blocking set of size p^3+p^2+p+1; hence, s lies in p^3 such planar minimal blocking sets. Every solid of Λ passing through M and s' is projected onto a planar minimal blocking set of size p^3+p^2+1 passing through s; thus giving p^2+p+1 extra planes through s intersecting Ω in a projected $PG(3,p)$.

Let s be a point of weight one of Ω which is the projection of a point s'

of $\mathcal{P}$. Then the plane $\langle M, s'\rangle$ lies in p^2 distinct 3-spaces of Λ not contained in $\mathcal{P}$ which are projected onto planar blocking sets of size p^3+p^2+1 through s.

Case 3. Suppose that Ω is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

Let $\mathcal{P} = \langle L, L^p, L^{p^2}\rangle \cap \Lambda$.

Every plane α through L in $\langle L, L^p, L^{p^2}\rangle$ has two conjugate planes α^p, α^{p^2} with respect to Λ, and these three planes intersect in at least one point of $\mathcal{P}$. Hence every plane through L in $\langle L, L^p, L^{p^2}\rangle$ contains at least one point of $\mathcal{P}$ and the projection of $\mathcal{P}$ is a line R of $PG(3, p^3)$. There are $p+1$ skew lines $L_1, \ldots, L_{p+1}$ in $\mathcal{P}$ which are projected onto points of weight $p+1$, and the remaining p^3-p points of $\mathcal{P}$ are projected onto points of weight one of the line R.

Then we call $\mathcal{P}$ the *special* 3-space of Λ, and its projection will always be denoted by the line R.

A point s' of $\Lambda \setminus \mathcal{P}$ is projected onto a point s lying on $p+1$ (p^2+1)-secants to Ω, which are the projections of $\langle s', L_i\rangle \cap \Lambda$, $i = 1, \ldots, p+1$. Each such (p^2+1)-secant through s lies in p^2 planes of $PG(3, p^3)$ containing a projected $PG(3, p)$ of Λ, which is a minimal blocking set of size p^3+p^2+1; hence, s lies in p^3+p^2 such planes. Considering these $PG(3, p)$ in Λ; these are the $PG(3, p)$ through a plane $\langle s', L_i\rangle$ only intersecting $\mathcal{P}$ in L_i.

Furthermore, through R, there are $p+1$ planes of $PG(3, p^3)$ containing $p^4+p^3+p^2+p+1$ projected points of Λ. The other planes through R contain p^3+p^2+p+1 projected weighted points of Λ; these all lie on R.

Hence, this projection forms a $\{(p^2+p+1)(p^3+1), p^2+p+1; 3, p^3\}$-minihyper containing the line R. Reducing the weight of every point on R by one yields a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega, w) \setminus R$.

Case 4. Suppose that Ω is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

Then this projection is a cone of p^2+p+1 lines; the vertex of the cone is a point having weight p^2+p+1, arising from the projection of the points of the plane $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$, and the base of the cone is a subplane $PG(2, p)$.

Remark 13 In this article, the symbols Ω, Λ and R will always have the following meaning. The symbol Ω will always denote the projection of a

$PG(5,p) \equiv \Lambda$ from a line L, and if Ω is the projection of Λ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 3$, then R will always denote the line contained in Ω.

3 $\{\delta(p^3+1), \delta; N, p^3\}$-minihypers for p non-square

Let (F, w) be a $\{\delta(p^3+1), \delta; 4, p^3\}$-minihyper.

We suppose that the total excess $\sum_{x\in F}(w(x)-1)$ is at most p^3-4p and that $\delta \leq 2p^2-4p$. Let $r \notin F$ be a point lying on at most $|F|(|F|-1)(p^3+1)/(2(|PG(4,p^3)|-|F|)) < 2p$ secants to (F,w); these latter secants contain at most $4p$ points of (F,w). So we can project (F,w) from r onto a solid Π to obtain a weighted minihyper (F', w'), and at most $2p$ points of (F', w') are the projection of at least two distinct points in F.

Lemma 14 *There is a bijective relation between the lines contained in* (F,w) *and the lines contained in* (F', w').

PROOF. A line of (F,w) is projected onto a line of (F',w'). No two lines are projected onto the same line. Assume that there is a line M contained in (F', w'). A point on M which only is the projection of one point in (F,w) defines one point of (F,w) in $\langle M, r\rangle$. The points on M which are the projections of at least two points of (F,w) define at most $4p$ points in $F \cap \langle M, r\rangle$. So, in $\langle M, r\rangle$ lie at most p^3+1+4p distinct points of (F,w) and they define a 1-fold blocking set in $\langle M, r\rangle$; hence there is a line of (F,w) in $\langle M, r\rangle$ (Theorems 6 and 11).

Theorem 15 ([5]) *If there is a line* M *contained in a* $\{\delta(q+1), \delta; N, q\}$*-minihyper* (F,w)*, with* $\delta \leq (q+1)/2$*, then* $(F,w)\setminus M$ *is a* $\{(\delta-1)(q+1), \delta-1; N, q\}$*-minihyper.*

The preceding lemma and theorem imply that from now on, we can assume that (F,w), and its projection (F',w') from r, do not contain any lines. Since by projecting from r, the excess increases by at most $4p$, (F',w') has at most excess p^3. Using Theorem 2, the only projections (F',w') we have to consider are the following $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers. Since these projected subgeometries $PG(5,p)$ lie in a 3-dimensional space Π, they are easily described as the projection of a subgeometry $PG(5,p) \equiv \Lambda$ from a line L.

(1) $\delta = p^2+p+1$ and (F',w') is the projection of a $PG(5,p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 5$. Then (F',w') only contains points of weight one.

(2) $\delta = p^2 + p + 1$ and (F', w') is the projection of a $PG(5, p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 4$. Then (F', w') contains one point of weight $p + 1$ and $p^5 + p^4 + p^3 + p^2$ points of weight one.

(3) $\delta = p^2 + p$ and $(F', w') = (\Omega, w) \setminus R$ is the projection of a $PG(5, p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, minus the line R contained in this projection. Then (F', w') contains exactly $p + 1$ points of weight p, lying on R, and all other points of (F', w') have weight one.

The goal now is to prove that (F, w) itself equals a projected subgeometry $PG(5, p)$ or a $\{(p^2 + p)(p^3 + 1), p^2 + p; 3, p^3\}$-minihyper $(\Omega, w) \setminus R$.

To achieve this goal, we discuss the preceding three possibilities one by one. The main difficulty is that we have to use certain properties of solids in projected subgeometries $PG(5, p)$. To make the ideas as clear as possible in Cases 2 and 3, we first describe these, rather obvious, properties originally in the non-projected subgeometry Λ. These are then described in the projected subgeometry (F', w') in Π, and by then describing them in the 4-space $PG(4, p^3)$, the characterization of the original minihyper (F, w) is obtained.

Case 1. Suppose that (F', w') is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Every plane of Π intersects (F', w') in $p^2 + p + 1$ or $p^3 + p^2 + p + 1$ points.

Then every solid of $PG(4, p^3)$ through r has p^3+p^2+p+1 or p^2+p+1 points of (F, w). If a solid π_3 through r has p^3+p^2+p+1 points of (F, w), then $F \cap \pi_3$ is a 1-fold blocking set with respect to the planes of π_3 (Theorem 11). By the results of Theorem 9, $F \cap \pi_3$ contains a non-projected $PG(3, p)$ or a minimal blocking set of size $p^3 + p^2 + p + 1$. For if it would share a minimal blocking set of size $p^3 + p^2 + 1$ with (F, w), we would have $p + 1$ different $(p^2 + 1)$-sets in the projection (F', w').

Let T be a $(p^2 + p + 1)$-secant to (F', w'). Consider two planes π_i, $i = 1, 2$, in Π through T containing $p^3 + p^2 + p + 1$ points of (F', w'). These planes π_i and r define solids containing $p^3 + p^2 + p + 1$ points of (F, w). The (projected) $PG(3, p)$ $\langle \pi_i, r\rangle \cap (F, w)$, $i = 1, 2$, sharing $p^3 + p^2 + p + 1$ points with (F, w), define a (projected) $PG(4, p) = \Omega_4$. Now, select a third plane π_3 of Π through T containing $p^3 + p^2 + p + 1$ points of (F', w') such that $F \cap \langle r, \pi_3\rangle$ does not lie in Ω_4. Note that there are p^2 choices for such a plane π_3. Then $\langle \pi_i, r\rangle \cap (F, w)$, $i = 1, 2, 3$, define a (projected) $PG(5, p) \equiv \Omega$.

We show that Ω is completely contained in (F, w). Consider a second $(p^2 + p + 1)$-secant M to (F', w'), skew to T, and consider all planes π_4 of Π through M containing $p^3 + p^2 + p + 1$ points of (F', w'). Then $\pi_i \cap \pi_4 \cap (F', w')$, $i = 1, 2, 3$, is a subline $PG(1,p)_i$ since both planes contain a projected $PG(3,p)$ of the projected $PG(5,p) \equiv (F',w')$. Then $\langle \pi_i, r\rangle \cap \langle \pi_4, r\rangle \cap (F,w) \equiv PG(1,p)_i$, $i = 1,2,3$, share one point with (F,w), projected from r onto T. Hence the (projected) $PG(3,p) \equiv F \cap \langle \pi_4, r\rangle$ shares three sublines $PG(1,p)_i$ with Ω. Hence, it also shares the plane $\langle PG(1,p)_1, PG(1,p)_2\rangle$ with Ω. Finally it is contained in Ω since it also shares $PG(1,p)_3$ with Ω and $PG(1,p)_3 \not\subset \langle PG(1,p)_1, PG(1,p)_2\rangle$. Letting vary π_4, we obtain that all $p^5+p^4+p^3+p^2+p+1$ points of Ω lie in (F,w). So, (F,w) is a (projected) $PG(5,p)$ of size $p^5+p^4+p^3+p^2+p+1$ only having points of weight one.

Case 2. Suppose that (F',w') is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

Then (F',w') consists of one point of weight $p+1$ and of $p^5+p^4+p^3+p^2$ points of weight one. Let M denote the line of Λ which is projected onto one point m of (F',w').

We first describe some properties of 3-spaces of Λ. These properties allow us to reconstruct Λ from carefully selected solids. We do this since these ideas will then be used to construct the projected $PG(5,p) \equiv \Omega$ contained in (F,w).

Let $\pi = \langle s, s^p, s^{p^2}\rangle \cap \Lambda$, with $s \in L \setminus M$, be a plane of Λ which is projected onto a (p^2+p+1)-set of (F',w'). Again we select $PG(3,p)_i$, $i = 1,2,3$, of Λ through π defining Λ. This time we select all $PG(3,p)_i$ outside the special $PG(4,p) \equiv \mathcal{P}$ of Λ. Consider a solid $PG(3,p)_4$ of Λ through M not lying in $\mathcal{P}$. This solid shares a unique point with π since this plane is skew to M. It shares a subline $PG(1,p)_i$ with $PG(3,p)_i$, $i = 1,2,3$, since they are two solids in Λ and since M is skew to the intersection. This implies that $PG(3,p)_4$ is contained in Λ. By letting vary the solid $PG(3,p)_4$, the subgeometry Λ is reconstructed.

We now describe the preceding observations in the subgeometry Λ in the projected subgeometry (F',w') to find the projected subgeometry Ω contained in (F,w).

Let T be a (p^2+p+1)-secant to (F',w'). Let π_1, π_2, π_3 be three distinct planes of Π through T intersecting (F',w') in 1-fold blocking sets $PG(3,p)_i$, $i = 1,2,3$, of size p^3+p^2+p+1 such that the 1-fold blocking sets of (F',w') in π_1, π_2, π_3 generate the projected subgeometry (F',w').

Then, as in Case 1, the corresponding solids $\langle \pi_i, r\rangle$ again intersect (F,w) in a non-projected subgeometry $PG(3,p)$ or in a minimal blocking set of size p^3+p^2+p+1. Let Ω be the projected subgeometry $PG(5,p)$ of $PG(4,p^3)$ defined by these 1-fold blocking sets of (F,w) in the solids $\langle \pi_i, r\rangle$, $i = 1,2,3$.

Let π be a plane of Π through m sharing a projected $PG(3,p)$ with (F',w'). Returning to $PG(4,p^3)$, $\langle \pi, r\rangle \cap (F,w)$ is either a (projected) $PG(3,p)$ of size p^3+p^2+p+1; or contains a minimal blocking set of size p^3+p^2+1 and maybe some extra points. In this latter case, since the projection m of M has weight $p+1$, the possible extra points lie on the line $\langle m, r\rangle$. In $PG(4,p^3)$, $F \cap \langle \pi, r\rangle$ shares a $PG(1,p)_i$ with $(F,w) \cap \langle \pi_i, r\rangle$, $i = 1,2,3$.

So the (projected) $PG(3,p)$ contained in $(F,w) \cap \langle \pi, r\rangle$ lies in Ω.

Letting vary π, this implies that already $p^5+p^4+p^3+p^2+1$ points of (F,w) lie in Ω. If Ω has $p^5+p^4+p^3+p^2+p+1$ distinct points, we will show further on that (F,w) coincides with this set of $p^5+p^4+p^3+p^2+p+1$ points. If Ω has one point of weight $p+1$, we show that (F,w) coincides with the $p^5+p^4+p^3+p^2$ points of weight one and one point of weight $p+1$. The only doubt which remains concerns the points on the line $\langle m, r\rangle =$(say) T'.

We project (F,w) from another point $r' \notin T'$, lying on at most $2p$ secants to F, onto a minihyper (F'',w''). Then (F'',w'') is the projection of a subgeometry $PG(5,p) \equiv \Lambda'$. We obtain at most $4p$ extra multiple points. Hence, the total excess is at most $5p$; and we are again in Case 1 or Case 2. If we are in Case 1, there is nothing left to prove. If we are in Case 2, let $m' \in F''$ be the point of weight $p+1$ of (F'',w''). All points of (F,w) on $\langle m,r\rangle \setminus \langle m', r'\rangle$ must lie in Ω. Note that there can only be one projected $PG(5,p)$ in $PG(4,p^3)$ containing at least $p^5+p^4+p^3+p^2$ points of (F,w). Since the points of $(\langle m', r'\rangle \setminus \langle m, r\rangle) \cap (F,w)$ have weight one, but the projection m' has weight $p+1$, $\langle m, r\rangle$ has no points of (F,w) outside Ω. Hence, either $\langle m, r\rangle \cap \Omega$ is one point of weight $p+1$ or $p+1$ points of weight one.

Case 3. Suppose that (F',w') is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

We will again describe properties of 3-spaces in Λ. The ideas following from these properties will then be translated into properties of projected 3-spaces in (F',w'). These latter properties will then be interpreted with respect to (F,w) in $PG(4,p^3)$ to construct a projected $PG(5,p) \equiv \Omega$ containing at least p^5+p^4 points of (F,w).

Let L_i, $i = 1,\ldots,p+1$, be the lines of $\mathcal{P} \subset \Lambda$ which contain p points projected onto one point of (F',w'). Select a plane π of Λ through L_1 not contained in $\mathcal{P} = \langle L, L^p, L^{p^2}\rangle \cap \Lambda$. Select three distinct 3-spaces $PG(3,p)_i$, $i = 1,2,3$, through π only sharing L_1 with $\mathcal{P}$ and generating Λ.

Every solid $PG(3,p)$ of Λ through L_2, only intersecting $\mathcal{P}$ in L_2, intersects the solids $PG(3,p)_i$, $i = 1,2,3$, in sublines $PG(1,p)_i$. These latter sublines

$PG(1,p)_i$, $i = 1,2,3$, generate this solid $PG(3,p)$ of Λ through L_2; hence by letting vary $PG(3,p)$ through L_2, we can prove that all points of $\Lambda \setminus \mathcal{P}$ lie in the 5-space generated by the solids $PG(3,p)_i$, $i = 1,2,3$.

We now interpret this in the projection (F', w'). Let $l_1, \dots, l_{p+1}$ be the projections of the lines $L_1, \dots, L_{p+1}$ of Λ.

A plane π of Λ through L_1, with $\pi \not\subset \mathcal{P}$, is projected onto a (p^2+1)-set T through l_1. The solids $PG(3,p)_i$ selected above are projected onto 1-fold blocking sets of size p^3+p^2+1 in planes π_1, π_2, π_3 through T.

This implies that in $\langle \pi_i, r\rangle$, $i = 1,2,3$, (F,w) shares a 1-fold blocking set of size p^3+p^2+p with $\langle \pi_i, r\rangle$, hence it contains a minimal blocking set B_i of size p^3+p^2+1 (Theorems 9 and 11). The point l_1 had weight p in (F', w'); so possible points of $(F,w) \cap \langle \pi_i, r\rangle$ not in B_i lie on the line $\langle l_1, r\rangle$.

Interpreting everything with respect to (F,w) in $PG(4,p^3)$, the minimal blocking sets B_i, $i = 1,2,3$, generate a (projected) subgeometry $PG(5,p) \equiv \Omega$. Note that B_i, $i = 1,2,3$, have the same vertex, since they share the (p^2+1)-set which is projected onto T.

Every projected $PG(3,p)$ of (F', w'), in a plane π_4, through l_j, $j = 2, \dots, p+1$, only intersecting the projection of $\mathcal{P}$ in l_j intersects T in exactly one point, and intersects $\pi_i \cap (F', w')$ in a $(p+1)$-secant. By the choice of the planes π_i, $i = 1,2,3$, these $(p+1)$-secants are not coplanar in Ω; hence this projected $PG(3,p)$ is completely contained in (F', w'), and interpreting this in $\langle \pi_4, r\rangle$, the corresponding minimal blocking set of size p^3+p^2+1 in $(F,w) \cap \langle \pi_4, r\rangle$ is completely contained in Ω.

Hence we have found a (projected) $PG(5,p) \equiv \Omega$ in $PG(4,p^3)$ having $p^5 + p^4 + p + 1$ points lying in (F,w).

Now Ω has $p+1$ points through which there pass (p^2+1)-secants. This is impossible for a $PG(5,p)$ projected from one point onto $PG(4,p^3)$. So Ω lies in a 3-dimensional subspace Π_3 of $PG(4,p^3)$, and Ω is a subgeometry $PG(5,p)$ projected from a line L' for which $\dim\langle L', L'^p, L'^{p^2}\rangle = 3$. Let R be the line contained in Ω.

Since the total weight of the points of (F,w) is equal to $(p^2+p)(p^3+1)$, we still need to determine the exact description of a total weight of order p^2-1. If there are points of (F,w) not belonging to Ω, then these points must lie on the lines $\langle l_i, r\rangle$, $i = 1, \dots, p+1$.

At most p^2-1 points of (F,w) do not belong to Π_3. Select new points $r' \notin F$, $r' \notin \Pi_3$. For all these points, the projection of (F,w) from r' onto Π_3 must be a projected $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega', w') \setminus R'$, where Ω'

is a projected $PG(5,p)$ in Π_3 from a line L'' with $\dim\langle L'', L''^p, L''^{p^2}\rangle = 3$ and where R' is the line contained in Ω'. Necessarily, $\Omega' = \Omega$.

Furthermore, the points of (F,w) not belonging to Π_3 must always be projected onto the points of weight p in $(\Omega,w) \setminus R$. This is not possible when (F,w) has points not belonging to Π_3. The only possibility is that (F,w) is this $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega,w) \setminus R$.

We now state the first concluding theorem of this article.

Theorem 16 *A $\{\delta(p^3+1), \delta; N, p^3\}$-minihyper (F,w), $N \geq 4$, p non-square, $p = p_0^h$, $p \geq 11$, p_0 prime, $h \geq 1$, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, and with excess $e \leq p^3 - 4p$, is either:*
(1) a sum of lines and of at most one (projected) $PG(5,p)$,
(2) a sum of lines and of a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega,w) \setminus R$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where R is the line contained in Ω.

PROOF. For $N = 4$, this follows from the preceding lemmas.

The result for $N > 4$ follows from analogous arguments, using an inductive proof with $N = 4$ as induction hypothesis.

Since for $N = 4$, the projection of (F,w) from r can increase the excess by at most $4p$, we only allow here $e \leq p^3 - 4p$ so that the projection (F', w') has excess $e' \leq p^3$.

4 $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihypers for p non-square

In this section, we classify $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihypers (F,w), $\delta \leq 2p^2 - 4p$, $N \geq 3$, p non-square, $p \geq 11$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h \geq 1$, with excess $e \leq p^2 + p$.

Consider a $PG(N-3, p^3)$ skew to (F,w). The $(N-2)$-dimensional subspaces through it intersect (F,w) in δ points (Theorem 12). Since the total excess of the points is at most p^2+p, there certainly is a $PG(N-2,p^3) \equiv \Delta$ intersecting (F,w) in δ points of weight one.

The hyperplanes through Δ will be denoted by $H_0, \ldots, H_{p^3}$. By Theorem 11, they intersect (F,w) in weighted $\{\delta(p^3+1), \delta; N-1, p^3\}$-minihypers satisfying the conditions of Theorem 16. So they intersect (F,w) in a weighted sum of lines and of at most one (projected) $PG(5,p)$, or at most one $\{(p^2+p)(p^3+$

$1), p^2+p; 3, p^3\}$-minihyper $(\Omega, w) \setminus R$, with Ω a projected $PG(5,p)$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 3$ and with R the line contained in Ω.

We want to classify the above mentioned $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihypers (F,w), $\delta \leq 2p^2-4p$, as a sum of planes and of at most one (projected) $PG(8,p)$.

We first show that if there are many lines in the intersections of the hyperplanes H_i with (F,w), then there are planes contained in (F,w).

Lemma 17 *If $r \in (F,w) \cap \Delta$ lies on two lines L_1, L_2 contained in (F,w), then $\langle L_1, L_2\rangle$ is contained in (F,w).*

PROOF. Assume that $\langle L_1, L_2\rangle$ is not contained in (F,w).

The plane $\langle L_1, L_2\rangle$ intersects (F,w) in a t-fold blocking set, with $t \geq 2$ (Theorem 11). If it intersects (F,w) in a t-fold blocking set, with necessarily $t \leq 2p^2$, by Theorem 11, it contains at most tp^3+2p^2 points of (F,w). On the other hand, it contains the two lines L_1 and L_2, and at least $t-1$ other points on every other line through r in $\langle L_1, L_2\rangle$ since r has weight one; yielding $tp^3+2p^2 \geq |(F,w) \cap \langle L_1, L_2\rangle| \geq (t+1)p^3 - t + 2$; a contradiction. Hence $\langle L_1, L_2\rangle$ is completely contained in (F,w).

Reducing the weights of all points of $\langle L_1, L_2\rangle$ by 1, yields a $\{(\delta-1)(p^6+p^3+1), (\delta-1)(p^3+1); N, p^3\}$-minihyper $(F,w) \setminus \langle L_1, L_2\rangle$ [4].

Hence, from now on, we assume that (F,w) does not contain planes.

Remark 18 (1) We can then assume that $\delta = p^2+p+1$ or $\delta = p^2+p$, since if $\delta > p^2+p+1$, then every hyperplane through Δ contains at least one line, and hence some point s of $(F,w) \cap \Delta$ lies on at least two lines of (F,w), and so s lies in a plane α contained in (F,w).

We assume that the total excess of the points of (F,w) is at most p^2+p. Since Δ intersects (F,w) in points of weight one, it is impossible that all p^3+1 hyperplanes $H_i, i=0,\ldots,p^3$, through Δ intersect (F,w) in a $\{(p^2+p)(p^3+1), p^2+p; N-1, p^3\}$-minihyper (F_i, w), which is a projected $PG(5,p)$ minus a line, since each such minihyper has $p+1$ points of weight p.

Hence, there is at least one hyperplane through Δ intersecting (F,w) in a $\{(p^2+p+1)(p^3+1), p^2+p+1; N-1, p^3\}$-minihyper (F_i, w) which is a projected $PG(5,p)$. Hence Δ contains p^2+p+1 points of weight one and we can assume that $\delta = p^2+p+1$.

(2) Also, every $(N-2)$-dimensional subspace Δ' sharing p^2+p+1 points of weight one with (F,w) must intersect (F,w) in either a non-projected $PG(2,p)$, or in a (p^2+p+1)-set.

For if $\Delta' \cap (F,w)$ would not be of one of these types, then this would imply that no hyperplane through Δ' would intersect (F,w) in a (projected) $PG(5,p)$. So, all hyperplanes H'_i through Δ' would share p^2+p+1 lines with (F,w), or they would share a minihyper (F'_i,w) with (F,w), where (F'_i,w) is the sum of a $\{(p^2+p)(p^3+1), p^2+p; N-1, p^3\}$-minihyper $(\Omega_i, w_i) \setminus R_i$ and a line R'_i, where Ω_i is a projected $PG(5,p)$ and R_i is the line contained in Ω_i. But then there is a point of $(F,w) \cap \Delta'$ lying on two lines of (F,w); so (F,w) would contain at least one plane, and this was excluded.

Lemma 19 *If there is a hyperplane H_0 through Δ containing p^2+p+1 lines of (F,w), then these lines form a cone with a (p^2+p+1)-set or a subplane $PG(2,p)$ as base.*

PROOF. Case 1. Assume first that the p^2+p+1 points in $\Delta \cap (F,w)$ form a non-projected $PG(2,p)$.

We show that all lines in $(F,w) \cap H_0$ are contained in a unique projected $PG(5,p) \equiv \Omega$ which is projected from a line L, for which $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

Consider a subline $PG(1,p) \equiv \tilde{M}$ in $\Delta \cap (F,w)$. Construct two $(N-2)$-dimensional subspaces Δ_1 and Δ_2 of H_0 through $\tilde{M}$, only sharing two distinct subplanes $PG(2,p)_1$ and $PG(2,p)_2$ with (F,w). Then there are exactly p^2 lines of (F,w) in H_0 intersecting $PG(2,p)_1$ and $PG(2,p)_2$ in distinct points. The two subplanes $PG(2,p)_i, i=1,2$, define a unique $PG(3,p) \equiv \Omega_3$ and these p^2 lines of $H_0 \cap (F,w)$ share already a subline $PG(1,p)$ with Ω_3.

Select a third $(N-2)$-dimensional subspace Δ_3 in H_0 through $\tilde{M}$ only sharing one subplane $PG(2,p)_3$ with (F,w), such that $PG(2,p)_3 \not\subseteq \Omega_3$. Then $PG(2,p)_3$ and Ω_3 define a (possibly projected) subgeometry $PG(4,p) \equiv \Omega_4$. Then the p^2 lines of (F,w) in H_0 intersecting $PG(2,p)_1$ and $PG(2,p)_2$ in distinct points share an other intersection point with Ω_4, and so they intersect Ω_4 in a (p^2+1)- or (p^2+p+1)-set.

By now selecting a fourth $(N-2)$-dimensional subspace Δ_4 in H_0 through $\tilde{M}$ only sharing one subplane $PG(2,p)_4$ with (F,w), such that $PG(2,p)_4 \not\subseteq \Omega_4$, and by repeating the arguments above, all lines of $H_0 \cap (F,w)$ intersecting $PG(2,p)_1$ and $PG(2,p)_2$ in distinct points now are completely contained in the projected $PG(5,p) \equiv \langle \Omega_4, PG(2,p)_4 \rangle \equiv \Omega_5$.

Repeating the arguments for other sublines than $\tilde{M}$, we get that all p^2+p+1

lines of $(F,w) \cap H_0$ are contained in Ω_5. Actually, Ω_5 is contained in a 3-dimensional space over $GF(p^3)$, namely the one generated by the planes over $GF(p^3)$ containing $PG(2,p)_1$ and $PG(2,p)_2$.

The only possibility is that Ω_5 is the projection of a subgeometry Λ from a line L, for which $\dim\langle L, L^p, L^{p^2}\rangle = 2$. The plane $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ is projected onto one point s of weight p^2+p+1 and the p^2+p+1 solids of Λ through this plane are projected onto lines through s.

Case 2. Assume that the p^2+p+1 points in $\Delta \cap (F,w)$ form a (p^2+p+1)-set $\{r_1, \ldots, r_{p^2+p+1}\}$ on a line L_1.

We may assume that every $PG(N-2,p^3)$ of H_0 containing p^2+p+1 points of weight one of $(F,w) \cap H_0$ intersects (F,w) in a (p^2+p+1)-set, since otherwise we are reduced to Case 1.

Consider a point r_1 of this (p^2+p+1)-set. By induction on the dimension of a subspace through r_1, it is possible to find an $(N-3)$-dimensional subspace of H_0 only sharing r_1 with (F,w). Then this $(N-3)$-dimensional subspace lies in at least a second $(N-2)$-dimensional subspace of H_0 only intersecting (F,w) in a (p^2+p+1)-set on a line L_2.

Then there are already p^2+p lines of (F,w) lying in the plane $\langle L_1, L_2\rangle$. Since r_1 is an arbitrary point of the (p^2+p+1)-set $(F,w) \cap L_1$, all p^2+p+1 lines $T_1, \ldots, T_{p^2+p+1}$ of $(F,w) \cap H_0$ are contained in $\langle L_1, L_2\rangle$.

The line T_1 has already p^2+p intersection points with the other lines $T_2, \ldots, T_{p^2+p+1}$, hence we have already the total excess p^2+p of (F,w) on the line T_1. This implies that all other intersections $T_i \cap T_j$, $2 \le i < j$, of the other lines T_i in $(F,w) \cap H_0$ have to coincide with $T_1 \cap T_2$. Hence, we have p^2+p+1 lines through one point.

We note that a cone of lines with vertex a point and base a (p^2+p+1)-set also is a projected subgeometry $PG(5,p)$. To obtain such a projected $PG(5,p)$, project first of all a subgeometry $PG(5,p)$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 2$. This projection is a cone with base a non-projected subplane $PG(2,p)$ in a plane Π skew to the vertex. Select a point r of Π only lying on tangents to the base of this cone. Project again, but now from r, then the projection is a cone with a (p^2+p+1)-set as base.

Lemma 20 *If there is a hyperplane H_0 through Δ intersecting (F,w) in a minihyper which is the sum of a $\{(p^2+p)(p^3+1), p^2+p; N-1, p^3\}$-minihyper $(\Omega, w) \setminus R$ and a line R', where Ω is a projected $PG(5,p)$ containing the line*

R, then $R = R'$.

PROOF. We know from Remark 18 (2) that every $(N-2)$-dimensional subspace Δ in H_0 intersecting $(F,w) \cap H_0$ in p^2+p+1 distinct points must intersect $(F,w) \cap H_0$ in a subplane $PG(2,p)$ or (p^2+p+1)-set. At least p^2+p of these points must belong to $\Omega \setminus R$, so also the latter point must belong to Ω. This latter point lies on R'. So R and R' share at least two points, so $R = R'$.

Consider again the $(N-2)$-dimensional space Δ containing $\delta = p^2+p+1$ points of weight one of (F,w). The preceding lemmas show that all hyperplanes through Δ intersect (F,w) in a projected $PG(5,p)$. This enables us to prove the following result.

Theorem 21 *There is a (projected) $PG(8,p)$ completely contained in (F,w).*

PROOF. Consider again a $PG(N-2,p^3) \equiv \Delta$ intersecting (F,w) in p^2+p+1 distinct points. Since $e \leq p^2+p$, there is at least one hyperplane H through Δ intersecting (F,w) in a (projected) $PG(5,p)$ only having points of weight one. It is then possible to select an $(N-2)$-dimensional space in H intersecting (F,w) in a $PG(2,p)$. Let this $(N-2)$-dimensional space in H play the role of Δ. Then the hyperplanes H_i, $i = 0,\ldots,p^3$, through Δ intersect (F,w) in a (projected) $PG(5,p)_i$. Since the total excess e of (F,w) is at most p^2+p, it is possible to find two hyperplanes H_1 and H_2 through Δ intersecting (F,w) in (projected) $PG(5,p)_1$ and $PG(5,p)_2$ without multiple points. These two subgeometries $PG(5,p)_1$ and $PG(5,p)_2$ define a (projected) $PG(8,p)$.

Consider a point r in $PG(5,p)_2 \setminus \Delta$ and consider a line $T \subset H_2$ through r containing a point of $\Delta \cap (F,w)$. Then T intersects (F,w) in $p+1$ or in p^2+p+1 distinct points. If T intersects (F,w) in p^2+p+1 points, consider a $PG(N-3,p^3)$ in H_2 through T only sharing the points on T with (F,w). There is at least one $PG(N-2,p^3) \equiv \Delta'$ in H_2 through this $PG(N-3,p^3)$ only sharing these p^2+p+1 points with (F,w); since if a $PG(N-2,p^3)$ shares more than $\delta = p^2+p+1$ points with (F,w), it shares at least p^3+p^2+p+1 points with (F,w) (Theorem 11). If T intersects (F,w) in $p+1$ points, we similarly find a $PG(N-2,p^3) \equiv \Delta'$ in H_2 through T intersecting (F,w) in p^2+p+1 distinct points. The points in $T \cap (F,w)$ generate together with $PG(5,p)_1$ a (projected) $PG(6,p)$ or $PG(7,p)$. We show that this subgeometry consists completely of points of (F,w).

Consider a hyperplane H through Δ'. Then H intersects (F,w) in a (projected) $PG(5,p)$. This hyperplane shares a subgeometry $PG(d,p)$, $d \geq 2$, with $PG(5,p)_1$, and this $PG(d,p)$ defines together with the points in $T \cap (F,w)$ a (projected) $PG(d',p)$, $d' \geq 3$, in $H \cap (F,w)$ containing r.

Letting vary H, all points of the (projected) subgeometry $\langle PG(5,p)_1, T\rangle$ are contained in (F,w).

Letting vary T, we obtain that the, possibly projected, 8-dimensional subgeometry $PG(8,p) = \langle PG(5,p)_1, PG(5,p)_2\rangle$ is contained in (F,w).

We present a new classification result on minihypers.

Theorem 22 *Let (F,w) be a $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihyper, $\delta \leq 2p^2-4p$, $N \geq 3$, $p \geq 11$ non-square, $p = p_0^h$, $h \geq 1$, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.*

Then (F,w) is a sum of planes and of at most one (projected) subgeometry $PG(8,p)$.

PROOF. This follows from the preceding arguments. We first discussed the possibility of planes in (F,w) and showed that it was possible to remove these planes from (F,w) by reducing in each such plane the weight of its points by one. This either characterized (F,w) completely, or in the other case, we proved that there is a (projected) subgeometry $\Pi_8 = PG(8,p)$ contained in (F,w). Assume that all planes have been removed from (F,w), then $\delta = p^2+p+1$. We now show that in this latter case, (F,w) coincides with Π_8. Let Δ be an $(N-2)$-dimensional space containing $\delta = p^2+p+1$ distinct points of (F,w). Every hyperplane through Δ intersects (F,w) in a $PG(5,p)$ and intersects Π_8 in a $PG(d,p)$, $d \geq 5$. These intersections must coincide, yielding that Π_8 is equal to (F,w).

There is also no problem regarding the weights of the points of (F,w) in this description of (F,w) as a sum of planes and of at most one (projected) $PG(8,p)$. For the planes, this again follows from the fact that it was possible to remove these planes from (F,w) by reducing in all these planes the weight of their points by one. For the possible remaining (projected) subgeometry $PG(8,p)$ in (F,w), everything is correct regarding the weights of the points by the definition of $(F,w)\cap H_i$, $i = 0,\ldots,p^3$ (Remark 10).

5 The general result for p non-square

We now prove by induction on μ the following characterization result.

Theorem 23 *Let (F,w) be a $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihyper, $\mu \geq 3$, $\delta \leq 2p^2-4p$, $N \geq 3$, $p \geq 11$ non-square, $p = p_0^h$, $h \geq 1$, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.*

Then (F, w) is a sum of μ-dimensional spaces $PG(\mu, p^3)$ and of at most one (projected) subgeometry $PG(3\mu + 2, p)$.

Let Δ be a $PG(N - 2, p^3)$ intersecting (F, w) in $\delta v_{\mu-1}$ points of weight one. Then $\Delta \cap (F, w)$ is a $\{\delta v_{\mu-1}, \delta v_{\mu-2}; N-2, p^3\}$-minihyper. Let $H_i, i = 0, \ldots, p^3$, be the hyperplanes through Δ. They intersect (F, w) in $\{\delta v_\mu, \delta v_{\mu-1}; N-1, p^3\}$-minihypers (F_i, w) (Theorem 12). By the induction hypothesis, these minihypers (F_i, w) are sums of $(\mu-1)$-dimensional spaces $PG(\mu-1, p^3)$ and of at most one (projected) subgeometry $PG(3\mu - 1, p)$. This then implies that $\delta \cap (F, w)$ is a union of pairwise disjoint $(\mu - 2)$-dimensional subspaces $PG(\mu - 2, p^3)$ and of at most one (projected) $PG(3\mu - 4, p)$.

Lemma 24 *If $\Delta \cap (F, w)$ contains a $(\mu - 2)$-dimensional space $PG(\mu - 2, p^3)$ Π, then (F, w) contains a μ-dimensional space $PG(\mu, p^3)$.*

PROOF. Consider all hyperplanes $H_i, i = 0, \ldots, p^3$, through Δ. Using the induction hypothesis, they either contain a $PG(\mu-1, p^3)$ of $(F, w) \cap H_i$ through Π or a (projected) $PG(3\mu - 1, p) \equiv \Omega$ of $(F, w) \cap H_i$ through Π.

Now assume that this latter possibility occurs. Then Π is the projection of a $PG(d, p) \equiv \pi$, $d \geq 3\mu - 6$, of Ω. Since $|\pi| > |\Pi|$, some points of Π are the projection of more than one point of π, so this would imply that there are multiple points in $\Delta \cap (F, w)$; a contradiction.

Hence, Π lies in p^3+1 subspaces $PG(\mu-1, p^3) \equiv \Pi_i$ of (F, w) in the respective hyperplanes H_i, $i = 0, \ldots, p^3$, through Δ. Every plane $\langle L_1, L_2 \rangle$ with $L_1 \subset \Pi_i$ and $L_j \subset \Pi_j$, $i \neq j$, is completely contained in (F, w). Namely, if this plane is not contained in (F, w), then it intersects (F, w) in an $\{m_1(p^3 + 1) + m_0, m_1; 2, p^3\}$-minihyper, with $m_1 + m_0 \leq 2p^2 - 4p$ (Theorem 11). But, again using the fact that this plane already contains two lines of (F, w), the arguments of Lemma 17 imply that this is impossible. We conclude that (F, w) contains a subspace $PG(\mu, p^3)$.

Removing the subspaces $PG(\mu, p^3)$ from (F, w) [4] by reducing for each such subspace $PG(\mu, p^3)$ the weight of its points by one shows that (F, w) is either a sum of subspaces $PG(\mu, p^3)$, or there remains a $\{(p^2 + p + 1)v_{\mu+1}, (p^2 + p + 1)v_\mu; N, p^3\}$-minihyper (F', w'), and the only case we still need to discuss is that $(F', w') \cap \Delta$ is a projected $PG(3\mu - 4, p)$, and all hyperplanes through Δ intersect (F', w') in a (projected) $PG(3\mu-1, p)$. Since there are at most p^2+p multiple points, we can select two hyperplanes H_1, H_2 through Δ intersecting (F', w') only in points of weight one. Both intersections contain a (projected) $PG(3\mu - 1, p)$, call them respectively $PG(3\mu - 1, p)_1$ and $PG(3\mu - 1, p)_2$. In $PG(3\mu-1, p)_i$, $i = 1, 2$, we do not have lines over $GF(p^3)$ since this would be a projection of a subgeometry $PG(t, p)$, $t \geq 3$, of size at least $p^3 + p^2 + p + 1$, and such a projection contains multiple points. So, a line T of H_2, intersecting $PG(3\mu-1,p)_2$ in at least two points, intersects $PG(3\mu-1,p)_2$ in $p+1$ or p^2+p+1 points forming respectively a $PG(1,p)$ or (p^2+p+1)-set.

Lemma 25 *The $(3\mu+2)$-dimensional (projected) subgeometry $PG(3\mu+2,p) = \langle PG(3\mu-1,p)_1, PG(3\mu-1,p)_2\rangle$ is completely contained in (F,w).*

PROOF. Consider a $(p+1)$- or a (p^2+p+1)-secant T to $H_2 \cap (F,w)$ containing one point of $\Delta \cap (F,w)$.

Consider a $PG(N-2,p^3) \equiv \Delta'$ in H_2 through T intersecting (F,w) in a $\{(p^2+p+1)v_{\mu-1},(p^2+p+1)v_{\mu-2}; N-2, p^3\}$-minihyper. Again the hyperplanes $H'_0,\ldots,H'_{p^3}$ through Δ' intersect (F,w) in a (projected) $PG(3\mu-1,p)$, call them respectively $PG(3\mu-1,p)'_i$, $i=0,\ldots,p^3$. Then $H_1 \cap PG(3\mu-1,p)'_i \equiv \mathcal{P}_i$ is a (projected) $PG(d'_i,p)$, $d'_i \geq 3\mu-4$. Then $\langle \mathcal{P}_i, T\cap(F,w)\rangle$ is completely contained in $PG(3\mu-1,p)'_i$. Hence $\langle PG(3\mu-1,p)_1, T\cap(F,w)\rangle$ lies completely in (F,w). Letting vary T, we obtain that $\langle PG(3\mu-1,p)_1, PG(3\mu-1,p)_2\rangle = PG(3\mu+2,p) = \Pi_{3\mu+2}$ lies completely in (F,w).

We now show that (F,w) coincides with $\Pi_{3\mu+2}$. Every hyperplane through Δ intersects (F,w) in a $PG(3\mu-1,p)$ and intersects $\Pi_{3\mu+2}$ in a $PG(d,p)$, $d \geq 3\mu-1$. These intersections must coincide, yielding that $\Pi_{3\mu+2}$ is equal to (F,w).

This now completes the proof of Theorem 23.

6 The case where p is a square

6.1 $\{\delta(p^3+1),\delta;4,p^3\}$*-minihypers*

Let $q=p^3$. We will project on a hyperplane Π from a point r lying on at most $2p$ secants to (F,w). This projection (F',w') is a sum of lines, (projected) $PG(3,p^{3/2})$ and of at most one projected $PG(5,p)$ or a $\{(p^2+p)(p^3+1),p^2+p;3,p^3\}$-minihyper $(\Omega,w)\setminus R$ (Theorem 3). As before, we can remove the lines of (F,w); now we will discuss the possible $PG(3,p^{3/2})$ in (F',w').

Lemma 26 *Every point s of (F,w) which is projected onto a point s' of weight one of (F',w'), lying in a $PG(3,p^{3/2})$ contained in (F',w'), lies in at least two Baer subplanes completely contained in (F,w).*

PROOF. We proceed as in [5, Lemma 2.10].

Lemma 27 *Through any point s of (F, w) which is projected onto s', a point of (F', w') of weight one lying in a $PG(3, p^{3/2})$ contained in (F', w'), there is a $PG(3, p^{3/2})$ completely consisting of points of (F, w).*

PROOF. We proceed as in [5, Lemma 2.11].

As in the proof of [5, Theorem 2.1], if (F, w) contains a subgeometry $D \equiv PG(3, p^{3/2})$, then reducing the weight of every point of D by one, gives a $\{(\delta - p^{3/2} - 1)(p^3 + 1), \delta - p^{3/2} - 1; 4, p^3\}$-minihyper (F', w').

The previous lemma, together with the previous sections, implies the following theorem. We wish to remark that the description of the minihypers can be done in different ways.

In the statement of the theorem, also the possibility of projected subgeometries $PG(3, p^{3/2})$ is included. In $PG(3, q)$, if one projects a subgeometry $PG(3, p^{3/2}) \equiv D$ from a point $s \notin D$, then a cone with base a Baer subline $PG(1, p^{3/2})$ is obtained. This cone is a $\{(p^{3/2} + 1)(p^3 + 1), p^{3/2} + 1; 3, p^3\}$-minihyper if the vertex is given the weight $p^{3/2} + 1$ and all other points are given weight one.

This cone is also a sum of lines, so it is also possible to simply not state explicitly these projected Baer subgeometries $PG(3, p^{3/2})$, and simply consider these lines as lines of the sum of lines inside the minihyper.

We however have written them in the formulation of the theorem since also in the general case of Theorems 31 and 33 projected subgeometries $PG(2\mu + 1, p^{3/2})$ can occur, and these projections are not equal to sums of spaces $PG(\mu, p^3)$ when $\mu \geq 2$.

Theorem 28 *A $\{\delta(p^3 + 1), \delta; N, p^3\}$-minihyper (F, w), $N \geq 4$, p square, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, with total excess $e \leq p^3 - 4p$, is a sum of either:*
(1) *lines, (projected) $PG(3, p^{3/2})$ (where the projection is from a point), and of at most one (projected) $PG(5, p)$,*
(2) *lines, (projected) $PG(3, p^{3/2})$, and of a $\{(p^2 + p)(p^3 + 1), p^2 + p; 3, p^3\}$-minihyper $(\Omega, w) \setminus R$, where Ω is a $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where R is the line contained in Ω.*

PROOF. The proof for $N = 4$ follows from the preceding lemmas and of the techniques of Section 3. To prove the result for $N > 4$, we use induction on

N, with $N = 4$ as induction basis.

6.2 $\{\delta(p^6 + p^3 + 1), \delta(p^3 + 1); N, p^3\}$*-minihypers*

Lemma 29 ([14]) *A Baer subline and a $(p^2 + p + 1)$-set in $PG(1, p^3)$ share at most $p + \sqrt{p} + 1$ points.*

Let Δ be a $PG(N - 2, p^3)$ intersecting (F, w) in δ points of weight one. As in Lemma 17, we can assume that a point of $(F, w) \cap \Delta$ lies on at most one line contained in (F, w). Since the total excess e is at most $p^2 + p$, it is also possible to find two distinct hyperplanes H_1 and H_2 through Δ intersecting (F, w) in unions of pairwise disjoint $PG(3, p^{3/2})$ and of at most one $\{(p^2 + p + 1)(p^3 + 1), p^2 + p + 1; N - 1, p^3\}$-minihyper which is a (projected) $PG(5, p)_i$, $i = 1, 2$, containing no multiple points. This implies that these latter projected $PG(5, p)_i$ only have $(p + 1)$- and $(p^2 + p + 1)$-secants.

This also implies that $\Delta \cap (F, w)$ is a union of pairwise disjoint Baer sublines $PG(1, p^{3/2})$ and of at most one subplane $PG(2, p)$ or $(p^2 + p + 1)$-set.

There are less than $2p^2/(p^{3/2} + 1) < 2\sqrt{p}$ Baer sublines in $\Delta \cap (F, w)$.

Consider a Baer subline in $\Delta \cap (F, w)$, and consider the $PG(3, p^{3/2})$ of (F, w) in H_1 and H_2 through this Baer subline. Call them $PG(3, p^{3/2})_1$ and $PG(3, p^{3/2})_2$ respectively.

Lemma 30 *The Baer subspace $PG(5, p^{3/2}) = \langle PG(3, p^{3/2})_1, PG(3, p^{3/2})_2\rangle$ is completely contained in (F, w).*

PROOF. Consider a Baer subline $\tilde{B}$ of $PG(3, p^{3/2})_2$ containing exactly one point of $\Delta \cap (F, w)$. The line T containing $\tilde{B}$ satisfies $|T \cap (F, w)| \leq \delta$, and has either a Baer subline, a point, or no points in common with a $PG(3, p^{3/2})$ contained in $F \cap H_2$, and it has either a $PG(1, p)$, a $(p^2 + p + 1)$-set, one point or no points in common with the possible projected $PG(5, p)$ in $(F, w) \cap H_2$. The Baer sublines on T which are the intersection of T with a subgeometry $PG(3, p^{3/2})$ contained in $H_2 \cap (F, w)$ are the only Baer sublines contained in $(F, w) \cap T$.

We now show that the subgeometry $PG(4, p^{3/2}) = \Pi_4 = \langle \tilde{B}, PG(3, p^{3/2})_1\rangle$ is completely contained in (F, w). Consider a $PG(N - 3, p^3)$ in H_2 through T not containing any other points of (F, w) than those of $T \cap (F, w)$, and consider a $PG(N - 2, p^3) \equiv \Delta'$ through this $PG(N - 3, p^3)$ only containing δ simple points of (F, w).

From the fact above stating that the only Baer sublines in $(F, w) \cap T$ arise from the intersections of T with the subgeometries $PG(3, p^{3/2})$ contained in $(F, w) \cap H_2$, if some hyperplane H' through Δ' contains lines of (F, w) through a point of $\tilde{B}$, then every point of $\tilde{B}$ lies on a line of $(F, w) \cap H'$.

So, consider all hyperplanes through Δ'; at most one of them contains lines of (F, w) through the points of $\tilde{B}$.

So at least p^3 hyperplanes through Δ' intersect (F, w) in a $\{\delta(p^3 + 1), \delta; N - 1, p^3\}$-minihyper containing a $PG(3, p^{3/2})$ intersecting Δ' in $\tilde{B}$. This latter $PG(3, p^{3/2})$ intersects H_1 in either a subline $PG(1, p^{3/2})$ or a subplane $PG(2, p^{3/2})$. Call this intersection B_1'. Then $\langle \tilde{B}, B_1' \rangle$ is a subgeometry over $GF(p^{3/2})$ completely contained in $(F, w) \cap H'$.

We conclude that Π_4 is completely contained in (F, w), up to maybe one hyperplane section Π_3.

We now show that Π_4 is completely contained in (F, w).

Consider a point r of $\Pi_4 \setminus F$, then all Baer sublines of Π_4 through r not lying in Π_3 share $p^{3/2}$ points with (F, w). Select such a Baer subline R of Π_4 through r intersecting (F, w) in $p^{3/2}$ points of weight one and such that the line T' through R only intersects (F, w) in points of weight one. Since T' is not contained in (F, w), it shares at most δ points with (F, w); see [5]. As for T, the only Baer sublines in $T' \cap (F, w)$ arise from the Baer subline intersections of T' with (F, w). It is impossible to partition the $p^{3/2}$ points of R in $(F, w) \cap \Pi_4$ over these Baer sublines and at most one $PG(1, p)$ or $(p^2 + p + 1)$-set. This however implies that the $p^{3/2}$ points of $(F, w) \cap \Pi_4$ on R are contained in a Baer subline contained in (F, w). Hence, $r \in F$.

Letting vary $\tilde{B}$ over $PG(3, p^{3/2})_2$ shows that the 5-dimensional Baer subgeometry $\langle PG(3, p^{3/2})_1, PG(3, p^{3/2})_2 \rangle$ is completely contained in (F, w).

Theorem 31 *Let (F, w) be a $\{\delta(p^6 + p^3 + 1), \delta(p^3 + 1); N, p^3\}$-minihyper, $\delta \leq 2p^2 - 4p$, $N \geq 5$, $p = p_0^h$, $h \geq 2$ even, $p_0 \geq 7$ prime, with excess $e \leq p^2 + p$.*

Then (F, w) is a sum of planes, (projected) $PG(5, p^{3/2})$, and of at most one (projected) subgeometry $PG(8, p)$.

PROOF. This follows from the preceding lemmas and the techniques of Section 4.

Remark 32 The Baer subgeometry can be at most projected from a point, since otherwise the total excess of the points would be too large.

We now prove by induction on μ the following characterization result.

Theorem 33 *Let (F, w) be a $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihyper, $\mu \geq 3$, $\delta \leq 2p^2 - 4p$, $N \geq 3$, $p = p_0^h$, $h \geq 2$ even, $p_0 \geq 7$ prime, with excess $e \leq p^2 + p$.*

Then (F, w) is a sum of μ-dimensional spaces $PG(\mu, p^3)$, (projected) $PG(2\mu + 1, p^{3/2})$, and of at most one (projected) subgeometry $PG(3\mu + 2, p)$.

Let Δ be a $PG(N-2, p^3)$ intersecting (F, w) in $\delta v_{\mu-1}$ points of weight one. Then $\Delta \cap (F, w)$ is a $\{\delta v_{\mu-1}, \delta v_{\mu-2}; N-2, p^3\}$-minihyper. Let $H_i, i = 0, \ldots, p^3$, be the hyperplanes through Δ. They intersect (F, w) in $\{\delta v_\mu, \delta v_{\mu-1}; N-1, p^3\}$-minihypers (F_i, w) (Theorem 12). By the induction hypothesis, these minihypers (F_i, w) are sums of $(\mu-1)$-dimensional spaces $PG(\mu-1, p^3)$, (projected) Baer subgeometries $PG(2\mu-1, p^{3/2})$, and of at most one (projected) subgeometry $PG(3\mu-1, p)$. This then implies that $\delta \cap (F, w)$ is a union of pairwise disjoint $(\mu-2)$-dimensional subspaces $PG(\mu-2, p^3)$, subgeometries $PG(2\mu-3, p^{3/2})$, and of at most one (projected) $PG(3\mu-4, p)$.

Lemma 34 *If $\Delta \cap (F, w)$ contains a $(\mu-2)$-dimensional space $PG(\mu-2, p^3)$ Π, then (F, w) contains a μ-dimensional space $PG(\mu, p^3)$.*

PROOF. This proof is similar to that of Lemma 24. We only need to consider the possibility that Π lies in a (projected) subgeometry $PG(2\mu-1, p^{3/2})$ contained in some hyperplane intersection $H_i \cap (F, w)$. If this occurs, then there are multiple points of this projected subgeometry in Π since $|PG(2\mu-3, p^{3/2})| > |PG(\mu-2, q)|$. This contradicts the fact that $(F, w) \cap \Delta$ contains no multiple points.

Removing the subspaces $PG(\mu, p^3)$ from (F, w) [4] by reducing for every such $PG(\mu, p^3)$ the weight of its points by one, leaves us with a minihyper (F', w') which intersects Δ in a union of pairwise disjoint $PG(2\mu-3, \sqrt{q})$, and of at most one $\{(p^2+p+1)v_{\mu-1}, (p^2+p+1)v_{\mu-2}; N-2, p^3\}$-minihyper Ω_5, where Ω_5 is a (projected) $PG(3\mu-4, p)$. Since there are at most p^2+p multiple points, we can select two hyperplanes H_1, H_2 through Δ intersecting (F', w') only in points of weight one. Both intersections contain (projected) $PG(2\mu-1, p^{3/2})$ and at most one (projected) $PG(3\mu-1, p)$.

Suppose that $(F', w') \cap \Delta$ contains a subgeometry $PG(2\mu-3, p^{3/2})$, then H_1 and H_2 share a subgeometry $PG(2\mu-1, p^{3/2})_1$ and $PG(2\mu-1, p^{3/2})_2$ with (F', w'), passing through this latter $PG(2\mu-3, p^{3/2})$.

Lemma 35 *The (projected) subgeometry* $PG(2\mu+1, p^{3/2}) = \langle PG(2\mu-1, p^{3/2})_1, PG(2\mu-1, p^{3/2})_2\rangle$ *is completely contained in* (F', w').

PROOF. Here the arguments of the proof of [5, Theorem 4.1] can be used.

If $\Delta \cap (F', w')$ contains a (projected) subgeometry $PG(3\mu-4, p)$, proceeding as in Section 5, it is possible to consider two hyperplanes H_1 and H_2 through Δ intersecting (F', w') in (projected) $PG(3\mu-1, p)_1$ and $PG(3\mu-1, p)_2$, only having points of weight one. These two subgeometries over $GF(p)$ define a $(3\mu+2)$-dimensional subgeometry over $GF(p)$ completely contained in (F', w').

Lemma 36 *The* $(3\mu+2)$*-dimensional (projected) subgeometry* $PG(3\mu+2, p) = \langle PG(3\mu-1, p)_1, PG(3\mu-1, p)_2\rangle$ *is completely contained in* (F', w').

The preceding lemmas now finish the proof of Theorem 33. There is no problem with the weights of the points of (F, w) in this description of (F, w) as a sum of μ-dimensional spaces, (projected) Baer subgeometries $PG(2\mu+1, p^{3/2})$, and of at most one (projected) subgeometry $PG(3\mu+2, p)$, since by induction, the weights are correct in the hyperplane intersections $(F, w) \cap H_i$, $i = 0, \ldots, p^3$, of (F, w) with the hyperplanes H_i through Δ.

7 Applications

The preceding classification results have many applications. They not only classify the corresponding linear codes meeting the Griesmer bound; they also can be used to obtain many results on substructures in finite incidence structures. For a detailed description of the use of $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers in finite incidence structures, we refer to [6].

We state explicitly the following results.

A *μ-spread* in $PG(N, q)$, $(\mu+1)|(N+1)$, is a set of $(q^{N+1}-1)/(q^{\mu+1}-1)$ μ-dimensional spaces partitioning the point set of $PG(N, q)$. A *partial μ-spread* $\mathcal{S}$ in $PG(N, q)$, $(\mu+1)|(N+1)$, is a set of pairwise disjoint μ-dimensional spaces. The *deficiency* δ of a partial μ-spread $\mathcal{S}$ in $PG(N, q)$, $(\mu+1)|(N+1)$, is the number $\delta = (q^{N+1}-1)/(q^{\mu+1}-1) - |\mathcal{S}|$. A *hole* of a partial μ-spread $\mathcal{S}$ is a point of $PG(N, q)$ not belonging to an element of $\mathcal{S}$. A *maximal* partial μ-spread $\mathcal{S}$ is a partial μ-spread not contained in a larger partial μ-spread.

Using a link [4] between minihypers and maximal partial μ-spreads in $PG(N, q)$, $(\mu+1)|(N+1)$, of deficiency $\delta < q$, the preceding classification results on the

$\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers imply the following result on maximal partial μ-spreads.

Theorem 37 (1) *A maximal partial μ-spread in $PG(N,p^3)$, $(\mu+1)|(N+1)$, $p = p_0^h$, $p \geq 11$, p_0 prime, $p_0 \geq 7$, $h \geq 1$ odd, of deficiency $0 < \delta \leq 2p^2 - 4p$, has deficiency $\delta = p^2+p+1$, and the set of holes is a (projected) subgeometry $PG(3\mu+2, p)$ of $PG(N,p^3)$.*

(2) *A maximal partial μ-spread in $PG(N,p^3)$, $(\mu+1)|(N+1)$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h \geq 2$ even, of deficiency $0 < \delta \leq 2p^2 - 4p$, has deficiency $\delta = r(p^{3/2}+1)+s(p^2+p+1)$, for a non-negative integer r and $s \in \{0,1\}$, and the set of holes is a union of r subgeometries $PG(2\mu+1, \sqrt{q})$ and s (projected) subgeometries $PG(3\mu+2, p)$ of $PG(N,p^3)$, which all are pairwise disjoint.*

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